TUTTE'S FIRST COLOUR-CYCLE CONJECTURE

by

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PREFACE

This thesis presents a proof of Conjecture I (see Section 35) of W. T. Tutte's paper "A contribution to the theory of chromatic polynomials" [15]. It is believed that this conjecture has not previously been resolved.

Sections 25 and 38 are original. The remainder of the thesis is a summary of the requisite graph theory and matroid theory. Most of the material in this summary is elementary. However, its inclusion makes the presentation self-contained.

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CONTENTS

Preface, ii
Acknowledgements, ii

CHAPTER 1  GRAPHS, 1

1  Graphs, 1
2  Valency, 2
3  Subgraphs, 3
4  Paths, 6
5  Connection, 9
6  Components, 10
7  Partitions, 11
8  \( \pi(G) \), 12
9  Cutsets, 16
10  Bonds, 20
11  Isthmuses, 26
12  Contractions, 26
13  Forests, 29
14  The principal forests of a graph, 33
15  The rank of a graph, 41
16  Polygons, 42
17  The polygons of a graph, 44

CHAPTER 2  MATROIDS AND GRAPHS, 50

18  Matroids, 51
19  Independent sets, 53
CHAPTER 2 MATROIDS AND GRAPHS (continued)
20 Rank, 55
21 Circuits, 56
22 Orthogonality, 58
23 The polygon matroid and the bond matroid of a graph, 62
24 Edmonds' theorem, 67
25 An application of Edmonds' theorem to graph theory, 79

CHAPTER 3 CYCLES AND COLOUR-CYCLES, 90
26 \( R, Z, \) and \( Z_n \), 90
27 Orientations, 91
28 Cycles, 92
29 Sums of cycles, 94
30 Circular paths and cycles, 95
31 Colour-cycles, 99
32 A one-one correspondence, 100
33 Existence of colour-cycles 1. Isthmuses, 102
34 Existence of colour-cycles 2. \( Z_n - Z - Z_{n+1} \), 104
35 Tutte's conjectures, 122
36 Existence of colour-cycles 3. Subgraphs, 124
37 Existence of colour-cycles 4. Polygons, 127
38 Existence of colour-cycles over \( Z_8 \), 137

References, 148
List of symbols, 152
Index of definitions, 155
CHAPTER 1 GRAPHS

(Sections 1 - 17)

This chapter is preliminary and consists of a summary of the graph theory needed in Chapters 2 and 3.

SECTION 1 GRAPHS

DEFINITION 1.1 [17, p. 3]

A graph $G$ consists of two finite sets $E(G)$ and $V(G)$ together with a function $I(G,-)$ which associates with each element $e$ of $E(G)$ a set $I(G,e)$ of one or two elements of $V(G)$.

The elements of $E(G)$ are called edges of $G$ and the elements of $V(G)$ are called vertices of $G$. The function $I(G,-)$ is called the incidence function of $G$.

If $e \in E(G)$ and $v \in I(G,e)$, then we say that $v$ is a G-end of $e$ and we also say that $v$ is G-incident with $e$ and that $e$ is G-incident with $v$. Unless confusion is likely to result, we use "end" instead of "G-end" and "incident" instead of "G-incident".

An edge of $G$ is a loop of $G$ if it has just one end and it is a link of $G$ if it has two ends.
DEFINITION 1.2 [17, p. 3]

A graph with no edges and no vertices is a **null graph**.

A graph with no edges and just one vertex \( v \) is a **vertex-graph** and is denoted by \([v]\) .

A graph consisting just of a loop and its single end is a **loop-graph**.

A graph consisting just of a link and its two ends is a **link-graph**.

SECTION 2 VALENCY

DEFINITION 2.1 [17, p. 4]

If a vertex \( v \) of a graph \( G \) is incident with \( m \) loops and \( n \) links of \( G \), then the valency \( \text{val}(G,v) \) of \( v \) in \( G \) is defined by the equation

\[
\text{val}(G,v) = 2m + n .
\]

DEFINITION 2.2 [17, p. 6]

The number of elements of a finite set \( S \) is denoted by \( |S| \).
**DEFINITION 2.3**

Let $G$ be a graph.

If $G$ is non-null, then

$$\sum_{v \in V(G)} \text{val}(G,v)$$

denotes the sum of the valencies of the vertices of $G$.

If $G$ is null, then we define

$$\sum_{v \in V(G)} \text{val}(G,v)$$

to be zero.

**PROPOSITION 2.4** [17, p. 5, 1.21]

If $G$ is a graph, then

$$\sum_{v \in V(G)} \text{val}(G,v) = 2|E(G)|$$

**SECTION 3  SUBGRAPHS**

**DEFINITION 3.1** [17, p. 5]

A graph $H$ is a subgraph of a graph $G$ if $E(H) \subseteq E(G)$, $V(H) \subseteq V(G)$, and, for every $e \in E(H)$, $I(H,e) = I(G,e)$. 
A subgraph of a graph \( G \) is a **proper subgraph** of \( G \) if it is not identical to \( G \).

If \( H \) is a subgraph (respectively proper subgraph) of a graph \( G \), then we say that \( H \) is **contained** (respectively **properly contained**) in \( G \) and write \( H \subseteq G \) (respectively \( H \subset G \)).

A subgraph \( H \) of a graph \( G \) is a **spanning subgraph** of \( G \) if \( V(H) = V(G) \).

**PROPOSITION 3.2** [17, p. 5, 1.32]

Let \( G \) be a graph.

If \( A \) is a subset of \( E(G) \) and \( X \) is a subset of \( V(G) \), then the following two conditions are equivalent.

(1) For every \( e \in A \), \( I(G,e) \subseteq X \).
(2) There is a subgraph \( H \) of \( G \) such that \( E(H) = A \) and \( V(H) = X \).

**DEFINITION 3.3** [17, p. 5]

Let \( G \) be a graph and let \( A \) be a subset of \( E(G) \).

By Proposition 3.2, there is a subgraph \( H \) of \( G \) such that \( E(H) = A \) and \( V(H) = V(G) \). We say that \( H \) is the **spanning subgraph** of \( G \) determined by \( A \) and denote it by \( G:A \).
There is also, by Proposition 3.2, a subgraph $K$ of $G$ such that $E(K) = A$ and $V(K)$ is the set of all vertices of $G$ which are ends of elements of $A$. We call $K$ the reduction of $G$ to $A$ and denote it by $G \cdot A$.

DEFINITION 3.4 \cite{17, p. 5}

Let $H_1, \ldots, H_p$ be subgraphs of a graph $G$.

By Proposition 3.2, there is a subgraph $H$ of $G$ such that

$$E(H) = \bigcup_{i=1}^{p} E(H_i)$$

and

$$V(H) = \bigcup_{i=1}^{p} V(H_i).$$

We call $H$ the union of the subgraphs $H_i$ and write

$$H = \bigcup_{i=1}^{p} H_i = H_1 \cup \cdots \cup H_p$$

DEFINITION 3.5

Let $\mathcal{H}$ be a collection of subgraphs of a graph $G$. 
If $\mathcal{H}$ is non-empty, then $\bigcup \mathcal{H}$ denotes the union of the members of $\mathcal{H}$.

If $\mathcal{H}$ is empty, then $\bigcup \mathcal{H}$ is defined to be the null subgraph of $G$.

**DEFINITION 3.6** \([17, \text{p. 6}]\)

Subgraphs $H_1, \ldots, H_p$ of a graph are said to be **disjoint** if no two distinct ones have a vertex in common and are said to be **edge-disjoint** if no two distinct ones have an edge in common.

By Proposition 3.2, disjoint subgraphs are edge-disjoint.

**SECTION 4** \ PATHS

**DEFINITION 4.1** \([17, \text{p. 28}]\)

A path $\alpha$ in a graph $G$ is a non-null finite sequence

$$\alpha = (x_0, a_1, x_1, \ldots, a_m, x_m)$$

which satisfies the following three conditions:

(1) for $i \in \{0, \ldots, m\}$, $x_i$ is a vertex of $G$;

(2) for $i \in \{1, \ldots, m\}$, $a_i$ is an edge of $G$; and

(3) for $i \in \{1, \ldots, m\}$, $I(G, a_i) = \{x_{i-1}, x_i\}$.
We say that the path $\alpha$ is a path from $x_0$ to $x_m$.

We write $E(\alpha)$ for the set of edges of $G$ which occur in $\alpha$ and $V(\alpha)$ for the set of vertices of $G$ which occur in $\alpha$. By Proposition 3.2, $E(\alpha)$ and $V(\alpha)$ determine a subgraph $G(\alpha)$ of $G$.

**DEFINITION 4.2** [17, p. 28]

Let

$$\alpha = (x_0, a_1, x_1, \ldots, a_m, x_m)$$

and

$$\beta = (y_0, b_1, y_1, \ldots, b_n, y_n)$$

be paths in a graph $G$ with $x_m = y_0$.

Then

$$(x_0, a_1, x_1, \ldots, a_m, x_m, b_1, y_1, \ldots, b_n, y_n)$$

is a path in $G$. We denote this path by $\alpha\beta$ and call it the product of the paths $\alpha$ and $\beta$.

**DEFINITION 4.3** [17, p. 30]

A path

$$(x_0, a_1, x_1, \ldots, a_m, x_m)$$

in a graph is simple if it satisfies the following
condition:

if $0 < i < j < m$, then $x_i \neq x_j$.

**PROPOSITION 4.4**

Let $v$ and $w$ be vertices of a graph $G$.

Then: there is a path from $v$ to $w$ in $G$ if and only if there is a simple path from $v$ to $w$ in $G$.

**PROOF**

The "if" part is trivial.

The "only if" part:

Suppose that there is a path from $v$ to $w$ in $G$. Then there is a path

$$
\alpha = (x_0, a_1, x_1, \cdots, a_m, x_m)
$$

from $v$ to $w$ in $G$ which has the smallest number of terms consistent with the property of being a path from $v$ to $w$ in $G$.

Suppose that $\alpha$ is not simple. Then there are integers $i$ and $j$ such that $0 < i < j < m$ and $x_i = x_j$. Let $\alpha'$ be the subsequence of $\alpha$ which has all the terms of $\alpha$ except those from $x_i$ to $a_j$. 
Then $\alpha'$ is a path from $v$ to $w$ in $G$ which has fewer terms than $\alpha$ has. Thus the choice of $\alpha$ is contradicted.

We conclude that $\alpha$ is simple.

The "only if" part of the proposition follows.

**DEFINITION 4.5** [17, p. 30]

A path

$$\alpha = (x_0, a_1, x_1, \cdots, a_m, x_m)$$

in a graph $G$ is a **circular path** if it satisfies the following four conditions:

1. $x_0 = x_m$;
2. if $0 \leq i < j \leq m - 1$, then $x_i \neq x_j$;
3. at least one edge of $G$ is a term of $\alpha$; and
4. if $1 \leq i < j \leq m$, then $a_i \neq a_j$.

**SECTION 5  CONNECTION**

**DEFINITION 5.1** [9, p. 13]

A graph $G$ is **connected** if it satisfies the following condition:

if $v$ and $w$ are vertices of $G$, then there is a path from $v$ to $w$ in $G$. 
PROPOSITION 5.2  [17, p. 13]

Every null graph, vertex-graph, link-graph, and loop-graph is connected.

SECTION 6  COMPONENTS

DEFINITION 6.1  [17, p. 13, 2.51]

A maximal non-null connected subgraph of a graph $G$ is called a component of $G$.

We denote the number of components of a graph $G$ by $c(G)$.

PROPOSITION 6.2  [17, p. 13, 2.53]

The components of a non-null graph $G$ are disjoint and their union is equal to $G$.

PROPOSITION 6.3  [15, p. 81]

Let $v$ and $w$ be vertices of a graph $G$.

Then: $v$ and $w$ belong to the same component of $G$ if and only if there is a path from $v$ to $w$ in $G$. 
SECTION 7  PARTITIONS

DEFINITION 7.1  [2, p. 15, Ex. 9]

A partition \( \rho \) of a non-empty set \( S \) is a collection of disjoint non-empty subsets of \( S \) whose union is \( S \).

We call the members of \( \rho \) the parts of \( \rho \).

DEFINITION 7.2  [2, p. 15, Ex. 9]

Let \( \rho \) and \( \tau \) be partitions of a non-empty set.

We say that \( \rho \) is a refinement of \( \tau \) and write \( \rho \preceq \tau \) if each part of \( \rho \) is contained in some part of \( \tau \).

We say that \( \rho \) is a proper refinement of \( \tau \) and write \( \rho \prec \tau \) if \( \rho \preceq \tau \) and \( \rho \neq \tau \).

PROPOSITION 7.3  [2, p. 15, Ex. 9 and p. 1, P2]

If \( \rho \) and \( \tau \) are partitions of a non-empty set, then \( \rho = \tau \) if and only if \( \rho \preceq \tau \) and \( \tau \preceq \rho \).
SECTION 8 \( \pi(G) \)

DEFINITION 8.1

If \( G \) is a non-null graph, then we define \( \pi(G) \) by the equation

\[
\pi(G) = \{ V(K) : K \text{ a component of } G \}.
\]

PROPOSITION 8.2 [15, p. 81]

If \( G \) is a non-null graph, then \( \pi(G) \) is a partition of \( V(G) \).

PROOF

This follows from Proposition 6.2.

PROPOSITION 8.3 [15, p. 81]

Let \( v \) and \( w \) be vertices of a graph \( G \).

Then: \( v \) and \( w \) belong to the same part of \( \pi(G) \) if and only if there is a path from \( v \) to \( w \) in \( G \).

PROOF

This follows from Proposition 6.3.
PROPOSITION 8.4

Let $K_1, \ldots, K_p$ be the components of a graph $G$ and let $A$ be a subset of $E(G)$.

Then

$$\pi(G:(E(G) - A)) = \bigcup_{i=1}^{p} \pi(K_i:(E(K_i) - A))$$

and

$$c(G:(E(G) - A)) = \sum_{i=1}^{p} c(K_i:(E(K_i) - A)).$$

PROOF

Since $G$ has at least one component, $G$ is non-null.

By definition

$$V(G:(E(G) - A)) = V(G)$$

and, for each $i \in \{1, \ldots, p\}$,

$$V(K_i:(E(K_i) - A)) = V(K_i).$$

Thus, since $G$ is non-null, $G:(E(G) - A)$ is non-null.

Also, for each $i \in \{1, \ldots, p\}$, $K_i$ is non-null so $K_i:(E(K_i) - A)$ is non-null.
Let

\[ \rho = \bigcup_{i=1}^{p} \pi(K_i:(E(K_i) - A)) \]

By Proposition 8.2,

\[ \pi(G) = \{V(K_1), \ldots, V(K_p)\} \]

is a partition of

\[ V(G) = V(G:(E(G) - A)) \]

and, for each \( i \in \{1, \ldots, p\} \),

\[ \pi(K_i:(E(K_i) - A)) \]

is a partition of

\[ V(K_i:(E(K_i) - A)) = V(K_i) \]

Thus \( \rho \) is a partition of \( V(G:(E(G) - A)) \).

We show that \( \rho = \pi(G:(E(G) - A)) \).

Suppose that \( v \) and \( w \) belong to the same part of \( \rho \). Then there is an \( i \in \{1, \ldots, p\} \) such that \( v \) and \( w \) belong to the same part of \( \pi(K_i:(E(K_i) - A)) \). Thus, by Proposition 8.3, there is a path \( \alpha \) from \( v \) to \( w \) in \( K_i:(E(K_i) - A) \).

But \( K_i:(E(K_i) - A) \) is a subgraph of \( G:(E(G) - A) \). Thus \( \alpha \) is a path from \( v \) to \( w \) in \( G:(E(G) - A) \). So, by Proposition 8.3, \( v \) and \( w \) belong to the same
part of \( \pi(G:(E(G) - A)) \). Thus \( \rho \leq \pi(G:(E(G) - A)) \).

On the other hand, suppose that \( v \) and \( w \) belong to the same part of \( \pi(G:(E(G) - A)) \). Then, by Proposition 8.3, there is a path \( \alpha \) from \( v \) to \( w \) in \( G:(E(G) - A) \). Since \( G:(E(G) - A) \) is a subgraph of \( G \), it follows that \( \alpha \) is a path from \( v \) to \( w \) in some component \( K_i \) of \( G \). Moreover, no element of \( A \) is a term of \( \alpha \). Thus \( \alpha \) is a path from \( v \) to \( w \) in \( K_i:(E(K_i) - A) \). So, by Proposition 8.3, \( v \) and \( w \) belong to the same part of \( \pi(K_i:(E(K_i) - A)) \). So \( v \) and \( w \) belong to the same part of \( \rho \). Thus \( \pi(G:(E(G) - A)) \leq \rho \).

Thus, by Proposition 7.3, \( \rho = \pi(G:(E(G) - A)) \).

Moreover,

\[
c(G:(E(G) - A)) = |\pi(G:(E(G) - A))| = |
\]

\[
\rho
\]

\[
= \sum_{i=1}^{p} |\pi(K_i:(E(K_i) - A))|
\]

\[
= \sum_{i=1}^{p} c(K_i:(E(K_i) - A)) .
\]

**COROLLARY 8.5**

Let \( G \) be a non-null graph.

If \( A \) is a subset of \( E(G) \), then
\[ \pi(G:(E(G) - A)) \leq \pi(G) \]

and

\[ c(G:(E(G) - A)) \geq c(G) . \]

SECTION 9  CUTSETS

DEFINITION 9.1  [9, p. 38], [21, p. 503]

Let \( G \) be a graph.

A subset \( A \) of \( E(G) \) is a cutset of \( G \) if the number of components of \( G:(E(G) - A) \) is greater than the number of components of \( G \).

PROPOSITION 9.2

Let \( G \) be a non-null graph.

If \( A \) is a subset of \( E(G) \), then the following four conditions are equivalent.

(1) \( A \) is a cutset of \( G \).
(2) \( c(G:(E(G) - A)) > c(G) \).
(3) \( \pi(G:(E(G) - A)) < \pi(G) \).
(4) For some pair of vertices \( v \) and \( w \) of \( G \), there is a path from \( v \) to \( w \) in \( G \) but no path from \( v \) to \( w \) in \( G:(E(G) - A) \).
PROOF

(1) $\iff$ (2): By Definition 9.1.

(2) $\iff$ (3): By Definition 8.1 and Corollary 8.5.

(3) $\iff$ (4): By Proposition 8.3 and Corollary 8.5.

PROPOSITION 9.3

Let $G$ be a non-null graph.

If $A$ is a subset of $E(G)$, then the following four conditions are equivalent.

(1) $A$ is not a cutset of $G$.
(2) $c(G:(E(G) - A)) = c(G)$.
(3) $\pi(G:(E(G) - A)) = \pi(G)$.
(4) For every pair of vertices $v$ and $w$ of $G$, if there is a path from $v$ to $w$ in $G$, then there is a path from $v$ to $w$ in $G:(E(G) - A)$.

PROOF

By Proposition 9.2 and Corollary 8.5.

PROPOSITION 9.4

Let $G$ be a graph.

Then: a subset $A$ of $E(G)$ is a cutset of $G$ if and only if there is a component $K$ of $G$ such
that \( A \cap E(K) \) is a cutset of \( K \).

**PROOF**

A null graph has no cutset and no component so we may assume that \( G \) is non-null.

Let \( K_1, \ldots, K_p \) be the components of \( G \) and let \( A \) be a subset of \( E(G) \). Then, by Proposition 8.4,

\[
\frac{c(G:(E(G) - A))}{p} = \sum_{i=1}^{p} \frac{c(K_i:(E(K_i) - A))}{1}.
\]

Thus:

\[
c(G:(E(G) - A)) > c(G) = p
\]

if and only if, for some \( i \in \{1, \ldots, p\} \),

\[
c(K_i:(E(K_i) - A)) > 1 = c(K_i).
\]

The proposition follows.

**PROPOSITION 9.5**

If \( H \) is a subgraph of a connected graph \( G \) and \( \emptyset \subset V(H) \subset V(G) \), then the set \( L \) of links of \( G \) with just one end in \( V(H) \) is a cutset of \( G \).
PROOF

Let $H$ be a subgraph of a connected graph $G$ such that $\emptyset \subseteq V(H) \subseteq V(G)$.

Let $L$ be the set of links of $G$ with just one end in $V(H)$.

Since $V(H) \neq \emptyset$, there is a vertex $v \in V(H)$.

Since $V(G) - V(H) \neq \emptyset$, there is a vertex $w \in V(G) - V(H)$. Moreover, since $G$ is connected, there is a path from $v$ to $w$ in $G$.

Let $\alpha = (x_0, a_1, x_1, \ldots, a_m, x_m)$ be a path from $v$ to $w$ in $G$. Then there is an $i \in \{1, \ldots, m\}$ such that $x_{i-1} \in V(H)$ and $x_i \in V(G) - V(H)$. Thus $a_i$ is a link of $G$ with just one end in $V(H)$. So $a_i \in L$. It follows that $\alpha$ is not a path in $G: (E(G) - L)$.

But, since $G: (E(G) - L) \subseteq G$, each path in $G: (E(G) - L)$ is a path in $G$.

It follows that there is no path from $v$ to $w$ in $G: (E(G) - L)$.

Thus, by Proposition 9.2, $L$ is a cutset of $G$. 
SECTION 10  BONDS

DEFINITION 10.1  [16, p. 5]

A bond of a graph $G$ is a minimal cutset of $G$.

PROPOSITION 10.2

Let $G$ be a graph.

Then: a subset $B$ of $E(G)$ is a bond of $G$ if and only if $B$ is a bond of some component of $G$.

PROOF

A null graph has no bond and no component so we may assume that $G$ is non-null.

Suppose that $A$ is a cutset of $G$. Then, by Proposition 9.4, $A$ contains a cutset of some component $K$ of $G$. Thus $A$ contains a bond $B$ of $K$. By Proposition 9.4, $B$ is a cutset of $G$. Since $B \subseteq A$, it follows that $A$ is a bond of $G$ only if $A = B$, i.e. only if $A$ is a bond of $K$.

Conversely, suppose that $A$ is a cutset of some component $K$ of $G$. Then, by Proposition 9.4, $A$ is a cutset of $G$. So $A$ contains a bond $B$ of $G$. Then, by the part of the proposition which has already been proved, $B$ is a bond of some component of $G$. 
Moreover, $B \subseteq A \subseteq E(K)$ and so, by Proposition 6.2, $B$ cannot be a bond of a component of $G$ other than $K$. Thus $B$ is a bond of $K$. Since $B \subseteq A$, it follows that $A$ is a bond of $K$ only if $A = B$, i.e. only if $A$ is a bond of $G$.

**Lemma 10.3**

Let $G$ be a graph and let $A$ be a subset of $E(G)$.

If $e$ is a link of $G$ whose two ends belong to different components of $G:(E(G) - A)$, then $e \in A$.

**Proof**

Suppose that $e \in E(G) - A$. Then $G \cdot \{e\} \subseteq G:(E(G) - A)$. Thus, since $G \cdot \{e\}$ is connected, $G \cdot \{e\}$ is contained in some component $K$ of $G:(E(G) - A)$. So $I(G,e) \subseteq V(K)$.

**Proposition 10.4** [16, p. 5]

Let $b$ be an edge of a graph $G$ and let $B$ be a bond of $G$.

Then: $b \in B$ if and only if $b$ is a link of $G$ whose two ends belong to different components of $G:(E(G) - B)$.
PROOF

The "if" part of the proposition follows directly from Lemma 10.3.

The "only if" part:

Since $B$ is a bond of $G$, it follows from Proposition 9.2 that

$$\pi(G:(E(G) - B)) < \pi(G).$$

Thus there is a component $K$ of $G:(E(G) - B)$ and a component $M$ of $G$ such that $V(K) \subset V(M)$. Since $K$ is a component of $G:(E(G) - B)$, $V(K) \neq \emptyset$.

Since $K$ is a non-null connected subgraph of $G$, $K$ is a subgraph of some component of $G$. Moreover, since $V(K) \subset V(M)$, it follows from Proposition 6.2 that $K$ is a subgraph of $M$. Thus $K$ is a subgraph of the connected graph $M$ and $\emptyset \subset V(K) \subset V(M)$.

Let $L$ be the set of links of $M$ with just one end in $V(K)$.

Then, by Proposition 9.5, $L$ is a cutset of $M$. So, by Proposition 9.4, $L$ is a cutset of $G$.

Moreover, each element of $L$ is a link of $G$ whose two ends belong to different components of $G:(E(G) - B)$. So, by Lemma 10.3, $L \subset B$.

Therefore, since $B$ is a minimal cutset of $G$,
Thus, if \( b \in B \), then \( b \in L \) and so \( b \) is a link of \( G \) whose two ends belong to different components of \( G:(E(G) - B) \).

**PROPOSITION 10.5**

If \( B \) is a bond of a graph \( G \), then the number of components of \( G:(E(G) - B) \) is just one greater than the number of components of \( G \).

**PROOF**

Let \( G \) be a graph.

Since a null graph has no bond, we may assume that \( G \) is non-null.

Let \( B \) be a bond of \( G \). Then, by Proposition 10.2, \( B \) is a bond of some component of \( G \).

Let \( K_1, \cdots, K_p \) be the components of \( G \). We may assume that \( B \) is a bond of \( K_1 \). Then \( B \subseteq E(K_1) \). So, by Proposition 6.2,

for \( i \in \{2, \cdots, p\} \), \( K_i:(E(K_i) - B) = K_i \).

Moreover, by Proposition 8.4,

\[
c(G:(E(G) - B)) = \sum_{i = 1}^{p} c(K_i:(E(K_i) - B)) .
\]
Thus
\[ c(G : (E(G) - B)) = c(K_1 : (E(K_1) - B)) + (p - 1) . \]

So we need only show that
\[ c(K_1 : (E(K_1) - B)) = 2 . \]

Since \( B \) is a bond of \( K_1 \) and since \( K_1 \) is a component,
\[ c(K_1 : (E(K_1) - B)) > c(K_1) = 1 . \]

Let \( b \in B \) and let
\[ H = K_1 : (E(K_1) - (B - \{b\})) . \]

Since \( B \) is a minimal cutset of \( K_1 \), \( B - \{b\} \) is not a cutset of \( K_1 \). Thus, by Proposition 9.3,
\[ c(H) = c(K_1) = 1 . \]

So \( H \) is connected. Since
\[ E(H) - \{b\} = E(K_1) - B , \]
it follows that
\[ H : (E(H) - \{b\}) = K_1 : (E(K_1) - B) . \]

Thus
\[ c(H : (E(H) - \{b\})) = c(K_1 : (E(K_1) - B)) . \]

Therefore
\[ c(H:(E(H) - \{b\})) > c(H) \]

So \( \{b\} \) is a bond of \( H \). Thus, by Proposition 10.4, \( b \) is a link of \( H \).

Let \( y_1 \) and \( y_2 \) be the ends of \( b \). Then \( y_1 \) and \( y_2 \) are vertices of \( H \).

Let \( v \) be a vertex of \( H \). Then, since \( H \) is connected, there is a path from \( v \) to \( y_1 \) in \( H \).

Thus, by Proposition 4.4, there is a simple path \( \alpha \) from \( v \) to \( y_1 \) in \( H \). If the path \( \alpha \) includes \( b \), then a subsequence of \( \alpha \) is a path from \( v \) to \( y_2 \) in \( H \) which does not include \( b \). Thus, either there is a path from \( v \) to \( y_1 \) in \( H:(E(H) - \{b\}) \) or there is a path from \( v \) to \( y_2 \) in \( H:(E(H) - \{b\}) \). So, by Proposition 6.3, either \( v \) and \( y_1 \) belong to the same component of \( H:(E(H) - \{b\}) \) or \( v \) and \( y_2 \) belong to the same component of \( H:(E(H) - \{b\}) \).

Thus it follows from Proposition 6.2 that

\[ c(H:(E(H) - \{b\})) \leq 2 \]

Therefore

\[ c(H:(E(H) - \{b\})) = 2 \]

So

\[ c(K_1:(E(K_1) - B)) = 2 \]
SECTION 11  ISTHMUSES

DEFINITION 11.1  [15, p. 82], [17, p. 17]

An edge $e$ of a graph $G$ is an isthmus of $G$ if $\{e\}$ is a bond of $G$.

PROPOSITION 11.2

If a component of a graph $G$ has an isthmus, then $G$ has an isthmus.

PROOF

This follows directly from Proposition 10.2.

SECTION 12  CONTRACTIONS

DEFINITION 12.1  [16, p. 6]

Let $G$ be a graph and let $A$ be a subset of $E(G)$.

The contraction $G$ ctr $A$ of $G$ to $A$ is a graph whose edges are the elements of $A$ and whose vertices are the components of $G: (E(G) - A)$. The incidence function of $G$ ctr $A$ is defined as follows:

for every $e \in A$ and each component $K$ of $G: (E(G) - A)$, $K$ is a $(G$ ctr $A)$-end of $e$ if and only if $V(K)$ contains a $G$-end of $e$. 
PROPOSITION 12.2

If a contraction of a graph $G$ has an isthmus, then $G$ has an isthmus.

PROOF

Let $G$ be a graph.

Since any contraction of a null graph is a null graph and a null graph has no isthmus, we may assume that $G$ is non-null.

Let $A$ be a subset of $E(G)$.

Suppose that $e$ is an edge of $G$ ctr $A$ which is not an isthmus of $G$.

Let

$$(K_0, a_1, K_1, \ldots, a_m, K_m)$$

be a path in $G$ ctr $A$.

Suppose that $i \in \{1, \ldots, m\}$. Then, since $K_{i-1}$ is a $(G$ ctr $A)$-end of $a_i$, $V(K_{i-1})$ includes a $G$-end $v_{i-1}$ of $a_i$. Similarly, $V(K_i)$ includes a $G$-end $w_i$ of $a_i$. Thus the sequence $\alpha_i = (v_{i-1}, a_i, w_i)$ is a path in $G$.

Suppose that $i \in \{1, \ldots, m-1\}$. Then $w_i$ and $v_i$ belong to $V(K_i)$. So, since $K_i$ is a component of $G:(E(G) - A)$, there is a path $\beta_i$ from $w_i$ to $v_i$.
\( w_i \) to \( v_i \) in \( G: (E(G) - A) \). Since \( G: (E(G) - A) \) is a subgraph of \( G \), \( \beta_i \) is a path from \( w_i \) to \( v_i \) in \( G \).

There is thus a path

\[ \alpha_1 \beta_1 \alpha_2 \beta_2 \cdots \alpha_{m-1} \beta_{m-1} \alpha_m \]

from \( v_0 \) to \( w_m \) in \( G \).

Since \( e \) is not an isthmus of \( G \), it follows from Proposition 9.3 that there is a path

\[ (z_0, c_1, z_1, \cdots, c_r, z_r) \]

from \( v_0 \) to \( w_m \) in \( G: (E(G) - \{e\}) \).

Let

\[ (c_{s_1}, \cdots, c_{s_t}) \]

be the subsequence of

\[ (c_1, \cdots, c_r) \]

whose terms belong to \( A \).

Then there are components \( M_0, \cdots, M_t \) of \( G: (E(G) - A) \) such that \( M_0 = K_0 \), \( M_t = K_m \), and, for \( i \in \{1, \cdots, t\} \), \( M_{i-1} \) and \( M_i \) are the \((G \text{ cont } A)\)-ends of \( c_{s_i} \). It follows that the sequence

\[ \gamma = (M_0, c_{s_1}, M_1, \cdots, c_{s_t}, M_t) \]
is a path from $K_0$ to $K_m$ in $G\text{ctr}A$ in which $e$ does not occur as a term. Thus $\gamma$ is a path from $K_0$ to $K_m$ in $(G\text{ctr}A):(E(G\text{ctr}A) - \{e\})$.

Using Proposition 9.3, we conclude that $e$ is not an isthmus of $G\text{ctr}A$.

The proposition follows.

SECTION 13 FORESTS

DEFINITION 13.1 [21, p. 503], [17, p. 17]

A graph $F$ is a forest if there is no circular path in $F$.

A connected forest is called a tree.

PROPOSITION 13.2 [17, p. 19, 3.31]

Every null graph, vertex-graph, and link-graph is a tree.

PROPOSITION 13.3 [17, p. 18, 3.12]

Every subgraph of a forest is a forest.

PROPOSITION 13.4 [17, p. 17]

Every edge of a forest $F$ is an isthmus of $F$. 
PROOF

Let $G$ be a graph with an edge $e$ which is not an isthmus of $G$.

If $e$ is a loop of $G$ and $v$ is the end of $e$, then $(v, e, v)$ is a circular path in $G$.

On the other hand, suppose that $e$ is a link of $G$ and that $v$ and $w$ are the ends of $e$. Then $(v, e, w)$ is a path from $v$ to $w$ in $G$. Thus, by Propositions 9.3 and 4.4, there is a simple path $\alpha$ from $v$ to $w$ in $G$:$(E(G) - \{e\})$. Let $\beta$ be the path $(w, e, v)$ in $G$. Since $e$ is not a term of $\alpha$ and since $\alpha$ is simple, it follows that $\alpha\beta$ is a circular path in $G$.

Thus, in either case, there is a circular path in $G$.

So $G$ is not a forest.

PROPOSITION 13.5 [17, p. 18, 3.23]

If a graph $F$ is a forest, then

$$|E(F)| = |V(F)| - c(F).$$

PROOF

Let $F$ be a forest.
If $E(F)$ is empty, then $c(F) = |V(F)|$ and so the proposition holds.

Assume that $E(F)$ is non-empty.

Let

$$E(F) = \{e_1, \ldots, e_m\}.$$ 

We define a sequence $(F_0, \ldots, F_m)$ of subgraphs of $F$ as follows: $F_0 = F$ and, for $i \in \{1, \ldots, m\}$, $F_i = F_{i-1} : (E(F_{i-1}) - \{e_i\})$. By Proposition 13.3, each $F_i$ is a forest.

Suppose that $i \in \{1, \ldots, m\}$. Then, by Proposition 13.4, $e_i$ is an isthmus of $F_{i-1}$. Thus, by Proposition 10.5, $c(F_i) = c(F_{i-1}) + 1$.

It follows that

$$c(F_m) = c(F_0) + m.$$ 

But

$$c(F_m) = |V(F_m)| = |V(F)|,$$

$$c(F_0) = c(F),$$ and

$$m = |E(F)|.$$ 

Thus

$$|V(F)| = c(F) + |E(F)|.$$
PROPOSITION 13.6 [17, p. 19, 3.35]

Every forest with at least one edge has a vertex of valency one.

PROOF

Let $F$ be a forest with at least one edge.

Let $H = F \cdot E(F)$. Then, for each $v \in V(H)$, $val(F,v) = val(H,v) \geq 1$. Moreover, $H$ is non-null and so $c(H) \geq 1$.

By Proposition 13.3, $H$ is a forest. Thus, by Proposition 13.5,

$$|E(H)| = |V(H)| - c(H).$$

By Proposition 2.4,

$$\sum_{v \in V(H)} val(H,v) = 2|E(H)|.$$

Therefore

$$\sum_{v \in V(H)} val(H,v) = 2(|V(H)| - c(H)).$$

Since, for each $v \in V(H)$, $val(H,v) \geq 1$ and since $c(H) \geq 1$, it follows that some vertex $w$ of $H$ must have valency one. But then $w$ is a vertex of $F$ and $val(F,w) = val(H,w) = 1$. So $F$ has a vertex of valency one.
SECTION 14  THE PRINCIPAL FORESTS OF A GRAPH

DEFINITION 14.1  [16, p. 6], [21, p. 503]

Let $G$ be a graph.

A forest of $G$ is a subgraph of $G$ which is a forest.

A spanning tree of $G$ is a spanning subgraph of $G$ which is a tree.

A principal forest of $G$ is a forest of $G$ which contains a spanning tree of each component of $G$.

PROPOSITION 14.2

A forest $F$ of a graph $G$ is a principal forest of $G$ if and only if $V(F) = V(G)$ and $c(F) = c(G)$.

PROOF

The "only if" part of the proposition follows easily from the definition of a principal forest together with Proposition 8.2 and Corollary 8.5.

The "if" part:

Let $F$ be a forest of a graph $G$.

If $G$ is null, then $F$ is a principal forest of $G$. Thus we may assume that $G$ is non-null.
Suppose that $V(F) = V(G)$ and $c(F) = c(G)$.

Let $K$ be a component of $G$. Then $K:(E(K) \cap E(F))$ is a spanning subgraph of $K$.

Moreover, since

$$E(K:(E(K) \cap E(F))) = E(K) \cap E(F) \subseteq E(F)$$

and

$$V(K:(E(K) \cap E(F))) = V(K) \subseteq V(G) = V(F),$$

it follows that $K:(E(K) \cap E(F))$ is a subgraph of $F$.

Thus, by Proposition 13.3, $K:(E(K) \cap E(F))$ is a forest.

Since $V(F) = V(G)$,

$$F = G:E(F) = G:(E(G) - (E(G) - E(F))).$$

Thus, since $c(F) = c(G)$, it follows from Proposition 9.3 that $E(G) - E(F)$ is not a cutset of $G$. So, by Proposition 9.4, $(E(G) - E(F)) \cap E(K)$ is not a cutset of $K$. Thus, by Proposition 9.3,

$$c(K:(E(K) \cap E(F))) = c(K).$$

So $K:(E(K) \cap E(F))$ is connected.

Thus $K:(E(K) \cap E(F))$ is a spanning tree of $K$ contained in $F$.

It follows that $F$ is a principal forest of $G$. 
PROPOSITION 14.3

If $F$ is a forest of a graph $G$, then there is a principal forest of $G$ which contains $F$.

PROOF

Let $G$ be a graph.

Since $E(G)$ and $V(G)$ are finite, we need only show that, if $F$ is a non-principal forest of $G$, then $F$ is properly contained in some forest of $G$.

Suppose that $F$ is a non-principal forest of $G$.

Let

$$H = G:E(F) = G:(E(G) - (E(G) - E(F)))$$

Then $E(F) = E(H)$ and $V(F) \subseteq V(G) = V(H)$. Thus $F \subseteq H$. Moreover, $H \cdot E(H) \subseteq F$. So, by Proposition 13.3, $H \cdot E(H)$ is a forest. Thus $H$ is a forest of $G$. Further, by Corollary 8.5, $c(H) \geq c(G)$.

If $H$ is a principal forest of $G$, then $F \neq H$ and so $F \subset H$.

On the other hand, suppose that $H$ is a non-principal forest of $G$. Then, by Proposition 14.2, $c(H) \neq c(G)$. Thus $c(H) > c(G)$. So, by Proposition 9.2, for some vertices $v$ and $w$ of $G$, there is a path $\alpha$ from $v$ to $w$ in $G$ but no path
from \( v \) to \( w \) in \( H \). Let

\[
\alpha = (x_0, a_1, x_1, \ldots, a_m, x_m).
\]

Then, since there is no path from \( v \) to \( w \) in \( H \), it follows that, for some \( i \in \{1, \ldots, m\} \), there is no path from \( x_{i-1} \) to \( x_i \) in \( H \).

Let

\[
K = H \cup G \cdot \{a_i\}.
\]

Then \( F \subseteq H \subseteq K \). We show that \( K \) is a forest.

Suppose that there is a circular path \( \beta \) in \( K \). Then \( V(\beta) \subseteq V(H) \) and \( E(\beta) \subseteq E(H) \cup \{a_i\} \). Thus, since there is no circular path in \( H \), \( a_i \) is a term of \( \beta \). But then there is a path from \( x_{i-1} \) to \( x_i \) in \( K : (E(K)\setminus\{a_i\}) = H \). Thus the choice of \( i \) is contradicted.

We conclude that there is no circular path in \( K \).

So \( K \) is a forest.

**COROLLARY 14.4**  
[17, p. 21, 3.41]

Every graph contains a principal forest.
PROPOSITION 14.5 \[16, \text{p. 3, 2.11}\]

If $F$ is a principal forest of a graph $G$ and $e \in (E(G) - E(F))$, then $e$ is a term of a circular path in $F \cup (G \cdot \{e\})$.

PROOF

Suppose that $F$ is a principal forest of a graph $G$ and that $e \in (E(G) - E(F))$.

If $e$ is a loop of $G$ and $v$ is the end of $e$, then $e$ is a term of the circular path $(v, e, v)$ in $F \cup (G \cdot \{e\})$.

On the other hand, suppose that $e$ is a link of $G$ and that $v$ and $w$ are the ends of $e$.

Let $K$ be the component of $G$ which contains $e$ and let $T$ be the spanning tree of $K$ contained in $F$. Then $v, w \in V(H) = V(T)$ and $T$ is connected. Thus there is a path from $v$ to $w$ in $T$. So, by Proposition 4.4, there is a simple path $\alpha$ from $v$ to $w$ in $T$. Since $e \notin E(T)$, $e$ is not a term of $\alpha$.

Let $\beta$ be the path $(w, e, v)$ in $G \cdot \{e\}$. Then $\alpha \beta$ is a circular path in $T \cup (G \cdot \{e\}) \subseteq F \cup (G \cdot \{e\})$. 

and e is a term of α. Thus e is a term of a circular path in F ∪ (G·{e}).

**COROLLARY 14.6** [21, p. 503, G3]

No principal forest of a graph G is properly contained in another principal forest of G.

**PROPOSITION 14.7** [21, p. 504, G4]

If F₁ and F₂ are principal forests of a graph G and e₁ ∈ E(F₁), then there is an e₂ ∈ E(F₂) with the property that

\[ G:((E(F₁) - \{e₁\}) ∪ \{e₂\}) \]

is a principal forest of G.

**PROOF**

Suppose that F₁ and F₂ are principal forests of a graph G and that e₁ ∈ E(F₁).

Then, by Proposition 14.2,

\[ V(F₁) = V(F₂) = V(G) \]

and

\[ c(F₁) = c(F₂) = c(G). \]
Let
\[ H = F_1 : (E(F_1) - \{e_1\}) = G : (E(F_1) - \{e_1\}) . \]

Then
\[ V(H) = V(F_1) = V(G) . \]

By Proposition 13.4, \( e_1 \) is an isthmus of \( F_1 \).
Thus, using Proposition 10.3,
\[ c(H) = c(F_1) + 1 = c(G) + 1 . \]

Let
\[ M = H \cup F_2 = G : (E(F_1) - \{e_1\}) \cup E(F_2) . \]

Then
\[ V(M) = V(G) . \]

Moreover, using Corollary 8.5,
\[ c(G) \leq c(M) \leq c(F_2) = c(G) . \]

Thus
\[ c(M) = c(G) . \]

Since \( H \subseteq F_1 \), it follows from Proposition 13.3 that \( H \) is a forest. Thus, by Proposition 14.3, \( H \) is contained in a principal forest \( K \) of \( M \). Using Proposition 14.2,
\[ V(K) = V(M) = V(G) \]
and 
\[ c(K) = c(M) = c(G). \]

Thus, by Proposition 14.2, \( K \) is a principal forest of \( G \).

By Proposition 13.5,
\[ |E(K)| = |V(K)| - c(K) \]
and
\[ |E(H)| = |V(H)| - c(H). \]

Thus
\[ |E(K)| = |V(K)| - c(K) = |V(G)| - c(G) \]
\[ = |V(H)| - (c(H) - 1) = |E(H)| + 1. \]

So there is an \( e_2 \in E(F_2) \) such that
\[ E(K) = (E(F_1) - \{e_1\}) \cup \{e_2\}. \]

Since \( V(K) = V(G) \), we have
\[ K = G:((E(F_1) - \{e_1\}) \cup \{e_2\}). \]

The proposition follows.
SECTION 15  THE RANK OF A GRAPH

DEFINITION 15.1 [21, p. 504]

The rank $r(G)$ of a graph $G$ is defined by the equation

$$r(G) = |V(G)| - c(G).$$

PROPOSITION 15.2 [17, p. 18, 3.23]

A graph $F$ is a forest if and only if $r(F) = |E(F)|$.

PROOF

The "only if" part of the proposition follows directly from Proposition 13.5.

The "if" part:

Suppose that $G$ is a graph which is not a forest. By Corollary 14.4, $G$ contains a principal forest $F$. By Proposition 14.2, $V(F) = V(G)$ and $c(F) = c(G)$. Since $G$ is not a forest, $F \subset G$. Thus, since $V(F) = V(G)$, it follows that $E(F) \subset E(G)$. So $|E(F)| < |E(G)|$. By Proposition 13.5, $|E(F)| = |V(F)| - c(F)$.

Therefore
\[ r(G) = |V(G)| - c(G) = |V(F)| - c(F) \]

\[ = |E(F)| < |E(G)|. \]

**Proposition 15.3** [21, p. 504]

If \( F \) is a principal forest of a graph \( G \), then \( r(F) = r(G) \).

**Proof**

This follows directly from Proposition 14.2.

**Section 16** Polygons

**Definition 16.1** [17, p. 30, 4.32 and p. 31, 4.36]

A graph \( P \) is a **polygon** if there is a circular path \( \alpha \) in \( P \) such that \( P = G(\alpha) \).

**Proposition 16.2** [17, p. 26]

Every loop-graph is a polygon.

**Proposition 16.3** [17, p. 27, 3.75 and p. 18, 3.23]

A graph is a forest if and only if it has no subgraph which is a polygon.
PROPOSITION 16.4  [17, p. 26, 3.71; p. 27, 3.75; and p. 18, 3.23]

Every proper subgraph of a polygon is a forest.

PROOF

Let $H$ be a proper subgraph of a polygon $P$. Then there is an $e \in E(P)$ such that $H \subseteq P : (E(P) - \{e\})$.

Since $P$ is a polygon, $|E(P)| = |V(P)|$ and $P : (E(P) - \{e\})$ is connected. Therefore

$r(P : (E(P) - \{e\}))$

$= |V(P : (E(P) - \{e\})| - c(P : (E(P) - \{e\}))$

$= |V(P)| - 1$

$= |E(P)| - 1$

$= |E(P : (E(P) - \{e\})|$

Thus, by Proposition 15.2, $P : (E(P) - \{e\})$ is a forest.

So, by Proposition 13.3, $H$ is a forest.
SECTION 17 THE POLYGONS OF A GRAPH

DEFINITION 17.1

A polygon of a graph $G$ is a subgraph of $G$ which is a polygon.

PROPOSITION 17.2 (Petersen) [9, p. 90, Thm 9.9]

If $G$ is a graph, then the following two conditions are equivalent.

(1) Every vertex of $G$ has even valency.

(2) There is a collection $\mathcal{P}$ of edge-disjoint polygons of $G$ such that $E(\bigcup \mathcal{P}) = E(G)$.

PROOF

$(1) \Rightarrow (2)$:

Suppose that $G$ is a graph with no edges. Let $\mathcal{P}$ be the empty collection of polygons of $G$. Then $\mathcal{P}$ is a collection of edge-disjoint polygons and $E(\bigcup \mathcal{P}) = \emptyset = E(G)$. Thus $G$ satisfies (2).

Assume that, if a graph $G$ satisfies (1) and $G$ has fewer than $n > 0$ edges, then $G$ satisfies (2).

Suppose that $G$ is a graph which satisfies (1) and has $n$ edges. Since $G$ has at least one edge and no vertex of $G$ has valency one, it follows from
Proposition 13.6 that \( G \) is not a forest. So \( G \) must contain a polygon \( P \).

Let

\[ H = G: (E(G) - E(P)) \].

Then \( H \) has fewer than \( n \) edges.

Suppose that \( v \in V(H) \). If \( v \in V(P) \), then \( \text{val}(H,v) = \text{val}(G,v) - 2 \). If \( v \notin V(P) \), then \( \text{val}(H,v) = \text{val}(G,v) \). Thus, in either case \( \text{val}(H,v) \) is even. So every vertex of \( H \) has even valency.

Thus, by the induction hypothesis, there is a collection \( \mathcal{P} \) of edge-disjoint polygons of \( G \) such that \( E(\bigcup \mathcal{P}) = E(H) \).

Since \( E(P) \cap E(H) = \emptyset \) and each polygon of \( H \) is a polygon of \( G \), \( \{P\} \cup \mathcal{P} \) is a collection of edge-disjoint polygons of \( G \). Since \( E(P) \cup E(H) = E(G) \), \( \cup (\{P\} \cup \mathcal{P}) = E(G) \). Thus \( G \) satisfies (2).

So, by induction, (1) implies (2).

(2) \( \Rightarrow \) (1):

Let \( \mathcal{P} \) be a collection of edge-disjoint polygons of a graph \( G \) such that \( E(\bigcup \mathcal{P}) = E(G) \).

If \( \mathcal{P} \) is empty, then \( E(G) = E(\bigcup \mathcal{P}) = \emptyset \) and so every vertex of \( G \) has even valency. Thus we may
assume that \( \mathcal{P} \) is non-empty.

Let \( \mathcal{P} = \{P_1, \ldots, P_m\} \) and, for each \( i \in \{1, \ldots, m\} \), let \( K_i = G : E(P_i) \). Then the \( K_i \) are edge-disjoint and

\[
\bigcup_{i=1}^{m} E(K_i) = E(G).
\]

Suppose that \( v \in V(G) \). Then

\[
\text{val}(G,v) = \sum_{i=1}^{m} \text{val}(K_i,v).
\]

But, for each \( i \in \{1, \ldots, m\} \), if \( v \in V(P_i) \), then \( \text{val}(K_i,v) = 2 \); and, if \( v \notin V(P_i) \), then \( \text{val}(K_i,v) = 0 \). Thus \( \text{val}(G,v) \) is even.

Therefore each vertex of \( G \) has even valency.

Thus (2) implies (1).

**PROPOSITION 17.3** [10, p. 163]

If \( Q_1, \ldots, Q_n \) are polygons of a graph \( G \), then there is a collection \( \mathcal{P} \) of edge-disjoint polygons of \( G \) such that

\[
E(U \mathcal{P}) = E(Q_1) + \cdots + E(Q_n)
\]

where \( E(Q_1) + \cdots + E(Q_n) \) is the symmetric difference of the \( E(Q_i) \).
PROOF

Let $Q_1, \ldots, Q_n$ be polygons of a graph $G$.

Let $H = G : (E(Q_1) + \cdots + E(Q_n))$ and, for each $i \in \{1, \ldots, n\}$, let $K_i = G : E(Q_i)$.

Suppose that $v \in V(H)$. Then

$$\text{val}(H, v) \equiv \sum_{i=1}^{n} \text{val}(K_i, v) \pmod{2}.$$  

But, for each $i \in \{1, \ldots, n\}$, if $v \in V(Q_i)$, then $\text{val}(K_i, v) = 2$; and, if $v \notin V(Q_i)$, then $\text{val}(K_i, v) = 0$. Thus

$$\sum_{i=1}^{n} \text{val}(K_i, v)$$

is even. So $\text{val}(H, v)$ is even.

Therefore each vertex of $H$ has even valency. Thus, by Proposition 17.2, there is a collection $\mathcal{P}$ of edge-disjoint polygons of $H$ such that $E(\cup \mathcal{P}) = E(H)$. But then $\mathcal{P}$ is a collection of edge-disjoint polygons of $G$ such that

$$E(\cup \mathcal{P}) = E(Q_1) + \cdots + E(Q_n).$$
PROPOSITION 17.4

If $F$ is a principal forest of a graph $G$, then there is a collection $\mathcal{P}$ of edge-disjoint polygons of $G$ such that

$$E(\bigcup \mathcal{P}) \supseteq (E(G) - E(F)).$$

PROOF

Let $F$ be a principal forest of a graph $G$.

Suppose that $E(G) - E(F)$ is empty and let $\mathcal{P}$ be the empty collection of polygons of $G$. Then $\mathcal{P}$ is a collection of edge-disjoint polygons of $G$ and $E(\bigcup \mathcal{P}) = \emptyset = E(G) - E(F)$.

On the other hand, suppose that $E(G) - E(F)$ is non-empty and let $E(G) - E(F) = \{e_1, \ldots, e_n\}$.

Suppose that $i \in \{1, \ldots, n\}$. Then, since $F$ is a principal forest of $G$ and $e_i \in E(G) - E(F)$, it follows from Proposition 14.5 that there is a circular path $\alpha_i$ in $G$ such that $e_i \in E(\alpha_i) \subseteq E(F) \cup \{e_i\}$. Let $Q_i = G(\alpha_i)$. Then $Q_i$ is a polygon of $G$ and $e_i \in E(Q_i) \subseteq E(F) \cup \{e_i\}$.

Thus, if $i, j \in \{1, \ldots, n\}$ and $i \neq j$, then $e_i \notin E(Q_j)$. So

$$\{e_1, \ldots, e_n\} \subseteq E(Q_1) + \cdots + E(Q_n).$$
Thus

$$E(G) - E(F) \leq E(Q_1) + \cdots + E(Q_n) .$$

Therefore, using Proposition 17.3, there is a collection $\mathcal{P}$ of edge-disjoint polygons of $G$ such that

$$E(\cup \mathcal{P}) = E(Q_1) + \cdots + E(Q_n) \geq E(G) - E(F) .$$
CHAPTER 2 MATROIDS AND GRAPHS

(Sections 18 - 25)

The proof of Conjecture 1, given in the next chapter, makes use of the following proposition:

(Proposition 25.3)

If every bond of a graph $G$ has at least three edges, then there are collections $P_1$, $P_2$, and $P_3$ of edge-disjoint polygons of $G$ such that

$$3 \sum_{i=1}^{3} E(P_i) = E(G).$$

The purpose of the present chapter is to prove this result. The proof involves applying a theorem of Edmonds' (Proposition 24.3) to the bond matroid of a graph (Definition 23.7). Thus: Sections 18 - 22 outline the required elements of matroid theory, Section 23 is concerned with the polygon and bond matroids of a graph, in Section 24 Edmonds' theorem is proved, and Section 25 gives a proof of Proposition 25.3.

For further information about matroids the reader is referred to the expository articles of Wilson [21]

SECTION 18  MATROIDS

DEFINITION 18.1  [21, p. 506]

A matroid $M$ consists of a finite set $E(M)$ together with a collection $\mathcal{B}(M)$ of subsets of $E(M)$ which satisfies the following three conditions:

(1) $\mathcal{B}(M)$ is non-empty;
(2) no member of $\mathcal{B}(M)$ properly contains another member of $\mathcal{B}(M)$; and
(3) if $B_1$ and $B_2$ are members of $\mathcal{B}(M)$ and $b_1 \in B_1$, then there is an element $b_2 \in B_2$ such that

$$(B_1 - \{b_1\}) \cup \{b_2\}$$

is a member of $\mathcal{B}(M)$.

The matroid $M$ is said to be on the set $E(M)$.

The elements of $E(M)$ are called elements of $M$.

The members of $\mathcal{B}(M)$ are called bases of $M$. 
PROPOSITION 18.2  [21, p. 506]

All the bases of a matroid have the same number of elements.

PROOF

Suppose that $B_1$ and $B_2$ are bases of a matroid $M$ and $|B_1| \leq |B_2|$. Choose a base $B_3$ of $M$ such that $|B_3| = |B_2|$ and

$$|B_3 \cap B_1| = \max \{|B \cap B_1| : B \text{ a base of } M \text{ and } |B| = |B_2|\}.$$ 

We show that $B_3 \subseteq B_1$.

Suppose that $B_3 \nsubsetneq B_1$. Let $b_3 \in B_3 - B_1$. Then, by 18.1 (3) there is an element $b_1 \in B_1$ such that the set

$$B_4 = (B_3 - \{b_3\}) \cup \{b_1\}$$

is a base of $M$. If $b_1 \in B_3$, then $B_4 \subseteq B_3$ contradicting 18.1 (2). On the other hand, if $b_1 \notin B_3$ then

$$|B_4 \cap B_1| > |B_3 \cap B_1|$$

contradicting the choice of $B_3$. 
It follows that $B_3 \subseteq B_1$. Thus

$$|B_1| > |B_3| = |B_2|.$$ 

Therefore, since $|B_1| \leq |B_2|$, it follows that $|B_1| = |B_2|$.

**SECTION 19  INDEPENDENT SETS**

**DEFINITION 19.1 [21, p. 506]**

Let $M$ be a matroid.

A subset of $E(M)$ is said to be **M-independent** if it is contained in some base of $M$. Unless confusion is likely to result, we use "independent" instead of "M-independent".

We also say that an independent subset of $E(M)$ is an **independent set of** $M$.

**PROPOSITION 19.2 [21, p. 507]**

If $X$ and $Y$ are independent sets of a matroid $M$ and $|X| < |Y|$, then there is an element $y \in Y - X$ such that $X \cup \{y\}$ is an independent set of $M$. 

PROOF

Suppose that X and Y are independent sets of a matroid M and |X| < |Y|.

Since X and Y are independent sets of M, each is contained in some base of M. Let B_2 be a base of M containing Y. Choose a base B_1 of M containing X and such that

|B_1 ∩ B_2| = max { |B ∩ B_2| : B a base of M and
X ⊆ B}.

We show that B_1 ∩ (Y - X) ≠ ∅.

Suppose that B_1 ∩ (Y - X) = ∅. Then

(B_1 - X) ∩ Y = ∅. Since |X| < |Y| and since, by Proposition 18.2, |B_1| = |B_2|, it follows that

|B_1 - X| > |B_2 - Y|. Thus B_1 - X ⊈ B_2 - Y. Since

(B_1 - X) ∩ Y = ∅, it follows that B_1 - X ⊈ B_2. Let

b_1 ∈ (B_1 - X) - B_2 = B_1 - (X ∪ B_2). Then, by

Proposition 18.1 (3), there is a b_2 ∈ B_2 such that

B_3 = (B_1 - \{b_1\}) ∪ \{b_2\}

is a base of M. If b_2 ∈ B_1, then B_3 ⊆ B_1 contradicting 18.1 (2). On the other hand, if

b_2 ⊈ B_1, then |B_3 ∩ B_2| > |B_1 ∩ B_2| contradicting the choice of B_1 since X ⊆ B_3. We conclude that B_1 ∩ (Y - X) ≠ ∅.
Let \( y \in B_1 \cap (Y - X) \). Then \( X \cup \{y\} \subseteq B_1 \).
Thus \( X \cup \{y\} \) is an independent set of \( M \).

**Proposition 19.3** [21, p. 507]

Let \( M \) be a matroid and let \( A \) be a subset of \( E(M) \).

If \( X \) and \( Y \) are maximal independent subsets of \( A \), then \( |X| = |Y| \).

**Proof**

Suppose that \( X \) and \( Y \) are independent subsets of \( A \) and \( |X| < |Y| \). Then, by Proposition 19.2,
There is a \( y \in Y - X \) such that \( X \cup \{y\} \) is an independent set of \( M \). But \( X \subseteq X \cup \{y\} \subseteq A \). Thus \( X \) is not a maximal independent subset of \( A \).

**Section 20 Rank**

**Definition 20.1** [7, p. 68]

Let \( M \) be a matroid.

If \( A \) is a subset of \( E(M) \), then the rank \( r_M(A) \) of \( A \) in \( M \) is defined to be the cardinality of a maximal independent subset of \( A \).
Proposition 19.3 ensures that $r_M$ is well-defined.

**SECTION 21  CIRCUITS**

**DEFINITION 21.1  [21, p. 507]**

Let $M$ be a matroid.

A subset of $E(M)$ is said to be \textit{M-dependent} if it is not independent. Unless confusion is likely to result, we use "dependent" instead of "M-dependent".

We also say that a dependent subset of $E(M)$ is a \textit{dependent set of $M$}.

A \textit{circuit} of $M$ is a minimal dependent subset of $E(M)$.

**PROPOSITION 21.2  [7, p. 70, Prop. 2]**

Let $M$ be a matroid.

Then: a subset of $E(M)$ is independent if and only if it contains no circuit of $M$. 
PROPOSITION 21.3  [7, p. 70, Prop. 1]

If $X$ is an independent set of a matroid $M$ and $e$ is an element of $M$, then $X \cup \{e\}$ contains at most one circuit.

**Proof**

Let $X$ be an independent set of a matroid $M$ and let $e$ be an element of $M$.

Suppose that $C_1$ and $C_2$ are distinct circuits of $M$ contained in $X \cup \{e\}$. Since $C_1$ and $C_2$ are distinct, there is an element $c_1 \in C_1 - C_2$. Note that $C_2 \subseteq (X \cup \{e\}) - \{c_1\}$. Since $C_1$ is a circuit of $M$ contained in $X \cup \{e\}$, it follows that $C_1 - \{c_1\}$ is an independent set contained in $X \cup \{e\}$. Thus $C_1 - \{c_1\}$ is contained in a maximal independent subset $Y$ of $X \cup \{e\}$. Since $Y$ is independent, $C_1 \notin Y$. So $c_1 \notin Y$. Thus $Y \subseteq (X \cup \{e\}) - \{c_1\}$.

Since $X$ is a maximal independent subset of $X \cup \{e\}$, it follows from Proposition 19.3 that $|Y| = |X|$. Thus $Y = (X \cup \{e\}) - \{c_1\}$. But then $C_2 \subseteq Y$ and so, by Proposition 21.2, the fact that $Y$ is independent is contradicted.

Thus $X \cup \{e\}$ contains at most one circuit.
PROPOSITION 21.4  [10, p. 160, Thm 7]

If $B$ is a base of a matroid $M$ and $e$ is an element of $M$ which does not belong to $B$, then there is a unique circuit $J(B,e)$ of $M$ such that $e \in J(B,e) \subseteq B \cup \{e\}$.

PROOF

Suppose that $B$ is a base of a matroid $M$ and $e \in E(M) - B$.

Since no base of $M$ properly contains another base of $M$, $B \cup \{e\}$ is dependent. Thus $B \cup \{e\}$ contains a circuit $C$ of $M$. If $e \not\in C$, then $C \subseteq B$. But $B$ is independent and so contains no circuit of $M$. Thus $e \in C$. Moreover, by Proposition 21.3, $C$ is the only circuit contained in $B \cup \{e\}$.

SECTION 22  ORTHOGONALITY

PROPOSITION 22.1  [21, p. 516]

If $M$ is a matroid, then

$$\{E(M) - B : B \text{ a base of } M\}$$

is the collection of bases of a matroid $M^*$ on $E(M)$.
We say that $M^*$ is the matroid orthogonal to $M$.

**PROOF**

Let $M$ be a matroid and let

$$\mathcal{B}(M^*) = \{E(M) - B : B \in \mathcal{B}(M)\}.$$ 

Since $\mathcal{B}(M)$ is non-empty, $\mathcal{B}(M^*)$ is non-empty.

Since no member of $\mathcal{B}(M)$ properly contains another member of $\mathcal{B}(M)$, no member of $\mathcal{B}(M^*)$ properly contains another member of $\mathcal{B}(M^*)$.

Suppose that $b_1$ and $b_2$ are members of $\mathcal{B}(M^*)$ and that $b_1 \in B_1$. Then $E(M) - B_1$ is a base of $M$ and $b_1 \notin E(M) - B_1$. Thus, by Proposition 21.4, there is a unique circuit $C$ of $M$ such that $b_1 \in C \subseteq (E(M) - B_1) \cup \{b_1\}$. Since $E(M) - B_2$ is $M$-independent, we have $C \notin E(M) - B_2$. Thus there is an element $b_2 \in C \cap B_2$. Let

$$X = ((E(M) - B_1) \cup \{b_1\}) - \{b_2\}.$$ 

Since $C$ is the only circuit of $M$ contained in $(E(M) - B_1) \cup \{b_1\}$ and $C$ is not contained in $X$, it follows that $X$ is $M$-independent. But $|X| = |E(M) - B_1|$. Thus, using Proposition 18.2, it follows that $X$ is a base of $M$. Now
\[ X = \left( (E(M) - B_1) \cup \{b_1\} \right) - \{b_2\} \]
\[ = \left( (E(M) - B_1) \cup \{b_1\} \right) \cap \left( E(M) - \{b_2\} \right) \]
\[ = \left( E(M) - (B_1 \cap (E(M) - \{b_1\})) \right) \cap \left( E(M) - \{b_2\} \right) \]
\[ = E(M) - \left( (B_1 - \{b_1\}) \cup \{b_2\} \right) \]
Thus \((B_1 - \{b_1\}) \cup \{b_2\}\) is a member of \(\mathcal{B}(M^*)\).

**PROPOSITION 22.2** [21, p. 516]

If \(M\) is a matroid, then the rank function \(r_{M^*}\) of the matroid \(M^*\) orthogonal to \(M\) satisfies the following condition:

for every subset \(A\) of \(E(M^*) = E(M)\),

\[ r_{M^*}(A) = |A| + r_M(E(M) - A) - r_M(E(M)) \]

**PROOF**

Let \(M\) be a matroid and let \(A\) be a subset of \(E(M^*) = E(M)\).

Suppose that \(X\) is a maximal \(M^*\)-independent subset of \(A\). Then, since \(X\) is \(M^*\)-independent, \(X\) is contained in some base \(B\) of \(M^*\). So \(X \subseteq A \cap B\). Since \(A \cap B\) is an \(M^*\)-independent subset of \(A\) and since \(X\) is a maximal
$M^*$-independent subset of $A$, $|A \cap B| \leq |X|$. Thus

$X = A \cap B$. Therefore

$$r_{M^*}(A) = \max \{|A \cap B| : B \text{ a base of } M^*\}.$$ 

Similarly

$$r_M(E(M) - A)$$

$$= \max \{|(E(M) - A) \cap B| : B \text{ a base of } M\}$$

$$= \max \{|(E(M) - A) \cap (E(M) - B)| : B \text{ a base of } M^*\}.$$ 

Let $B$ be a base of $M^*$. Then

$$|A \cap B|$$

$$= |A| + |B| - |A \cup B|$$

$$= |A| + (|E(M)| - |A \cup B|) - (|E(M)| - |B|)$$

$$= |A| + |E(M) - (A \cup B)| - |E(M) - B|$$

$$= |A| + |(E(M) - A) \cap (E(M) - B)| - r_M(E(M)).$$

Therefore

$$r_{M^*}(A) = \max \{|A \cap B| : B \text{ a base of } M^*\}$$

$$= \max \{|A| + |(E(M) - A) \cap (E(M) - B)| - r_M(E(M)) :$$

$B \text{ a base of } M^*\}
\[ |A| + \max \{|(E(M) - A) \cap (E(M) - B)| : B \text{ a base of } M^*\} - r_M(E(M)) \]

\[ = |A| + r_M(E(M) - A) - r_M(E(M)). \]

SECTION 23  THE POLYGON MATROID AND THE BOND MATROID OF A GRAPH

PROPOSITION 23.1  [21, p. 509, (2)]

If \( G \) is a graph, then the edge-sets of the principal forests of \( G \) are the bases of a matroid \( M(G) \) on the set \( E(G) \).

PROOF

This proposition is an easy consequence of Corollaries 14.4 and 14.6 and Proposition 14.7.

PROPOSITION 23.2  [21, p. 509, (2)]

If \( G \) is a graph, then the edge-sets of the polygons of \( G \) are the circuits of the matroid \( M(G) \).

PROOF

Let \( G \) be a graph.
Suppose that $P$ is a polygon of $G$. By Proposition 16.3, $E(P)$ is not contained in the edge-set of any principal forest of $G$. Thus $E(P)$ is a dependent set of $M(G)$. By Proposition 16.4, every proper subset of $E(P)$ is the edge-set of a forest of $G$. Thus, by Proposition 14.3, every proper subset of $E(P)$ is contained in the edge-set of some principal forest of $G$ and so is an independent set of $M(G)$. Thus $E(P)$ is a circuit of $M(G)$.

Conversely, suppose that $C$ is a circuit of $M(G)$. Then $C$ is not contained in the edge-set of any principal forest of $G$. Thus, by Proposition 14.3, $C$ is not contained in the edge-set of any forest of $G$. Thus $G\cdot C$ is not a forest. So, by Proposition 16.3, $G\cdot C$ contains a polygon $P$ of $G$. But then $E(P)$ is a circuit of $M(G)$ and $E(P) \subseteq C$. Thus, since $C$ is a circuit of $(G)$, $E(P) = C$.

**Definition 23.3** [21, p. 509, (2)]

The matroid $M(G)$ is called the *polygon matroid* of the graph $G$. 

PROPOSITION 23.4 [21, p. 509, (2)]

The rank function $r_{M(G)}$ of the polygon matroid $M(G)$ of a graph $G$ satisfies the following condition:

for every subset $A$ of $E(G)$,

$$r_{M(G)}(A) = r(G:A).$$

PROOF

Let $G$ be a graph and let $A$ be a subset of $E(G)$.

Suppose that $X$ is a maximal $M(G)$-independent subset of $A$. We show that $G:X$ is a principal forest of $G:A$. Since $X \subseteq A$, $G:X$ is a subgraph of $G:A$. Since $X$ is $M(G)$-independent, it follows from Propositions 23.1 and 14.3 that $G:X$ is a forest. Thus $G:X$ is a forest of $G:A$. So, by Proposition 14.3, $G:X$ is contained in a principal forest $F$ of $G:A$. But $G:A \subseteq G$. So $F$ is a forest of $G$. Thus, by Proposition 14.3, $F$ is contained in a principal forest of $G$. Thus $E(F)$ is $M(G)$-independent. But $X \subseteq E(F) \subseteq A$ and $X$ is a maximal $M(G)$-independent subset of $A$. Thus $X = E(F)$. Moreover, using Proposition 14.2, $V(F) = V(G:A) = V(G)$. So $G:X = F$. Thus $G:X$ is a principal forest of $G:A$. 
Therefore, using Propositions 15.2 and 15.3, 

\[ r_{M(G)}(A) = |X| = |E(G:X)| = r(G:X) = r(G:A). \]

**Definition 23.5** [21, p. 509, (3)]

Let \( G \) be a graph.

We denote the matroid orthogonal to \( M(G) \) by \( M^*(G) \).

**Proposition 23.6** [21, p. 509, (3)]

If \( G \) is a graph, then the bonds of \( G \) are the circuits of the matroid \( M^*(G) \).

**Proof**

Let \( G \) be a graph and let \( A \) be a subset of \( E(G) = E(M^*(G)) \).

Suppose that \( A \) is not a cutset of \( G \). Then, by Proposition 9.3, \( c(G:(E(G) - A)) = c(G) \). Let \( F \) be a principal forest of \( G:(E(G) - A) \). Then \( F \) is a forest of \( G \). Moreover, using Proposition 14.2, we have

\[ V(F) = V(G:(E(G) - A)) = V(G) \]
and
\[ c(F) = c(G:(E(G) - A)) = c(G) . \]

Thus, by Proposition 14.2, \( F \) is a principal forest of \( G \). Since \( E(F) \subseteq E(G) - A \), \( A \subseteq E(G) - E(F) \).
Thus \( A \) is \( M^*(G) \)-independent.

Conversely, suppose that \( A \) is \( M^*(G) \)-independent. Then there is a principal forest \( F \) of \( G \) such that \( A \subseteq E(G) - E(F) \). Thus \( E(F) \subseteq E(G) - A \). Since \( F \) is a principal forest of \( G \), it follows from Proposition 14.2 that \( V(F) = V(G) \) and \( c(F) = c(G) \). Since \( V(F) = V(G) \) and \( E(F) \subseteq E(G) - A \),
\[ F = G:(E(F)) \subseteq G:(E(G) - A) \subseteq G . \]
Thus, using Corollary 8.5,
\[ c(F) = c(G:E(F)) \leq c(G:(E(G) - A)) \leq c(G) . \]
But \( c(F) = c(G) \). So
\[ c(G:(E(G) - A)) = c(G) . \]
Thus, by Proposition 9.3, \( A \) is not a cutset of \( G \).

Therefore: \( A \) is a cutset of \( G \) if and only if \( A \) is \( M^*(G) \)-dependent.

Since the circuits of \( M^*(G) \) are the minimal
M*(G)-dependent subsets of \( E(M^*(G)) = E(G) \), the proposition follows.

**DEFINITION 23.7** [21, p. 509, (3)]

The matroid \( M^*(G) \) is called the **bond matroid** of the graph \( G \).

**SECTION 24** **EDMONDS' THEOREM**

**LEMMA 24.1** [16, p. 3, 2.22]

If \( B \) is a base of a matroid \( M \) and \( e \) is an element of \( M \) which does not belong to \( B \), then, for each element \( c \) of \( J(B,e) \), \( (B - \{c\}) \cup \{e\} \) is a base of \( M \).

**PROOF**

Suppose that \( B \) is a base of a matroid \( M \) and \( e \in E(M) - B \).

Let \( c \in J(B,e) \).

If \( c = e \), then \( (B - \{c\}) \cup \{e\} = B \) and so \( (B - \{c\}) \cup \{e\} \) is a base of \( M \).

On the other hand, suppose that \( c \neq e \). Then
J(B,e) ∉ (B - {c}) U {e}. By Proposition 21.4, J(B,e) is the only circuit of M contained in B U {e}. Thus, since (B - {c}) U {e} ⊆ B U {e}, it follows that (B - {c}) U {e} contains no circuit of M. So (B - {c}) U {e} is an independent set of M. Thus (B - {c}) U {e} is contained in a base of M. But, since |(B - {c}) U {e}| = |B|, it follows from Proposition 18.2 that no base of M properly contains (B - {c}) U {e}. So (B - {c}) U {e} is a base of M.

**Lemma 24.2**

Let B be a base of a matroid M.

Let \{a_1, \ldots, a_n\} be a subset of E(M) which is disjoint from B.

Let \{b_1, \ldots, b_n\} be a subset of B which satisfies the following two conditions:

1. If 1 ≤ i ≤ n, then \(b_i \in J(B, a_i)\); and
2. If 1 ≤ i < j ≤ n, then \(b_j \notin J(B, a_i)\).

Then

\[(B - \{b_1, \ldots, b_n\}) U \{a_1, \ldots, a_n\}\]

is a base of M.
PROOF

By Lemma 24.1, the proposition holds if \( n = 1 \).

Assume that \( n > 1 \).

By Lemma 24.1, the set

\[ B' = (B - \{b_n\}) \cup \{a_n\} \]

is a base of \( M \).

Suppose that \( 1 \leq i \leq n-1 \). Then, since

\( b_n \notin J(B, a_i) \subseteq B \cup \{a_i\} \), we have

\( J(B, a_i) \subseteq B' \cup \{a_i\} \). But, since \( a_i \notin B' \), it follows from Proposition 21.4 that \( J(B', a_i) \) is the only circuit contained in \( B' \cup \{a_i\} \). Thus

\( J(B', a_i) = J(B, a_i) \).

Thus \( \{b_1, \ldots, b_{n-1}\} \) is a subset of \( B' \) which satisfies the conditions obtained from Conditions (1) and (2) by replacing "n" with "n - 1" and "B" with "B'".

Moreover, \( B' \) is a base of \( M \) and

\( \{a_1, \ldots, a_{n-1}\} \) is a subset of \( E(M) \) which is disjoint from \( B' \).

Thus, since \( n - 1 < n \), we may assume as an induction hypothesis that
(B' - \{b_1, \ldots, b_{n-1}\}) \cup \{a_1, \ldots, a_{n-1}\}

is a base of M. But

(B - \{b_1, \ldots, b_n\}) \cup \{a_1, \ldots, a_n\}

= (B' - \{b_1, \ldots, b_{n-1}\}) \cup \{a_1, \ldots, a_{n-1}\}.

So

(B - \{b_1, \ldots, b_n\}) \cup \{a_1, \ldots, a_n\}

is a base of M.

PROPOSITION 24.3 (Edmonds) [7, p. 69, Thm 1]

The set E(M) of elements of a matroid M can be expressed as the union of as few as k bases of M if and only if,

for every subset A of E(M),

|A| \leq k \cdot r_M(A).

PROOF

Let M be a matroid. Then: E(M) can be expressed as the union of no bases of M if and only if E(M) is empty. Thus we may assume that k > 0.
The "only if" part: [7, p. 69]

Suppose that \( B_1, \ldots, B_k \) are bases of \( M \) and

\[
E(M) = \bigcup_{i=1}^{k} B_i.
\]

Let \( A \) be a subset of \( E(M) \). Then

\[
A \subseteq \bigcup_{i=1}^{k} B_i.
\]

So

\[
|A| \leq \sum_{i=1}^{k} |A \cap B_i|.
\]

Moreover, for \( i \in \{1, \ldots, k\} \), \( A \cap B_i \) is an independent subset of \( A \) and so

\[
|A \cap B_i| \leq r_M(A).
\]

Thus

\[
|A| \leq k \cdot r_M(A).
\]

The "if" part:

The proof which follows is an adaptation of proofs of Bruno and Weinberg [3, p. 29, 5.2 and p. 34, 5.3]
Let \((B_1, \cdots, B_k)\) be a \(k\)-tuple of bases of \(M\) such that

\[
\begin{vmatrix}
\mathcal{E}(M) - \bigcup_{i=1}^{k} B_i \\
\end{vmatrix}
\]

is minimal.

We show that, if

\[
\begin{vmatrix}
\mathcal{E}(M) - \bigcup_{i=1}^{k} B_i \\
\end{vmatrix} \geq 1 ,
\]

then there is a subset \(A_t\) of \(\mathcal{E}(M)\) such that

\[
|A_t| > k \cdot r_M(A_t).
\]

Suppose that

\[
\begin{vmatrix}
\mathcal{E}(M) - \bigcup_{i=1}^{k} B_i \\
\end{vmatrix} \geq 1 .
\]

Let

\[
A_0 = \mathcal{E}(M) - \bigcup_{i=1}^{k} B_i.
\]

For \(n \geq 0\), we define \(A_{n+1}\) in terms of \(A_n\) by the equation
\[ A_{n+1} = \bigcup_{i=1}^{k} \left( \mathcal{U} \{ J(B_i, e) : e \in A_n \text{ and } e \notin B_i \} \right) \]

Since no member of \( A_0 \) belongs to any \( B_i \), it follows that, for \( n \geq 0 \), \( A_n \subseteq A_{n+1} \subseteq E(M) \). Thus, since \( E(M) \) is finite, there is a least positive integer \( t \) such that, for \( n \geq t \), \( A_n = A_t \).

Let

\[ X = \{ e \in E(M) : e \text{ is an element of at least two coordinates of } (B_1, \ldots, B_k) \} \]

Then \( A_0 \cap X = \emptyset \). We show that \( A_t \cap X = \emptyset \).

Suppose that there is a least positive integer \( s \) such that \( A_s \cap X \neq \emptyset \). Let \( e_s \in A_s \cap X \). Then, by the construction of the sequence \( A_n \), there is a sequence

\[ (e_0, \ldots, e_s) \]

of elements of \( E(M) \) and a sequence

\[ (D_1, \ldots, D_s) \]

of coordinates of \( (B_1, \ldots, B_k) \) such that

1. if \( 0 \leq i \leq s \), then \( e_i \in A_i \); and
2. if \( 1 \leq i \leq s \), then \( e_{i-1} \notin D_i \) and \( e_i \in J(D_i, e_{i-1}) \).
Suppose that there are integers $i$ and $j$ such that $1 \leq i < j \leq s$ and $e_j \in J(D_i, e_{i-1})$. Then there is an $n < s$ such that $e_s \in A_n$. But then $e_s \in A_n \cap X$, contradicting the supposition that $s$ is the least integer such that $A_s \cap X \neq \emptyset$.

Thus, if $1 \leq i < j \leq s$, then $e_j \notin J(D_i, e_{i-1})$. It follows that the $e_i$ are distinct and that, for $1 \leq i \leq s$, $e_i \in D_i$.

We proceed to construct a new $k$-tuple $(B'_1, \ldots, B'_k)$ of bases of $M$.

Suppose that $1 \leq i \leq k$.

If no term of $(D_1, \ldots, D_s)$ is equal to $B_i$, then let $B'_i = B_i$.

Otherwise, let

$$(D_{i_1}, \ldots, D_{i_m})$$

be the maximal subsequence of $(D_1, \ldots, D_s)$ all of whose terms are equal to $B_i$. Then

$$\{e_{i_1-1}, \ldots, e_{i_m-1}\} \subseteq E(M)$$

and

$$\{e_{i_1-1}, \ldots, e_{i_m-1}\} \cap B_i = \emptyset.$$
Moreover,
\[
\{e_{i_1}, \ldots, e_{i_m}\}
\]
is a subset of \(B_i\) such that

1. \(i_1 \leq p \leq m\), then \(e_p \in J(B_i, e_{i_p - 1})\); and
2. \(1 \leq p < q \leq m\), then \(e_q \notin J(B_i, e_{i_p - 1})\).

Thus it follows from Lemma 24.2 that the set
\[
B'_i = (B_i - \{e_{i_1}, \ldots, e_{i_m}\}) \cup \{e_{i_1 - 1}, \ldots, e_{i_m - 1}\}
\]
is a base of \(M\).

Consider the \(k\)-tuple
\[
(B'_1, \ldots, B'_k).
\]

It follows from the construction of the \(B'_i\) that
\[
\left( \bigcup_{i=1}^{k} B_i \right) - \{e_1, \ldots, e_s\} \cup \{e_0, \ldots, e_{s-1}\}
\]

Moreover, there is only one \(i\) such that \(e_s \in B_i\) and \(e_s \notin B'_i\). But \(e_s \notin X\) and so \(e_s\) belongs to at least two coordinates of \((B_1, \ldots, B_k)\). Thus
Thus
\[ k \quad k \]
\[ E(M) - \bigcup_{i=1}^{k} B'_i < E(M) - \bigcup_{i=1}^{k} B_i. \]

Therefore
\[ \left| E(M) - \bigcup_{i=1}^{k} B'_i \right| < \left| E(M) - \bigcup_{i=1}^{k} B_i \right|. \]

But this contradicts the minimality of
\[ \left| E(M) - \bigcup_{i=1}^{k} B_i \right|. \]

So we conclude that, for every \( n \geq 0 \), \( A_n \cap X = \emptyset \).
Thus $A_t \cap X = \emptyset$.

It follows that the sets

$$A_t \cap B_1, \ldots, A_t \cap B_k$$

are disjoint subsets of $A_t$. So

$$|A_t| = \left( \sum_{i=1}^{k} |A_t \cap B_i| \right) + \left| E(M) - \bigcup_{i=1}^{k} B_i \right|. \quad (5)$$

We show that, for $1 \leq i \leq k$, $A_t \cap B_i$ is a maximal independent subset of $A_t$.

Suppose that $1 \leq i \leq k$.

Since $B_i$ is a base of $M$ and $A_t \cap B_i \subseteq B_i$, it follows that $A_t \cap B_i$ is an independent subset of $A_t$.

Suppose that $e \in A_t - (A_t \cap B_i)$. Then, by the construction of the sequence $A_n$, $J(B_i, e) \subseteq A_{t+1}$.

But, by the choice of $t$, $A_{t+1} = A_t$. So

$J(B_i, e) \subseteq A_t$. Since $J(B_i, e) \subseteq B_i \cup \{e\}$, it follows that

$J(B_i, e) \subseteq A_t \cap (B_i \cup \{e\}) = (A_t \cap B_i) \cup \{e\}$.

Thus $(A_t \cap B_i) \cup \{e\}$ is not an independent set of $M$.

Thus, for $1 \leq i \leq k$, $A_t \cap B_i$ is a maximal independent subset of $A_t$. 
Therefore
\[ k \cdot r_M(A_t) = \sum_{i=1}^{k} |A_t \cap B_i| . \]

So, using (5),
\[ |A_t| = k \cdot r_M(A_t) + \left| \sum_{i=1}^{k} E(M) - \bigcup_{i=1}^{k} B_i \right| . \]

But, by supposition,
\[ \left| \sum_{i=1}^{k} E(M) - \bigcup_{i=1}^{k} B_i \right| \geq 1 . \]

So
\[ |A_t| > k \cdot r_M(A_t) . \]
SECTION 25  AN APPLICATION OF EDMONDS' THEOREM TO
GRAPH THEORY

PROPOSITION 25.1

If \( G \) is a connected graph and every bond of \( G \) has at least three edges, then there are spanning trees \( T_1, T_2, \) and \( T_3 \) of \( G \) such that

\[
\sum_{i=1}^{3} (E(G) - E(T_i)) = E(G).
\]

PROOF

Let \( G \) be a connected graph.

Suppose that \( G \) is null. Then \( G \) is a spanning tree of itself. Let \( T_1 = T_2 = T_3 = G \). Then

\[
\sum_{i=1}^{3} (E(G) - E(T_i)) = 0 = E(G).
\]

Thus we may assume that \( G \) is non-null.

Consider the following five conditions on \( G \):

(1) Every bond of \( G \) has at least three edges.

(2) For every subset \( A \) of \( E(G) \),

\[
3(c(G;(E(G) - A)) - 1) \leq 2|A|.
\]
(3) For every subset $A$ of $E(M^*(G))$,

$$|A| \leq 3 \cdot r_{M^*(G)}(A).$$

(4) $E(M^*(G))$ can be expressed as the union of as few as three bases of $M^*(G)$.

(5) There are spanning trees $T_1$, $T_2$, and $T_3$ of $G$ such that

$$3 \bigcup_{i=1}^{3} (E(G) - E(T_i)) = E(G).$$

We show that $(1) \Rightarrow (2) \iff (3) \iff (4) \Rightarrow (5)$.

$(1) \Rightarrow (2)$:

Let $A$ be a subset of $E(G)$ and let $m = c(G:(E(G) - A))$. Since $G$ is non-null, $m \neq 0$. If $m = 1$, then we have

$$3(c(G:(E(G) - A)) - 1) = 3(m - 1) = 0 \leq 2|A|.$$

and so, in this case, $(1)$ implies $(2)$. Thus we may assume that $m \geq 2$.

Consider the contraction $G_{\text{ctr}}A$ of $G$ to $A$. By the definition of a contraction,

$$E(G_{\text{ctr}}A) = A.$$
and

\[ V(G \text{ ctr } A) = \{K_1, \ldots, K_m\} \]

where \( K_1, \ldots, K_m \) are the components of \( G: (\Sigma(G) - A) \).

Suppose that \( 1 \leq i \leq m \). Since components are non-null, \( \emptyset \subseteq V(K_i) \). Since \( G: (\Sigma(G) - A) \) has at least two components, it follows from Proposition 6.2 that \( V(K_i) \subseteq V(G: (\Sigma(G) - A) = V(G) \). Thus \( \emptyset \subseteq V(K_i) \subseteq V(G) \). Let \( L_i \) be the set of links of \( G \) with just one end in \( K_i \). Then, by Proposition 9.5, \( L_i \) is a cutset of \( G \). Thus \( L_i \) contains a bond of \( G \). So \( |L_i| \geq 3 \). Suppose that \( e \in L_i \). Then \( e \) is a link with one end in \( V(K_i) \) and the other end not in \( V(K_i) \). Thus, by Lemma 10.3, \( e \in A \). So \( e \) is an edge of \( G \text{ ctr } A \) and \( K_i \) is a \( (G \text{ ctr } A) \)-end of \( e \). Thus

\[ \text{val}(G \text{ ctr } A, K_i) \geq |L_i| \geq 3. \]

Therefore, using Proposition 2.4, we have

\[ 3(c(G: (\Sigma(G) - A)) - 1) = 3(m - 1) \leq 3m \]

\[ < \sum_{K_i \in V(G \text{ ctr } A)} \text{val}(G \text{ ctr } A, K_i) \]

\[ = 2|E(G \text{ ctr } A)| = 2|A|. \]
By the definition of $M^*(G)$, $E(M^*(G)) = E(G)$.

Let $A$ be a subset of $E(M^*(G))$. By Proposition 22.2,

$$r_{M^*(G)}(A) = |A| + r_M(G)(E(G) - A) - r_M(G)(E(G)).$$

By Proposition 23.4,

$$r_M(G)(E(G) - A) = r(G:(E(G) - A))$$
$$= |V(G:(E(G) - A))| - c(G:(E(G) - A))$$
$$= |V(G)| - c(G:(E(G) - A))$$

and

$$r_M(G)(E(G)) = r(G:(E(G)))$$
$$= r(G) = |V(G)| - c(G).$$

But $G$ is connected so $c(G) = 1$. Thus

$$r_M(G)(E(G)) = |V(G)| - 1.$$ 

Therefore:

$$|A| < 3 \cdot r_{M^*(G)}(A)$$

if and only if

$$|A| < 3 \left( |A| + r_M(G)(E(G) - A) - r_M(G)(E(G)) \right)$$
if and only if

$$3 \left( r_{M(G)}(E(G)) - r_{M(G)}(E(G) - A) \right) \leq 2|A|.$$  

if and only if

$$3 \left( |V(G)| - 1 \right) - \left( |V(G)| - c(G:(E(G) - A)) \right) \leq 2|A|.$$  

if and only if

$$3 \left( c(G:(E(G) - A)) \right) \leq 2|A|.$$  

(3) $\iff$ (4):

This is a corollary of Edmonds' theorem (Proposition 24.3).

(4) $\Rightarrow$ (5):

Let $B$ be a base of $M^*(G)$, then $E(G) - B$ is a base of $M(G)$. By the definition of $M(G)$, there is thus a principal forest $F$ of $G$ such that $E(F) = E(G) - B$. By Proposition 14.2, $V(F) = V(G)$ and $c(F) = c(G)$. Since $c(G) = 1$, $c(F) = 1$. Thus $F$ is a spanning tree of $G$. Therefore:

if $B$ is a base of $M^*(G)$, then there is a spanning tree $T$ of $G$ such that $E(G) - E(T) = B$. Thus (4) implies (5).
PROPOSITION 25.2

If every bond of a graph $G$ has at least three edges, then there are principal forests $F_1$, $F_2$, and $F_3$ such that

$$
3 \bigcup_{i=1}^3 (E(G) - E(F_i)) = E(G).
$$

PROOF

Let $G$ be a graph with $n$ components.

If $n = 0$ or $n = 1$, then $G$ is connected and the proposition follows directly from Proposition 25.1.

Suppose that $n \geq 2$ and that every bond of $G$ has at least three edges.

Let $K_1, \ldots, K_n$ be the components of $G$.

Suppose that $1 \leq i \leq n$. Then, since $K_i$ is a component of $G$, $K_i$ is connected. Moreover, by Proposition 10.2, every bond of $K_i$ is a bond of $G$. Thus $K_i$ satisfies the conditions of Proposition 25.1. So there are spanning trees $T_{i1}$, $T_{i2}$, and $T_{i3}$ of $K_i$ such that

$$
3 \bigcup_{j=1}^3 (E(K_i) - E(T_{ij})) = E(K_i).
$$
Suppose that $1 \leq j \leq 3$. Let

$$F_j = \bigcup_{i=1}^{n} T_{ij}.$$ Then

$$V(F_j) = \bigcup_{i=1}^{n} V(T_{ij}) = \bigcup_{i=1}^{n} V(K_i) = V(G).$$

Moreover,

$$F_j = G:\left(\bigcup_{i=1}^{n} E(T_{ij})\right)$$

and

$$T_{ij} = K_i : E(T_{ij}).$$

So, by Proposition 8.4,

$$c(F_j) = \bigcup_{i=1}^{n} c(T_{ij}).$$

But the $T_{ij}$ are connected and so it follows that

$$c(F_j) = c(G).$$

By Proposition 6.2, the $K_i$ are disjoint. Thus the $T_{ij}$ are disjoint. So
\[ |E(F_j)| = \sum_{i=1}^{n} |E(T_{ij})| \]

and

\[ |V(F_j)| = \sum_{i=1}^{n} |V(T_{ij})| . \]

Therefore, using Proposition 15.2,

\[ |E(F_j)| = \sum_{i=1}^{n} |E(T_{ij})| \]

\[ = \sum_{i=1}^{n} (|V(T_{ij})| - c(T_{ij})) \]

\[ = |V(F_j)| - c(F_j) . \]

Thus, by Proposition 15.2, \( F_j \) is a forest. But \( F \subseteq G \), \( V(F_j) = V(G) \), and \( c(F_j) = c(G) \), so, by Proposition 14.2, \( F_j \) is a principal forest of \( G \).

Since the \( K_i \) are disjoint and, for \( 1 \leq i \leq n \), \( E(T_{ij}) \subseteq E(K_i) \), we have, for \( 1 \leq j \leq 3 \),

\[ \bigcup_{i=1}^{n} (E(K_i) - E(T_{ij})) \]

\[ = \left( \bigcup_{i=1}^{n} E(K_i) \right) - \left( \bigcup_{i=1}^{n} E(T_{ij}) \right) . \]
Moreover,

\[ n \bigcup_{i=1}^{n} E(K_i) = E(G) \]

and, for \( 1 \leq j \leq 3 \),

\[ n \bigcup_{i=1}^{n} E(T_{ij}) = E(F_j) . \]

Thus

\[ E(G) = \bigcup_{i=1}^{n} E(K_i) \]

\[ = \bigcup_{i=1}^{n} \left( \bigcup_{j=1}^{3} E(K_i) - E(T_{ij}) \right) \]

\[ = \bigcup_{j=1}^{3} \left( \bigcup_{i=1}^{n} E(K_i) - E(T_{ij}) \right) \]

\[ = \bigcup_{j=1}^{3} \left( \bigcup_{i=1}^{n} E(K_i) - (\bigcup_{i=1}^{n} E(T_{ij})) \right) \]

\[ = \bigcup_{j=1}^{3} \left( E(G) - E(F_j) \right) . \]
PROPOSITION 25.3

If every bond of a graph $G$ has at least three edges, then there are collections $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_3$ of edge-disjoint polygons of $G$ such that

$$\bigcup_{i=1}^{3} \mathcal{P}_i = E(G).$$

PROOF

Let $G$ be a graph.

Suppose that every bond of $G$ has at least three edges. Then, by Proposition 25.2, there are principal forests $F_1$, $F_2$, and $F_3$ such that

$$\bigcup_{i=1}^{3} (E(G) - E(F_i)) = E(G).$$

It follows from Proposition 17.4 that, for each $i \in \{1, 2, 3\}$, there is a collection $\mathcal{P}_i$ of edge-disjoint polygons of $G$ such that

$$E(\bigcup_{i=1}^{3} \mathcal{P}_i) \geq E(G) - E(F_i).$$

Thus

$$\bigcup_{i=1}^{3} \mathcal{P}_i \geq \bigcup_{i=1}^{3} (E(G) - E(F_i)) = E(G).$$
So

\[ \sum_{i=1}^{3} E(U \mathcal{F}_i) = E(G). \]
CHAPTER 3  CYCLES AND COLOUR-CYCLES

(Sections 26 - 38)

Sections 26 - 35 are concerned with the content of pages 80 - 84 of W. T. Tutte's paper [15]. In the last of these sections Tutte's conjectures are discussed and it is shown that in order to prove Conjecture 1 it is sufficient to prove the following proposition:

(Proposition 38.4)

If a graph $G$ has no isthmus, then it has a colour-cycle over $\mathbb{Z}_8$.

In Sections 36 and 37 two results concerning the existence of colour-cycles are proved. These results are used in Section 38 together with Proposition 25.3 to prove Proposition 38.4.

SECTION 26  $\mathbb{R}$, $\mathbb{Z}$, AND $\mathbb{Z}_n$

In the sequel

$\mathbb{R}$ denotes a commutative ring with a unit element;
denotes the ring of integers; and

\[ \mathbb{Z}_n \] denotes the ring of integers modulo \( n \), where \( n \) is an integer greater than or equal to 2. We consider \( \mathbb{Z}_n \) to consist of the subset \( \{0, \ldots, n - 1\} \) of \( \mathbb{Z} \) together with integral addition modulo \( n \) and integral multiplication modulo \( n \).

SECTION 27 \hspace{1cm} ORIENTATIONS

DEFINITION 27.1 \hspace{1cm} [15, p. 81]

Let \( G \) be a graph.

An orientation of \( G \) over \( R \) is a map \( \eta \) from \( E(G) \times V(G) \) to \( R \) which satisfies the following two conditions:

(1) for \( e \in E(G) \) and \( v \in V(G) \), if \( e \) is a loop of \( G \) or if \( v \) is not an end of \( e \), then \( \eta(e,v) = 0 \); and

(2) for \( e \in E(G) \) and \( v, w \in V(G) \), if \( e \) is a link of \( G \) and \( v \) and \( w \) are its ends, then either \( \eta(e,v) = 1 \) and \( \eta(e,w) = -1 \) or \( \eta(e,v) = -1 \) and \( \eta(e,w) = 1 \).
PROPOSITION 27.2

If $G$ is a graph with no edges, then there is just one orientation of $G$ over $\mathbb{R}$.

PROOF

Let $G$ be a graph with no edges. Then $E(G) \times V(G)$ is empty and so there is just one map from $E(G) \times V(G)$ to $\mathbb{R}$. Moreover, Conditions (1) and (2) of Definition 27.1 are vacuous when $E(G)$ is empty and so this map is an orientation of $G$ over $\mathbb{R}$.

PROPOSITION 27.3

If $G$ is a graph, then there is an orientation of $G$ over $\mathbb{R}$.

SECTION 28  CYCLES

DEFINITION 28.1

Let $f$ be a map from a finite set $E$ to $\mathbb{R}$.

If $A$ is a non-empty subset of $E$, then

$$\sum_{a \in A} f(a)$$
denotes the sum of the images of the elements of $E$.

If $A$ is the empty subset of $E$, then we define

$$\sum_{a \in A} f(a)$$

to be the zero element of $R$.

**DEFINITION 28.2** [15, p. 82]

Let $G$ be a graph and let $\eta$ be an orientation of $G$ over $R$.

An $\eta$-cycle of $G$ over $R$ is a map $f$ from $E(G)$ to $R$ which satisfies the following condition:

$$\sum_{e \in E(G)} \eta(e,v) f(e) = 0,$$

for every $v \in V(G)$,

**DEFINITION 28.3** [15, p. 82]

Let $G$ be a graph.

A cycle of $G$ over $R$ is a map from $E(G)$ to $R$ which is an $\eta$-cycle of $G$ over $R$ for some orientation $\eta$ of $G$ over $R$. 
PROPOSITION 28.4

If $G$ is a graph with no edges, then there is just one cycle of $G$ over $\mathbb{R}$.

SECTION 29 SUMS OF CYCLES

PROPOSITION 29.1 [18, p. 16]

Let $E$ be a set.

If $f_1, \ldots, f_m$ are maps from $E$ to $\mathbb{R}$ and if $\rho_1, \ldots, \rho_m$ are elements of $\mathbb{R}$, then there is just one map $f$ from $E$ to $\mathbb{R}$ which satisfies the condition:

for every $e \in E$,

$$f(e) = \rho_1 f_1(e) + \cdots + \rho_m f_m(e).$$

We denote this map $f$ by $\rho_1 f_1 + \cdots + \rho_m f_m$.

PROPOSITION 29.2 [18, p. 16]

Let $G$ be a graph and let $\mathfrak{n}$ be an orientation of $G$ over $\mathbb{R}$.

If $f_1, \ldots, f_m$ are $\mathfrak{n}$-cycles of $G$ over $\mathbb{R}$ and if $\rho_1, \ldots, \rho_m$ are elements of $\mathbb{R}$, then $\rho_1 f_1 + \cdots + \rho_m f_m$ is an $\mathfrak{n}$-cycle of $G$ over $\mathbb{R}$. 
PROOF

Let $f_1, \ldots, f_m$ be $n$-cycles of $G$ over $R$ and let $\rho_1, \ldots, \rho_m$ be elements of $R$.

If $G$ has no edge, then $\rho_1 f_1 + \cdots + \rho_m f_m$ is the unique cycle of $G$ over $R$.

Suppose that $E(G) \neq \emptyset$. Then $V(G) \neq \emptyset$. Let $v \in V(G)$. Then

$$\sum_{e \in E(G)} \eta(e,v) (\rho_1 f_1 + \cdots + \rho_m f_m)(e)$$

$$= \sum_{i=1}^{m} \rho_i \left( \sum_{e \in E(G)} \eta(e,v) f_i(e) \right)$$

$$= 0.$$ 

Thus $\rho_1 f_1 + \cdots + \rho_m f_m$ is an $n$-cycle of $G$ over $R$.

SECTION 30 CIRCULAR PATHS AND CYCLES

DEFINITION 30.1

If $v$ is a vertex of a graph $G$, then we define $L(G,v)$ to be the set of links of $G$ which are incident with $v$. 
PROPOSITION 30.2 \[12, \text{p. 364}\]

Let $G$ be a graph, let $\eta$ be an orientation of $G$ over $\mathbb{R}$, and let $f$ be a map from $E(G)$ to $\mathbb{R}$. If $v$ is a vertex of $G$, then

$$\sum_{e \in E(G)} \eta(e,v) f(e) = \sum_{e \in L(G,v)} \eta(e,v) f(e).$$

PROPOSITION 30.3

Let $G$ be a graph and let $\eta$ be an orientation of $G$ over $\mathbb{R}$. Let

$$\alpha = (x_0, a_1, x_1, \ldots, a_m, x_m)$$

be a circular path in $G$ with at least two edge-terms.

Define a map $f$ from $E(G)$ to $\mathbb{R}$ by the following two conditions:

(1) for $i \in \{1, \ldots, m\}$, $f(a_i) = \eta(a_i, x_i)$; and

(2) for $e \in E(G) - E(\alpha)$, $f(e) = 0$.

Then $f$ is an $\eta$-cycle of $G$ over $\mathbb{R}$.

PROOF

Let $i \in \{1, \ldots, m\}$. Then it follows from the
definition of a circular path that $a_i$ and $a_{i+1}$ are
links of $G$ incident with $x_i$. Thus
\[ \{ a_i, a_{i+1} \} \subseteq L(G, x_i) \]. So
\[
\sum_{e \in L(G, x_i)} \eta(e, x_i) f(e)
\]
\[
= \sum_{e \in L(G, x_i) - \{ a_i, a_{i+1} \}} \eta(e, x_i) f(e)
+ \eta(a_i, x_i) f(a_i) + \eta(a_{i+1}, x_i) f(a_{i+1})
\]
Now, since $a_i$ and $a_{i+1}$ are the only edges in $E(\alpha)$
which are incident with $x_i$,
\[ L(G, x_i) \cap E(\alpha) \subseteq \{ a_i, a_{i+1} \} \]
Thus, if $e \in L(G, x_i) - \{ a_i, a_{i+1} \}$, then $f(e) = 0$.
So
\[
\sum_{e \in L(G, x_i) - \{ a_i, a_{i+1} \}} \eta(e, x_i) f(e) = 0
\]
Moreover
\[ \eta(a_i, x_i) f(a_i) = \eta(a_i, x_i) \eta(a_i, x_i) = 1 \]
and
\[ \eta(a_{i+1}, x_i) f(a_{i+1}) = \eta(a_{i+1}, x_i) \eta(a_{i+1}, x_{i+1}) = -1 \]
Thus
\[
\sum_{e \in L(G, x_1)} \eta(e, x_1) f(e) = 0 .
\]

So, by Proposition 30.2,

\[
\sum_{e \in E(G)} \eta(e, x_1) f(e) = 0 .
\]

Similarly

\[
\sum_{e \in E(G)} \eta(e, x_0) f(e) = 0 .
\]

Finally, suppose that \( v \in V(G) - V(\alpha) \). Then \( L(G, v) \cap E(\alpha) = \emptyset \). Thus, if \( e \in L(G, v) \), then \( f(e) = 0 \). So

\[
\sum_{e \in L(G, v)} \eta(e, v) f(e) = 0 .
\]

Thus, by Proposition 30.2,

\[
\sum_{e \in E(G)} \eta(e, v) f(e) = 0 .
\]
SECTION 31 COLOUR-CYCLES

DEFINITION 31.1 [18, p. 82]

Let G be a graph and let \( \eta \) be an orientation of G over R.

An \textit{n-colour-cycle} of G over R is an \( n \)-cycle of G over R such that, for every \( e \in E(G) \), \( f(e) \neq 0 \).

DEFINITION 31.2 [18, p. 82]

Let G be a graph.

A \textit{colour-cycle} of G over R is a map from \( E(G) \) to R which is an \( n \)-colour-cycle of G over R for some orientation \( \eta \) of G over R.

PROPOSITION 31.3

If G is a graph with no edges, then the unique cycle of G over R is a colour-cycle of G over R.
PROPOSITION 32.1 \([18, \text{ p. } 82]\)

Let \(G\) be a graph and let \(\eta\) and \(\lambda\) be orientations of \(G\) over \(\mathbb{R}\).

Then there is a one-one correspondence from the set of \(\eta\)-cycles of \(G\) over \(\mathbb{R}\) to the set of \(\lambda\)-cycles of \(G\) over \(\mathbb{R}\).

Moreover, the appropriate restriction of this correspondence is a one-one correspondence from the set of \(\eta\)-colour-cycles of \(G\) over \(\mathbb{R}\) to the set of \(\lambda\)-colour-cycles of \(G\) over \(\mathbb{R}\).

PROOF

Let \(f\) be a map from \(E(G)\) to \(\mathbb{R}\). Then there is just one map \(g\) from \(E(G)\) to \(\mathbb{R}\) which satisfies the following two conditions:

(1) if \(e\) is a loop of \(G\), then \(g(e) = f(e)\); and

(2) if \(e\) is a link of \(G\) and \(v\) and \(w\) are the ends of \(e\), then

\[
g(e) = \lambda(e, v) \eta(e, v) f(e) = \lambda(e, w) \eta(e, w) f(e) .
\]
Suppose that $v \in V(G)$. Then, by Proposition 30.2,

$$\sum_{e \in E(G)} \lambda(e,v) g(e) = \sum_{e \in L(G,v)} \lambda(e,v) g(e).$$

Now, if $e \in L(G,v)$, then $e$ is a link of $G$ and $v$ is an end of $e$. Thus

$$\sum_{e \in L(G,v)} \lambda(e,v) g(e)$$

$$= \sum_{e \in L(G,v)} \lambda(e,v) \left( \lambda(e,v) \eta(e,v) f(e) \right)$$

$$= \sum_{e \in L(G,v)} \eta(e,v) f(e).$$

Finally, using Proposition 30.2 again,

$$\sum_{e \in L(G,v)} \eta(e,v) f(e) = \sum_{e \in E(G)} \eta(e,v) f(e).$$

It follows that $g$ is a $\lambda$-cycle of $G$ over $R$ if and only if $f$ is an $\eta$-cycle of $G$ over $R$.

If $e$ is a link of $G$ and $v$ is an end of $e$, then either $\lambda(e,v) = 1$ or $\lambda(e,v) = -1$, and either $\eta(e,v) = 1$ or $\eta(e,v) = -1$. Thus, if $e$ is a link of $G$ and $v$ is an end of $e$, then either $\lambda(e,v) \eta(e,v) = 1$ or $\lambda(e,v) \eta(e,v) = -1$. So, for every $e \in E(G)$, either $g(e) = f(e)$ or
\[ g(e) = -f(e) \]. Therefore, for every \( e \in E(G) \),
\[ g(e) = 0 \] if and only if \( f(e) = 0 \).

It follows that \( g \) is a \( \lambda \)-colour-cycle of \( G \) over \( \mathbb{R} \) if and only if \( f \) is an \( \eta \)-colour-cycle of \( G \) over \( \mathbb{R} \).

SECTION 33  EXISTENCE OF COLOUR-CYCLES 1. ISTMUSES

PROPOSITION 33.1 (Tutte) [18, p. 82]

If a graph \( G \) has an isthmus, then there is no colour-cycle of \( G \) over \( \mathbb{R} \).

PROOF

Let \( b \) be an isthmus of a graph \( G \). Then \( \{b\} \) is a bond of \( G \).

Let

\[ H = G : (E(G) - \{b\}) \] .

Then, by Proposition 10.4, \( b \) is a link of \( G \) whose two ends belong to different components of \( H \).

Let \( y \) be an end of \( b \) and let \( K \) be the component of \( H \) which includes \( y \).

Let \( \eta \) be an orientation of \( G \) over \( \mathbb{R} \) and
let $f$ be an $n$-cycle of $G$ over $R$. Then, for every $v \in V(G)$,

$$
\sum_{e \in E(G)} \eta(e,v) f(e) = 0.
$$

Thus

$$
\sum_{v \in V(K)} \sum_{e \in E(G)} \eta(e,v) f(e) = 0.
$$

So

$$
\sum_{e \in E(G)} \left( f(e) \sum_{v \in V(K)} \eta(e,v) \right) = 0.
$$

Suppose that $e \in E(G) - \{b\}$. Then $e \in E(H)$. Thus, since $K$ is a component of $H$, it follows from Propositions 6.2 and 3.2 that either $I(G,e) \cap V(K) = \emptyset$ or $I(G,e) \subseteq V(K)$. In either case it follows from the definition of an orientation that

$$
\sum_{v \in V(K)} \eta(e,v) = 0.
$$

Therefore

$$
f(b) \sum_{v \in V(K)} \eta(b,v) = 0.
$$

But, if $v \in V(K) - \{y\}$, then $v$ is not an end of $b$ and so $\eta(b,v) = 0$. Thus

$$
f(b) \eta(b,y) = 0.
$$
However, \( b \) is a link of \( G \) and \( y \) is an end of \( b \) so, by the definition of an orientation, either \( n(b,y) = 1 \) or \( n(b,y) = -1 \). So

\[
 f(b) = 0.
\]

Thus \( f \) is not an \( n \)-colour-cycle of \( G \) over \( R \).

SECTION 34  EXISTENCE OF COLOUR-CYCLES 2.

\[
\mathbb{Z}_n - \mathbb{Z} - \mathbb{Z}_{n+1}
\]

REMARK 34.1

In this section the concept of an integral flow in a graph is introduced and used to prove a result of Tutte's (Proposition 34.13).

The word "flow" is used by Berge [1, p. 76] and Rota [12, p. 364] to mean cycle. This is not the sense in which it is used here. In defence of this nonconformity, the usage of Ford and Fulkerson [8, p. 4] may be cited.

DEFINITION 34.2

An integral flow \((n,f)\) in a graph \(G\) consists of an orientation \( n \) of \( G \) over \( \mathbb{Z} \) together with a map \( f \) from \( E(G) \) to \( \mathbb{Z} \).
DEFINITION 34.3 [8, p. 4]

If \((n,f)\) is an integral flow in a graph \(G\), then an \((n,f)\)-source is a vertex \(v\) of \(G\) such that
\[
\sum_{e \in E(G)} \eta(e,v) f(e) > 0.
\]
and an \((n,f)\)-sink is a vertex \(v\) of \(G\) such that
\[
\sum_{e \in E(G)} \eta(e,v) f(e) < 0.
\]

PROPOSITION 34.4

Let \((\eta,f)\) be an integral flow in a graph \(G\).

Then: there is an \((\eta,f)\)-source if and only if there is an \((\eta,f)\)-sink.

PROOF

Let
\[
S = \{ v \in V(G) : v \text{ is an } (\eta,f)\text{-source} \}
\]
and
\[
T = \{ v \in V(G) : v \text{ is an } (\eta,f)\text{-sink} \}.
\]

Then \(S\) and \(T\) are disjoint. So
\[
\sum_{v \in S} \sum_{e \in E(G)} \eta(e,v) f(e) \\
+ \sum_{v \in T} \sum_{e \in E(G)} \eta(e,v) f(e) \\
+ \sum_{v \in V(G) - (S \cup T)} \sum_{e \in E(G)} \eta(e,v) f(e) \\
= \sum_{v \in V(G)} \sum_{e \in E(G)} \eta(e,v) f(e) \\
= \sum_{e \in E(G)} \left( f(e) \sum_{v \in V(G)} \eta(e,v) \right).
\]

But, for every \( v \in V(G) - (S \cup T) \),
\[
\sum_{e \in E(G)} \eta(e,v) f(e) = 0.
\]

Further, by the definition of an orientation, for every \( e \in E(G) \),
\[
\sum_{v \in V(G)} \eta(e,v) = 0.
\]

Thus
\[
\sum_{v \in S} \sum_{e \in E(G)} \eta(e,v) f(e) \\
= - \sum_{v \in T} \sum_{e \in E(G)} \eta(e,v) f(e).
\]

So \( S \) is non-empty if and only if \( T \) is non-empty.
**DEFINITION 34.5**

Let \((\eta, f)\) be an integral flow in a graph \(G\).

A path

\[(x_0, a_1, x_1, \ldots, a_m, x_m)\]

in \(G\) is an \((\eta, f)\)-path if, for every \(i \in \{1, \ldots, m\}\),

\[\eta(a_i, x_i) f(a_i) < 0.\]

**PROPOSITION 34.6**

If \((\eta, f)\) is an integral flow in a graph \(G\) and \(v\) is an \((\eta, f)\)-source, then there is a simple \((\eta, f)\)-path in \(G\) from \(v\) to some \((\eta, f)\)-sink.

**PROOF**

Let \(G\) be a graph and let \(\eta\) be an orientation of \(G\) over \(\mathbb{Z}\).

Let \(F'\) be the set of all maps \(f'\) from \(E(G)\) to \(\mathbb{Z}\) which satisfy the following condition:

there is an \((\eta, f')\)-source \(v\) with the property that there is no simple \((\eta, f')\)-path in \(G\) from \(v\) to an \((\eta, f')\)-sink.
Let $F$ be the subset of $F'$ which satisfies the following condition:

$$\sum_{e \in E(G)} |f(e)| \leq \sum_{e \in E(G)} |f'(e)|.$$ 

Then: $F'$ is empty if and only if $F$ is empty.

We show that $F$ is empty.

Suppose that $f \in F$. Then there is an $(\eta,f)$-source $x_0$ with the property that there is no simple $(\eta,f)$-path in $G$ from $x_0$ to an $(\eta,f)$-sink.

Since $x_0$ is an $(\eta,f)$-source,

$$\sum_{e \in E(G)} \eta(e, x_0) f(e) > 0.$$ 

So there is a link $a_1$ of $G$ such that $x_0$ is an end of $a_1$ and

$$\eta(a_1, x_0) f(a_1) > 0.$$ 

Let $x_1$ be the other end of $a_1$. Then

$$\eta(a_1, x_1) f(a_1) < 0.$$ 

Thus the sequence

$$a_1 = (x_0, a_1, x_1)$$
is a simple \((n,f)\)-path in \(G\) from \(x_0\).

Suppose that \(m \geq 1\) and that the sequence

\[
\alpha_m = (x_0, a_1, x_1, \ldots, a_m, x_m)
\]

is a simple \((n,f)\)-path in \(G\) from \(x_0\). Then

\[
\eta(a_m, x_m) f(a_m) < 0.
\]

Moreover, it follows from the choice of \(x_0\) that \(x_m\) is not an \((n,f)\)-sink. Thus there is a link \(a_{m+1}\) of \(G\) such that \(x_m\) is an end of \(a_{m+1}\) and

\[
\eta(a_{m+1}, x_m) f(a_{m+1}) > 0.
\]

Let \(x_{m+1}\) be the other end of \(a_{m+1}\). Then

\[
\eta(a_{m+1}, x_{m+1}) f(a_{m+1}) < 0.
\]

It follows that the sequence

\[
\alpha_{m+1} = (x_0, a_1, x_1, \ldots, a_m, x_m, a_{m+1}, x_{m+1})
\]

is an \((n,f)\)-path in \(G\) from \(x_0\).

Suppose that, for every \(k \in \{0, \ldots, m-1\}\), \(x_{m+1} \neq x_k\). Then, since \(\alpha_m\) is simple, it follows that \(\alpha_{m+1}\) is simple. Thus \(\alpha_{m+1}\) is a simple \((n,f)\)-path in \(G\) from \(x_0\) and \(V(\alpha_m) \subset V(\alpha_{m+1})\).

On the other hand, suppose that there is a \(k \in \{0, \ldots, m-1\}\) such that \(x_{m+1} = x_k\). Let
\[ \alpha_{m+1}^k = (x_k, a_{k+1}, x_{k+1}, \ldots, a_m, x_m, a_{m+1}, x_{m+1}) \].

Then, since \( \alpha_{m+1} \) is an \((\eta, f)\)-path in \( G \), it follows that \( \alpha_{m+1}^k \) is an \((\eta, f)\)-path in \( G \). Since \( \alpha_m \) is simple we have:

for \( k \leq i < j \leq m \), \( x_i \neq x_j \); and,

for \( k + 1 \leq i < j \leq m \), \( a_i \neq a_j \).

Moreover,

for \( k + 1 \leq i \leq m - 1 \), \( x_m \) is not an end of \( a_i \).

But \( x_m \) is an end of \( \alpha_{m+1} \). Thus,

for \( k + 1 \leq i \leq m - 1 \), \( \alpha_{m+1} \neq a_i \).

Further, since

\[ \eta(a_m, x_m) f(a_m) < 0 \]

and

\[ \eta(a_{m+1}, x_m) f(a_{m+1}) > 0 \],

it follows that

\[ \alpha_{m+1} \neq \alpha_m \).

Thus \( \alpha_{m+1}^k \) is a circular \((\eta, f)\)-path in \( G \) with at least two edge-terms.
Since, for any path \( a \) in \( G \), \( V(a) \subseteq V(G) \) and since \( V(G) \) is finite, we conclude that there is a circular \((\eta,f)\)-path

\[
\beta = (y_0, b_1, y_1, \ldots, b_n, y_n)
\]

in \( G \) with at least two edge-terms.

Let \( g \) be the map from \( E(G) \) to \( \mathbb{Z} \) such that

for \( i \in \{1, \ldots, n\} \), \( g(b_i) = \eta(b_i, y_i) \); and,

for \( e \in E(G) - E(\beta) \), \( g(e) = 0 \).

Then, by Proposition 30.3, \( g \) is an \( n \)-cycle of \( G \) over \( \mathbb{Z} \).

Let \( h = f + g \). Then, since \( g \) is an \( n \)-cycle, it follows that, for every \( v \in V(G) \),

\( v \) is an \((\eta,h)\)-source (respectively \((\eta,h)\)-sink)

if and only if

\( v \) is an \((\eta,f)\)-source (respectively \((\eta,f)\)-sink).

Suppose that \( i \in \{1, \ldots, n\} \). Then, since \( \beta \) is an \((\eta,f)\)-path,

\[
\eta(b_i, y_i) f(b_i) < 0 .
\]

Thus

\[
|f(b_i) + \eta(b_i, y_i)| < |f(b_i)| .
\]
So

\[ |h(b_i)| < |f(b_i)|. \]

Moreover, for \( e \in E(G) - E(\beta) \),

\[ h(e) = f(e). \]

Therefore

\[ \sum_{e \in E(G)} |h(e)| < \sum_{e \in E(G)} |f(e)|. \]

Thus, since \( f \in F \), it follows that \( h \notin F' \).

So, since \( x_0 \) is an \((\eta, h)\)-source, there is a simple \((\eta, h)\)-path

\[ \gamma = (z_0, c_1, z_1, \ldots, c_p, z_p) \]

in \( G \) from \( x_0 \) to an \((\eta, h)\)-sink.

Suppose that \( c_i = b_j \) for some \( i \in \{1, \ldots, p\} \) and \( j \in \{1, \ldots, n\} \). Then either \( z_i = y_{j-1} \) or \( z_i = y_j \).

Suppose that \( z_i = y_{j-1} \). Then

\[ \eta(c_i, z_i) = \eta(b_j, y_{j-1}) = -\eta(b_j, y_j) \]

and

\[ h(c_i) = h(b_j) = f(b_j) + g(b_j) \]

\[ = f(b_j) + \eta(b_j, y_j). \]
Thus

\[ \eta(c_i, z_i) h(c_i) = -\eta(b_j, y_j) f(b_j) - 1. \]

But \( \beta \) is an \((\eta, f)\)-path so

\[ \eta(b_j, y_j) f(b_j) < 0. \]

Thus

\[ \eta(c_i, z_i) h(c_i) \geq 0. \]

However, this contradicts the fact that \( \gamma \) is an \((\eta, h)\)-path.

We conclude that \( z_i = y_j \). Then

\[ \eta(c_i, z_i) = \eta(b_j, y_j) \]

and

\[ h(c_i) = h(b_j) = f(b_j) + g(b_j) \]

\[ = f(b_j) + \eta(b_j, y_j) \]

Thus

\[ \eta(c_i, z_i) h(c_i) = \eta(b_j, y_j) f(b_j) + 1. \]

So

\[ \eta(c_i, z_i) f(c_i) = \eta(b_j, y_j) f(b_j) \]

\[ = \eta(c_i, z_i) h(c_i) - 1 < 0. \]
On the other hand, if \( c_i \notin E(\beta) \), then
\[
h(c_i) = f(c_i)
\]
and so
\[
\eta(c_i, z_i) f(c_i) = \eta(c_i, z_i) h(c_i) < 0.
\]

Thus \( \gamma \) is an \((n,f)\)-path in \( G \) from \( x_0 \) to an \((n,f)\)-sink. But this contradicts the choice of \( x_0 \).

We conclude that \( F \) is empty.

**DEFINITION 34.7**

If \((n,f)\) is an integral flow in a graph \( G \), then we define \( S(n,f) \) and \( \sigma(n,f) \) as follows:

\[
S(n,f) = \{ v \in V(G) : v \text{ is an } (n,f)\text{-source} \}
\]

and

\[
\sigma(n,f) = \left\{ \sum_{v \in S(n,f)} \sum_{e \in E(G)} \eta(e,v) f(e) \right\}.
\]

**PROPOSITION 34.8**

If \((n,f)\) is an integral flow in a graph \( G \) and \( \sigma(n,f) = 0 \), then \( f \) is an \( n \)-cycle of \( G \) over \( Z \).

**PROOF**

Let \((n,f)\) be an integral flow in a graph \( G \).
Suppose that \( \sigma(\eta, f) = 0 \). Then \( S(\eta, f) \) is empty. That is, there are no \((\eta, f)\)-sources. Thus, by Proposition 34.4, there are no \((\eta, f)\)-sinks. Thus \( f \) must be an \( \eta \)-cycle.

**Proposition 34.9**

Let \( \lambda \) be an orientation of a graph \( G \) over \( \mathbb{Z} \) and let \( \eta \) be an integer greater than or equal to 2.

If there is a map \( g \) from \( E(G) \) to \( \mathbb{Z} \) such that

1. for every \( v \in V(G) \),
   \[ \sum_{e \in E(G)} \lambda(e,v)g(e) \equiv 0 \pmod{n} \]
2. for every \( e \in E(G) \), \( |g(e)| < n \); and
3. \( \sigma(\lambda, g) > 0 \);

then there is a map \( h \) from \( E(G) \) to \( \mathbb{Z} \) such that

4. for every \( v \in V(G) \),
   \[ \sum_{e \in E(G)} \lambda(e,v)h(e) \equiv 0 \pmod{n} \]
5. for every \( e \in E(G) \), \( h(e) \equiv g(e) \pmod{n} \);
6. for every \( e \in E(G) \), \( |h(e)| < n \); and
7. \( \sigma(\lambda, g) > \sigma(\lambda, h) \).
PROOF

Let $g$ be a map from $E(G)$ to $\mathbb{Z}$ which satisfies Conditions (1), (2), and (3). Then, since $\sigma(\lambda, g) > 0$, there is a $(\lambda, g)$-source $x_0$. Thus, by Proposition 34.6, there is a simple $(\lambda, g)$-path

$$\alpha = (x_0, a_1, x_1, \ldots, a_m, x_m)$$
in $G$ from $x_0$ to a $(\lambda, g)$-sink $x_m$.

Define a map $h$ from $E(G)$ to $\mathbb{Z}$ as follows:

- for $i \in \{1, \ldots, m\}$,
  $$h(a_i) = g(a_i) + \lambda(a_i, x_i) n;$$
- for $e \in E(G) - E(\alpha)$,
  $$h(e) = g(e).$$

Then it is clear that $h$ satisfies Conditions (5) and (4).

If $e \in E(G) - E(\alpha)$, then $h(e) = g(e)$ and so $|h(e)| < n$.

Suppose that $i \in \{1, \ldots, m\}$. Then, since $\alpha$ is a $(\lambda, g)$-path,

$$\lambda(a_i, x_i) g(a_i) < 0.$$ 

Thus $g(a_i)$ and $\lambda(a_i, x_i)$ are of opposite sign.

Since $|g(a_i)| < n$, it follows that
\[|g(a_i) + \lambda(a_i, x_i) n| < n.\]

So \(|h(a_i)| < n.\)

Thus \(h\) satisfies Condition (6).

If \(v \in V(G) - V(\alpha)\), then \(L(G, v) \cap E(\alpha) = \emptyset\)
and so, using Proposition 30.2,

\[\sum_{e \in E(G)} \lambda(e, v) h(e) = \sum_{e \in E(G)} \lambda(e, v) g(e).\]

Suppose that \(i \in \{1, \ldots, m-1\}\). Then \(L(G, x_i) \cap E(\alpha) = \{a_i, a_{i+1}\}\). Thus

\[\sum_{e \in L(G, x_i) - \{a_i, a_{i+1}\}} \lambda(e, x_i) h(e) = \sum_{e \in L(G, x_i) - \{a_i, a_{i+1}\}} \lambda(e, x_i) g(e).\]

But

\[\lambda(a_i, x_i) h(a_i) + \lambda(a_{i+1}, x_i) h(a_{i+1}) = \lambda(a_i, x_i)(g(a_i) + \lambda(a_i, x_i) n) + \lambda(a_{i+1}, x_i)(g(a_{i+1}) + \lambda(a_{i+1}, x_{i+1}) n) = (\lambda(a_i, x_i) g(a_i) + n) + (\lambda(a_{i+1}, x_i) g(a_{i+1}) - n) = \lambda(a_i, x_i) g(a_i) + \lambda(a_{i+1}, x_i) g(a_{i+1}).\]
So

\[ \sum_{e \in L(G,x_1)} \lambda(e,x_1) h(e) = \sum_{e \in L(G,x_1)} \lambda(e,x_1) g(e) \]

Thus, by Proposition 30.2,

\[ \sum_{e \in E(G)} \lambda(e,x_1) h(e) = \sum_{e \in E(G)} \lambda(e,x_1) g(e). \]

Now, \( L(G,x_0) \cap E(a) = \{a_1\} \). Thus

\[ \sum_{e \in L(G,x_0) - \{a_1\}} \lambda(e,x_0) h(e) = \sum_{e \in L(G,x_0) - \{a_1\}} \lambda(e,x_0) g(e). \]

Moreover,

\[ \lambda(a_1,x_0) h(a_1) = \lambda(a_1,x_0) \left( g(a_1) + \lambda(a_1,x_1) n \right) = \lambda(a_1,x_0) g(a_1) - n. \]

Thus

\[ \sum_{e \in E(G)} \lambda(e,x_0) h(e) = \left( \sum_{e \in E(G)} \lambda(e,x_0) g(e) \right) - n. \]

Since \( g \) satisfies Conditions (1) and (3), it follows that
\[ \sum_{e \in E(G)} \lambda(e, x_0) g(e) > \sum_{e \in E(G)} \lambda(e, x_0) h(e) \geq 0. \]

Similarly
\[ \sum_{e \in E(G)} \lambda(e, x_m) g(e) < \sum_{e \in E(G)} \lambda(e, x_m) h(e) \leq 0. \]

Therefore
\[ \sigma(\lambda, g) > \sigma(\lambda, h). \]

Thus \( h \) satisfies Condition (7).

**PROPOSITION 34.10**

Let \( \lambda \) be an orientation of a graph \( G \) over \( \mathbb{Z} \) and let \( n \) be an integer greater than or equal to 2.

If there is a map \( g \) from \( E(G) \) to \( \mathbb{Z} \) such that

1. for every \( v \in V(G) \),
   \[ \sum_{e \in E(G)} \lambda(e, v) g(e) \equiv 0 \pmod{n}; \text{ and} \]
2. for every \( e \in E(G) \), \( |g(e)| < n \);

then there is a \( \lambda \)-cycle \( h \) of \( G \) over \( \mathbb{Z} \) such that

3. for every \( e \in E(G) \), \( h(e) \equiv g(e) \pmod{n} \) and \( |h(e)| < n \).
PROOF

If \((\eta, f)\) is an integral flow in \(G\), then \(\sigma(\eta, f)\) is finite and positive. Thus the proposition follows easily from Propositions 34.9 and 34.8.

PROPOSITION 34.11 (Tutte) [15, p. 83, Thm] and [14, p. 478, Thm IV]

Let \(G\) be a graph and let \(n\) be an integer greater than or equal to 2.

If there is a cycle \(f\) of \(G\) over \(\mathbb{Z}_n\), then there is a cycle \(h\) of \(G\) over \(\mathbb{Z}\) such that, for every \(e \in E(G)\), \(h(e) \equiv f(e) \pmod{n}\) and \(|h(e)| < n\).

PROOF

Let \(f\) be a cycle of \(G\) over \(\mathbb{Z}_n\). Then \(f\) is an \(n\)-cycle of \(G\) over \(\mathbb{Z}_n\) for some orientation \(n\) of \(G\) over \(\mathbb{Z}_n\).

Let \(i\) be the inclusion map from \(\mathbb{Z}_n\) to \(\mathbb{Z}\). Let \(\lambda = in\) and \(g = if\). Then \(\lambda\) and \(g\) satisfy the conditions of Proposition 34.10.

The conclusion of the proposition follows easily.
PROPOSITION 34.12

Let $G$ be a graph and let $n$ be an integer greater than or equal to 2.

Then: there is a colour-cycle of $G$ over $\mathbb{Z}_n$ if and only if there is a colour-cycle $h$ of $G$ over $\mathbb{Z}$ such that, for every $e \in E(G)$, $|h(e)| < n$.

PROOF

The "only if" part of the proposition follows directly from Proposition 34.11.

The "if" part:

Let $h$ be a colour-cycle of $G$ over $\mathbb{Z}$ such that, for every $e \in E(G)$, $|h(e)| < n$. Then there is a map $f$ from $E(G)$ to $\mathbb{Z}_n$ which satisfies the following condition:

for every $e \in E(G)$,
if $0 < h(e) < n$, then $f(e) = h(e)$; and,
if $-n < h(e) < 0$, then $f(e) = h(e) + n$.

Moreover, since $h$ is a colour-cycle of $G$ over $\mathbb{Z}$, $f$ is a colour-cycle of $G$ over $\mathbb{Z}_n$. 
PROPOSITION 34.13 (Tutte) [15, p. 83, Thm]

Let $G$ be a graph and let $n$ be an integer greater than or equal to 2.

If there is a colour-cycle of $G$ over $\mathbb{Z}_n$, then there is a colour-cycle of $G$ over $\mathbb{Z}_{n+1}$.

PROOF

Suppose that there is a colour-cycle of $G$ over $\mathbb{Z}_n$. Then, by Proposition 34.12, there is a colour-cycle $h$ of $G$ over $\mathbb{Z}$ such that, for every $e \in E(G)$, $|h(e)| < n < n + 1$. Thus, again by Proposition 34.12, there is a colour-cycle of $G$ over $\mathbb{Z}_{n+1}$.

SECTION 35  TUTTE'S CONJECTURES

Let $G$ be a graph, let $n$ be an integer greater than or equal to 2, and let $\eta$ be an orientation of $G$ over $\mathbb{Z}_n$.

We define $\phi(G,\eta)$ to be the number of $\eta$-colour-cycles of $G$ over $\mathbb{Z}_n$.

It follows from Proposition 32.1 that $\phi(G,\eta)$ does not depend on the orientation $\eta$ used in the
definition and, moreover, $\phi(G,n) > 0$ if and only if there is a colour-cycle of $G$ over $\mathbb{Z}_n$. Thus it follows from Proposition 33.1 that $\phi(G,n) = 0$ if $G$ has an isthmus.

The following two conjectures concerning $\phi(G,n)$ have been advanced by W. T. Tutte [15, p. 83].

**CONJECTURE I**

There exists a positive integer $m$ such that $\phi(G,n) > 0$ whenever $n \geq m$ and $G$ has no isthmus.

**CONJECTURE II**

$\phi(G,n) > 0$ whenever $n \geq 5$ and $G$ has no isthmus.

H. H. Crapo [5, p. 16] refers to Conjecture I as follows:

"Tutte's conjecture, that every graph is $n$-cocolorable, has remained untouched for 15 years now. The statement is false for $n = 4$ (consider the Petersen graph). But it is conceivably true for $n = 5$.

In an earlier paper [4, p. 212] Crapo says that Conjecture II "... has now remained unsettled for fifteen years."
G.-C. Rota [13, p. 231] mentions a similar conjecture.

Tutte [15, p. 83, Thm] has shown that, if \( \phi(G,n) > 0 \), then \( \phi(G,n+1) > 0 \). (i.e., Proposition 34.13.) Thus, in order to prove Conjecture I it is sufficient to exhibit an integer \( m \) such that \( \phi(G,m) > 0 \) whenever \( G \) has no isthmus. We shall show that \( \phi(G,8) > 0 \) whenever \( G \) has no isthmus. In other words: we shall show that, if a graph \( G \) has no isthmus, then there is a colour-cycle of \( G \) over \( \mathbb{Z}_8 \).

**SECTION 36  EXISTENCE OF COLOUR-CYCLES 3. SUBGRAPHS**

**PROPOSITION 36.1**

Let \( H_1, \ldots, H_m \) be edge-disjoint subgraphs of a graph \( G \) such that

\[
\bigcup_{i=1}^{m} E(H_i) = E(G) .
\]

If, for each \( i \in \{1, \ldots, m\} \), there is a colour-cycle of \( H_i \) over \( \mathbb{R} \), then there is a colour-cycle of \( G \) over \( \mathbb{R} \).
Proof

It is clear that the proposition holds if $E(G)$ is empty. Thus we may assume that $E(G)$ is not empty.

Let $\eta$ be an orientation of $G$ over $\mathbb{R}$.

Suppose that $i \in \{1, \ldots, m\}$. Then, since $H_i$ is a subgraph of $G$, we may define an orientation $\eta_i$ of $H_i$ over $\mathbb{R}$ as follows:

$$\eta_i(e,v) = \eta(e,v).$$

Then, since there is a colour-cycle of $H_i$ over $\mathbb{R}$, it follows from Proposition 32.1 that there is an $\eta_i$-colour-cycle $h_i$ of $H_i$ over $\mathbb{R}$. We define a map $f_i$ from $E(G)$ to $\mathbb{R}$ as follows:

$$f_i(e) = h_i(e);$$

for $e \in E(H_i)$, $f_i(e) = h_i(e)$; and,

$$f_i(e) = 0$$

for $e \in E(G) - E(H_i)$. Then, for every $v \in V(H_i)$,

$$\sum_{e \in E(G)} \eta(e,v) f_i(e) = \sum_{e \in E(H_i)} \eta_i(e,v) h_i(e) = 0.$$
If \( v \in V(G) - V(H_1) \) and \( e \in E(H_1) \), then, by Proposition 3.2, \( v \) is not an end of \( e \) and so \( \eta(e,v) = 0 \). Thus, for every \( v \in V(G) - V(H_1) \),

\[
\sum_{e \in E(G)} \eta(e,v) f_i(e) = 0.
\]

Thus \( f_i \) is an \( \eta \)-cycle of \( G \) over \( R \).

Let

\[
f = f_1 + \cdots + f_m.
\]

Then, by Proposition 29.2, \( f \) is an \( \eta \)-cycle of \( G \) over \( R \).

But the \( H_i \) are edge-disjoint, and

\[
\bigcup_{i=1}^{m} E(H_i) = E(G).
\]

Thus, if \( e \in E(G) \), then there is an \( i \in \{1, \ldots, m\} \) such that

\[
f(e) = f_i(e) = h_i(e).
\]

Therefore, since, for every \( i \in \{1, \ldots, m\} \), \( h_i \) is
a colour-cycle of $H_i$ over $R$, it follows that $f$ is an $\eta$-colour-cycle of $G$ over $R$.

SECTION 37  EXISTENCE OF COLOUR-CYCLES 4. POLYGONS

REMARK 37.1

The source of the main result of this section (Proposition 37.5) is to be found in discussions of the critical problem by Crapo and Rota [6, Section 16] (see also Wilson [21, p. 524]) and Tutte [18].

PROPOSITION 37.2  [9, p. 38], [1, p. 90, Cor.]

Let $G$ be a graph and let $\eta$ be an orientation of $G$ over $R$.

If $\mathcal{P}$ is a collection of edge-disjoint polygons of $G$, then there is an $\eta$-cycle $f$ of $G$ over $R$ such that:

(1) for $e \in E(G) - E(\mathcal{P})$, either $f(e) = 1$ or $f(e) = -1$; and

(2) for $e \in E(G) - E(\mathcal{P})$, $f(e) = 0$. 

PROOF

Let $\mathcal{P}$ be a collection of edge-disjoint polygons of $G$.

Suppose that $\mathcal{P}$ is empty. Then $E(\bigcup\mathcal{P})$ is empty. Let $f$ be the map from $E(G)$ to $R$ such that, for every $e \in E(G)$, $f(e) = 0$. Then $f$ is an $n$-cycle of $G$ over $R$ which satisfies Conditions (1) and (2).

Suppose that $\mathcal{P}$ is not empty.

Then, for each $P \in \mathcal{P}$, we may define an $n$-cycle $f_P$ of $G$ over $R$ as follows.

(a) Suppose that $P \in \mathcal{P}$ and $P$ is a loop-graph.
Let $f_P$ be the map from $E(G)$ to $R$ such that

- for $e \in E(P)$, $f_P(e) = 1$; and,
- for $e \in E(G) - E(P)$, $f_P(e) = 0$.

Then it is easily shown that $f_P$ is an $n$-cycle of $G$ over $R$.

(b) Suppose that $P \in \mathcal{P}$ and $P$ is not a loop-graph.
Then there is a circular path $\alpha$ in $G$ such that $\alpha$ has at least two edge-terms and $G(\alpha) = P$. Let $f_P$ be the $n$-cycle of $G$ over $R$ determined by $\alpha$ according to
Proposition 30.3. Then, for \( e \in E(P) \), either \( f_P(e) = 1 \) or \( f_P(e) = -1 \).

Let

\[
f = \sum_{P \in \mathcal{P}} f_P.
\]

Then, by Proposition 29.2, \( f \) is an \( n \)-cycle of \( G \) over \( \mathbb{R} \). Moreover, since the members of \( \mathcal{P} \) are edge-disjoint, \( f \) satisfies Conditions (1) and (2).

**PROPOSITION 37.3**

Let \( G \) be a graph.

If \( f \) is a cycle of \( G \) over \( \mathbb{Z} \), then there is a collection \( \mathcal{P} \) of edge-disjoint polygons of \( G \) such that, for every \( e \in E(G) \), \( e \in E(\cup \mathcal{P}) \) if and only if \( f(e) \) is odd.

**PROOF**

Let \( f \) be a cycle of \( G \) over \( \mathbb{Z} \). Then \( f \) is an \( \eta \)-cycle of \( G \) over \( \mathbb{Z} \) for some orientation \( \eta \) of \( G \) over \( \mathbb{Z} \).

Let

\[
H = G : \{ e \in E(G) : f(e) \text{ is odd} \}.
\]
Suppose that $v$ is a vertex of $H$. Then

$$\sum_{e \in E(G)} \eta(e,v) f(e) = 0.$$ 

Thus, by Proposition 30.2,

$$\sum_{e \in L(G,v)} \eta(e,v) f(e) = 0.$$ 

But

$$L(H,v) = \{e \in L(G,v) : f(e) \text{ is odd}\}.$$ 

So

$$\sum_{e \in L(H,v)} \eta(e,v) f(e)$$

is even. Thus $|L(H,v)|$ is even. So, by the definition of valency, $\text{val}(H,v)$ is even.

Therefore every vertex of $H$ has even valency.

Thus, by Proposition 17.2, there is a collection $\mathcal{P}$ of edge-disjoint polygons of $H$ such that

$$E(\cup \mathcal{P}) = E(H).$$

Since $H$ is a subgraph of $G$, $\mathcal{P}$ is a collection of edge-disjoint polygons of $G$. And, by the definition of $H$, for every $e \in E(G)$, $e \in E(H) = E(\cup \mathcal{P})$ if and only if $f(e)$ is odd.
PROPOSITION 37.4

Let $G$ be a graph and let $m$ be a positive integer.

If $f$ is a cycle of $G$ over $\mathbb{Z}$ such that, for every $e \in E(G)$, $|f(e)| < 2^m$, then there are collections $P_1, \ldots, P_m$ of edge-disjoint polygons of $G$ such that, for every $e \in E(G)$,

$$m \quad e \in \bigcup_{i=1}^{m} E(U \cup P_i) \quad \text{if and only if} \quad f(e) \neq 0.$$ 

PROOF

Note that we do not require the collections $P_i$ to be non-empty or to be distinct.

Let $f$ be a cycle of $G$ over $\mathbb{Z}$ such that, for every $e \in E(G)$, $|f(e)| < 2^m$. Then $f$ is an $n$-cycle of $G$ over $\mathbb{Z}$ for some orientation $n$ of $G$ over $\mathbb{Z}$.

We define a sequence

$$(f_1, \ldots, f_m)$$

of $n$-cycles of $G$ over $\mathbb{Z}$ and a sequence

$$(P_1, \ldots, P_m)$$
of collections of edge-disjoint polygons of $G$ such that, for $i \in \{1, \ldots, m\}$ and $e \in E(G)$,

$$e \in E(U_1) \text{ if and only if } f_i(e) \text{ is odd.}$$

Let $f_1 = f$.

Suppose that $i \in \{1, \ldots, m\}$ and that $f_i$ has been defined. Then, by Proposition 37.3, there is a collection $\mathcal{P}_i$ of edge-disjoint polygons of $G$ such that, for every $e \in E(G)$,

$$e \in E(U_{i+1}) \text{ if and only if } f_i(e) \text{ is odd.}$$

Suppose that $i \in \{1, \ldots, m - 1\}$ and that $f_i$ and $\mathcal{P}_i$ have been defined. Then, by Proposition 37.2, there is an $n$-cycle $g_i$ of $G$ over $\mathbb{Z}$ such that,

for $e \in E(U_{i+1})$, either $g_i(e) = 1$ or $g_i(e) = -1$; and,

for $e \in E(G) - E(U_{i+1})$, $g_i(e) = 0$.

Now, by Proposition 29.2, $f_i + g_i$ is an $n$-cycle of $G$ over $\mathbb{Z}$. Moreover, for every $e \in E(G)$,

$$(f_i + g_i)(e) \text{ is even. Thus } \frac{1}{2}(f_i + g_i) \text{ is an } n\text{-cycle of } G \text{ over } \mathbb{Z}. \text{ Let } f_{i+1} = \frac{1}{2}(f_i + g_i).$$

Note that, for $e \in E(G)$, if $f_i(e)$ is even then $f_{i+1}(e) = \frac{1}{2}f_i(e)$. 
We show that the sequence \((P_1, \ldots, P_m)\) satisfies the condition:

for every \(e \in E(G)\),

\[
\begin{align*}
\forall e \in \bigcup_{i=1}^{m} E\left(U P_i\right) \text{ if and only if } f(e) \neq 0 .
\end{align*}
\]

Let \(e \in E(G)\).

Suppose that \(f(e) \neq 0\). Then, since \(|f(e)| < 2^m\), there is a \(k \in \{1, \ldots, m-1\}\) such that \(f(e)\) is divisible by \(2^{k-1}\) but not divisible by \(2^k\). Since \(f(e)\) is divisible by \(2^{k-1}\), it follows that \(f_k(e) = (\frac{1}{2})^{k-1}f(e)\). But \((\frac{1}{2})^{k-1}f(e)\) is odd and so \(f_k(e)\) is odd. Thus \(e \in E\left(U P_k\right)\). So

\[
\begin{align*}
\forall e \in \bigcup_{i=1}^{m} E\left(U P_i\right) .
\end{align*}
\]

On the other hand, suppose that \(f(e) = 0\). Then, for \(i \in \{1, \ldots, m\}\), \(f_i(e) = 0\) and so \(e \notin E\left(U P_i\right)\). Thus

\[
\begin{align*}
\forall e \notin \bigcup_{i=1}^{m} E\left(U P_i\right) .
\end{align*}
\]
PROPOSITION 37.5

Let $G$ be a graph and let $m$ be a positive integer.

Then: there is a colour-cycle $f$ of $G$ over $Z$ such that, for every $e \in E(G)$, $|f(e)| < 2^m$ if and only if there are collections $P_1, \ldots, P_m$ of edge-disjoint polygons of $G$ such that

$$m \bigcup_{i=1}^{m} E(P_i) = E(G).$$

PROOF

The "only if" part of the proposition follows directly from Proposition 37.4.

The "if" part:

Let $P_1, \ldots, P_m$ be collections of edge-disjoint polygons of $G$ such that

$$m \bigcup_{i=1}^{m} E(P_i) = E(G).$$

Let $\eta$ be an orientation of $G$ over $Z$.

Suppose that $i \in \{1, \ldots, m\}$. Then, by Proposition 37.2, there is an $\eta$-cycle $f_i$ of $G$
over \( \mathbb{Z} \) such that:

for \( e \in E(\cup \mathcal{P}_i) \), either \( f_i(e) = 1 \) or \( f_i(e) = -1 \); and,

for \( e \in E(G) - E(\cup \mathcal{P}_i) \), \( f_i(e) = 0 \).

Let

\[
    f = \sum_{i = 1}^{m} 2^{i-1} f_i.
\]

Then, by Proposition 29.2, \( f \) is an \( n \)-cycle of \( G \) over \( \mathbb{Z} \).

Suppose that \( e \in E(G) \). Then, since

\[
    \bigcup_{i = 1}^{m} E(\cup \mathcal{P}_i) = E(G),
\]

it follows that \( e \) belongs to at least one of the \( E(\cup \mathcal{P}_i) \). Thus there is an integer \( k \) such that

\[
    k = \max \{ i : 1 \leq i \leq m \text{ and } e \in E(\cup \mathcal{P}_i) \}.
\]

Therefore

\[
    |f(e)| = \left| \sum_{i = 1}^{m} 2^{i-1} f_i(e) \right| = \left| \sum_{i = 1}^{k} 2^{i-1} f_i(e) \right|
\]
Moreover,

\[ |f(e)| = \left| \sum_{i=1}^{m} 2^{i-1} f_i(e) \right| \]

\[ < \sum_{i=1}^{m} 2^{i-1} = 2^m - 1 < 2^m . \]

Thus \( f \) is a colour-cycle of \( G \) over \( \mathbb{Z} \) such that, for every \( e \in E(G) \), \( |f(e)| < 2^m \).

**PROPOSITION 37.6**

Let \( G \) be a graph and let \( m \) be a positive integer.

Then: there is a colour-cycle of \( G \) over \( \mathbb{Z}_{2^m} \) if and only if there are collections \( P_1, \ldots, P_m \) of edge-disjoint polygons of \( G \) such that

\[ m \cup \bigcup_{i=1}^{m} P_i = E(G) . \]
This proposition follows directly from Propositions 34.12 and 37.5.

DEFINITION 38.1

We say that a graph $G$ is vertex-critical if it satisfies the following two conditions:

(1) $G$ has no isthmus and there is no colour-cycle of $G$ over $\mathbb{Z}_8$; and

(2) if $H$ is a graph which has fewer vertices than $G$ has, then either $H$ has an isthmus or there is a colour-cycle of $H$ over $\mathbb{Z}_8$.

PROPOSITION 38.2

If a graph is vertex-critical, then it is connected.

PROOF

Suppose that $G$ is a vertex-critical graph which is not connected.
Let $K_1, \ldots, K_m$ be the components of $G$. Then, by Proposition 6.2, the $K_i$ are edge-disjoint and

$$\bigcup_{i=1}^{m} E(K_i) = E(G).$$

Suppose that $i \in \{1, \ldots, m\}$. Then, since $G$ has at least two components and since the components of a graph are non-null and disjoint,

$$|V(K_i)| < |V(G)|.$$  

Also, since $G$ has no isthmus, it follows from Proposition 11.2 that $K_i$ has no isthmus. Thus, since $G$ is vertex-critical, there is a colour-cycle of $K_i$ over $Z_8$.

Thus it follows from Proposition 36.1 that $G$ has a colour-cycle over $Z_8$. But this contradicts the supposition that $G$ is vertex-critical.

We conclude that every vertex-critical graph is connected.

**PROPOSITION 38.3**

Every bond of a vertex-critical graph has at least three edges.
PROOF

Let $B$ be a bond of a vertex-critical graph $G$.

Since a bond must have at least one edge and since $G$ has no isthmus, it follows that $B$ has at least two edges.

Suppose that $B$ has just two edges, $a$ and $b$ say.

Since $B$ is a bond of $G$, it follows from Proposition 10.5 that the number of components of $G : (E(G) - B)$ is just one greater than the number of components of $G$. But, by Proposition 38.2, $G$ is connected. Thus $G : (E(G) - B)$ has just two components, $M_1$ and $M_2$ say.

By Proposition 10.4, both $a$ and $b$ are links of $G$ and neither $a$ nor $b$ has both of its $G$-ends contained in the same component of $G : (E(G) - B)$.

Let $x_1$ (respectively $x_2$) be the $G$-end of $a$ contained in $M_1$ (respectively $M_2$).

Let $y_1$ (respectively $y_2$) be the $G$-end of $b$ contained in $M_1$ (respectively $M_2$).

Note that, in the argument which follows, we make no assumption as to whether or not $x_1 = y_1$ (respectively $x_2 = y_2$).
Let
\[ H = G \text{ctr} (E(G) - \{a\}) . \]

Then \( E(H) = E(G) - \{a\} \). Moreover, the vertices of \( H \) are the components of \( G:[a] \), that is,
\[ V(H) = \left\{ [v] : v \in V(G) - \{x_1, x_2\} \right\} \cup \{G \cdot \{a\}\} . \]

Note that \( |V(H)| < |V(G)| \).

Since \( G \) has no isthmus, it follows from Proposition 12.2 that \( H \) has no isthmus. Further, \( |V(H)| < |V(G)| \) and \( G \) is vertex-critical. Thus there is a colour-cycle of \( H \) over \( Z_8 \).

Let \( \eta \) be an orientation of \( G \) over \( Z_8 \). Since \( H \) is a contraction of \( G \), \( e \) is a link of \( H \) and \( K \) is an \( H \)-end of \( e \) only if \( e \) is a link of \( G \) and just one \( G \)-end of \( e \) is contained in \( K \). Thus we may define an orientation \( \lambda \) of \( H \) over \( Z_8 \) as follows:

for \( e \in E(H) \) and \( K \in V(H) \),

if \( e \) is a loop of \( H \) or if \( K \) is not an \( H \)-end of \( e \), then \( \lambda(e,K) = 0 \); and,

if \( e \) is a link of \( H \) and \( K \) is an \( H \)-end of \( e \), then \( \lambda(e,K) = \eta(e,v) \) where \( v \) is the \( G \)-end of \( e \) which is contained in \( K \).
Since there is a colour-cycle of $H$ over $\mathbb{Z}_8$, it follows from Proposition 32.1 that there is a $\lambda$-colour-cycle $g$ of $H$ over $\mathbb{Z}_8$.

Define a map $f$ from $E(G)$ to $\mathbb{Z}_8$ as follows:

For $e \in E(G) - \{a\} = E(H)$, $f(e) = g(e)$; and

$f(a) = -\eta(a,x_1)\eta(b,y_1)\ f(b) = -\eta(a,x_2)\eta(b,y_2)\ f(b)$.

We shall show that $f$ is an $\eta$-colour-cycle of $G$ over $\mathbb{Z}_8$ and so obtain a contradiction.

Let $v \in V(G) - \{x_1, x_2\}$. If $e \in L(G,v)$, then $I(G,e) \neq \{x_1, x_2\}$. But, if $e$ is a link of $G$ and a loop of $H$, then $I(G,e) = \{x_1, x_2\}$. Thus, if $e \in L(G,v)$, then $e \in L(H,[v])$. On the other hand, if $e \in L(H,[v])$, then $e \in L(G,v)$. Therefore $L(G,v) = L(H,[v])$. It follows that, if $e \in L(G,v)$, then $\eta(e,v) = \lambda(e,[v])$. Moreover, since $I(G,a) = \{x_1, x_2\}$, $a \notin L(G,v)$ and so, if $e \in L(G,v)$, then $f(e) = g(e)$. Thus

$$\sum_{e \in E(G)} \eta(e,v)\ f(e) = \sum_{e \in E(H)} \lambda(e,[v])\ g(e).$$

Therefore, using Proposition 30.2 and the fact that $g$ is a $\lambda$-cycle of $H$ over $\mathbb{Z}_8$,

$$\sum_{e \in E(G)} \eta(e,v)\ f(e) = \sum_{e \in E(H)} \lambda(e,[v])\ g(e) = 0.$$
It remains to show that

\[(1) \sum_{e \in E(G)} \eta(e, x_1) f(e) = 0 \]

and that

\[(2) \sum_{e \in E(G)} \eta(e, x_2) f(e) = 0 . \]

We show that (1) holds. If \( v \in V(M_1) - \{x_1\} \), then \( v \in V(G) - \{x_1, x_2\} \). So

\[
\sum_{v \in V(M_1) - \{x_1\}} \sum_{e \in E(G)} \eta(e, v) f(e) = 0 .
\]

Thus

\[
\sum_{e \in E(G)} \eta(e, x_1) f(e)
\]

\[
= \sum_{e \in E(G)} \eta(e, x_1) f(e)
\]

\[
+ \sum_{v \in V(M_1) - \{x_1\}} \sum_{e \in E(G)} \eta(e, v) f(e)
\]

\[
= \sum_{v \in V(M_1)} \sum_{e \in E(G)} \eta(e, v) f(e)
\]

\[
= \sum_{e \in E(G)} \sum_{v \in V(M_1)} \eta(e, v) f(e) .
\]

Now, \( E(G) \) is the union of the disjoint sets \( E(M_1) \), \( E(M_2) \), and \( B \). If \( e \in E(M_1) \), then
I(G, e) ≤ V(M_1). Thus, if e ∈ E(M_1), then
\[ \sum_{v \in V(M_1)} \eta(e, v) f(e) = 0. \]

If e ∈ E(M_2) and v ∈ V(M_1), then e is not G-incident with v and so \( \eta(e, v) = 0 \). Thus, if e ∈ E(M_2), then
\[ \sum_{v \in V(M_1)} \eta(e, v) f(e) = 0. \]

Therefore
\[ \sum_{e \in E(G)} \sum_{v \in V(M_1)} \eta(e, v) f(e) \]
\[ = \sum_{e \in E(M_1)} \sum_{v \in V(M_1)} \eta(e, v) f(e) \]
\[ + \sum_{e \in E(M_2)} \sum_{v \in V(M_1)} \eta(e, v) f(e) \]
\[ + \sum_{e \in E(M_2)} \sum_{v \in V(M_1)} \eta(e, v) f(e) \]
\[ = \sum_{e \in E(M_1)} \sum_{v \in V(M_1)} \eta(e, v) f(e) \]
\[ = \sum_{v \in V(M_1)} \eta(a, v) f(a) \]
\[ + \sum_{v \in V(M_1)} \eta(b, v) f(b). \]
Now

\[ V(M_1) \cap I(G,a) = \{x_1\} \]

and

\[ V(M_1) \cap I(G,b) = \{y_1\} . \]

Thus

\[
\sum_{v \in V(M_1)} \eta(a,v) f(a) + \sum_{v \in V(M_1)} \eta(b,v) f(b) = \eta(a,x_1) f(a) + \eta(b,y_1) f(b) .
\]

But, by the definition of \( f \),

\[ f(a) = -\eta(a,x_1) \eta(b,y_1) f(b) . \]

So

\[ \eta(a,x_1) f(a) = -\eta(b,y_1) f(b) . \]

Thus

\[ \eta(a,x_1) f(a) + \eta(b,y_1) f(b) = 0 . \]

We have thus shown that (1) holds.

A similar argument, with \( x_1, y_1, M_1 \), and \( M_2 \) replaced by \( x_2, y_2, M_2 \), and \( M_1 \) respectively, shows that (2) holds.
Therefore f is an \( \eta \)-cycle of \( G \) over \( \mathbb{Z}_8 \).

Now, if \( e \in E(G) - \{a\} = E(H) \), then
\[
f(e) = g(e) \neq 0.
\]
Moreover, either \( \eta(a,x_1) = 1 \) or \( \eta(a,x_1) = -1 \), and either \( \eta(b,y_1) = 1 \) or \( \eta(b,y_1) = -1 \). Thus either \( \eta(a,x_1) \eta(b,y_1) = 1 \) or \( \eta(a,x_1) \eta(b,y_1) = -1 \). So, by the definition of \( f \), either \( f(a) = f(b) \) or \( f(a) = -f(b) \). Thus \( f(a) \neq 0 \).

Thus \( f \) is an \( \eta \)-colour-cycle of \( G \) over \( \mathbb{Z}_8 \).

But this contradicts the fact that \( G \) is vertex-critical. So we conclude that \( B \) has at least three edges.

**PROPOSITION 38.4**

If a graph \( G \) has no isthmus, then there is a colour-cycle of \( G \) over \( \mathbb{Z}_8 \).

**PROOF**

If the proposition is false, then there is a vertex-critical graph.

Suppose that \( G \) is a vertex-critical graph. Then, by Proposition 38.3, every bond of \( G \) has at least three edges. Thus, by Proposition 25.3, there are collections \( \mathcal{P}_1 \), \( \mathcal{P}_2 \), and \( \mathcal{P}_3 \) of edge-disjoint
polygons of \( G \) such that

\[
\bigcup_{i=1}^{3} E(\cup \mathcal{P}_i) = E(G).
\]

So, by Proposition 37.6, there is a colour-cycle of \( G \) over \( \mathbb{Z}_8 \). But this contradicts the supposition that \( G \) is vertex-critical.

Thus there is no vertex-critical graph and so the proposition holds.

**REMARK 38.5**

By virtue of the considerations of Section 35, it follows that Conjecture I is true.

As a bonus we obtain a strengthening of Proposition 25.3, namely:

**PROPOSITION 38.6**

If a graph \( G \) has no isthmus, then there are collections \( \mathcal{P}_1, \mathcal{P}_2, \) and \( \mathcal{P}_3 \) of edge-disjoint polygons of \( G \) such that

\[
\bigcup_{i=1}^{3} E(\cup \mathcal{P}_i) = E(G).
\]

i = 1
PROOF

This follows from Propositions 38.4 and 37.6.
REFERENCES


LIST OF SYMBOLS

1.1 \( E(G) \)
\( V(G) \)
\( I(G,-) \)

1.2 \([v]\)

2.1 \( \text{val}(G,v) \)

2.2 \( |S| \)

2.3 \( \sum_{v \in V(G)} \text{val}(G,v) \)

3.1 \( \subseteq \) (subgraphs)
\( \subset \) (subgraphs)

3.3 \( G:A \)
\( G\cdot A \)

\( P \)

3.4 \( \bigcup_{i=1}^{p} H_i \) (subgraphs)

3.5 \( \bigcup H \) (subgraphs)

4.1 \( E(\alpha) \)
\( V(\alpha) \)
\( G(\alpha) \)

4.2 \( \alpha \beta \)
6.1 $c(G)$

7.2 $< \text{(partitions)}$

7.2 $< \text{(partitions)}$

8.1 $\pi(G)$

12.1 $G \text{ctr A}$

15.1 $r(G)$

18.1 $E(M)$

$\mathcal{B}(M)$

20.1 $r_M(A)$

21.4 $J(B,e)$

22.1 $M^*$

23.1 $M(G) \text{ (see also 23.3)}$

23.5 $M^*(G) \text{ (see also 23.7)}$

26 $R$

$\mathbb{Z}$

$\mathbb{Z}_n$

27.1 $\eta(e,v)$

28.1 $\sum_{a \in A} f(a)$

29.1 $\rho_1 f_1 + \ldots + \rho_m f_m$
30.1 \( L(G,v) \)

34.2 \((n,f)\)

34.7 \( S(n,f) \)
\( \sigma(n,f) \)

35 \( \phi(G,n) \)
INDEX OF DEFINITIONS

Base, 18.1
Bond, 10.1
Bond matroid, 23.7

Circuit, 21.1
Circular path, 4.5
Colour-cycle, 31.1, 31.2
Component, 6.1
Connected graph, 5.1
Contained (subgraph), 3.1
Contraction, 12.1
Cutset, 9.1
Cycle, 28.2, 28.3

Dependent, 21.1
Dependent set of a matroid, 21.1
Disjoint subgraphs, 3.6

Edge, 1.1
Edge-disjoint subgraphs, 3.6
Element of a matroid, 18.1
End, 1.1

Flow, 34.1, 34.2
Forest, 13.1
Forest of a graph, 14.1
Graph, 1.1
Incidence function, 1.1
Incident, 1.1
Independent, 19.1
Independent set of a matroid, 19.1
Isthmus, 11.1
Integral flow, 34.2
Link, 1.1
Link-graph, 1.2
Loop, 1.1
Loop-graph, 1.2
Matroid, 18.1
Null graph, 1.2
Orientation, 27.1
Orthogonal matroid, 22.1
Part, 7.1
Partition, 7.1
Path, 4.1, 34.5
Polygon, 16.1
Polygon of a graph, 17.1
Polygon matroid, 23.3
Principal forest of a graph, 14.1
Product of paths, 4.2
Proper refinement, 7.2
Proper subgraph, 3.1
Properly contained (subgraph), 3.1

Rank in a matroid, 20.1
Rank of a graph, 15.1
Reduction, 3.3
Refinement, 7.2

Simple path, 4.3
Sink, 34.3
Source, 34.3
Spanning subgraph, 3.1
Spanning subgraph determined by a set of edges, 3.3
Spanning tree of a graph, 14.1
Subgraph, 3.1

Tree, 13.1

Union of subgraphs, 3.4, 3.5

Valency, 2.1
Vertex, 1.1
Vertex-critical graph, 38.1
Vertex-graph, 1.2