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X-RAY MOTION ANALYSIS OF CHARGE PARTICLES IN A LABORATORY MILL

by

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Thesis presented for the Degree of
Doctor of Philosophy
in the Department of Mechanical Engineering
University of Cape Town

August 2005

Abstract

The work reported herein is of an automated X-ray vision system used to track the 3D trajectories of a typical bulk charge particle within an experimental Perspex mill, the intention of which is to simulate the grinding motion of rock found in typical industrial mills. The experimental rig is constructed through an optimisation scheme that determines the maximum allowed dimensions of the mill that can be imaged by the X-ray system, however, the optimisation principle is not limited to the current study and can be employed to maximise any volume being investigated with the X-ray system.

The raw data for each position of the tracked particle comprises of two X-ray images of the tumbling mill generated orthogonal to each other, with a phase lag between them. The correction of the phase lag between the biplanar images is a new addition to the usual usage of the system resulting in an effective sampling rate of 100 frames per second, thereby ensuring that the resolution is sufficient to conduct detailed kinematic studies.

The processing of the raw images are achieved with a dynamic template matching algorithm followed by a modified, and improved, implementation of the Canny edge detector, while the centering of the edge images are based on an adapted conic fitting routine, resulting in an overall subpixel centering accuracy. The processed images are then mapped to object space using the direct linear transformation (DLT), equipped with a physically valid variance model that is shown to improve the standard implementation even when robust solvers are employed. The final reconstruction accuracy of the tracked particle was shown to be 0.15 mm and is achieved without iteration due to the appropriateness of the variance model.

The high accuracy data was initially used to benchmark the discrete element method (DEM), providing the first numerical comparisons that surpassed the usual end-window snapshots employed by other investigators for the purpose of validation. The analysis of the data was not restricted to DEM verification, and in some sense, surpassed the initial objective by yielding trends useful to comminution practices. Amongst these analyses was the finding that the circulation rate of the charge is not once per mill revolution but greater, depicting a linear trend with mill speed. The slip between charge layers was shown to follow a linear pattern, with the degree of slip increasing linearly with mill speed. The phenomenon of charge surging was quantified, resulting in a trend for the variation of the surge amplitude with mill speed. A mechanism for the surging phenomenon was also proposed in this study. A particularly useful outcome of the data analyses was the formulation of a power model through heuristic trends of the center of mass (CoM) and center of circulation (CoC) of the charge. The methodology outlined by the model was shown to be robust, providing a correct approach to obtaining a truly fundamental power model based on generally applicable principles.

List of Publications

The following is a list of the authors published technical papers on the research described in this PhD thesis. Papers 1 and 2 deal with the first objective of this thesis, section (1.1.1). Of the various imaging techniques and photogrammetric mapping algorithms investigated in chapters (4, 5 and 6), the most efficient in terms of automated analysis and highest accuracy are included in papers 1 and 2. The third publication is a preliminary look at some of the outcomes of the 3D particle tracking system described in papers 1 and 2. The next two articles (papers 4 and 5) highlight the use of accurate experimental data in the validation of complex numerical tools like the Discrete Element Method (DEM). These papers clearly point out the need for statistically sound validation techniques that surpass the standard snapshot comparisons of other investigators. The formulation of scalar charge descriptors are also included.

1. **I. Govender**, M.S. Powell and G.N. Nurick, "Automated imaging to track the 3D motion of particles", *Journal of Experimental Mechanics*, volume 42, number 2, pp. 153 - 160, 2002
2. **I. Govender**, A.T. McBride and M.S. Powell, "Improved experimental tracking techniques for validating discrete element method simulations of tumbling mills", *Journal of Experimental Mechanics*, volume 44, number 6, pp. 593 - 607, 2004
3. **I. Govender**, M.S. Powell and G.N. Nurick, "3D particle tracking: A rigorous technique for verifying DEM", *Minerals Engineering*, volume 14, number 10, pp. 1329 - 1340, 2001
4. **I. Govender**, M.S. Powell, V. Balden and G.N. Nurick, "Validated DEM - Potential major improvements to SAG mill modelling", *Proceedings of the 3rd International Conference on Autogenous and Semiautogenous Grinding Technology*, volume IV, pp. 101 - 114, 2001
5. A.T. McBride, **I. Govender**, M.S. Powell and T.J. Cloete, "Contributions to the experimental validation of the discrete element method applied to tumbling mills", *Engineering Computations: International Journal for Computer-Aided Engineering*, volume 21, number 2, pp. 119 - 136, 2004

ACKNOWLEDGEMENTS

The task of monitoring the automated image analysis took several months, requiring patience and endurance that far surpassed my capabilities - For this I would sincerely like to thank my wife, Ayesha, the epitome of unwavering discipline. To my son Owen, your arrival inspired me, providing the main impetus in seeing the write up to fruition. The early days of a PhD student are fraught with insecurity and a lacking sense of confidence. It is only with the support of truly caring people that these dark days are endured - Mum and Dad, thank you for providing this place of solace. Finally, my supervisor, Dr M.S. Powell, is kindly acknowledged for allowing me the leeway to explore ideas, sometimes peripheral and never ending, that in hindsight have formed the key components of this PhD.

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Chapter 1

Introduction

The key component to mineral beneficiation in the world is the comminution of ores. The process entails the crushing of large rocks (25-300 mm) down to fine particles (38-2000 μ m). Rotary mills are commonly used comminution devices, filled to between 20 and 45% of the volume with balls or rocks to form a grinding media. The essentially cylindrical geometry of rotary mills span a wide variety of configurations, ranging from the 'pancake' shape, typical of South American mills, to the long mills of South Africa. A variety of empirical models have been employed to elucidate the relationship between energy input and the particle size made from a given feed size, none of which is entirely satisfactory.

The greatest problem with comminution devices lies in the fact that most of the energy input to the machine is absorbed by the machine itself, and only a small fraction of the total energy is available for breaking the material; the power cost may represent more than half of the total processing cost. Considering the dwindling resources of high-grade minerals and the economic incentive for lowering energy in mineral beneficiation processes, interest in investigating better ways to improve grinding efficiency is only natural. The major thrust in comminution research today lies in understanding the grinding mechanism, power draft estimation, modelling and scale-up studies, and the design of the mill shell. The development of these mechanisms has mainly been via step-wise increments, based on operator experience and empirical results derived from test-work on both full-scale and pilot mills. While these methods are generally successful and currently indispensable, they provide little insight into the mechanics of the charge motion and scale poorly as one moves away from the window of operating conditions in which they were formulated.

Currently it is still not possible to design mills in detail from first principles due to a lack of knowledge of the charge behaviour. A good approach to the above problems clearly resides in the understanding of the dynamic motion of the particles within the mill, which is the genesis of the forces experienced by the particles to be ground.

Most fundamental investigations into particle behaviour studies include either high-speed photography of a 2D mill (a mill with negligible length) or 3D mill (a mill with a realistic length) with one axial end fitted with a Plexiglas window. These experiments cannot track the particles deep within the mill and are subject to the end window effect, which influences the particle trajectories. In addition

the current power draft models are derived from empirical studies, and always include an unphysical scaling factor to fit the empirical model to the plant data.

In the past decade computational techniques like the Discrete Element Method (DEM) have been utilised to meet the challenges of predicting particle motion in rotary mills. The DEM is a promising numerical tool that should be capable of simulating the complex dynamic particle-particle motion and interactions within tumbling mills. It is envisioned that DEM will eventually be used in conjunction with empirical methods to better optimise the mill design process. However, prior to this occurring DEM must be rigorously validated from its most fundamental level upwards. This is particularly important in light of the computational expense and lack of sound experimental validation currently available.

In a seminal research project, Powell [1] described the use of diagnostic X-rays for the study of particle motion in an experimental mill. These investigations were a step forward from the usual end window experiments, shedding some light on the movement of particles deep within the bulk of the charge. However, the resulting data was still mainly 2D in nature and provided limited information as to the position of the tracked particles along the length of the mill. The uncertainty associated with the particle's position, arising from unquantified lag between the front and side view X-ray images, coupled with insufficient data, precluded any statistical analyses of the particle trajectories. The analyses resulting from Powell's work was to establish a set of trends from the observed 2D profiles.

The research reported herein is concerned with automated 3D tracking of particles in an experimental mill, using X-ray image analysis and photogrammetric techniques, for the purpose of DEM verification and a broader understanding of charge kinematics and dynamics. This work seeks to fill the vital gap linking computational results to rigorous experimental data. The tracked particle behaviour is described for varying mill conditions, and includes kinematic analyses and scalar charge descriptors like the center of circulation, shoulder and toe positions.

1.1 Objectives

The basis for this thesis, including the formulation of the hypothesis, is described at the end of the literature review chapter, section (2.4). Formally stated, the hypothesis of this thesis is:

The use of biplanar X-ray imaging techniques to reconstruct accurate 3D trajectory data of a tracer particle within a simplified, laboratory tumbling mill provides unique and valid information for understanding charge kinematics and dynamics, and consequently to benchmark the predictions of DEM through statistically based comparisons.

The hypothesis is investigated in three stages: The first stage, which is also the primary objective of this thesis, was to design, develop, and implement algorithms for an automated 3D tracking system to monitor the tumbling motion of charge particles in a laboratory Perspex mill. The next stage was to use the 3D particle trajectory data as the basis for validating the discrete element method. The final stage entailed the analysis of the 3D trajectory data in order to discern meaningful relationships for the various milling configurations investigated.

1.1.1 Objective one: Automated image processing and image mapping

The key stages of the automated tracking procedure are:

1. The formulation, development and coding of automated imaging tools, including the mapping of image space to 3D real space from two X-ray images of the tumbling Perspex mill; and
2. A testing process to benchmark the imaging tools and improve the centering accuracy in locating particle centroids.

The digitally acquired data of the tracked particles necessitated (and facilitated) the development of automated imaging techniques (modification of existing algorithms and the development of new ones). A mapping algorithm known as the Direct Linear Transformation (DLT) is formulated and modified along with the imaging tools, and together constitute the automated 3D tracking procedure. The governing objectives and constraints in designing the system and its algorithms are:

1. The applications are close-range and specific to X-ray imaging (conventional photogrammetric problems can be solved with the current system but the results may not be optimal).
2. The tracked objects have fairly smooth surfaces; however the resulting images may be non-continuous in texture (intensity distribution) with variations in shape across the sampling range.
3. The system is not feature specific although the target centering routines only guarantee sub-pixel accuracy for elliptical projected patterns.
4. All measurement stages of the tumbling mill involving high contrasting objects (steel or lead balls) are fully automated while the tracking of low contrast objects (silver painted plastic beads) may be classified as semi-automated with user intervention required when the template matching fails. The calibration stages which precede the particle tracking require minimal target identification in the calibration routines.
5. The algorithms used ensure high accuracy measurement of the tracked particles in the Lagrangian system. The resulting precision in the Eulerian system are highly dependent on the amount and quality of data used to reconstruct the mill outline and central axis i.e. to obtain the transformation matrix from Lagrangian (control frame) to Eulerian (mill shell) system.

The second stage of this objective was to establish the validity of the imaging tools and the DLT mapping algorithm. The ultimate objective for rigorously testing these routines is to establish the most accurate and thus suitable procedure for use in the automated image processing. The validity of the DLT mapping process involved tests against a control system known as a control frame. The control frame was designed to facilitate the optimisation of the 3D mapping transformation parameters. An investigation into the various measurement systems used in surveying 3D surfaces is presented in Appendix (B). The feature extraction routines are tested with synthetically generated image targets amidst simulated noise; it is also tested with the control frame. Integral to improved accuracy is target centering. Four target centering routines are formulated and coded, and included in the testing process.

1.1.2 Objective two: Experimental comparisons with DEM

The particle trajectory data from the experimental mill provides accurate data against which to validate DEM simulations. The identification of unique features within the charge allowed direct comparisons to be made between the experimental and DEM data. The features used in the comparisons include the Center of Circulation (CoC), the equilibrium surface, angles of departure at the shoulder regions, and the position of the toe. These verification tools form part of a larger project to develop a validation toolbox tailored to rigorously test the outputs of DEM. In the context of this work the validation is achieved by comparing DEM outputs to trajectory data derived from the particle tracking experiments.

1.1.3 Objective three: Analysis of 3D trajectory data

The large amounts of data accumulated from the tracking experiments was sufficient to conduct detailed analysis of the charge motion. The analysis includes:

1. Finite Difference approximations for calculating velocity and acceleration from positional data
2. Slip between charge layers
3. Longitudinal motion of charge
4. Toe and shoulder estimates
5. Modelling the motion of the center of mass as a function of the center of circulation and mill speed with a view to developing a power model

Chapter 2

Review of Literature

The literature surveyed in this thesis may be divided into two sections: The first section takes a historical look at charge motion and power draw studies with particular focus given to the fundamentals of charge behaviour and their influence on power formulations. Research of this nature includes simplified mathematical models of single particle motion in laboratory mills, semi-empirical descriptions of charge motion based on non-invasive techniques of monitoring particle motion deep within the charge of an experimental mill, and numerical modelling of charge motion based on simplified contact theories. In some instances empirical studies have informed the thinking of fundamental research and are included in this historical survey.

Section two discusses the various image processing and 3D reconstruction techniques applicable to biplanar digital X-ray photogrammetry. The large body of research in this field preclude a historical review and will therefore be restricted to works deemed pertinent to the problem of automated 3D particle tracking.

2.1 Charge motion analysis: An historical survey

One of the earliest works in charge motion analysis was in 1905 by White [2]. He assumed that particles along the mill periphery move without slip until a gravitational and centrifugal force balance is reached. At this point the particle is projected into parabolic motion, strikes the “impact point” and is then drawn into the bulk of the charge. His assumption of no slip led to an overprediction of the “equilibrium point”¹ (when compared to experiments) which he argued was due to insufficient ball loading. Essentially, a higher charge filling increases the normal force exerted on the mill lining which in turn increases the frictional force between the balls and lining. The insufficient loading explanation led to the conclusion of undue liner wear caused by slip and direct impacts. His conclusions on regulating water level in order to generate useful impact work is a valuable contribution to comminution.

¹equilibrium point refers to the point at which the particle is projected into free flight

Davis [3] formulated a similar mathematical description for a single particle moving along the mill shell without slip. Verification of the theoretical results were achieved by taking end window images of an experimental mill. The agreement between experiment and theory was said to compare well except for the intersecting trajectory paths of cataracting particles observed in experiments. This was argued away as accidental and amplified by a low charge filling. His work clearly highlights the value of mathematical modelling of charge motion while elucidating the shortcomings of simplified assumptions.

A decade later other researchers ([4] and [5]) refined the work of Davis and White by including friction in the force balance, however they encountered problems in reproducing the experimental trajectories, especially in short length laboratory mills due to the interference of the end walls. During this same period Barth [6] took a similar mathematical approach which led to the prediction of an “equilibrium surface” within the charge that seems to keep particles in a dynamic equilibrium under the action of centripetal, gravitational and frictional forces. The “equilibrium surface” defined a plane that splits the charge into two distinct regions: an inner static region that does not move relative to the shell and a shear region where balls either flow down the free surface of the charge (cascading) or follow parabolic paths as they are projected into free-flight (cataracting)².

Three decades later Rose & Sullivan [7] presented a critical summary on the mathematical descriptions of particle motion in tumbling mills in which they criticised those simplified models that ignored the frictional characteristics of charge in predicting particle trajectories. In particular they described the behaviour of the charge as mill speed increases to a critical value. At this critical speed the outermost layer of balls centrifuge and are said to be stationary with respect to the inner wall of the drum. The critical speed was calculated as:

$$N_c = \frac{1}{2\pi} \sqrt{\frac{2g}{D - 2r}} \quad (2.1)$$

where g is acceleration due to gravity, D is the inner diameter of the drum and r the radius of the ball. The critical speed was calculated in revolutions per second. This simple formulation has become one of the most universal charge descriptors because it provides a common basis for comparing different size mills.

The next step forward in charge motion analysis was by McIvor [8]. After considering the problems encountered by Gow et al [4] and Fahrenwald & Lee [5], he decided to conduct a theoretical analysis of single particle behaviour inside a rotating drum. His model improved on the existing ones in that it also included lifter effects. The theoretical analysis was based on an assumed coefficient of friction and an unobstructed free fall path. In this formulation the particle take-off point is characterised by the angle φ which is a function of mill rotation speed ω , liner tilt β and friction angle θ (see figure 2.1).

$$\varphi = \theta - \beta + \arcsin \left[\frac{\omega^2 R}{g} \cos(\theta - \beta) \right] \quad (2.2)$$

²Powell's description of the equilibrium surface is very similar though just as incomplete. The work by McBride et al gives a more rigorous description of the equilibrium surface.

$$\theta = \arctan(\mu_s) \quad (2.3)$$

Essentially, this relationship describes the point at which gravitational, centrifugal and frictional³ forces acting on a given ball are in equilibrium. The lifter is implicit in this formulation as it is defined by the tilt angle β . Below this point of equilibrium, the ball is held securely in the charge by the centrifugal acceleration. Above this point, gravity takes over pulling the ball into free flight.

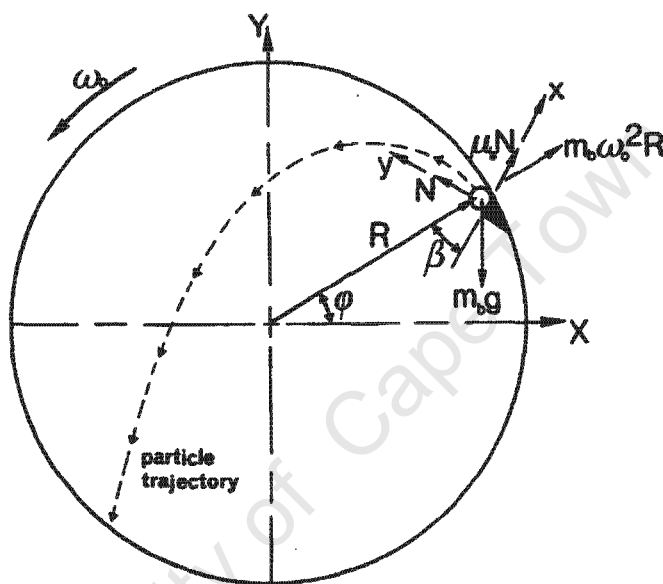


Figure 2.1: Point of flight for a single particle in a ball mill (McIvor [8])

The conclusion that a particle's trajectory is independent of the mill size is valuable because it implies that the critical speed at which a mill operates is independent of the mill size. Other interesting conclusions, though not directly revealed in his calculations, were that particle trajectory and power draw vary with lifter height and profile respectively.

The next five years saw interesting developments in the understanding of charge surging and its contribution to liner wear. In 1986 Vermeulen & Howat [9] published their findings on the charge surging phenomenon encountered in smooth lined rod mills (i.e. mills that do not incorporate lifter bars). The dynamical theory formulated propounds that the fluctuations in charge slippage are due to alternating cataracting and cascading phases. Essentially, when the angle of repose is at a maximum, a comparatively large portion of the charge is in free flight. This reduces the pressure on the outermost layer of charge and consequently slippage of this layer becomes more pronounced. The slip of the peripheral charge layer lowers the angle of repose and results in an increase in cascading. This progressively increases the *en-masse* charge thereby increasing the charge pressure felt by the

³Friction described by McIvor is sliding friction which is negligible for spherical particles as the gravitational force will cause it to roll down the lifter.

outermost layer. Increased pressure leads to increased friction between the rods and lining, resulting in a reduction in slip and an increase in the angle of repose, thereby repeating the cycle of events. The large fluctuations in slip induce massive wear on the lining and clearly illustrates the value of using lifter bars.

In a follow up study Vermeulen & Howat [10] investigated the influence of lifter bars on the motion of *en-masse* grinding media. The lack of quantitative results motivated them to measure the mean effects of lifter bars at the interface of contiguous layers of rods in a laboratory mill. Using the approximations for finite changes in R , ω and p , equation (2.4), they derived a relation for the dynamic pressure in the *en-masse* charge and used this to determine the relative slip between layers, equation (2.5).

$$\frac{\Delta p}{p} = \left(\frac{\partial p}{\partial R} \Delta R + \frac{\partial p}{\partial \omega} \Delta \omega \right) p^{-1} \quad (2.4)$$

$$p = \alpha \rho g R + \beta \rho \omega^2 R^2 \quad (2.5)$$

Where p is the mean dynamic pressure, ρ is the mean dynamic density, ω is the angular velocity of *en masse* rods at R from the mill center, and g is the gravitational acceleration; α and β are dimensionless constants.

Their key findings may be summarised as:

1. Slip occurs at the inter-facial surfaces between layers.
2. The two outer-most layers for the mill with low lifters are keyed into the rotary motion of the mill.
3. Slip is greatest for the mill with smooth lining.

The next wave of comminution research came as a result of advances in technology and computing, producing novel approaches to experimentally track and mathematical model the motion of particles in a tumbling mill. Mishra and Rajamani ([11] and [12]) adapted the numerical schemes of Cundall & Strack [13] to fit the geometric configuration of a tumbling mill. This numerical technique known as the discrete element method (DEM) considers Newtonian force interactions between balls, and balls and lining. The fundamental basis of these interactions is equated to a pair of normal and tangential spring-dashpots, figure (2.2), at every contact to simulate a viscous-contact damping model. The DEM was revolutionary in that it modelled the individual particles in a discrete system separately and thus represented the first truly numerical model. In a more recent publication, Venugopal & Rajamani [14] describe the DEM as a numerical scheme that allows for finite rotations and displacements of rigid bodies, where complete loss of contacts and formation of new contacts between bodies take place as the calculation cycle progresses. The physical system consists of a shell, which is a complex polygonal assembly of rectangular plates rotating at high velocities, enclosing a large number of spheres that

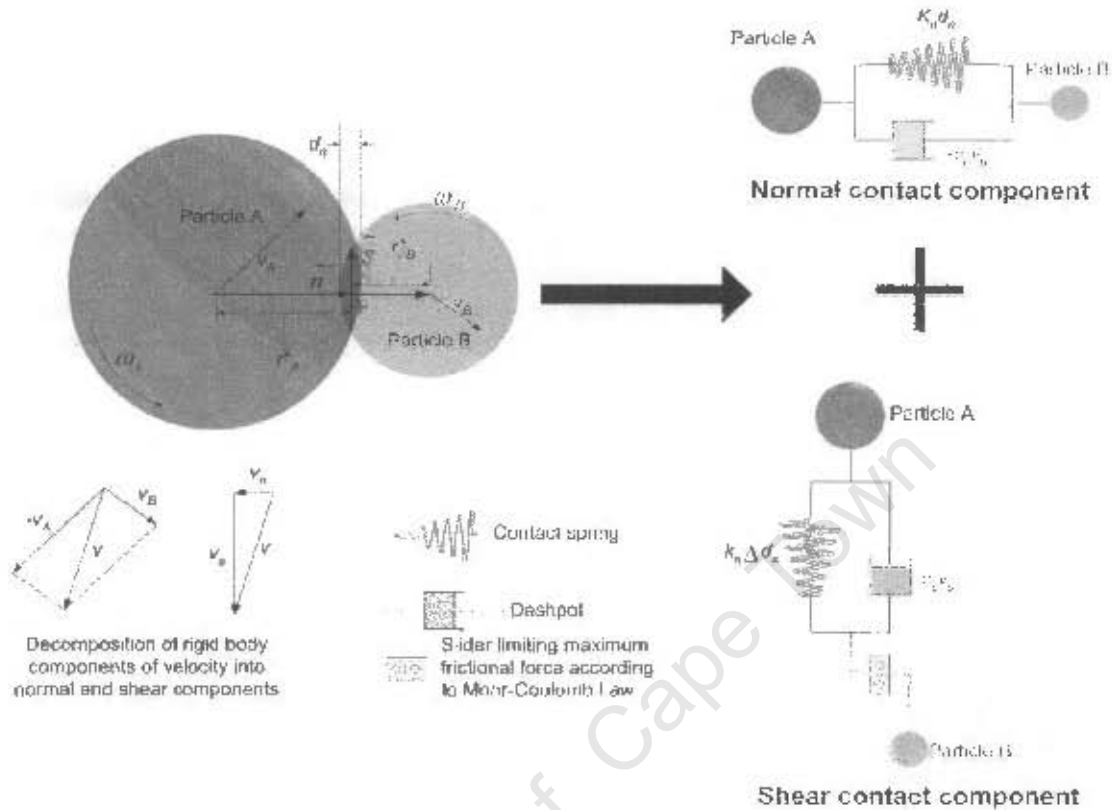


Figure 2.2: Schematic of linear spring and dashpot force law within the soft-contact model, McBride et al [15]

have different physical properties. Here “spheres” denote steel balls as well as rocks. As the mill shell rotates causing the ball charge to cascade and cataract, each sphere collides with numerous other spheres as well as the rectangular plates constituting the mill shell. All of these collisions affect the trajectory of each sphere and have to be accounted for when modelling the entire system. First, the net unbalanced force on each sphere is determined by summing up the individual contact forces. From the unbalanced force, the acceleration, velocity and displacement of each sphere are computed. Then, in the selected Δt duration of time the walls are moved as per the angular velocity of the mill shell and the spheres are moved as per the computed velocities. As a result, new contacts are generated and some of the previously existing contacts are broken. This information is updated and stored in a contact-list, which is referred to when the cycle of calculations is repeated for the next step in time. The relative positions of the balls and the walls are also computed and stored in order to facilitate contact detection.

The initial 2D model of Mishra and Rajamani provided some insights into general charge motion but was still far too simplified to predict the complex movements and forces within industrial mills. Even with the extension to superquadric particles by Cleary [16] and using a particle size distribution, the 2D models were fundamentally different with regards to particle packing, percolation of smaller particles through the charge and energy distribution when compared to expected results from real

mills. Cleary's work [16] did however provide clear evidence that particle shape and size influences charge behaviour producing a higher toe region and a surface profile that was distinctly bi-linear in shape. The power draw was also noticeably lower. The extension to 3D was formulated by Archarya [17] for a limited number of particles and using a 25×28 cm mill. A full 3D implementation using industrial scales, non-spherical particles and a representative particle size distribution was successfully developed by Cleary [18]. Venugopal & Rajamani [14] also developed a 3D code but for spherical particles. The current status of DEM is to couple the discrete particles with a fluid model. The 2003 ICRA meeting in Cape Town, South Africa saw Cleary's and Nordell's implementation of coupled modelling using a technique called smooth particle hydrodynamics (SPH).

The last decade has seen DEM develop from an academic interest into a viable tool that promises to surpass the current semi-empirical models. The advantage of such a tool will be its predictive and extrapolation potential for any comminution device. One noticeable drawback is the lack of usable publications in the literature. Most are advertisements based on the promise of solving certain problems and reinforced with high end graphics renderings of selected simulations. These promises have seduced many industries, however, it is only with rigorous experimental validation that any confidence can be given to the DEM. Most validation work appear to be not as popular as their numerical counterpart, probably because it is not directly related to the improved comminution potential that DEM promises. Despite this, experimental validation is still key to the DEM's survival and acceptance. In particular, accurate verification data of charge characteristics, kinematics, dynamics and measured DEM input parameters are needed. A selection of experimental techniques for studying tumbling mills is now briefly described. Some of these articles were not specifically designed for DEM validation although hindsight reveals their potential value to the problem. An example is the work by Powell & Nurick ([19], [20] and [21]), which is also the basis for the current research.

A novel experimental program to track the motion of particles deep within the tumbling charge of an experimental mill was conducted by Powell & Nurick ([19], [20] and [21]). The first part of the study extends Powell's theory of a single particle rotating along the inner periphery of the mill. Powell's [22] initial solution of a particle's projection into free flight is very similar to the work by McIvor [8] in that the equilibrium point is in a comparable form to McIvor's equation (2.2). The difference in Powell's work is however due to the explicit integration of the lifter into the formulation. Thus when the equilibrium point is reached the particle remains in contact with the lifter until it slides off into a trajectory defined by the gravitational force. This description was extended in Powell & Nurick [19] to incorporate new and improved definitions of the various regions of the mill charge. These included the description of an equilibrium surface, a technique for uniquely obtaining the centre of circulation (CoC) and an estimate of the angle of repose that is valid for all mill speeds. The equilibrium surface⁴ is defined as a plane that divides the ascending, *en masse* charge from the descending charge. This surface is said to start at the toe and end at the shoulder of the charge, and passes through the turning point of all ball paths at the top and bottom of each concentric path of the charge. The CoC is located at the point where a tangent line to the equilibrium surface intersects a radial line at exactly 90 degrees. The CoC is said to be uniquely located because its curvature is different to that of the mill shell. The angle of repose, which also locates the CoC, is then obtained as in figure (2.3).

⁴Powell's definition of an equilibrium surface is similar to the one put forward by Barth [6]

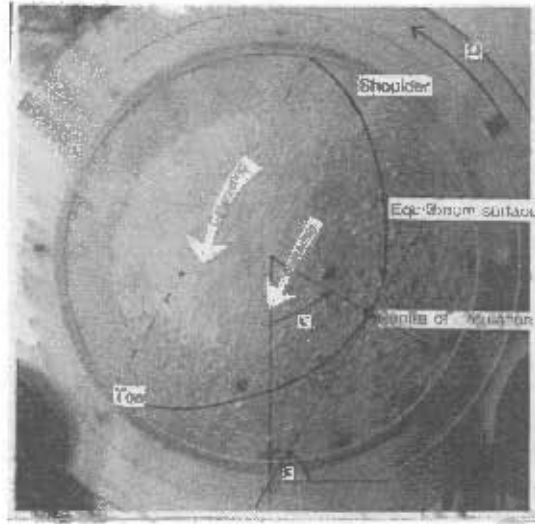


Figure 2.3: Definitions of the regions of the mill charge according to Powell [1]

The novel experimental approach taken by Powell employed a X-ray machine used in cardiac diagnostics and semi-invasive surgery. He notes that the particular advantage of the equipment was its ability to film in two planes simultaneously at rates of up to fifty frame per second. This provides the minimum amount of data for a 3D photogrammetric reconstruction of points common to both image scenes. A Perspex mill ($\varnothing 190$ mm and length 97 mm) fitted with lifters ranging from 1.8 to 10 mm and charged with 6 mm plastic beads was used to investigate charge motion. The particles to be tracked were steel, lead and glass balls of varying sizes. The relatively higher density of the tracked particles produced suitable contrast images - tracked particles showed up as dark circular spots while the bulk charge and mill shell appeared as light regions in the image. Unfortunately, a 3D reconstruction was never achieved by the authors and a 2D analysis based on front view images was conducted instead. The projection errors associated with X-ray images and the limited amount of data precluded a statistical analysis of the data. A wide range of trends were observed including segregation effects, charge slippage, average accelerations as a function of speed (% critical), impact force and energy estimates and work done per circulation by the tracked particle. The net work done per circulation was computed as the sum of the work over each 0.02 s time interval between filming frames. By assuming a constant acceleration between frames the work done between the $(n-1)^{th}$ and $(n)^{th}$ frame was computed as:

$$Work_n = mass_{ball} \left[\left(\frac{d_n - d_{n-1}}{0.2} \right)^2 - g \right] d_n \quad (2.6)$$

where d_n is the displacement between the $(n-1)^{th}$ and $(n)^{th}$ position and g is gravitational acceleration.

Powell's work laid the foundation for this thesis. The accurate 3D coordinates of the tracked particle's position as a function of time could provide useful data for experimentally validating DEM. The

simplified milling conditions are well suited to simulation while the potential to accumulate large amounts of data facilitate a statistical comparison between experiment and simulation. The full realisation of the 3D data is however a non-trivial task as revealed in this thesis.

The behaviour of a single isolated particle within a tumbling mill has been investigated by Dong & Moys [23] using multi-exposed photographs with a stroboscope as the light source. The behaviour of the single isolated particle allows the value of the coefficient of restitution and friction to be determined for subsequent incorporation into a DEM simulation. Their work is limited to 2D, requiring the user to discern the particle trajectory path from a photograph thereby limiting the amount of data that can be gathered from an experiment, and would not be able to capture the behaviour of a particle within the bulk charge.

Another powerful technique used in tracking the motion of a single particle within the bulk charge is positron emission particle tracking (PEPT). This non-invasive technique has similar capabilities to the biplanar X-ray filming method employed in this work in that the trajectory path of a single positron emitting tracer particle within the bulk charge can be determined. PEPT is based on the detection of nearly collinear gamma rays emitted during the process of positron decay and subsequent annihilation of the positrons with electrons within the tracer particle or surrounding material. The back-to-back gamma rays are detected using a positron camera consisting of two gamma ray detectors and the position of the annihilation event determined using triangulation. In practice, the position of the annihilation event is determined using multiple gamma ray pairs. The uncertainties in the positional location of the tracer particle and the detection frequency are dependent on various factors, including the velocity of the tracer. A tracer moving at 1 m.s^{-1} can be located within 5 mm 250 times per second while a tracer moving at 0.1 m.s^{-1} can be located within 2 mm 25 times per second, Parker et al [24]. The measured particle velocities in the current work range from 0.7 m.s^{-1} to 1.2 m.s^{-1} . The accuracy of the X-ray filming method employed in this work is therefore more than 30 fold greater than what would be obtained using PEPT. An advantage of the PEPT method compared to the biplanar X-ray filming is the extended duration over which the tracer is tracked; approximately 1 – 2 h compared to 100 s in this work. The PEPT method also claims the ability to study realistic systems (scaled versions of industrial type equipment) although the published works use low density rigs and particles that are very similar to the requirements of the X-ray equipment. The PEPT method has been used by various researchers for the purpose of validating DEM ([25] and [26]).

The contact parameters needed to conduct DEM simulations have to be obtained independently for the Perspex mill and charge used in the current work. These parameters are the coefficient of restitution, material stiffness, and coefficient of friction. Material stiffness is required to correctly establish the forces generated in the springs. The coefficient of restitution is related to the damping property of the material and hence can be related to the damping parameters, which in turn are used to establish the forces developed in the dashpot. Lastly, the coefficient of friction is used to check whether sliding occurs. The experimental procedures for obtaining these parameters will now be discussed in principle.

Material stiffness

Hofler & Herbst [27] studied microscale comminution processes by means of a drop weight apparatus, called the Ultra fast load cell. In these experiments the material stiffness is determined by dropping a ball on an anvil, with and without a few layers of particles on the anvil. These tests cover both metal-metal and metal-particle-metal type collisions. Mishra [28] used this principle by imbedding a strain gauge in the anvil to record the strain waves. The analysis of the data gave stiffness. Since Mishra was not studying the forces that cause particles to break, nor the mode of breakage, the force-displacement behaviour was simply plotted and the normal stiffness, K_n , computed. It should be noted that the choice of the stiffness coefficient greatly influences the stability and accuracy of the simulation since it is directly related to the critical time step. Therefore, it is prudent to choose a value for K_n which will not allow significant contact overlap, while allowing a reasonable time step.

From Hertzian contact theory for a sphere, it is known that $\frac{2}{3} \leq \frac{K_s}{K_n} \leq 1$ for the Poisson's ratio varying between 0 to 0.5; K_s is the tangential stiffness. Cundall and Strack [13] showed that their simulation result did not change when $\frac{2}{3} \leq \frac{K_s}{K_n} \leq 1$.

The force in the dashpot is related to the velocity by the damping constant. The damping constant can be estimated by means of the coefficient of restitution and spring constant. Corkum [28] showed that for a given coefficient of restitution e the damping constant C_n is:

$$C_n = -\frac{2 \ln(e) \sqrt{Km}}{\sqrt{[\ln(e)]^2 + \pi^2}} \quad (2.7)$$

where $m = \frac{m_1 m_2}{m_1 + m_2}$ and m_1 and m_2 are the masses of two colliding discs. If m_1 is very large compared to m_2 , as in the case of a moving wall, then m is the mass of the disc only.

According to Mishra [28] motion analysis and, to a lesser extent, energy loss in the mill is not dependent on material stiffness, however a correct force distribution does vary with material stiffness. He thus concludes that the choice of stiffness is not crucial.

Coefficient of Restitution

The Newtonian coefficient of normal restitution is defined as

$$e = \frac{\int R dt}{\int P dt} \quad (2.8)$$

where $\int R dt$ and $\int P dt$ are the deformation and restitution impulses respectively. The coefficient of restitution may be physically interpreted as a measure of the energy loss in a collision and is the ratio of the relative velocities of separation after impact to the relative velocities of approach before impact.

Thus for two moving discs before and after impact, the coefficient of restitution can be derived, using the above result, as:

$$e = -\frac{V_{a2} - V_{b2}}{V_{a1} - V_{b1}} \quad (2.9)$$

where V_a and V_b are the velocities of the discs a and b , and 1 and 2 refer to the situation just before and after impact, respectively. We now discuss experimental procedures for measuring these velocities under different types of collisions observed in a ball mill.

The charge profile may be broadly classified into cataracting and cascading. Within the cataracting regime the collisions may include high velocity ball-wall and ball-ball collisions. The cascading region is dominated by ball-ball interactions at relatively lower velocities. Mishra [28] points out that the cascading region has no impacts, making it "hard to determine the coefficient of restitution". The commonly used experiments in simulating these collisions involve simple drop-ball tests and colliding pendulums, Mishra [28].

The drop ball tests, used to investigate ball-wall collisions, entail dropping balls of different sizes from fixed heights which would correspond to an average drop height in the mill for a given mill speed. By measuring the rebound height the coefficient of restitution is easily derived in the form

$$e = \sqrt{\frac{h_1}{h_2}} \quad (2.10)$$

where h_1 and h_2 are the rebound and fall heights respectively. Mishra used an average fall height equal to half the mill diameter.

The pendulum experiment is the classic collision experiment for identical particles. By allowing one pendulum to stay at rest while the other is pulled to varying distances and released, the height of the balls before and after impact can be measured. In this case the restitution coefficient is modified as follows:

$$e = 2\sqrt{\frac{h_1}{h_2}} - 1 \quad (2.11)$$

where h_1 is the rebound height for the ball which was originally at rest, and h_2 is the fall height for the impacting ball.

There are numerous techniques for acquiring the rebound heights. The common method involves high speed photography with a grid background for instantaneous position coordination. Alternately X-ray filming in biplanar mode also produces reasonably accurate 3D location of the particles. The advantage of photogrammetric methods is that it does not restrict experiments to two dimensions.

2.2 Power Draw

Aside from the last decade that saw modern technology harnessing the impetus of researchers in tackling the fundamentals of charge motion and dynamics, the general lack of published literature on the subject is evident. The focus it seems, motivated by the lack of mathematical tools at the time, shifted away from charge motion studies in favour of mill power draw. However, even this path was afflicted by the general ignorance surrounding the internal dynamics of the tumbling load. Harris [29] commented that, in the 40 years since 1945, few attempts had been made to dispel the notion that the net power of a grinding mill cannot be analytically determined because of this general ignorance.

Many of those that took on the challenge employed the torque-arm principle to establish power equations. This method assumes that the total charge mass lies below the chord joining the shoulder and toe, and that there is a center of gravity through which the mass acts, as shown in figure (2.4). The torque required to rotate the offset of the center of gravity is then given by:

$$T = MgR_{CoM}\sin(\alpha) \quad (2.12)$$

where T is torque, M the mass of the charge, g is the acceleration due to gravity, R_{CoM} the distance from the mill center to the center of mass of the charge (CoM), and α the angle of repose.

The power delivered to the charge is then:

$$P = 2\pi T\phi \quad (2.13)$$

where ϕ is the rotation rate of the mill in revolutions per second (*rps*).

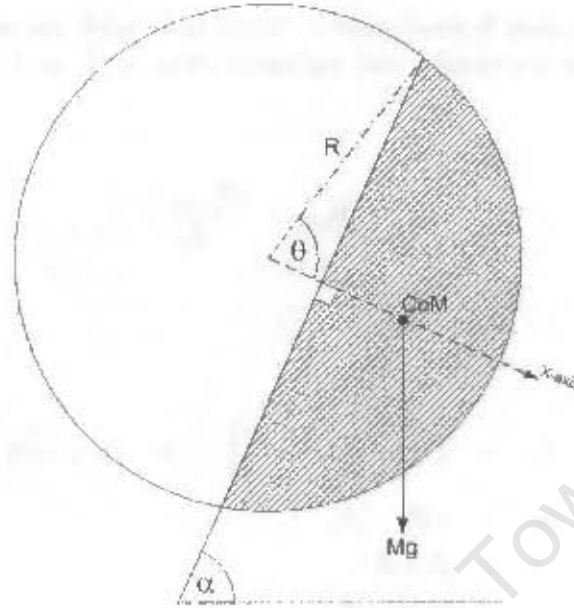


Figure 2.4: Basic geometry of charge used in the torque-arm implementation of power draw. A similar shape was employed by Hogg & Fuerstenau [30] and adopted by Harris et al [29] in determining power draw

In general, the radial arm to the CoM of the shaded area is given by

$$R_{CoM} = \frac{\int (\rho_i V_i) X_i}{\int (\rho_i V_i)} \quad (2.14)$$

where ρ_i is the density of volume element V_i , which can vary across the charge body.

Replacing ρ_i with the average bulk charge density ρ_B allows R_{CoM} to be calculated as the geometric centroid:

$$R_{CoM} = \frac{2R \sin^3(\theta)}{3[\theta - \sin(\theta) \cos(\theta)]} \quad (2.15)$$

$$\theta = \frac{1}{2} \left\{ \cos^{-1} \left(\frac{2R^2 - B^2}{2R^2} \right) \right\} \quad (2.16)$$

where B is the length of the repose slope which can be calculated once the toe and shoulder positions are known.

Implicit to equation (2.15) is that the length of the repose slope is less than the mill diameter, which is not true for high mill speeds with appreciable cataracting. The geometric centroid corresponding

to an aggressive configuration is illustrated in figure (2.5) with the unshaded area representing the charge body. By denoting the shaded and unshaded areas as A_1 and A_2 respectively, the radial arm to the CoM is given by

$$R_{CoM}^2 = \frac{R_{CoM}^1 A_1}{A_2} \quad (2.17)$$

where

$$A_1 = R^2 \cos^{-1} \left[\frac{R-h}{R} \right] - (R-h) \sqrt{2Rh - h^2} \quad (2.18)$$

$$A_2 = \pi R^2 - A_1 \quad (2.19)$$

$$h = R - d \quad (2.20)$$

$$d = R \cos \left(\frac{\theta}{2} \right) \quad (2.21)$$

and R_{CoM}^1 is according to equation (2.15).

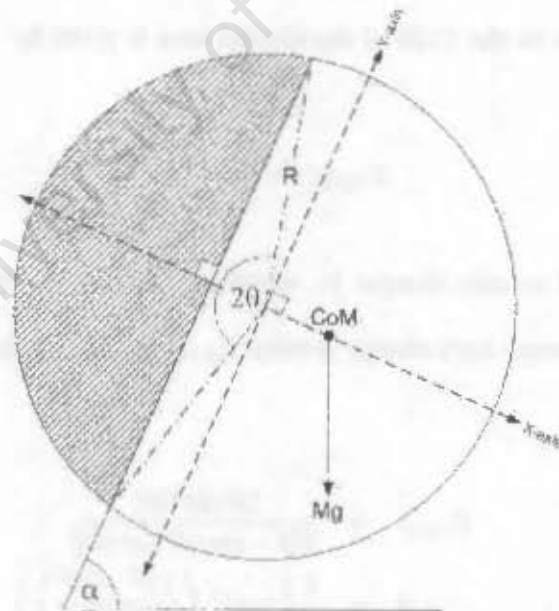


Figure 2.5: Geometric centroid of the simplified charge shape for appreciable cataracting. NB: unshaded region represents charge body

Bond (1962)

One of the earliest empirical power models based on the torque-arm principle was due to Bond [31, 32, 33]. In its most general form, the power draw was given by:

$$P = K \rho \sin(\alpha) J(1 - \beta J) L D^{2.3} \phi \left[1 - \frac{0.1}{2^{\gamma - \zeta \phi}} \right] W \quad (2.22)$$

where

ρ = bulk density of the load

W = weight of the charge

α = angle of repose of the charge

L = mill interior length

D = mill interior diameter

ϕ = fraction of critical speed

J = fraction of mill volume occupied by the charge

K = a constant which is said to be strongly affected by liner design and slurry properties, Moys [34]

β = a parameter that is related to the load fraction

γ and ζ are parameters.

A popular form of equation (2.6), suggested by Bond, is

$$P = 12.262 \rho J(1 - 0.937J) L D^{2.3} \phi \left[1 - \frac{0.1}{2^{9 - 10\phi}} \right] \quad (2.23)$$

The speed function $\phi \left[1 - \frac{0.1}{2^{9 - 10\phi}} \right]$ ensures that power reaches a maximum with respect to speed, figure (2.6), while the filling term $J(1 - \beta J)$ allows for maximum power at a particular filling, and is achieved as follows:

$$P \propto J(1 - \beta J) \quad (2.24)$$

Differentiating with respect to J gives:

$$\frac{dP}{dJ} = 1 - 2\beta J \quad (2.25)$$

P is a maximum when $\frac{dP}{dJ} = 0$, therefore the filling (J) at maximum power is given by

$$J = \frac{1}{2\beta} \quad (2.26)$$

where β is a parameter that must be fitted. Alternately, if the filling fraction is known, β can be determined.

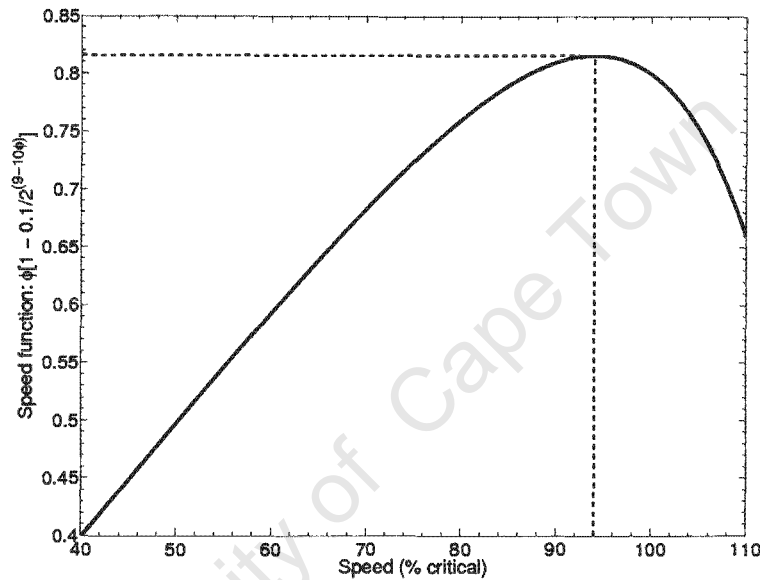


Figure 2.6: Relationship between % of critical speed and Bond's speed function

The parameters K , β , γ and ζ must be determined, presumably through calibration, before the model can be used. From an optimisation perspective, this requires knowledge of the power, preferably measured, at a minimum of four different speeds, for the same charge, liner and slurry properties, in order to fit the parameters optimally. The model should then be able to predict a power curve that yields the optimum speed and filling for say maximum power draw at that configuration. Moys [34] suggests Powell's [35] non-derivative regression technique for this optimisation. Bond's pragmatic modelling methodology has formed the basis for most subsequent empirical endeavors. Unfortunately, such approaches lead to poor predictive capabilities, especially outside the window of operation for which they were formulated, which is probably why technological advancement in comminution remains distinctly conservative. Notwithstanding, Bond's power equation remains widely used and adapted.

Arbiter and Harris (1982); Harris, Schnock and Arbiter (1985)

Arbiter & Harris [36] developed a semi-empirical power equation that employed the torque-arm principle coupled with empirical results. The charge shape used in their formulation was similar to that

of Hogg & Fuerstenau [30] who employed the angle of repose of the charge in an energy based power formulation, figure (2.4). The power equation was given as:

$$P = \frac{1}{6}\pi\rho NLD^3 \sin^3(\theta) \sin(\alpha) \quad (2.27)$$

where

ρ = mean bulk charge density

L = mill interior length

D = mill interior diameter

N = rotation rate

L_f = load fraction

α = charge angle of repose

θ = angle related to mill filling (figure 2.4)

By replacing $\sin^3(\theta)$ with the approximation $4L_f(1 - L_f)$ and lumping parameters, equation (2.27) was simplified to:

$$P = kWND(1 - L_f) \quad (2.28)$$

with k said to characterise mill type, table (2.1), and W the weight of the charge.

Type of mill	k
grate	0.13
Tube	0.12
Autogenous	0.115
Overflow	0.11
Rod	0.09

Table 2.1: Values of k for different mill types (after Harris et al [29])

Equation (2.28) was further generalised by Harris et al [29] to incorporate different operational conditions and presented as:

$$P = KD^n L_f Y \lambda F (1 - aF) \quad (2.29)$$

where

f = fraction of critical speed

L = mill length

λ = charge density

F = load fraction

Y = speed correction factor

K, n, a = parameters, similar to those used by Bond.

A common form of equation (2.29) is:

$$P = 1.88mgfL^{0.5}(1 - F)\sin(\alpha) \quad (2.30)$$

The absence of the speed function leads to a linear trend with speed. It is interesting to note that even after investigating 39 operating mills, the simplified charge shape, which clearly does not account for cataracting, was left intact. It is therefore not surprising that they obtained poor predictions at high speeds when cataracting is significant.

Liddell and Moys (1988)

Liddell & Moys [37] adopted equation (2.30) with a few adjustments. The speed correction factor was replaced with the normalised, measured torque, which they presented in tabular form. An average value of $\alpha = 42^\circ$, determined by Harris et al [29] after observing 39 operating mills, was used for the repose angle together with the substitution of ρJV_{mill} for the weight of the charge, where V_{mill} is the mill volume. The final equation, which varied linearly with speed, was given as:

$$P = 9.69\rho fLD^{2.5}F(1 - 1.06F)Y \quad (2.31)$$

with Y at a given speed, computed as:

$$Y(s) = \frac{T(s)}{T_{max}} \quad (2.32)$$

where

$T(s)$ = measured torque at speed s

T_{max} = maximum torque

Their comparison of power models based on the torque-arm principle [29, 30, 38] indicate reasonable agreement for these models at low mill speeds becoming progressively poorer and quite arbitrary, as the speed increases. Liddell & Moys [37] found the Harris et al [29] model to be the most accurate while the Bond [38] model, equation (2.23), consistently overestimated power by approximately 50%. An interesting conclusion from their work was that 'Until the motion of the mill load can be characterised mathematically, the power drawn by the mill cannot be realistically calculated, and the simplistic models will continue to be used'.

Feurstenau, Kapur and Velamakanni (1990)

Feurstenau et al [39] and later Kapur et al [40] addressed the issue of simplistic charge profiles by incorporating the cataracting charge into their power formulation. The charge was partitioned into cascading and cataracting, and the energy consumption in both regions incorporated into an expression for power draw, figure (2.7). The power equation also included a minor friction component after Pietsch [41].

$$P(t) = P_{cs}(t) + P_{ct}(t) + P_f(t) \quad (2.33)$$

where P_{cs} and P_{ct} are the cascading and cataracting components respectively and P_f is due to internal friction.

The cascading component of the power was estimated using the model of Hogg & Fuerstenau [30] coupled with the empirical volume approximation for $\sin^3(\theta)$ introduced by Harris et al [29],

$$P = \frac{2NW_1g(D-d)}{3J_1} \phi(J) \sin(\alpha) \quad (2.34)$$

$$\phi(J) = \sin^3(\theta) \quad (2.35)$$

$$= 4J_1(1 - J_1) \quad 0.35 \leq J_1 \leq 0.5 \quad (2.36)$$

$$= 4J_1(1.05 - 1.33J_1) \quad 0.2 \leq J_1 \leq 0.35 \quad (2.37)$$

while the cataracting component was due to the torque experienced by a centrifuging layer of charge,

$$P_{ct} = \pi Ng(D-d)W_2 \sin\left(\alpha + \theta + \frac{\beta}{2}\right), \quad (2.38)$$

and said to go negative for those portions of the 'stuck' layer that extended beyond the topmost (12 o'clock) position of the mill. The friction component was represented empirically as:

$$P_f = C \exp(-Kt) \quad (2.39)$$

where C and k are constants, and t represents the grinding time. The weight fractions, $1 - \epsilon$ and ϵ , that partition the charge W into cascading and cataracting portions respectively, were estimated by

$$\epsilon = \frac{4NBd^2\Psi(d)}{JD} \left[1 - \frac{\Psi(d)}{D} \right] \eta, \quad (2.40)$$

where

η = is the absolute viscosity of the slurry, said to be obtained from detailed measurements by the authors, given by

$$\eta = \eta_{\infty} \{1 - \exp[-X(t - t_0)]\} \quad (2.41)$$

$\Psi(d)$ = some function of ball diameter d (unspecified)

B = a lumped parameter

N = rotational rate

η_{∞} = limiting viscosity after an extended period of grinding t_0

Venugopal & Rajamani [14] note that “Although better than the torque-arm formula, a perfect partitioning of the mill charge into cascading and cataracting regions is highly idealised”. Morrell [42] comments that “The model contained six parameters which required to be fitted Its usefulness in most practical situations where limited data are available, must therefore be considered to be limited”. Notwithstanding the frictional component, equations (2.34 and 2.38) can be directly tested against unique data, such as that derived from 3D tracking experiments, like the work of Powell & Nurick [20, 21] and the proponents of positron emission particle tracking (PEPT) [43, 44, 24, 45]. In particular the end points of the arc spanned by angle β can be equated to the upper and lower shoulder positions identified by Powell & McBride [46], the CoM can be calculated as the time-averaged centroid from the particle positions, thus yielding θ , and the weight fractions can be directly computed. The only drawback is the accuracy in position achieved by these authors - a pertinent issue as the scale of the experiment is reduced.

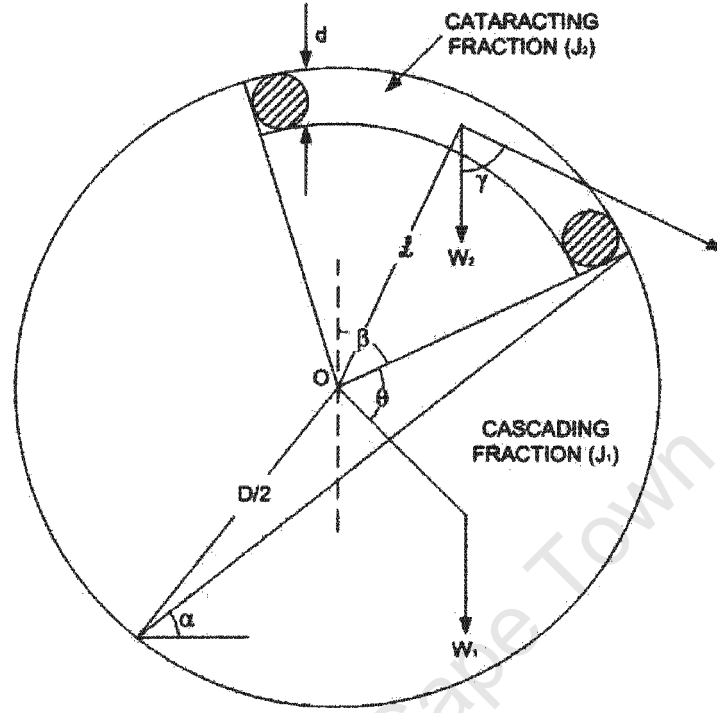


Figure 2.7: Partitioning of grinding charge between cascading and cataracting regimes (after Feurstenau et al [39])

Moys (1993)

Moys [34], motivated by the inadequacies of the Bond model at high speeds, variable mill loads and high slurry densities, which are commonplace in the South African ROM mills, proposed a semi-phenomenological model of mill power. The model partitioned the charge into two regions: a centrifuging portion that draws no power and an active, non-centrifuging portion that draws power according to the Bond model. The centrifuging layer reduces the mill diameter (D) to an effective diameter (D_{eff}) which consequently reduces the load fraction and percentage of critical speed to the effective values of (J_{eff}) and (N_{eff}) respectively. The effective quantities, for a load fraction of 0.5, were incorporated into a simplified form of the Bond model (i.e. without the empirical speed function) and given as:

$$P = K D_{eff}^{2.3} \sin(\alpha) \rho J_{eff} (1 - \beta J_{eff}) N_{eff} \quad (2.42)$$

where

δ_c = thickness of the centrifuging layer

$$D_{eff} = (1 - 2\delta_c) D$$

α = angle of repose

ρ_L = bulk density of the load

$N_{eff} = \frac{\text{rpm} \sqrt{D_{eff}}^{100}}{42.3}$ (presumably)

K and β are constants similar to those used by Bond.

The thickness of the centrifuging layer was empirically related to the operating variables by

$$\delta_c = J^{\Delta_J} \exp \left[-\frac{N^* - N}{\Delta_N} \right] \quad (2.43)$$

with N^* and Δ_N said to be strong functions of liner profile and slurry viscosity while Δ_J was said to govern the strength of the dependence of δ_c on J , and is a strong function of liner profile. The model was compared to that of Bond by fitting the relevant parameters at non-standard operating conditions and using them to predict power at normal ranges of operation. The parameter fits were achieved with Powell's [35] non-derivative regression technique. The comparisons showed that Bond's proposed parameters ($\beta = 0.937$, $\gamma = 9$, and $\zeta = 10$) result in large errors between theory and practice, while the regression fit produced significant improvements.

Morrell (1992)

Morrell [42] extended the torque-arm principle to an integral form in which he computed the lever-arm to every 'element of charge' visible from the end window and summed this over an annular-ringed profile, figure (2.8), to obtain the power delivered to the charge.

$$P_{net} = 2\pi g L \rho \int_{r_i}^{r_m} \int_{\theta_T}^{\theta_S} N_r r^2 \cos(\theta) d\theta dr \quad (2.44)$$

where

L = mill length

ρ = density of total charge

N_r = rotation rate of particle at radial distance, r , from center of mill (rev.s^{-1})

r = radial position (m)

θ = angular position of element

θ_T = angular displacement of shoulder position (radians)

θ_S = angular displacement of toe position (radians)

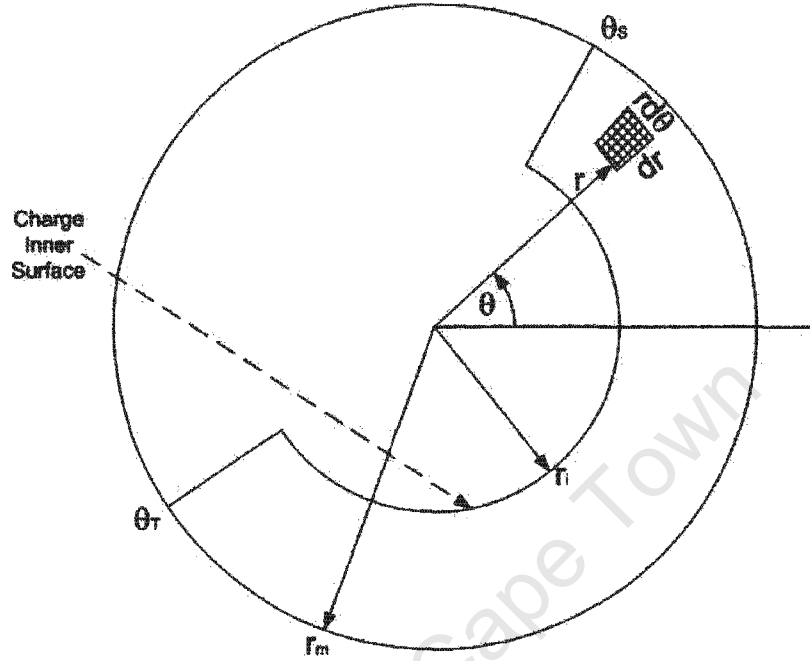


Figure 2.8: Schematic diagram of simplified charge shape (redrawn after Morrell [42])

Frictional power loss, which was said to be a direct consequence of slip within the charge, was modelled from laboratory measurements as the variation of the particles's angular velocity N_r with radial location r . According to him "The transfer of energy from the mill shell to the charge is influenced by the frictional forces acting on the concentric layers of the charge. The degree of slip between these layers depends on the magnitude of the frictional forces. The latter are, in turn, affected by the type, shape and size of the ore and by the rheological characteristics of the slurry. This is manifested in the charge as a change in angular velocity with radial position." The relationship was obtained through linear regression of laboratory measurements and given as

$$N_r = \frac{N_m r_m (r - r_i)}{r (r_m - r_i)} \quad (2.45)$$

where N_m is the angular velocity of the mill, r_m the radius of the mill and r_i the radial position of the charge surface. Equation (2.45) coupled with empirical relationships for θ_S and θ_T , expressed as functions of the load fraction (J_t) and fraction of critical speed (ϕ), were incorporated into equation (2.44) to give:

$$P_{net} = 0.5g^{1.5} L \rho \phi r_m^{0.5} \left(\frac{(2r_m^3 - 3r_m^2 r_i + r_i^3)}{3(r_m - r_i)} \right) [\sin \theta]_{\theta_T}^{\theta_S} \quad (2.46)$$

with

$$\tau_i = \tau_m \left(1 - \frac{(2\pi J_t)}{(2\pi + \theta_S - \theta_T)} \right)^{0.5} \quad (2.47)$$

Morrell's work denotes the seminal deviation from the simplistic charge shape defined by the angle of repose with marked improvement in power prediction, highlighting the value of charge shape in determining power. Except for the empirical relations defining the toe and shoulder angular displacements, the model contains physical quantities that can be measured on industrial operations with reasonable accuracy. Morrell's model, unlike Bond, has predictive capability that does not depend on *a priori* knowledge of power. A particularly attractive aspect of the model is the absence of parameters that are difficult to quantify. A drawback of the model is the lack of sensitivity to lifter geometry, implying that two mills with completely different lifter geometry (everything else identical) will produce the same power draw.

Powell (1996)

Powell [1, 21] used the trajectory of a single particle within the charge to compute the center of mass per circulation, with the average over all circulations yielding the center of mass of the charge. This time-averaged estimate was said to incorporate all regions of the charge (cascading, cataracting and *en-masse* motion) if the circulations spanned all layers of the charge. By formulating an empirical linear relationship, equation (2.48), for the variation of angular location of the center of mass, θ_{com} , with mill speed, the torque was calculated using the simple torque-arm principle.

$$Angle(^{\circ}) = \theta_{com} = 27 + 0.42 \times speed(\%crit) \quad (2.48)$$

$$T = Mg.R.\sin(\theta_{com}) \quad (2.49)$$

The realistic torque curve derived from his work reaffirms that the simple torque-arm principle is fundamentally correct and can yield good predictions if the polar coordinates of the center of mass are accurately characterised. Unfortunately, no relationship for the radial-arm (R) was given, making the model practically difficult to implement, given the sensitivity of power to R at high speeds. Equation (2.48) is also charge specific.

2.2.1 Discussion on power models

Morrell [42], Liddell & Moys [37] and Powell [21, 1] obtained realistic torque/power curves that increased with mill speed up to a maximum power, falling off with further increases in speed. Their

contributions are encouraging in light of models that have found acceptance in plant design despite their “invariant power peaks with respect to filling and speed or no power maxima at all”, Morrell [42].

According to the fundamental laws of mechanics "The centre of mass of a system of particles moves like a single particle, whose mass is the total mass of the system, acted on by a force equal to the total external force acting on the system", Symon [47]. The definition of torque as used in the lever-arm principle is defined in all introductory physics texts including Halliday et al [48] and Reese [49]. According to them, the torque and hence power draw, using the illustration in figure (2.4), is:

$$T = MR_g \sin(\alpha) \quad (2.50)$$

$$P = 2\pi T\phi \quad (2.51)$$

The difficulty in determining the angular position of the CoM seems to be the primary reason for inventing the angle of repose, which is at best ill-defined. Such complexities are intimately linked to the lack of mathematical tools and experimental techniques needed to locate the center of mass, and notwithstanding, the angle of repose. Similar sentiments are echoed by Venugopal & Rajamani [14] who note that “... power draft approaches had to evolve gradually since the cataracting part of the charge could not be modeled in any simplified manner and there was much uncertainty in fixing the shoulder position of the charge.” The understanding of power delivered to the charge seems intimately linked with a better understanding of charge motion fundamentals.

2.3 A review of selected X-ray photogrammetric and machine vision articles

2.3.1 Introduction

The scope of image processing prevented an historical look at the field. The review is broken up into four broad categories. The first category deals with edge detection algorithms that are classified as state of the art in the vision community, Heath [50]. Thresholding techniques were also formulated as part of the image processing tools developed in this thesis and constitutes the second category. To avoid duplication, only the paper on optimal thresholding is included with the details of thresholding theory being deferred to chapter (4). The third category presents the three basic schemes that describe the mapping of image space into object space of the observations. The photogrammetric literature included here are biased towards reconstruction formulations that take into account the constraints of the biplanar angiographic equipment or those relating to X-ray photogrammetry. Category four looks at the conic fitting techniques employed in this thesis for the purpose of accurate centroid location of the tracked particles in image space. Again, the reader is referred to chapter (5) for more detailed descriptions on the target centering techniques employed in this thesis.

2.3.2 Edge detection

Edge detection has been widely studied in the machine vision literature. The governing principle of most edge detectors is to find the appropriate operator that readily distinguishes the edge pixels from the noise. The operators are usually convolved with the image in Fourier space. These operators vary from the simple derivative approximations used by Roberts [51] in discerning edges as the maxima of the gradient images to the more sophisticated implementation of Canny's [52] robust Gaussian filtering which has become the benchmark of any good edge detector. Another popular edge detector usually credited to Marr & Hildreth [53] is the Laplacian of Gaussian method (LoG) in which the Laplacian of a Gaussian filter is convolved with the image and the edges identified as the zero crossings in the image; see section (4.3.2) for an illustrated description of zero crossing.

Non-linear filtering methods like the Nalwa algorithm [54] is also considered to be a state of the art edge detector although the resulting edges are often dilated beyond the actual object. Such spurious detail can lead to undue bias in subsequent centering stages which is undesirable for the current task of developing an accurate, automated 3D tracking system. Nalwa suggests thresholding to thin the edges - a solution less elegant than Canny's non-maximal suppression algorithm. The Iverson algorithm [55] which uses logical checks to identify edges is claimed to improve linear edge detection algorithms, however it also suffers similar problems to the Nalwa algorithm [54] and is therefore unsuitable to the problem of this thesis.

The edge detection algorithm formulated by Canny [52] is generally considered as a "standard method" by most researchers. The full implementation of the Canny method is given in section (4.3.4) with only a descriptive account presented here. The Canny method uses linear filtering by convolving a Gaussian kernel (mathematical operator) with the image in the x-direction to smooth the noise. The edge intensity and slope direction is then computed for each pixel in the smoothed image. This is achieved by first differentiating the smoothed image in both the x and y directions. The magnitude and direction of the gradient for a given pixel is then computed according to equations (4.17) and (4.18) respectively. The next stage, known as non-maximal suppression, compares the edge strength of each candidate pixel to its two nearest neighbours along the slope direction. Essentially the pixel is set to zero if it is less than or equal to the edge strength of the aforementioned neighbouring pixels. This process is sometimes referred to as thinning. The resultant image is then made binary by an adaptive thresholding with hysteresis in order to eliminate the streaking of edge contours. The process of hysteresis requires two edge strength thresholds. All candidate edge pixels below the lower threshold are labelled as non-edges and set to zero while those pixels above the lower threshold and connected to pixels above the high threshold via a chain of edge pixels are labelled as edge pixels and assigned the value of one. The edge detector described by Canny requires three input parameters. The first is *sigma*, the standard deviation of the Gaussian filter specified in pixels. The second is the low threshold value, *low*, which is usually specified as a fraction of the high threshold. The third input parameter is the high threshold value, *high*.

Many investigators have implemented variations of the Canny method with comparable successes. These approaches generally differ in the smoothing stage and are generally preferred if sufficient knowledge of the problem is known and the expected outcomes are clear. A popular variant is to perform convolution with the Gaussian in two orthogonal directions, usually along the rows and

columns of the image. Some researchers convolve row-wise with a Gaussian and column-wise with the derivative of the Gaussian. These types of smoothing show very little differences in practice. A year after Canny published his algorithm, Bergholm [56] proposed an edge focussing technique. His technique begins by searching a Gaussian blurred image for pixels having a local gradient maximum that is higher than some preset threshold. The edges are then “focused” by tracking them back through scale space to finer resolutions - essentially, by reducing the Gaussian blurring and edge tracking according to the above-mentioned technique, an edge is deemed valid if it also featured in the previous edge search with greater blurring. The overall idea is not dissimilar to Canny’s method except for the final edge discrimination; it also requires more computational time than most of the other edge tracking techniques discussed.

A drawback of directional operators like the Gaussian kernel is that it requires two convolutions which increase the computational expense. The implementation of the Canny algorithm in this thesis indicate that four convolutions are needed to prevent asymmetry of the resultant edges caused by subpixel shift, see section (4.3.5.1) for details.

2.3.3 Thresholding

In the context of feature extraction algorithms, the thresholding techniques appear less sophisticated than edge detectors with a notable lack of sensitivity to noise contaminated pixels of interest. Most thresholding seems to take on a complimentary role with edge detectors. There are however instances where optimal thresholding algorithms have proven to be just as adequate as their edge discerning counterparts. The work by Chow & Kaneko [57] is a particular case in point which embodies an optimal solution to extracting meaningful boundary information from X-ray images. The similarity to the X-ray tracking problem in this thesis demanded an investigation into this work.

The aim of the authors was to obtain the outline of a heart image using boundary detection methods on low quality cineangiograms. The usual mode based methods were said to be inadequate for these low quality images. They opted instead for dynamic thresholding based on local characteristics. The assumption, which they claim to be empirically verified, is that the probability distribution of the intensity for any small region of the picture consists solely of the object or the background - a unimodal distribution. The preprocessing stages involve logarithmic, subtraction and averaging operations. By considering subimages, after preprocessing, consisting of light and dark regions, a mixed probability distribution function can be fitted to the data. The distribution is then tested for bimodality. Those regions depicting clear bimodality are analysed for an estimate of the threshold by the method of maximum likelihood⁵. These thresholds are then interpolated twice, first regionwise and then pointwise, to obtain a threshold for each and every image point. These thresholds are then used in a binary decision process for the final segmentation. Their underlying assumption for fitting a bimodal probability distribution is that the distributions are normal for both object and background. The method described has become commonplace in image processing and is often described as the “optimal thresholding algorithm”. The general form of this technique - the probability of classifying (erroneously) object pixels as background - is implemented in this thesis. The nature of the data made it unnecessary to divide the image.

⁵Similar minimisations are conducted in this thesis; see sections (4.2.2 and A.4) for details.

2.3.4 Photogrammetric reconstruction

The reconstruction of a scene from image data has three basic formulations. (1) Using the actual geometric parameters associated with the camera, a pair of equations, often referred to as the collinearity equations, can be formulated for each image point. The collection of equation pairs constituting the various points can then be solved using the bundle adjustment method as proposed by Brown [58], provided that at least two images of the scene are taken from different orientations for the same instant in time. The bundle adjustment employs an elegant algebraic scheme called partitioning and solves for all parameters simultaneously in an iterative network adjustment. The technique is however dependent on good provisional estimates for the zeroth iteration. (2) The direct linear transformation (DLT), after Abdel-Aziz & Karara [59] is a direct solution scheme that uses eleven linearly dependent parameters instead of the nine linearly independent camera parameters in setting up the collinearity equations. The eleven parameter formulation ensures that a solution is always achieved, even without iteration, provisional estimates of the parameters, and fiducial markings. The relationship to the camera parameters are trivially obtained, see section (6.2.3), and the incorporation of higher order optical distortions into an iterative least squares solution scheme, see sections (6.3 and 6.4) for details, ensures that the accuracies are comparable to the bundle adjustment. Adams [60] presents an alternate derivation of the direct linear transformation using X-ray imaging geometry. His work builds on the seminal ideas of Klein [61] who used homogeneous coordinates to formulate the mapping of 3D space onto a plane. The value of Klein's [61] work clearly lies in the geometric validity of the DLT solution to X-ray imaging. (3) The use of control frames in photogrammetric applications is by far the most reliable method of parametrising the scene space. However, as 3D reconstruction techniques become more widely used in industrial and medical applications, the need for reconstruction techniques that require minimal variation to existing application also arose. One such application is in X-ray angiography. Cheriet & Meunier [62] developed a "self-calibration" imaging system with the biplanar angiographic equipment used traditionally in the diagnostics of heart valves. Their formulation uses the rotation and focal length of the equipment (rotation and focal length is displayed on the gantry control circuit) to parametrise the scene space. The geometric parameters are optimised using a non-linear optimisation scheme based on the Levenberg-Marquard algorithm [63]. The technique used is very efficient with regards to minimal interference of the complex clinical protocol. It does, however, require at least twelve point correspondences to be made and it has not been demonstrated for moving objects in their experimental tests. It is hoped that they also realised the phase lag issue associated with the angiographic equipment - a fact that is pertinent to the study of moving objects, see section (7.7) for details. As far as can be gathered from their investigation of accuracy, the error associated with the reconstruction of a plane grid is a thickness error of 0.3 mm.

It is interesting to note that both the bundle adjustment and the DLT as presented in the literature do not explicitly model variances (variances are intimately linked to the weight matrices used in the solution schemes) in the iterative implementations. Both schemes begin by assuming either a uniform weighting or zero variance at the zeroth iteration - linearly dependent or ill-conditioned formulations produce quickly diverging solutions with these models. Another drawback of these formulations is the dependence on a control frame in order to bound the solution. The proponents of the bundle adjustment claim that a control frame is not necessary. This is only true if good provisional estimates of unknown points are available along with an a priori calibration procedure. The X-ray filming work conducted in this thesis prevent reasonable estimates of the unknown points being made without

some sort of prior calibration with control frames.

2.3.5 Conic fitting by the least squares method

A conic section is formed as the perspective projection of a circle. It is also known that the projection of a sphere onto a plane, as in X-ray imaging, yields elliptical shapes [64]. These projections are often encountered in machine vision making them important to any feature extraction and centering algorithm. The research described in this thesis deals solely with the problem of discerning accurate centering information from the projections of a plastic bead for the purpose of reconstructing its 3D position as a function of time. The image data vary from near circular to moderately eccentric ellipses with varying degrees of noise. The elucidation of the curve that belies the digital projection data can be modelled in two ways: a parametric or implicit curve. The latter is preferred because of its relatively simple form, a biquadratic implicit polynomial, while the former finds commonplace use in computer graphics and computer aided design. The review begins with some mathematical preliminaries needed to facilitate the subsequent discussion. The basis for optimisation, the error of fit metric, is then briefly described as an introduction to the actual minimisation and constraint implementations of conic fitting. Only three metric structures are discussed and used in this thesis. The reader is referred to Fitzgibbons [65] for a more thorough account of the different metrics available. The discussion then continues with the various conic fitting routines investigated in this thesis. The wide range of techniques employed in the literature necessitate that only selected works as regards the three metric forms are discussed. The robustness of fit for each of the six algorithms investigated is done in chapter (5).

Mathematically, the parametric curve is represented in the form

$$C(\mathbf{a}) = \{c(\mathbf{a}; t) \mid t_{min} \leq t \leq t_{max}\} \quad (2.52)$$

while the implicit curve is defined as the zero set of scalar functions $F(\mathbf{x})$:

$$C(\mathbf{a}) = \{\mathbf{x} \mid F(\mathbf{a}; \mathbf{x}) = 0\} \quad (2.53)$$

The problem of conic fitting may be described as the minimisation of a distance metric by the method of least squares fitting. To facilitate the ensuing discussion, some mathematical preliminaries are necessary. In particular the problem may be stated as an optimisation problem in the following manner, Fitzgibbons [65]. Given:

1. A set of $2D$ points $P = \{\mathbf{x}_i\}_{i=1}^n$, where $\mathbf{x}_i = (x_i, y_i)$ are the digitally sampled data points;
2. A family of curves $C(\mathbf{a})$ parametrised by vector $\mathbf{a}$, where the $\mathbf{a}_i$ are the parameters of the conic section;

3. A distance (error of fit) metric relating C and P through $\mathbf{a}$, which measures the likelihood that the curve $C(\mathbf{a})$ explains the data points P ;

find the value $\mathbf{a}_{min}$ for which the error function attains its global minimum. The curve $C(\mathbf{a}_{min})$ is then said to best fit the data. The optimisation schemes that employ the least squares methodology generally only differ in the form of the distance metric and a related constraint. From the seminal work of Bookstein [66] to the robust metric of Fitzgibbons [65], the more effective distance metrics as regards computational efficiency and range of applicability are the geometric distance, algebraic distance and Kanatani's 'statistical distance' which are now briefly described.

1. The *geometric distance* is the maximum-likelihood estimator under Gaussian noise assumptions. Geometrically, it is the distance between the sampled point $\mathbf{x}$ and the nearest point on the modelled curve C

$$D_G(\mathbf{x}, C) = \min_{p \in C} \|\mathbf{p} - \mathbf{x}\| \quad (2.54)$$

The *geometric distance* has been cited by many investigators ([67], [68] and [69]) to be the optimal metric to minimise from the least squares point of view. Statistically this implies that the curve which achieves the minimisation of the mean geometric distance most closely models the ideal curve corrupted by gaussian noise. Implicit to this assumption is that each sample point is an image of only one curve point⁶.

2. The *algebraic distance* is an approximation for implicit functions like the conic section. These scalar functions introduce a natural distance metric between some point on the plane and the curve. The algebraic distance between some point $\mathbf{x}_i$ and the curve defined by $F(\mathbf{a}, \mathbf{x}) = 0$ is simply the value of F at $\mathbf{x}_i$, where $(x_i, y_i) \in \mathbf{x}$ and $\mathbf{a}_i \in \mathbf{a}$.

$$D_A(\mathbf{x}_i, C) = F(\mathbf{a}; \mathbf{x}_i) \quad (2.55)$$

The algebraic distance will not in general exhibit a simple relationship to the geometric distance and may depend on the scaling of the parameters.

3. Kanatani [70] contends that the geometric distance is not the maximum-likelihood (*maximum a-posteriori*) estimator when the data used to fit the curve is digitally sampled and not a straight line. He argues that the sampled data represent the probability of being from the imaging model. His model assumes that a true curve is digitally sampled (i.e. the resultant edge image of a digitally sampled projection is the model for the true curve), with each point on the sampled curve being corrupted by a random error vector taken from an isotropic gaussian distribution. In terms of a probability distribution function (pdf), the sampled edge data constituting the curve is a gaussian blurr of the original curve. Thus by constructing the appropriate digital model, we can estimate the probability at each pixel by convolving it with the appropriate gaussian. To illustrate Kanatani's method, consider figure (2.9). The white line represents the 'true' digital ellipse after edge detection by the Canny method [52]. The ellipse is then

⁶Kanatani argues that each sample point represents the probability of being from the digital model

convolved with a Gaussian resulting in the blurring of the ellipse (the black region). The two black dashed lines denote the 68% confidence interval. The argument against the geometric distance being the maximum likelihood estimator is made clear in this figure. Notice along the high curvature region, the two white points, denoted by the '+', are equi-distant from the sampled curve and have distinctly different probabilities of being from the original curve while the two white points along the low curvature region, denoted by the '*', are also equi-distant from the sampled curve and have approximately equal probabilities. Thus by minimising the sum of the geometric distances the weighting is greater for points outside the sampled ellipse than inside, especially along the high curvature regions. Mathematically, if the digital model of the ellipse is chosen to be the implicit curve $F(\mathbf{a}; \mathbf{x}) = 0$ and the normalised noise pdf is $G(\mathbf{x} - \mathbf{p})$ then the probability that point $\mathbf{p}$ is from the original digital curve is equivalent to the convolution

$$P(\mathbf{p} | F) = \iint [\delta F(\mathbf{a}; \mathbf{x})] G(\mathbf{x} - \mathbf{p}) dA \quad (2.56)$$

where δ is the Kroneka delta and $\mathbf{x}$ is the digitally sampled data.

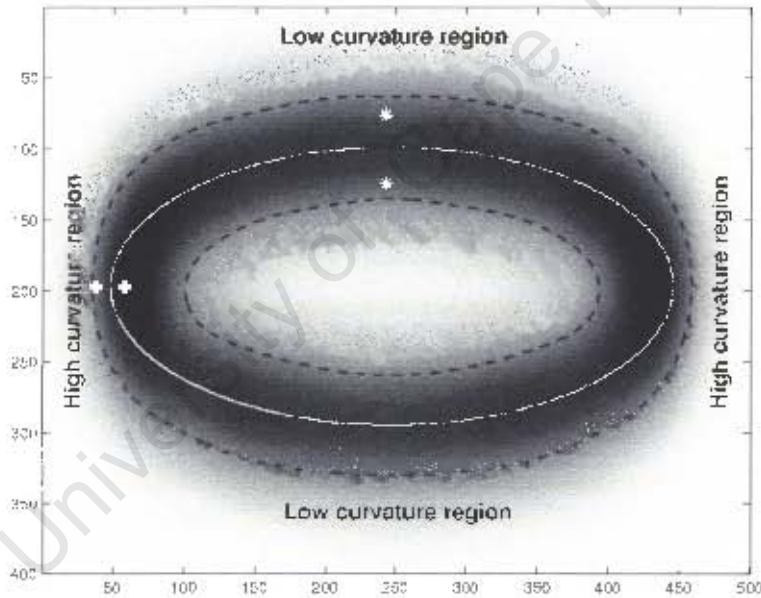


Figure 2.9: Image of digital ellipse corrupted by gaussian noise. Each pixel represents a probability of being from the original digital curve (Fitzgibbons [65])

We now discuss a selection of constraints coupled with the distance metrics described. Using the mathematical structure described above, the problem of conic fitting is reduced to the formulation and subsequent minimisation of the distance metric subject to certain constraints. These constraint forms are essentially what distinguishes the various conic fitting formulations. For the sake of uniformity we represent a general conic by an implicit second order polynomial

$$F(\mathbf{a}, \mathbf{x}) = ax^2 + bxy + cy^2 + dx + ey + f = 0 \quad (2.57)$$

where $\mathbf{a} = [a \ b \ c \ d \ e \ f]^T$ and $\mathbf{x} = (x, y)^T$. $F(\mathbf{a}, \mathbf{x}_i)$ is called the algebraic distance of a point $\mathbf{x}_i$ to the conic $F(\mathbf{a}, \mathbf{x}) = 0$. In matrix form:

$$F(\mathbf{a}, \mathbf{x}) = \mathbf{D}\mathbf{a} \quad (2.58)$$

$$\text{with } \mathbf{D} = \begin{bmatrix} x_1^2 & x_1 y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2 y_2 & y_2^2 & x_2 & y_2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_n^2 & x_n y_n & y_n^2 & x_n & y_n & 1 \end{bmatrix} \text{ and } \mathbf{a} = [a \ b \ c \ d \ e \ f]^T.$$

The standard method, sometimes referred to as the linear algorithm, minimises the algebraic distance

$$\sum_{i=1}^n F(\mathbf{a}, \mathbf{x}_i)^2 = \|\mathbf{D}\mathbf{a}\|^2 \quad (2.59)$$

subject to the constraint $\|\mathbf{a}\|^2 = 1$. The solution scheme is straight forward and explicitly outlined in Golub & van Loan [71]. Their solution scheme employs Lagrange multipliers to analytically minimise the constrained objective function (the algebraic distance) to form an eigenvector problem which is solved in the usual way. According to Fitzgibbons [72] the linear algorithm fares poorly on many real data sets due to its inherent statistical bias, especially when the images curve and are partially occluded. Similar observations on occluded and noisy images were made in this thesis, the results of which are given in tables (5.2 and 5.3) and figures (5.1 and 5.2) respectively. The algorithm is however the fastest and easiest to implement. It is interesting to note that most authors of new conic fitting routines choose to compare their results with the linear algorithm.

Kanatani's [70] approach improves on the linear algorithm by calculating the statistical bias described in figure (2.9) and subtracting it from the resulting minimisation. The solution requires an iterative procedure because a priori knowledge of the true minimum and Gaussian noise level is needed to achieve the desired correction. The scheme suggested by Kanatani [70] was to iterate until the correction for the noise level converges to zero or some acceptable limit. Kanatani [70] referred to this correction metric as the "statistical distance". The iterative requirement makes the algorithm computationally expensive while the tensor formulation complicates implementation. Despite the rigorous formulation and a seemingly sound bias correction, the algorithm does not perform adequately for the X-ray image data. One possible reason might be attributed to the noise in the X-ray data not being simply Gaussian in nature. Kanatani's statistical bias correction fared poorly under highly occluded data and failed completely with noisy data. These results are demonstrated in tables (5.2 and 5.3) and figures (5.1 and 5.2).

The work by Bookstein [66] is widely regarded as the seminal work on conic fitting with constraints. His aim was to reduce the statistical bias introduced by the linear formulation by imposing a constraint on the conic parameters. He minimised the algebraic distance subject to a quadratic constraint

$$a^2 + \frac{b^2}{2} + c^2 = 1 \quad (2.60)$$

The formulated generalised eigensystem was shown to be rank deficient, hence a solution by block decomposition was followed. Bookstein's [66] algorithm performed adequately for highly occluded data and poorly for noisy data.

In keeping with the objective of finding conic fitting routines that are adapted for ellipses, the algebraic distance with linear constraint

$$a + c = 1 \quad (2.61)$$

was also investigated. The slight bias towards ellipses have made this algorithm popular in the machine vision community ([73], [74], [65] and [75]). The constraint is implemented by eliminating a from the equations and solving for $[b, c, d, e, f]$. The eliminated parameter is then calculated using the constraint $a = 1 - c$. The linear algorithm produced unsatisfactory results for both highly occluded and noisy images in the current investigations. It should be noted that the preference expressed by the above-mentioned authors are in most instance based on theoretical assessments that lack sufficient experimental testing. The rationale employed in the investigations of this thesis was simple - if the routine is not robust enough to cope with the distortions offered by real data it is deemed unsatisfactory.

The geometric distance metric is used to fit conic sections to image data that are normally distributed about the curve. The geometric distance was proposed by Nakagawa & Rosenfeld [69] who defined it as the orthogonal distance between the sampled data and a corresponding point on the curve. According to them the minimisation of the geometric distance is only valid if each sample point is an image of only one curve point. Following their solution scheme, the geometric distance at sample point $\mathbf{x}$ is obtained by solving the simultaneous equations

$$\begin{aligned} \mathbf{p} + \lambda \nabla F(\mathbf{a}; \mathbf{p}) &= \mathbf{x} \\ F(\mathbf{a}; \mathbf{p}) &= 0 \end{aligned} \quad (2.62)$$

for $\mathbf{p}$ subject to the minimisation of the geometric distance.

Of all the conic fitting routines investigated, the one that optimally suited the image data of the plastic bead was the ellipse specific algorithm by Fitzgibbons [65]. He minimised the algebraic distance subject to the constraint $4ac - b^2 = 1$ thereby incorporating the ellipticity constraint into the solution scheme. The aforementioned constraint obviates the difficulty associated with the usual iterative tests against the constraint $b^2 - 4ac < 0$ by using a relatively simple equality constraint. Mathematically,

the global minimisation problem is formulated as the minimisation of $E = \|\mathbf{D}\mathbf{a}\|^2$ subject to the

$$\text{constraint } \mathbf{a}^T \mathbf{C} \mathbf{a} = 1, \text{ where } \mathbf{C} = \begin{bmatrix} 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The minimisation scheme followed is according to Bookstein [66]. The tests conducted on highly occluded and noisy data show the Fitzgibbons algorithm to be the optimal choice for the X-ray imaging problem. He does however note that the ellipse specific algorithm is biased towards low eccentricity. The conic fitting algorithms described in this review were tested against two sample X-ray images depicting high occlusion and noise, and is illustrated in chapter (5).

2.4 Concluding remarks and hypothesis

The elucidation of the fundamentals governing charge motion has been a slow journey that has spanned a century of investigation and is still no closer to any real understanding than those endeavours by White [2] and Davis [3]. The effective use of highly tweaked recipes to troubleshoot most problem scenarios on the plant has sustained the rate of this journey. Not surprisingly, the available literature lacks the creativity in experimentally investigating charge motion, with the end-window filming technique becoming the common thread that weaves the same incomplete, sometimes incorrect, picture of the motion of particles within a tumbling mill. The use of novel technologies has been one of the most refreshing aspects of recent investigators, with notable mention going to the proponents of PEPT and tomography. Advances in computing has also brought to the fore powerful numerical schemes for tackling the highly complex business of modelling a tumbling mill. Despite these efforts, there is still a need for more fundamental experimental investigations that surpass the traditional end-window studies. It is with this sentiment that the current research project evolved. The context in which this thesis gains impetus is based on the premise that good quality experimental trajectory data of charge particles within a tumbling mill provides sufficient information to begin the elucidation process. The immediate value of such data would be to statistically validate numerical schemes like the DEM, whose primary objective will be to complement, and in the long term replace, black box empiricism with a fundamental model of charge dynamics in a tumbling mill. Looking beyond DEM validation, the data affords a unique opportunity to accurately quantify features like the center of mass (CoM), toe and shoulder angles, which are key ingredients to many comminution power models, and unfortunately equally difficult to determine with typical comminution data.

2.4.1 Hypothesis

The use of biplanar X-ray imaging techniques to reconstruct accurate 3D trajectory data of a tracer particle within a simplified, laboratory tumbling mill provides unique and valid information for understanding charge kinematics and dynamics, and consequently to benchmark the predictions of DEM through statistically based comparisons.

The following ideas and assumptions encapsulate the hypothesis of this thesis:

1. 3D trajectory information of a typical charge particle in an experimental tumbling mill can be reconstructed from biplanar X-ray images using photogrammetric techniques.
2. Based on statistical sampling theory of normally distributed data, the motion of a mono-sized charge in a simplified experimental mill may be represented by the trajectory of a typical charge particle.
3. Statistically comparing DEM simulations with experimental data of a typical charge particle according to particle kinematics and average dynamics provides an alternative, and improvement, to current verification techniques.
4. Beyond DEM verification, the trends derived from the kinematic and dynamic analyses are useful to better understanding comminution.

2.4.2 Techniques to be used in this study

The state of the art in feature extraction from image data revolves around Canny's technique [52] which is widely accepted as the standard method. Thresholding techniques are also reliable when the data is distinct. The probabilistic optimisation scheme used by Chow & Kaneko [57] do much to extract areas of interest in less distinct data, however they lack the robustness needed in the presence of noise and occlusion, especially when a fair degree of automation is required. Edge detection and thresholding techniques do well to localise the regions of interest in the X-ray image data but do not eliminate noise completely. In fact a large amount of automated data processing, like that encountered in this thesis, always brings in varying degrees of noise and occlusion which must be adequately dealt with in the subsequent centering stages. The range of target centering algorithms surveyed in this literature review⁷ tend to cover most cases encountered in the thesis. Like most mature centering techniques (Adams [76], Heipke et al [77] and Godding [78]), it is always useful to include some level of manual intervention to deal with anomolous cases.

The mapping of 2D image space into 3D real space is based on three formulations: the DLT [59], the bundle adjustment [58] and, more recently, self-calibration techniques. These methods are equally valid and accurate if used correctly. Integral to the attainment of this objective is a well structured transformation matrix. The DLT is useful when *apriori* calibration of the camera is unavailable although the guaranteed solution can sometimes be spurious. The problem of poor solutions may be attributed to the non-orthogonality of the transformation matrix that results with the standard DLT implementation. The modified DLT solution proposed by Hatze [79] remedies this skewness by accounting, in some sense, for the experimental errors but does not eliminate it altogether. A guaranteed orthogonal transformation can only be achieved if the error model has some physical resemblance to the actual experimental errors. The bundle solution is a more physically representative photogrammetric model that only converges with good initial estimates - poor estimates make the

⁷the papers included in the literature survey do not cover all techniques investigated. See chapter (5) for more techniques used.

transformation matrix unstable due to non-orthogonality. The luxury afforded to the bundle solution - that of good initial estimates - was not available to the X-ray equipment used in the current work making the DLT the preferred choice. Protagonists of the bundle adjustment methodology do have successful implementations when the DLT solution is used to obtain the initial estimates. The problem does however hinge on an accurately calibrated control frame to ensure stability of subsequent stages. The introduction of sophisticated imaging technology to the various industries, most notably the medical field, have necessitated alternative imaging models that obviate the need for calibration frames. The transformation matrix arising from these self-calibration techniques are tailored to the equipment and require robust numerical methods for an accurate solution. The use of these techniques is however only valid for stationary objects and have not been tested on systems with phase lag between the two cameras.

2.4.3 Tools for comparison of experiment with DEM

Despite the decade long relationship between DEM and comminution, literature on DEM validation are few and far between. Amongst these endeavors is the development of validation tools for the benchmarking of DEM - A project of the comminution research group at the University of Cape Town, of which the author is a member, that ran parallel with this research project. A recent outcome of the validation toolbox was the development of numerical schemes to determine the center of circulation (CoC) of the charge based on particle positions. This particular technique was used to benchmark the DEM against the trajectory data derived from the particle tracking experiments of this thesis. The comparison with the initial sets of 3D trajectory data derived from this thesis was done by McBride et al [15], and is presented in chapter (8). Additional numerical schemes for identifying charge features like the toe and shoulder were conducted by Powell & McBride [46] and employed in chapter (9).

It is anticipated that the analysis of the trajectory data will yield further tools of comparison, however, the key contribution of this work will be of a robust image analysis technique, followed by the mapping of the two planar images into 3D space. This process needs to be automated to collect sufficient data to provide statistically useful information for the validation of DEM.

2. The maximum angle of the beam subtended by the intensifying screen, figure (3.4)
3. The separation distance between the X-ray source and intensifying screen (also known as the Focal Film Distance or FFD)

The mill, tumbling media and drive rig must also satisfy the following criteria:

1. There must be a digitally discernable contrast between the tracked particle and the rest of the charge including the mill
2. Both the media and shell have to be sufficiently strong to withstand typical particle interactions within a mill run at speeds of 60 – 120% of critical
3. The media have to depict realistic particle behaviour
4. The mill shell and media must have a low cumulative mass absorption coefficient such that an aluminium-filtered beam will pass through the mill and still be sufficiently strong to expose the X-ray film.
5. The motor and support rig have to lie outside the path of the intersecting beams due to their relatively high cumulative mass absorption coefficients.
6. The support and drive mechanism have to be portable to optimise the time spent on the BAE.

The structure used in this chapter is as follows: Section (3.2) deals with the penetrative properties of the X-ray beam and its influence on the selection of the bulk charge and material properties of the mill. The next section, section (3.3), describes the design of the support rig and drive train. Section (3.4) addresses the geometric constraints of the angiographic equipment. These constraints are used to formulate a set of parametric equations for optimising the selection of mill dimensions.

3.2 Material selection based on the penetrative properties of the X-ray beam

The penetrative properties of X-rays are well described by an exponential decay function:

$$I_x = I_0 \exp \left[- \left(\frac{\mu}{\rho} \right)_m \rho_m x \right] \quad (3.2)$$

where

H = lifter bar height

The lifter parameters used in the Skega formula are illustrated in figure (3.2).



Figure 3.2: Schematic of lifter geometry used in Skega formula

The general design principles described above are used in the current work and may be summarised as:

1. Aspect ratio (length - to - diameter ratio) must be in the range $0.5 \leq \frac{L}{D} \leq 2$
2. The ball diameter - to - mill diameter ratio must be in the range $0.0125 \leq \frac{B}{D} \leq 0.0286$
3. Lifter profile selection will be constrained by equation (3.1)

Coupled with these design criteria are the constraints imposed by the biplanar angiographic equipment (BAE) - An effect that dominates the mill and rig design. In order to guarantee that the entire mill volume is captured by the X-ray imaging system it is imperative that the mill and rig conform to the X-ray beam characteristics. This includes both the penetrative properties and geometry of the beam. The beam characteristics that influence the design are:

1. The maximum energy (intensity) of the X-ray beam produced at the X-ray tube; see Appendix (C) for details

X-rays are produced within a contained unit, marked with the letters A in figure (3.1), pass through the patient (in our case the Perspex mill) resulting in scintillations of varying contrast on the input screen of the image intensifier, denoted by the letters B in figure (3.1). The scintillations are then recorded by a high speed digital camera, positioned directly behind the image intensifier, which is eventually sampled at effective rates of 12, 25, 50 or 75 frames per second depending on the requirements of the cardiologist. The sampled images are also relayed to an on-line television screen, indicated by the letter C in figure (3.1), providing a quick in-situ diagnosis of the heart valves. The usual procedure following the angiographic assessment is to download the relevant X-ray image scenes to a personal computer for further diagnostics, including writing the data to compact disc.

The biplanar mode of the angiographic unit is used to obtain two, orthogonal and nearly simultaneous, images of the heart (denoted by the two perpendicular image intensifiers)¹. The images resulting from biplanar filming satisfies the minimum conditions for reconstructing 3D positions of corresponding points. The asynchronicity of the sampling between the two planes produce images that are rotated in space and time, thus precluding a natural frame by frame comparison. The correction of the phase lag between the X-ray images is a post-processing stage and therefore not dealt with in this chapter. The reader is referred to chapter (7) for a treatment of phase lag, while details pertaining to the functions of the BAE is addressed in Appendix (C).

This chapter deals with the design of the mill and support rig shown in figure (3.1). The primary objective of the design is to immerse the maximum possible volume (biggest mill possible) within the conical, intersecting X-ray beams while ensuring that the radiation captured by the intensifying screen contains all pertinent information regarding the tracked particles.

Due to the complex nature of tumbling mills, various design criteria exist for the numerous operational conditions used around the world. For example, Austin et al [80] point out that typical semi-autogenous grinding or SAG mills have length-to-diameter ratios of 0.5 whereas a ratio of 2 is more appropriate for autogenous grinding (AG) mills. Bond [38] obtained an empirical relationship for calculating the ball diameter based on a selection of factors coupled with extensive plant experience. Typically, ball diameters of 100 mm can be used in mills with diameters ranging from 3.5 m to 5.5 m whereas balls with diameters of 125 mm go into mills having diameters from 4.5 m to 10 m. Another important requirement of comminution is to optimise the cascading/cataracting effect for ball on rock interaction. To achieve this, the correct balance between mill speed and lifter geometry must be determined. A useful relationship for achieving this balance is the formula by Moller & Brough [81], given by

$$H = \left(1 - \frac{N}{100}\right) W \quad (3.1)$$

where

N = percentage of critical speed

W = width of the lifter

¹ The synchronicity of the images are not necessary for the general diagnostic purposes

Chapter 3

Experimental Design

3.1 Introduction

The equipment used to track the 3D motion of particles deep within the bulk charge of a tumbling mill is the biplanar angiographic equipment (BAE). The BAE is a medical diagnostic tool used to assess the condition of heart valves after a contrast medium (a semi-opaque dye) has been injected into the patient's heart valves. Consequently, the image produced has the heart valves clearly marked as black tubes with a lighter grey for the surrounding portions of the heart. In normal angiographic mode a person would be lying on the bed where the experimental rig is positioned, figure (3.1).

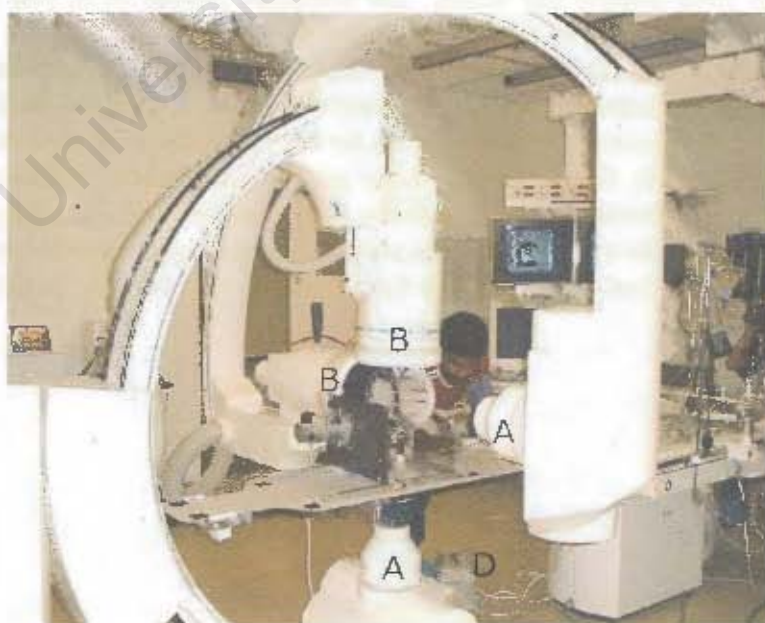


Figure 3.1: Angioscope in fully interlinked position

I_0	—	initial intensity of the X-ray beam
I_χ	=	residual beam intensity
$\left(\frac{\mu}{\rho}\right)_m$	=	cumulative mass absorption coefficient for absorber m
ρ_m	=	loose bulk density of the absorber
χ	=	average length of the absorber

The material used in the manufacture of the experimental mill had to satisfy the following criteria:

1. The bulk of the charge should be transparent to X-rays
2. The mill shell should be transparent to X-rays

Investigations indicate that 300 mm thick material with a density less than 1.5 g.cm^{-3} results in a residual beam that is sufficiently intense to produce the required X-ray images. Perspex was found to best satisfy this criterion; see Appendix (C) for details. However, the dimensions of the mill had to be further reduced to satisfy the beam geometry. The central projection, as referred to by Klein [61] is characteristic of X-ray beam generation. The beam appears to diverge from a central point (the focal spot) in a cone-like fashion, figure (3.4). Accordingly the mill had to be optimised such that it lies completely within the intersecting X-ray cones illustrated in figure (3.8). Based on these constraints, a Perspex mill (density $\rho = 1.2 \text{ g.cm}^{-3}$) with a diameter and length of 142 mm was constructed. The bulk of the balls used were 6.1 mm diameter plastic beads (transparent to X-rays) while the tracked particle was a typical plastic bead with a thin coating of silver paint. The standard deviation of the ball diameter was determined from a random sample of 300 balls and shown to be 0.3 mm.

3.3 Design of support rig and drive train

The high demand of the BAE for its usual angiographic application afforded limited access to the equipment. The support rig was therefore purpose designed to be portable while still maintaining adequate rigidity in order to reduce vibrations. The constituent parts of the rig were made of aluminium and held together by screws and bolts in order to facilitate alignment of the steel shafts and drive wheels. The design of the rig and drive train had to satisfy the following conditions:

1. the orthogonal X-ray beams had to pass through uninhibited to the scintillating screens
2. the height of the rig should be chosen such that the mill is concentric with one of the scintillating screens and the mill itself is completely immersed in the intersecting X-ray beams

The drive mechanism also satisfied the above-mentioned criteria. A variable speed motor with a calibrated speed controller, denoted by the letter D in figure (3.1), was connected directly to one of the drive shafts. The motor was carefully matched with the power requirements of the mill, ensuring that all speeds appropriate to the milling experiments were easy to set and maintain for the duration

of each experimental run. The speed of the mill was also measured using a hand - held tachometer. The speed measurements were taken before and after each experimental run, at 5 second intervals for a duration of 1 minute, and the averages compared to ensure that the mill speed was according to the setting on the speed controller.

3.4 Mill dimensions based on the geometric constraints of the X-ray beam

The dimensions of the Perspex mill are determined by the X-ray beam geometry (primary constraint) and the aspect ratio (secondary constraint). In this section we formulate a parametric model for selecting the optimal mill dimensions. The aim of the optimisation process is to determine the maximum mill size that can be completely immersed within the intersecting X-ray beams. The maximum mill size will not necessarily imply maximum volume, nonetheless the maximum volume will also be investigated as a possible criterion. The model is based on three configurations of the X-ray beam, figure (3.3). Each arrow shown in figure (3.3) indicates the possible direction of the X-ray beam.

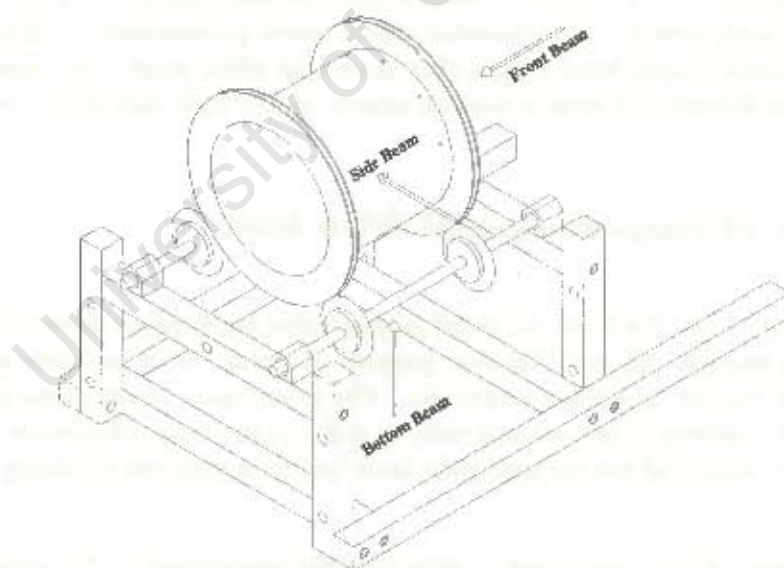


Figure 3.3: CAD model of experimental rig illustrating the 3 filming orientations

All other configurations (orientations of the beam) are considered to be either non-optimal or equivalent and are therefore not discussed any further. An X-ray experiment employing the BAE will use any two of the three configurations shown. The following criteria are used in describing the geometry:

1. The maximum diameter of the scintillation screen available for imaging; denoted as S in figure (3.4)
2. The maximum Focal Film Distance (FFD) that can be achieved in biplanar mode
3. The minimum distance achievable between the experimental mill and the scintillation screens in biplanar mode; denoted as d in figure (3.4)
4. The flexibility of the C-arms in biplanar mode

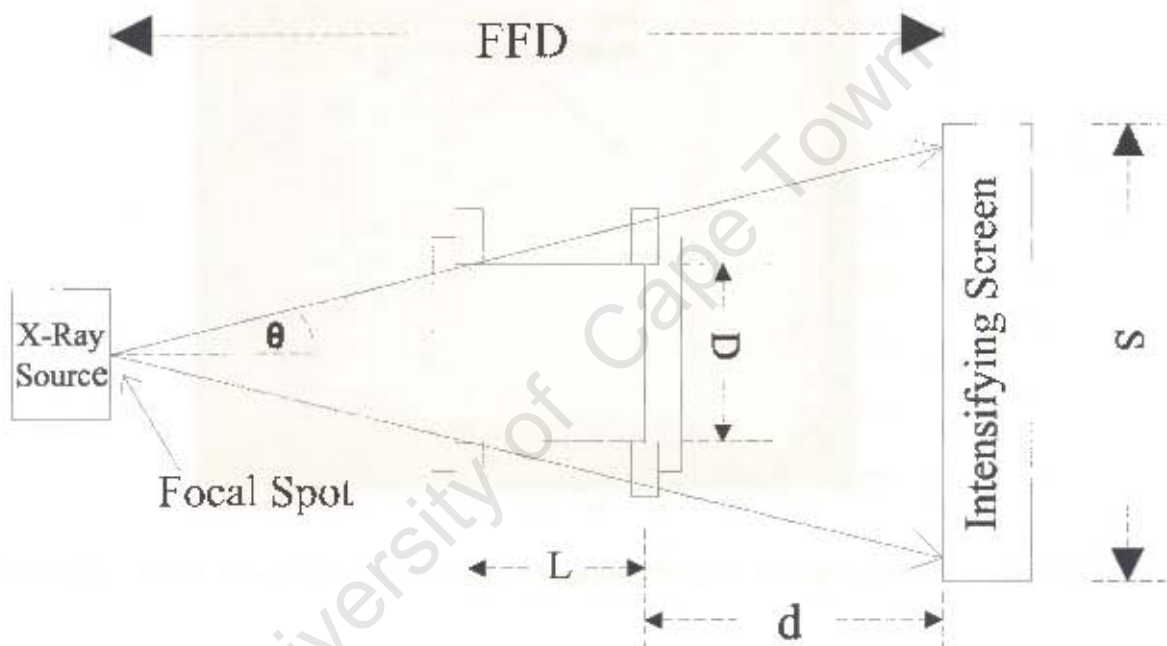


Figure 3.4: Illustration of parameters used in optimising the mill design according to the geometric constraints of the BAE (front view filming)

The scintillation screen has a diameter of 23 cm, from which a maximum of 18.7 cm is available during the image capture process. The reduced imaging area is due to the automated collimators that reduce scatter by 'chopping' the X-ray beam. The reduced dimensions are calculated with the aid of scaling markers introduced by Siemens technicians - Essentially two points are marked on the intensifying screen and the distance between them measured. The technicians take several measurements and an average value is then computed. Figure (3.5) is an X-ray image of the intensifying screen showing the scaling markers as two small, dark spots. In image space the distance between these points is computed as 164.3 pixels (image spatial coordinate) whereas the physical distance as measured by the Siemens technician is 6.0 cm. Combining this with the known image resolution (512×512 pixels), a simple proportionality can be computed for the physical distance spanned by 512 pixels:

$$S \leq \frac{512 \times 6.0}{164.3} = 18.7 \text{ cm} \quad (3.3)$$

where S is the diameter of the intensifying screen used in the X-ray imaging process.

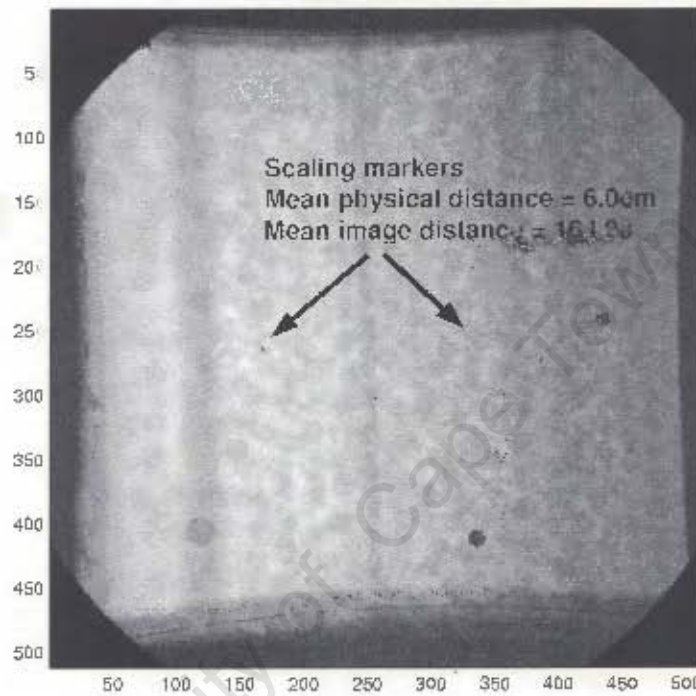


Figure 3.5: Scaling markers for calibrating the intensifying screen and image coordinates

The maximum focal film distance for the BAE is 100 cm. However, in biplanar mode the limited manoeuvrability restricts this to approximately 95 cm. This value is obtained from several investigations with the BAE. Another important constraint is the ability to get the mill as close as possible to the scintillation screen. Figure (3.4) shows this constraint (d) for the X-ray beam passing through the front end of the mill. This distance can vary depending on the filming orientation used in the experiment, figure (3.3). The spacing between the intensifying screen and the mill shell is denoted by d_F , d_S and d_B and illustrated in figure (3.6). The additional 7 cm shown in the figure accounts for the 5 cm protective cap on the intensifying screen and the 1 cm mandatory space between the screen cap and object being X-rayed (If the spacing is less than 1 cm the BAE automatically notifies you and prevents any further use until it is rectified). The final spacing used in the optimisation is thus given by (d_1 , d_2 and d_3), see figure (3.6) and table (3.1), where

$$\begin{aligned} d_1 &= d_F + 7 \\ d_2 &= d_S + 7 \\ d_3 &= d_B + 7 \end{aligned} \quad (3.4)$$

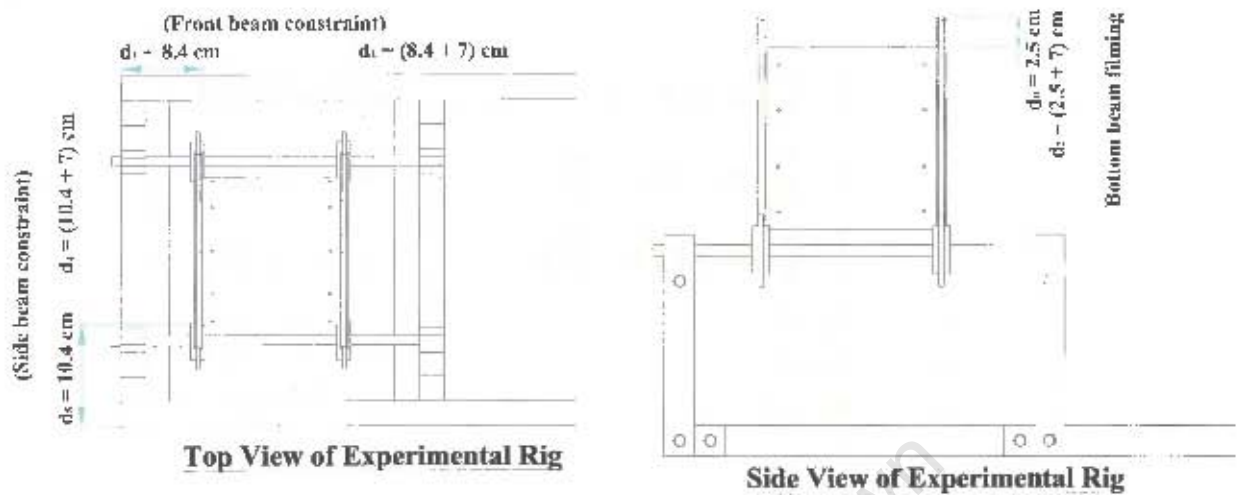


Figure 3.6: Illustration of distance between mill shell and intensifying screen

The constraint parameters are summarised in table (3.1).

Description of Constraint parameters	constraint values (cm)
allowed FFD (F)	95
allowed screen size (S)	18.7
screen to shell spacing for front beam (d_1)	15.40
screen to shell spacing for side beam (d_2)	17.36
screen to shell spacing for bottom beam (d_3)	9.50

Table 3.1: Constraint parameters of the BAE used to formulate parametric equations

Using these constraints, the following parametric equations were developed for a given mill diameter(D):

$$L \leq \left(-\frac{F}{S}\right) D + (F - d_1) \quad \text{if } L \geq D \quad (3.5)$$

$$L \leq \left(-\frac{S}{F}\right) D + S \left(1 - \frac{d_2}{F}\right) \quad \text{if } L \geq D \quad (3.6)$$

$$L \leq \left(-\frac{S}{F}\right) D + S \left(1 - \frac{d_3}{F}\right) \quad \text{if } L \geq D \quad (3.7)$$

$$d_1 = d_F + 7 \quad (3.8)$$

$$d_2 = d_S + 7 \quad (3.9)$$

$$d_3 = d_B + 7 \quad (3.10)$$

$$L \leq D \cos(\theta) \quad \text{if } 0 < L < L_{max} < D \leq D_{max} \quad (3.11)$$

$$\theta = \tan^{-1} \left[\frac{S}{2F} \right] \quad (3.12)$$

$$D_{max} = \frac{S(F - d_i)}{F \cos(\theta) + \frac{S}{2} [1 + \sin(\theta)]} \quad i = 2, 3 \quad (3.13)$$

$$L_{max} = D_{max} \cos(\theta) \quad (3.14)$$

$$0.5 \leq \frac{L}{D} \leq 2.0 \quad (3.15)$$

where D = diameter of mill, L = mill length for the given diameter, (d_1 , d_2 or d_3) depend on the beam orientation and the other terms are defined in table (3.1) and shown in figures (3.4 and 3.7).

Equation (3.5) represents the front beam configuration and is derived using the geometry of figure (3.4). Equations (3.6 and 3.7) represent the side and bottom beam configurations respectively and are derived using the vertical beam geometry illustrated in figure (3.8). Constraint equation (3.11) is derived for the bottom and side beam geometry when ($L < D$) and based on figure (3.7); The general case of $L < D$ is illustrated by the small circle while the larger circle indicates the configuration for the maximum allowed diameter.

Plots of the linearised parametric equations are employed to facilitate the ensuing discussion. The plots show the variation of mill length (L) and volume (V) as a function of mill diameter (D). The following classification are used for the three X-ray beam configurations:

- Front beam configuration will be referred to as Case 1
- Side beam configuration will be called Case 2
- Bottom beam configuration will be denoted as Case 3

In addition three combinations of these cases, corresponding to the three possible configurations for biplanar filming, will be discussed:

1. Combination 1 implies Cases 1 and 2
2. Combination 2 implies Cases 1 and 3
3. Combination 3 implies Cases 2 and 3

3.4.1 Combination 1: Front and Side beam filming with the BAE

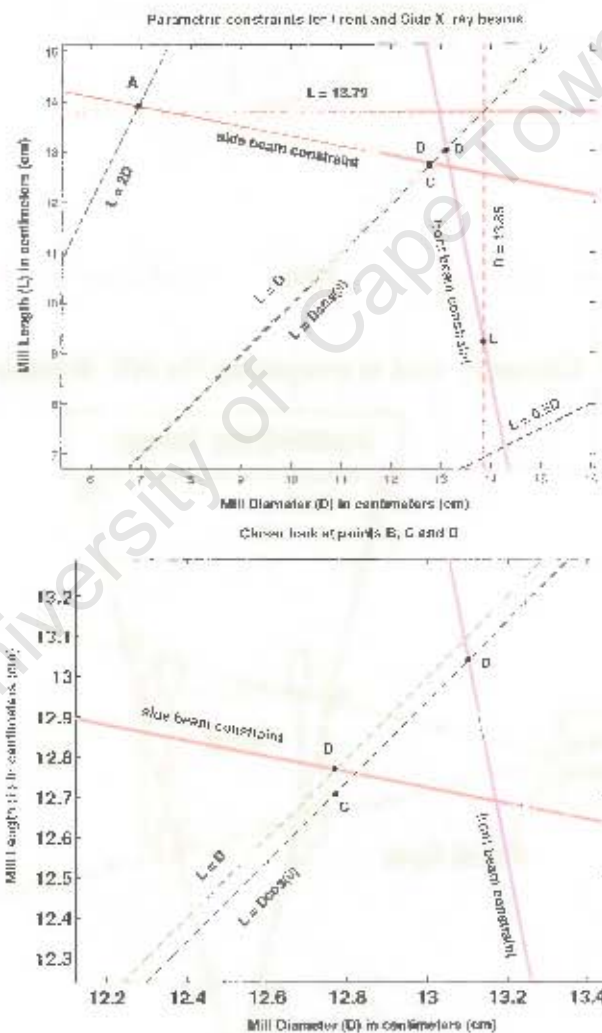


Figure 3.9: Parametric constraints for Combination 1

The parametric constraints for Combination 1 is described by equations (3.5, 3.6 and 3.11) with d replaced by d_2 where necessary. These equations are plotted in figure (3.9). The bottom plot in the

figure is simply a zoom in on regions that are not clearly distinguishable in the plot above it. The discussion will thus shift from one to the other if necessary.

The lower bound for the diameter is at point *A*. As the diameter is increased, the mill length (*L*) decreases along the side beam constraint, equation (3.6), until it intersects the $L = D$ line at point *B*, see bottom plot in figure (3.9). Beyond this point ($L < D$) equation (3.6) is no longer valid. The constraint is then taken up by equation (3.11) for points beyond point *C*. Further increases in diameter now produces a corresponding increase in length according to equation (3.11). This persists up to point *D* at which the constraint shifts to equation (3.5). Continuing to increase the diameter beyond point *D* now results in a sharp decrease in length according to equation (3.5). The drop in length is maintained up to point *E*, which is the upper bound to the diameter for $L < D$. In summary, the path that satisfies the parametric constraints simultaneously is *ABCDE*.

For the path discussed (*ABCDE*) the appropriate diameter and length can be obtained and the mill volume calculated. The internal volume of the mill is given by:

$$V = \frac{\pi D^2 L}{4} \quad (3.16)$$

A plot of the variation of mill volume with diameter clearly indicates the maximum volume for Combination 1, figure (3.10). Note that only the relevant regions of the plot is shown for reasons of clarity. The maximum volume is obtained at point *D* (same point in figure (3.9)) with $D = 13.10$ cm and $L = 13.04$ cm.

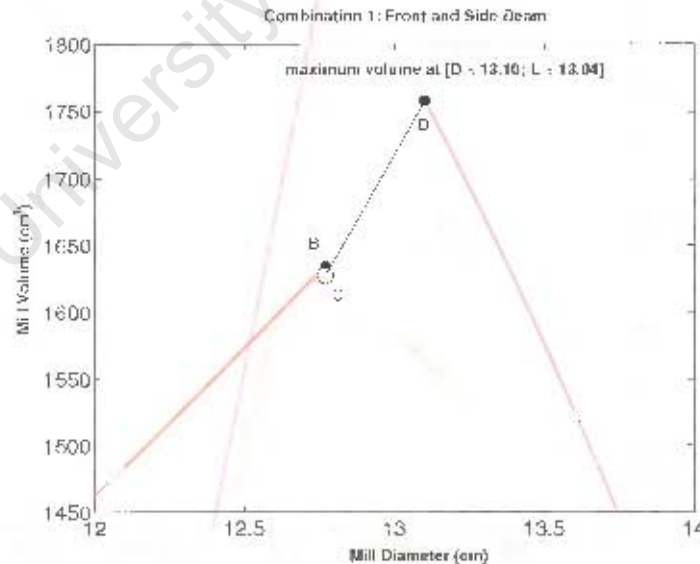


Figure 3.10: Combination 1: Variation of mill volume with diameter

3.4.2 Combination 2: Front and Bottom beam filming with the BAE

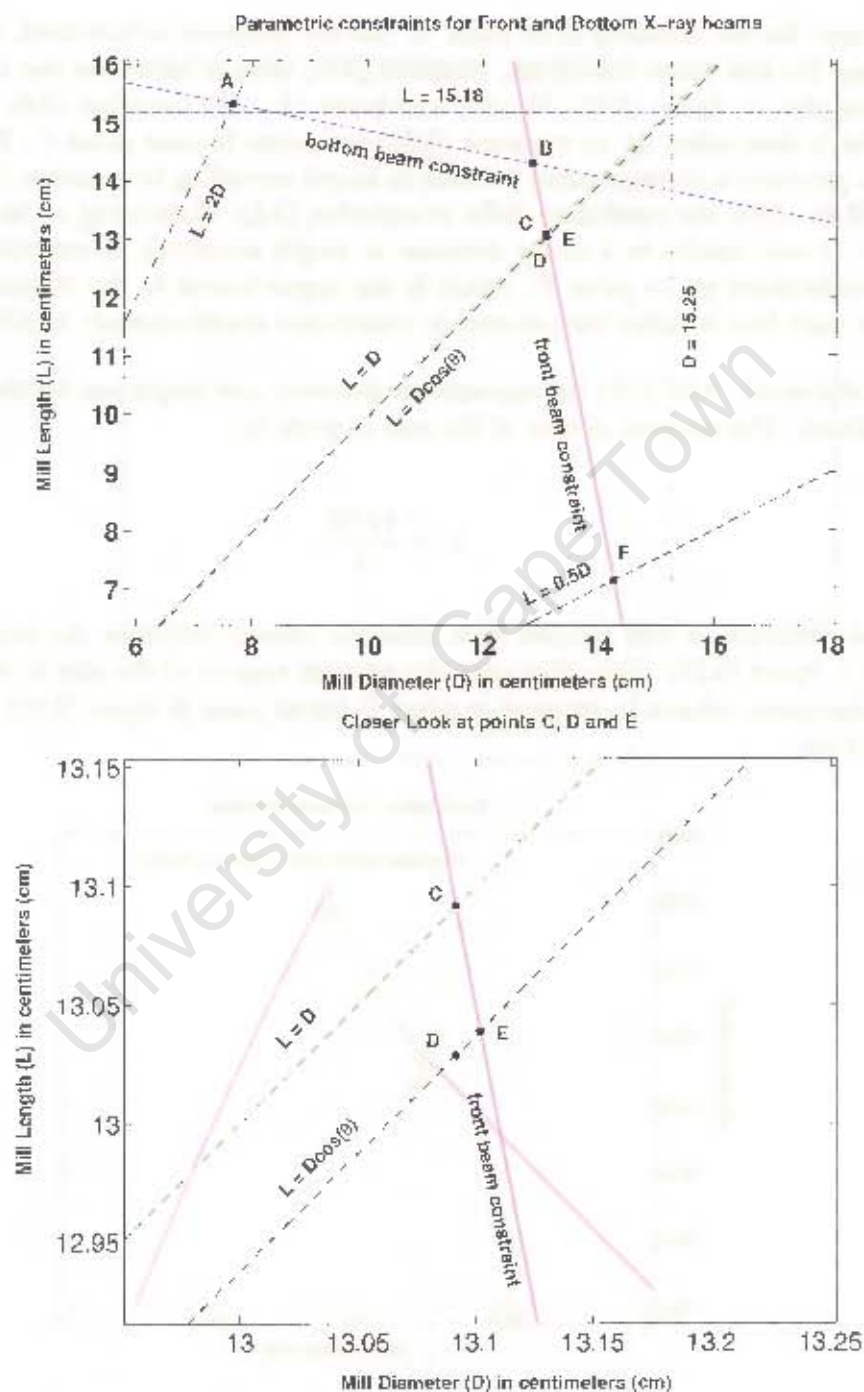


Figure 3.11: Parametric constraints for Combination 2

The parametrisation for Combination 2 is achieved with equations (3.5, 3.7 and 3.11) with $d = d_3$ where appropriate. The intersection of $L = 2D$ and equation (3.7) is denoted as point A in figure (3.11) and marks the lower bound on the diameter. The length is observed to increase along the bottom beam constraint, equation (3.7), up to the intersection with equation (3.5) at point B . It then begins to drop sharply along equation (3.5) until the $L = D$ line is reached at point C . Equation (3.11) is then invoked for diameters beyond point D . The length now increases with increase in diameter, from D to E , according to equation (3.11). Further increases in diameter causes a drop in length along equation (3.5) until it intersects the lower bound of the aspect ratio constraint at point F . Again we can gather the volumes along the path ($ABCDEF$) and plot this against diameter to obtain the maximum volume dimensions, figure (3.12). The maximum volume, denoted by the peak in the graph at point B , corresponds to $D = 12.85$ cm and $L = 14.29$ cm.

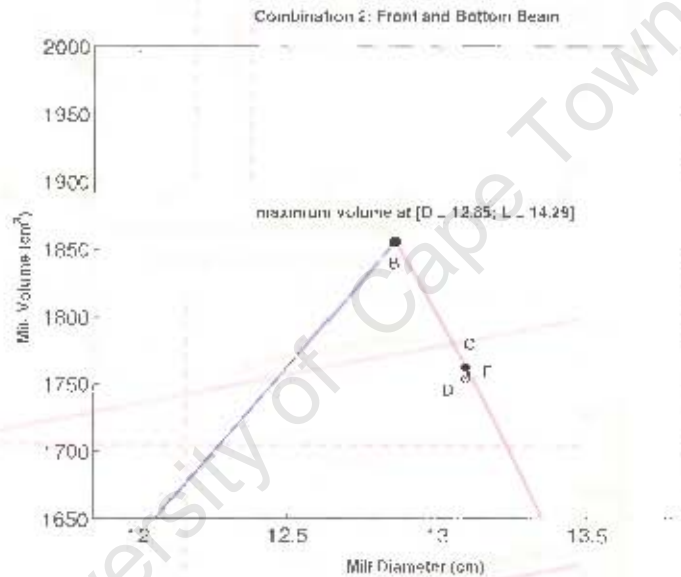


Figure 3.12: Combination 2: Variation of mill volume with diameter

3.4.3 Combination 3: Side and Bottom beam filming with the BAE

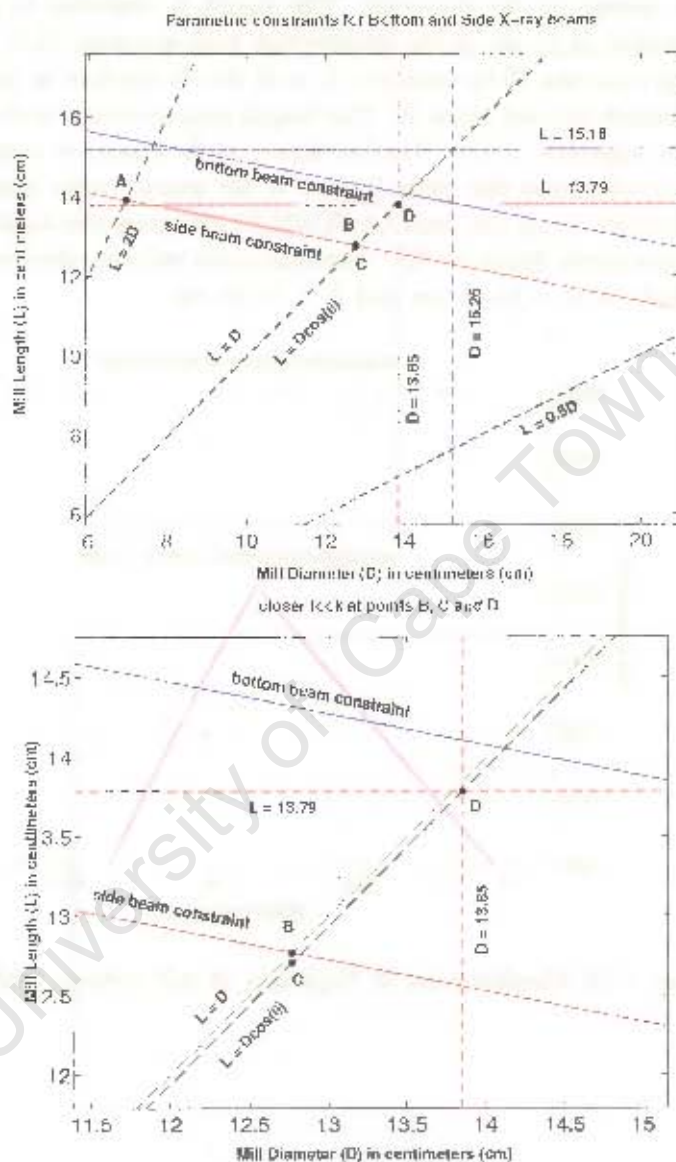


Figure 3.13: Parametric constraints for Combination 3

The parametric description for Combination 3 is determined by equations (3.6, 3.7 and 3.11) with $d = d_2$ for side beam filming and $d = d_3$ for bottom beam filming. Since the side beam constraint is below the bottom beam constraint in figure (3.13), the lower bound to the diameter (denoted by point A) is obtained at the intersection of the $L = 2D$ aspect ratio constraint and equation (3.6). The length decreases gradually from A to B according to equation (3.6) where it is abruptly shifted to the path of $(L = D \cos \theta)$ from point C onwards. It continues to increase steadily along this line until the upper bound of the diameter for $L < D$ is reached at point D, which is also the point of

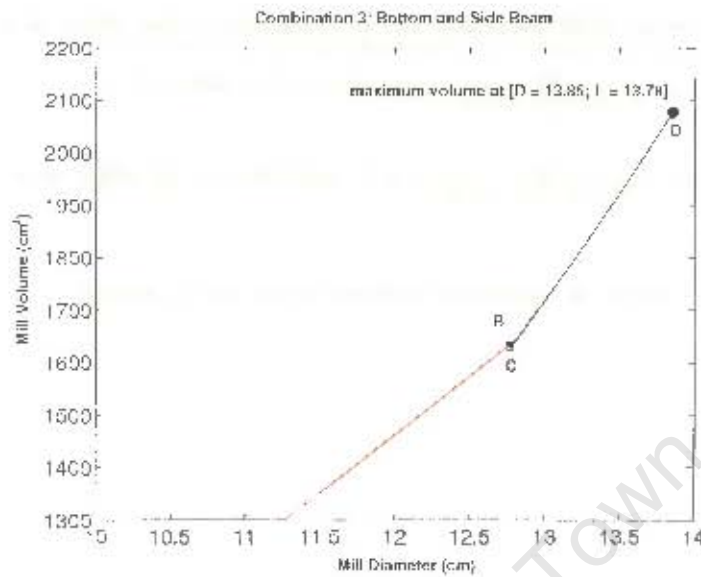


Figure 3.14: Combination 3: Variation of mill volume with diameter

maximum volume, figure (3.14). The corresponding mill dimensions at this point are $D = 13.85$ cm and $L = 13.78$ cm.

Combination 3 was used in the current work due to the front view obstruction caused by the drive train (motor and mounting plate).

3.4.4 Special cases of parametrisation scheme

The only term that can be adjusted in the parametric equations is the distance between the mill and intensifying screen (d_1 , d_2 and d_3). As noted in figure (3.6), this distance was defined as:

$$\begin{aligned} d_1 &= d_F + 7 \\ d_2 &= d_S + 7 \\ d_3 &= d_B + 7 \end{aligned} \tag{3.17}$$

where d_F , d_S and d_B are constraints imposed by the support rig for the mill and the rest of the d_i ($i = 1, 2, 3$) is associated with a fixed protective layer between the actual intensifying screen and the outside. The following effects are observed by changing d_i by an amount x ; $x \in \mathbb{R}^+$:

- Case 1 curve shifts vertically by x , opposite to the effect of x (i.e. if $d_i \rightarrow d_i + x$, curve shifts vertically downwards, and if $d_i \rightarrow d_i - x$, curve shifts vertically upwards)

- Case 2 and 3 curves shift vertically by $\frac{Sx}{F}$, opposite to the effect of x
- D_{max} changes by $\frac{Sx}{F \cos \theta + \frac{S}{2}(1+\sin \theta)}$, opposite to the effect of x
- L_{max} changes by $\left[\frac{Sx}{F \cos \theta + \frac{S}{2}(1-\sin \theta)} \right] \cos \theta$, opposite to the effect of x

The optimal effect in terms of maximum volume would be to reduce:

1. d_1 by d_F
2. d_2 by d_S
3. d_3 by d_B

which would result in:

- Case 1 curve shifting vertically upwards by d_F
- Case 2 and 3 curves shifting vertically upwards by $\frac{Sd_S}{F}$ and $\frac{Sd_B}{F}$ respectively
- D_{max} increasing by $\frac{Sd_S}{F \cos \theta + \frac{S}{2}(1+\sin \theta)}$ for Case 2 and $\frac{Sd_B}{F \cos \theta + \frac{S}{2}(1+\sin \theta)}$ for Case 3
- L_{max} increasing by $\left[\frac{Sd_S}{F \cos \theta + \frac{S}{2}(1+\sin \theta)} \right] \cos \theta$ for Case 2 and $\left[\frac{Sd_B}{F \cos \theta + \frac{S}{2}(1+\sin \theta)} \right] \cos \theta$ for Case 3

The effect of reducing the d_i terms is equivalent to setting ($d_F = d_S = d_B = 0$). This describes an ideal geometric configuration for the tumbling mill system where the influence of the support rig and drive system is negligible (This is equivalent to filming a mill without the support rig and drive system). Table (3.2) shows a summary of the optimal dimensions for both the ideal and actual geometry used.

actual geometry	D in cm	L in cm	V in cm^3	d_1 in cm	d_2 in cm	d_3 in cm
Combination 1	13.10	13.04	1757.56	15.4	17.36	
Combination 2	12.85	14.29	1853.23	15.4		7.5
Combination 3	13.85	13.78	2076.61		17.36	7.5
ideal geometry						
Combination 1	14.47	14.47	2379.55	7	7	
Combination 2	14.47	14.47	2379.55	7		7
Combination 3	15.70	15.63	3025.86		7	7

Table 3.2: Optimised mill dimensions for the various beam combinations

3.5 Discussion of results

The following observations are derived from table (3.2):

1. Combination 3 produces the largest volume mill for the actual geometry used.
2. Combination 3 obtains the largest diameter mill.
3. The longest length mill is achieved with combination 2.
4. All combinations result in essentially square mills.

From these observations, it is clearly apparent that Combination 3 is the optimal mill configuration. The closest manufactured Perspex tube to combination 3 is one with an inner diameter (ID) = 14.2 cm and outer diameter (OD) = 14.8 cm. The effect of using a tube with an inner diameter of 14.2 cm is illustrated in figure (3.15): The tracked particle is moving along the inner wall of the mill with the shaded region indicating that portion of the particle that is not captured by the BAE. The illustration (not drawn to scale) assumes that the mill and intensifying screen are perfectly concentric. In such instances the centering accuracy of the tracked particle is compromised. However, the occlusion effect can be minimised by using side and bottom beam filming (combination 3), figure (3.16). In this case the ball only suffers from partial occlusion when moving through the shaded region as shown in figure (3.16) (not drawn to scale). The problem of determining the centroid of an occluded object in the X-ray image is dealt with in chapter (5).

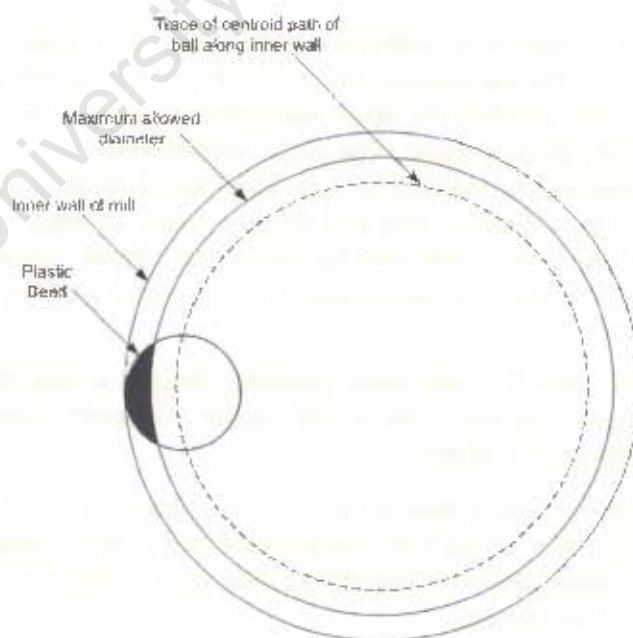


Figure 3.15: Illustration of that portion of the tracked ball chopped out of X-ray image (shaded portion) when moving along the inner mill periphery: front view filming (not drawn to scale)

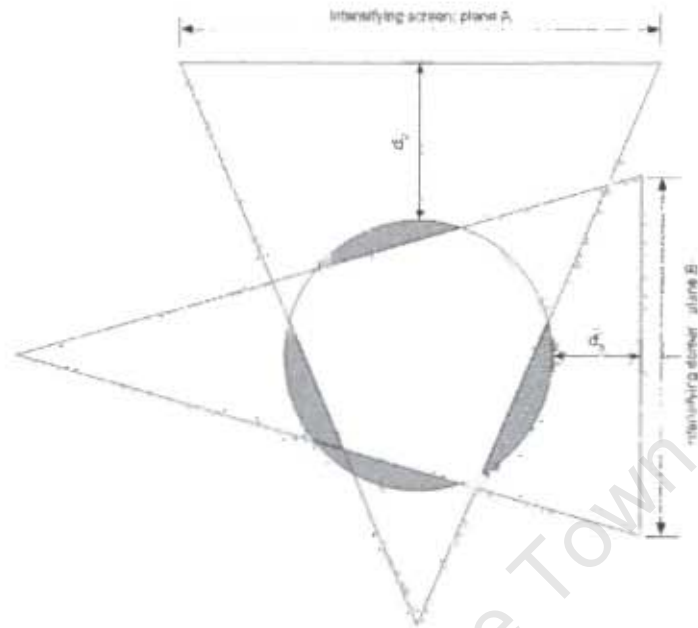


Figure 3.16: Illustration of the portion of the mill chopped out of X-ray image (shaded portion) when moving along the inner mill periphery: side and bottom view filming

3.6 Mill Geometry

The basic geometry used in this thesis satisfies a square mill. This shape is a variation on the pancake mill used by Powell [1] and the dimensions are governed pre-dominantly by the X-ray characteristics. In addition the lowest mass possible for these dimensions is desired in order to ensure that the mill and charge are realistically proportioned. The major contribution to the mill mass used by Powell [1] were the flanges. The flanges are reduced by half in thickness (from 12 mm to 6 mm) and 100 mm in diameter (from 300 mm to 200 mm). The mill lid is a screw on type. This modification effectively halves the thickness of the bolt-on-lid used by Powell [1]. These changes reduce the mill mass by 2.03 kg (from 2.72 kg to 0.67 kg). The advantage of the reduced mass is:

1. The charge-to-mill ratio is made more realistic; though a ratio that is comparable to industrial milling is impractical since this would require increased charge density and hence greatly reducing X-ray penetrative depth.
2. The torque fluctuations are relatively greater and therefore easier to measure. Unfortunately, attempts to measure the torque were unsuccessful due to the relatively small mass of the charge - Essentially, the power drawn by the charge (less than 1 Watt) is not easily discernable amidst the overall noise of the system.

Figure (3.17) illustrates a staggered CAD model of the mill. The flanges are glued onto the mill shell forming one single unit. The lifter bars are then attached to the shell via plastic screws (screws

not shown in figure) and finally, after the plastic beads are put into the mill, the lid is screwed on. The screw on lid is essential to the design because of the limited time frame allocated for conducting experiments - When changing lifter configuration the lid is easily unscrewed.

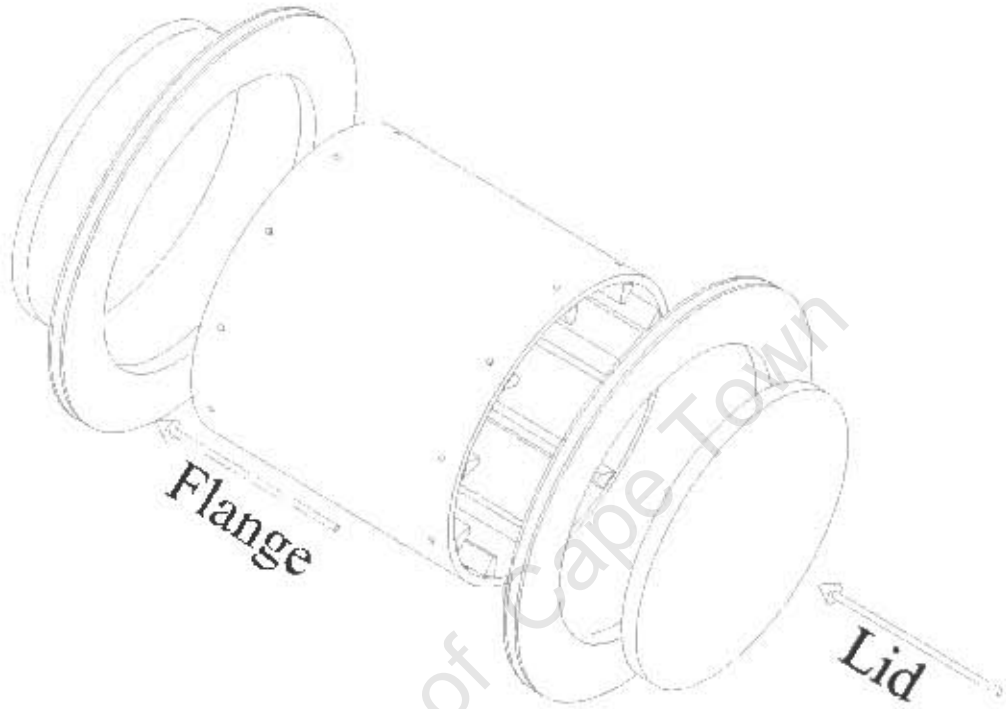


Figure 3.17: Staggered CAD model of mill showing the assembly process

The generic shape of the lifter bar is illustrated in figure (3.2). The full range of lifter profiles used are described in table (3.3).

Number	Height (H) in (mm)	Width (W) in (mm)	Lifter angle in (deg)	Length (L) in (mm)
1	5	9	45	142
2	7	9	45	142
3	7	9	90	142
4	9	9	60	142
5	9	9	curved	142

Table 3.3: Range of lifter profiles manufactured for the Perspex mill

The curved angular face indicated in table (3.3) is illustrated in figure (3.18). The curved face is intended to simulate a worn lifter.

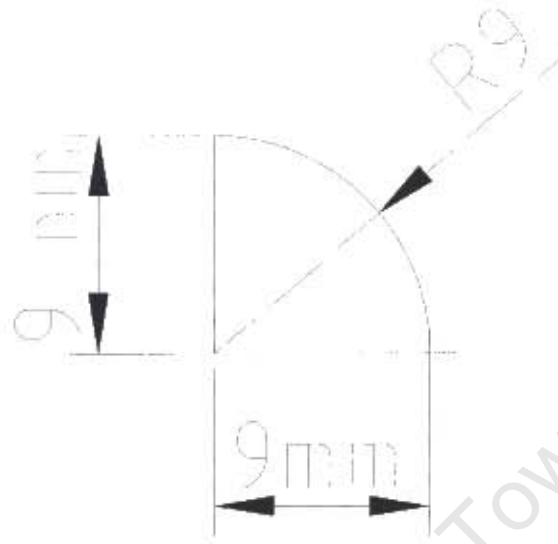


Figure 3.18: Illustration of the lifter profile used to simulate a worn lifter

3.7 Summary

The mill dimension optimisation scheme was designed to satisfy the X-ray beam characteristics and aspect ratio constraints. The final dimensions is a compromise between the predictions of the optimisation scheme and the availability of material. The choice of materials used were also influenced by the portability of the experimental rig which is the main reason for choosing aluminium beams for the support rig. The optimisation is problem specific and therefore not suited to all geometric configurations. The equations formulated assume collinearity between the mill axes and the scintillating screens.

Chapter 4

Image Processing

4.1 Introduction

In most cases of close-range photogrammetry, artificial targets are imaged and used as either control or object points. These targets can be crosses, circles, or spheres which can be extracted from an image. For control points the (previously) surveyed coordinates normally refer to the centroid of these targets. The present work uses steel ball bearings on a Perspex frame, figure (4.1).



Figure 4.1: Control frame used to parametrise X-ray space

Spheres have the property of becoming ellipses under a projective transformation, section (7.3), and thus circular features (circles and ellipses) can be identified by their elliptical shape on an image. In X-ray photogrammetry highly attenuating targets, like ball bearings, provide a high contrast with the background, which eases the detection task and improves the centering precision. The disadvantage of spherical targets is that the centre of the sphere cannot be directly surveyed. In addition, the centering, in image space, is subject to the errors of digitisation, penumbra and vibration, chapter (7).

The first stage of image processing is the location or detection of the targets, which is described in this chapter. The input is a digital X-ray image and the output is a list of target positions and geometric features (area, perimeter, eccentricity, pixel coordinates). The final stage, chapter (5), is the accurate determination of the target's position (target centering), using these geometric features.

There are many different methods of approximately locating targets. Many measurement systems or once-off measurement projects use interactive pointing of a graphics cursor with the mouse or keyboard, such as is done by Adams [76], Heipke et al [77] and Godding [78]. This method is simple to implement and very reliable especially in difficult cases, but it does have the disadvantage of requiring the assistance of an operator.

Most semi- or fully-automated photogrammetric systems use grey level thresholding to locate the positions and perimeter of high contrasting control targets. Beyer et al [82], El-Hakim [83], Ruther & Parkyn [84], Van den Heuvel [85] and Van der Vlugt & Ruther [86] all use some form of grey level thresholding for locating bright targets. Most implementations use global thresholding, some use local thresholding (El-Hakim [83]) and still others utilise multiple thresholding (Van den Heuvel [85]).

Convolution with different types of filters can also be utilised for locating target areas but some idea of the geometry of the target is required in this case. Convolution, or more correctly correlation with an edge enhancement filter [87] and subsequent thresholding is often successfully used for locating distinct target edges. The case of low contrasting objects provides the most complex case of any feature extraction process and is best solved by first localising the region of interest before any feature extraction and centering can be conducted. The current work employs Dynamic Template Matching or DTM (section 4.4) to provisionally locate the low contrasting object.

Both thresholding and edge detection techniques were investigated in this work. Two thresholding techniques, a modified edge detection method, and template matching were developed and coded. These feature extraction processes were incorporated into automated image processing algorithms and are described in sections (4.2, 4.3 and 4.4) respectively. It should be noted that the original experiments involved steel ball bearings, a glass bead, a lead ball and a plastic bead with a lead rod inserted through it, Govender et al [88]. These were selected to be consistent with the initial work of Powell [22]. In addition they help explore the range of the feature extraction routines investigated in this work. The final tracer used to map the motion of a typical bulk charge particle was a plastic bead, randomly selected from the bulk charge, and given a surface layer of silver laquer, Govender et al [89].

4.1.1 Pre-filtering

The following discussion assumes that the noise in the image is a direct consequence of extraneous objects in the actual scene being imaged. Implicit to this assumption is that the image plane is flat. A model of the intensifying screen shape and its influence on the resultant image is discussed in the section on future work, section (10.4).

The goal of digital X-ray photogrammetry is to obtain objects from stereo images in an attempt to understand the scene being viewed. A fundamental requirement of this goal is to find object

boundaries. Sharp contrast in images generally corresponds to object boundaries or phenomena such as shadow. It is, therefore, the goal of the early vision process in digital photogrammetry to identify these boundaries. To remove noise, which generally shows up as high contrast areas, a smoothing filter is utilised.

There are numerous smoothing methods that can be employed on digital images. The main consideration when choosing a smoothing operator, Marr & Hildreth [53], is to ensure that it is localised in the frequency and spatial domain; where localised implies to reduce the range of scales. In addition it should not generate spurious detail when smoothing from finer to coarser scales. It is well known that the one operator that optimally comes close to satisfying these conditions is the Gaussian operator. Gaussian smoothing has been successfully combined with gradient operators for edge detection purposes. The smoothed images, $I(x, y, \sigma)$, are generated by convolving the original image, $I_0(x, y)$ with a Gaussian kernel of variance σ .

$$I(x, y, \sigma) = I_0(x, y) * G(x, y, \sigma) \quad (4.1)$$

where

$$G(x, y, \sigma) = \frac{1}{2\pi\sigma^2} e^{\left(-\frac{x^2+y^2}{2\sigma^2}\right)} \quad (4.2)$$

and '*' represents the convolution integral. The variance is proportional to the width (standard deviation) of the filter and consequently determines the amount of smoothing (dislocalisation). The Gaussian, while being popular, has some limitations. Firstly, it dislocalises and blurs edges as greater amounts of smoothing are performed. Secondly, the Gaussian does not consider homogeneity or isotropy when smoothing and this sometimes causes spectrally different regions to be inappropriately merged. Thus a variance greater than one pixel is undesirable for accurate target centering to pixel or subpixel accuracy. Unfortunately, Gaussian smoothing, even with $\sigma = 1$, does not remove the vertical bands (caused by lifter bars) observed in the X-ray images, or the intensity variation from frame to frame. The top, left image in figure (4.2) shows the vertical bands formed by the lifter bars. The effects of the noise results in a multimodal intensity image with large variations in the background, which complicates feature extraction. Investigations revealed that the intensity fluctuations across the X-ray image scene are fundamental to the angiographic equipment and could not be removed *a priori*. The lifter bars are also integral to the study of charge motion and had to be incorporated into the experimental setup despite the negative impact on the image processing. A smoothing technique is proposed as a pre-filter to all subsequent feature extraction discussions.

By subtracting column-wise the vertical average from each pixel, a smoothing effect, top right image in figure (4.2), was achieved. The smoothing calculation is given in Equation (4.3).

$$I(x_i, y_j) = I_0(x_i, y_j) - \frac{\sum_j I_0(x_i, y_j)}{r} \quad (4.3)$$

where i is fixed during the summation, implying a column-wise summation, and

$I_0(x_i, y_j)$ is the i^{th} pixel in row j of the original intensity image,

$I(x_i, y_j)$ is the i^{th} pixel in row j of the intensity flattened image, and

r is the number of rows in the image

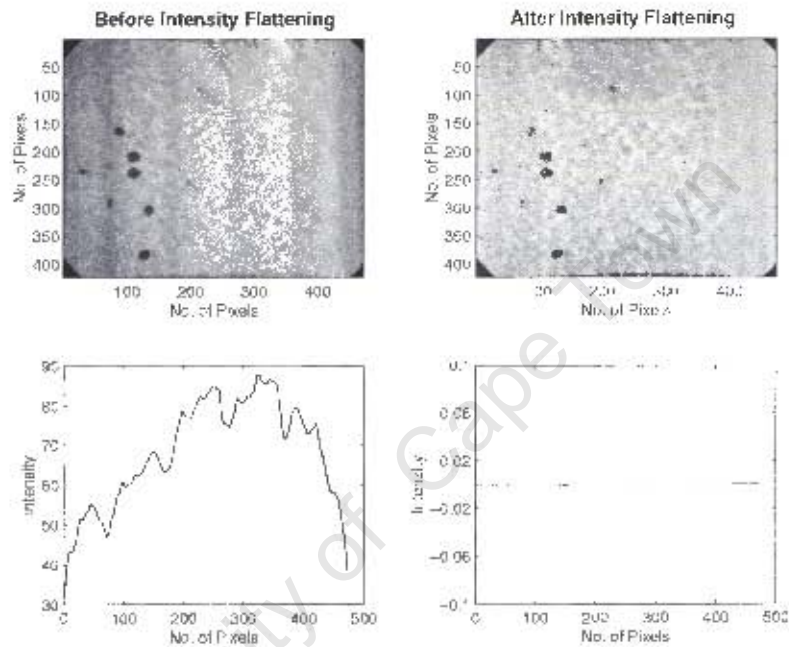


Figure 4.2: Intensity flattening algorithm applied to a typical X-ray image

The intensity flattening algorithm effectively normalises the mean column-wise intensity to zero, figure (4.2). The vertical bands have been blended into the higher intensity values resulting in a uniform background. This effectively reduces the number of distinct (dominant) modes thus facilitating the detection of global thresholds and edges, figure (4.3).

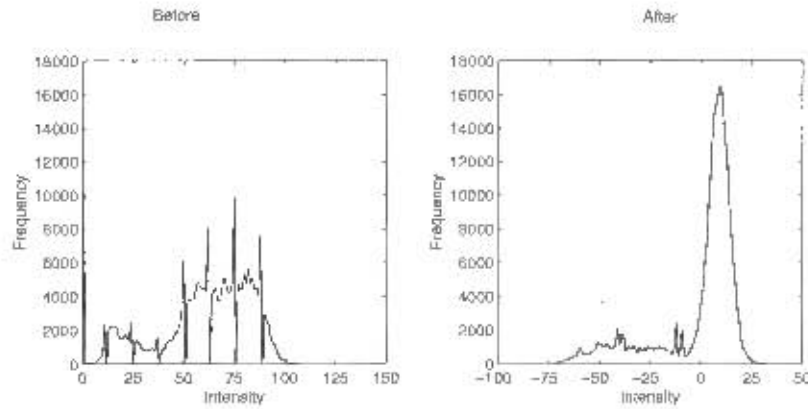


Figure 4.3: Histogram plot of typical X-ray image before and after intensity flattening algorithm

4.2 Thresholding Techniques

Consider a grey-level histogram corresponding to an image, $f(x, y)$, composed of light objects and a dark background in such a way that object and background pixels have grey levels grouped into two dominant modes. One obvious way to extract the objects from the background is to find a threshold T that separates these modes. Then any point, (x, y) , for which $f(x, y) > T$ is called an object point; otherwise, the point is called a background point.

Mathematically, thresholding may be viewed as an operation that involves tests against a function T of the form

$$T = T[x, y, p(x, y), f(x, y)] \quad (4.4)$$

where $f(x, y)$ is the grey level of point (x, y) , and $p(x, y)$ denotes some local property of this point; in the X-ray image case, the average grey level of a neighbourhood centered on (x, y) . A thresholded image $g(x, y)$ is defined as:

$$g(x, y) = \begin{cases} 1 & \text{if } f(x, y) > T \\ 0 & \text{if } f(x, y) \leq T \end{cases} \quad (4.5)$$

Thus pixels labelled 1 (or any other convenient intensity level) correspond to objects, whereas pixels labelled 0 correspond to the background. The problem, of course, is where to place the threshold. One approach is to specify the number and location of the threshold by trial and error. This method is satisfactory if the number of different images to be processed is small. In situations where automatic setting of the threshold (as in the X-ray image scenes) is required, the problem becomes one of characterising a given histogram in some invariant manner. This problem, and proposed solutions, are now discussed.

4.2.1 Percentage Threshold method

The percentage threshold method may be broadly classified as an adaptation of the *p-tile method*, [87]. As an example, if f is a picture of a printed page, we may know that the characters should occupy a certain fraction θ of the space on the page. Knowing this, we should threshold at that value T such that just fraction θ of the picture points have grey values $\geq T$; this fraction is called the $(1 - \theta)$ -tile of the grey level histogram. This *p-tile method* is rarely applicable in practice, since the area occupied by the objects of interest are hardly ever known in advance. However analysis of the X-ray data revealed that all images in the data set depicted similar, if not identical, characteristics as regards the objects of interest that are distinct. Such distinction is usually achieved with steel or lead balls which formed part of the initial investigations. The following key assumptions describe these common properties.

1. The tracked ball bearings appear in all X-ray images. This implies that the number of pixels occupied by the tracked ball bearings will be the same in all images.
2. The grey levels occupied by the tracked balls are the same in all images. Implicit to this assumption is that the X-ray beam intensity is constant for a given experimental run.
3. A reasonably accurate estimate of the p^{th} -tile can be obtained from a trial and error process.

Procedure for implementing percentage threshold technique

1. Crop images to remove the noisy ends. This is typically 50 pixels from the top and bottom, left and right of the image.
2. Apply intensity flattening algorithm to cropped images.
3. Estimate percentage of pixels occupied by tracked particles. This can be done by applying simple thresholds to randomly selected images in the image scene.
4. Establish a raster reordered histogram of intensity image. The number of bins correspond to the number of grey-values.
5. Compute cumulative sum of histogram bins from lowest to highest bin.
6. Find maximum cumulative sum $<$ estimated area occupied by tracked particles. Mathematically stated:

$$S_i = \sum_{i=1}^i B_i \quad (4.6)$$

$$(S_i)_{\max} < P \times C \times R \quad (4.7)$$

where

S_i is the i^{th} cumulative sum of the raster reordered histogram,

B_i is the i^{th} bin in the histogram,

P is the estimated percentage of pixels occupied by the tracked particles,

C & R are the number of columns and rows in the image respectively,

The index of $(S_i)_{\max}$ corresponds to the i^{th} grey-level bin, with intensity T_{thresh} .

7. Segment images according to the following thresholding criterion:

$$I \leq T_{thresh} \quad (4.8)$$

All pixels (object pixels) with intensity values less than or equal to T_{thresh} are set to one; the rest are set to zero. Figure (4.4) illustrates the outcome of this procedure applied to a typical X-ray image of the experimental tumbling mill. The original image is shown in figure (4.2); the dark, circular areas are the tracked particles.

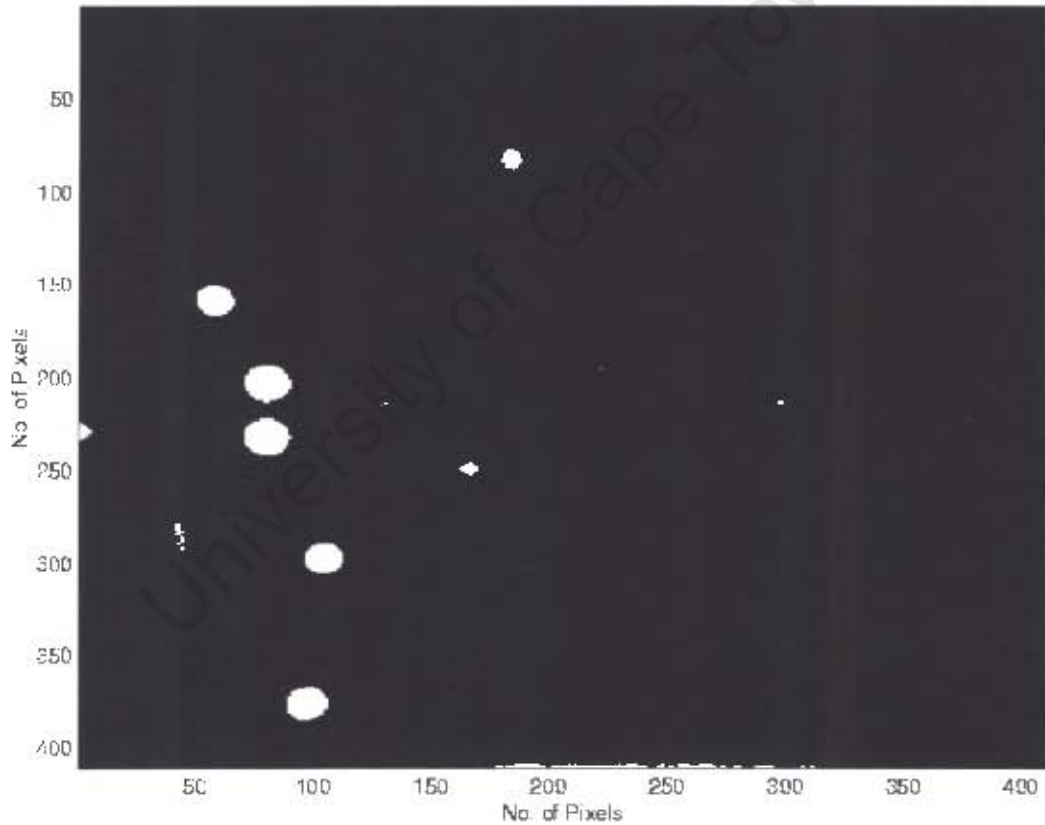


Figure 4.4: Segmentation using the percentage method

4.2.2 Optimal thresholding

An image containing an object on a contrasting background has an essentially bimodal grey-level histogram, figure (4.5). The two dominant peaks correspond to the relatively large numbers of points

inside and outside the object. The dip between the two peaks - the valley - corresponds to the relatively few points around the edge of the object. If the image or region of the image containing the object is noisy and not large, the histogram itself will be noisy. Unless the dip is uncommonly sharp, the noise will make the edge location obscure, or at least unreliable from one image to the next. The following histogram plot of an object in a typical X-ray image illustrates the overlapping modes (obscured edge). The image has been pre-filtered with the intensity flattening algorithm.

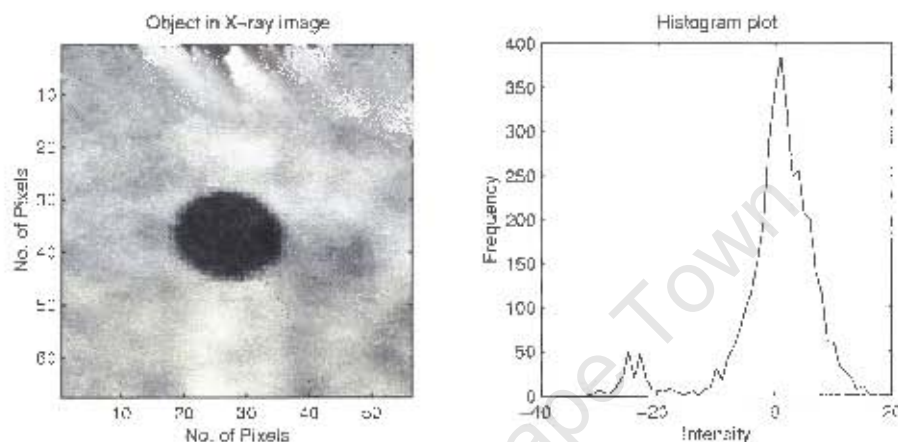


Figure 4.5: Modal nature of object in X-ray image

The distortions can be overcome to some extent by smoothing the histogram, using a Gaussian filter of variance equal to one pixel. The result is a distinctly bimodal image, figure (4.6) - a desirable condition for segmenting an image.

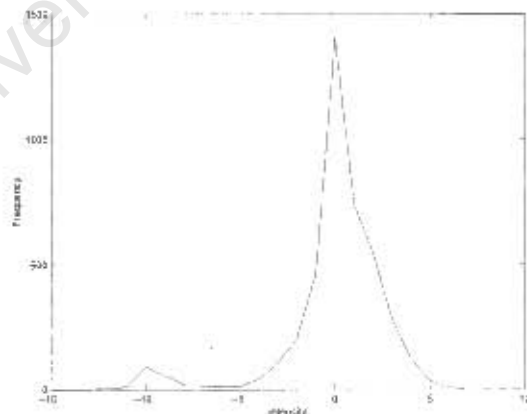


Figure 4.6: Histogram plot of a Gaussian filtered object in a X-ray image

It is clear from the histogram plot of the Gaussian filtered object, figure (4.6), that the threshold region lies somewhere in the valley. Thresholding in this region is relatively insensitive to the exact choice of the threshold level, since the grey levels at a valley bottom are relatively unpopulated.

Intuitively a purely bimodal histogram has the correct threshold, in terms of minimal error, at the bottom of the valley, between the two modes. In general, the correct, and hence optimal, threshold is not always distinguished simply by the lowest point in a valley. The complete solution to such an optimisation problem is generally the roots to a quadratic, Appendix (A.4). This implies that the optimal solution can require two thresholds. Special knowledge of the problem is generally required to physically interpret the solutions before choosing the correct threshold. Depending on the problem requirements, it might be sufficient to use either of the solutions.

We now consider the histogram plot of the entire X-ray image, after intensity flattening and Gaussian filtering, to illustrate the problem specific criteria needed to facilitate an optimal thresholding process, figure (4.7).

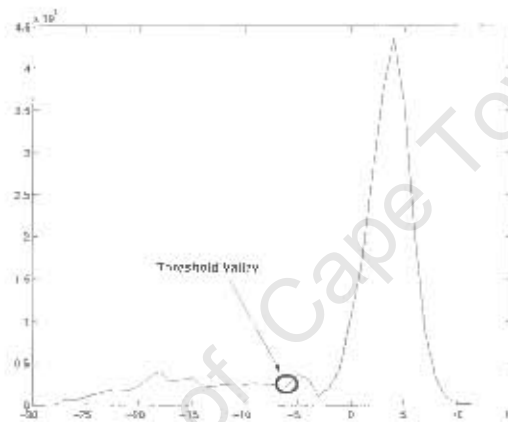


Figure 4.7: Localised threshold region based on the initial guess

Although the dominant mode corresponding to the background, is clearly distinguishable, other minor, and somewhat obscured modes, are still visible. Thus an obvious choice as to the desired edge that separates objects from background is not possible.

It is known that the penumbra effect [90] although essentially symmetric, creates an artificial, 'halo-like' edge around the 'true' edge, figure (4.8). In particular the intensity gradient between the shadowy region and the object are steeper than that formed between the penumbra region and the background. The steeper gradient across the 'true' edge implies a greater probability of correctly classifying pixels as either objects or background along this region. In terms of error probabilities, there is a lower probability of (erroneously) classifying object points as background points and vice versa along the 'true' edge.

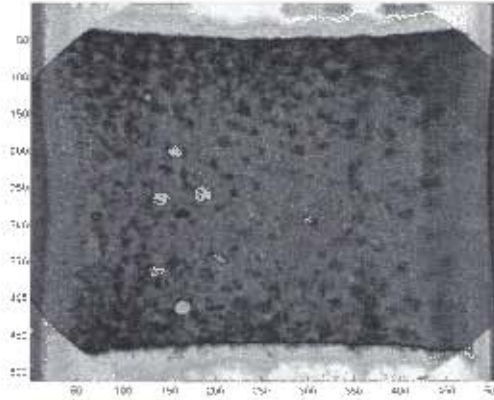


Figure 4.8: Illustration of penumbra effect around objects in X-ray image

In such cases an optimal threshold, in terms of minimum error, is required. In particular, the existence of an absolute minimum error probability implies that a single threshold solution would be sufficient, Appendix (A.4).

$$T = \frac{\mu_1 + \mu_2}{2} + \frac{\sigma^2}{\mu_1 - \mu_2} \ln \left(\frac{P_1}{P_2} \right) \quad (4.9)$$

where T is the optimal threshold, μ_1 and μ_2 are the mean values of the two brightness levels, σ_1 and σ_2 are the standard deviations about the means, and P_1 and P_2 are the *a priori* probabilities (mixing parameters) of the two levels.

For completeness the quadratic solution is also investigated, Appendix (A.4). Based on the above discussion, the two optimal solutions should correspond to the 'true' edge, and the 'soft' edge formed by the penumbra region and background.

The optimal thresholding routine is designed to minimise the error probability density of interpreting an object pixel as background and vice versa, Appendix (A.4). Both cases can be solved only by iterative numerical methods, and thus require an initial guess to either the threshold or probability density parameters.

4.2.2.1 Optimal thresholding algorithm for a single solution

The single threshold solution is a special case of the general quadratic formulation. The solution obtained is for the absolute minimum error probability, where the prior probabilities are assumed equal, ($P_1 = P_2$). The optimal threshold solution therefore reduces to:

$$T = \frac{\mu_1 + \mu_2}{2} \quad (4.10)$$

In general the single solution is not suited to edge enhancement, but rather to processes that subsequently generate binary images for the calculation of certain geometric properties, like in X-ray photogrammetry. Due to the formulation of the solution, the provisional value (the first guess) is a grey level. The initial guess is specific to the nature of the X-ray data and for this reason not suited to all intensity images in general. Two, equally accurate, guessing techniques were used in the present work. The first method looks at histogram plots of isolated objects, figure (4.5), in the X-ray images. The mean of all valleys (grey levels between the two peaks) are then computed as the initial guess to the thresholding process. This approach is very similar to that adopted by Chow & Kaneko [57], although the computational demands are less. The second, and more elegant guessing process is to use the mean intensity of the image corners. The average of the corners has two advantages:

1. The intensity of the corners are very similar to the intensity of the objects of interest.
2. Image cropping, as in the percentage method, can overlap into the path traversed by the objects - No cropping is required for optimal thresholding.

The implementation of the optimal thresholding routine for a single solution is now outlined.

1. Establish the mean intensity for each of the four corners in the image. The sides of the corners are defined as:

$$Length = \frac{R}{100} \quad (4.11)$$

$$Breadth = \frac{C}{100} \quad (4.12)$$

where R and C are the number of rows and columns in the image.

The choice of 100 ensures that the corner comprises more low intensities than high. Compute the mean of the four values obtained.

$$T_{guessed} = mean(corners) \quad (4.13)$$

Using $T_{guessed}$ as the initial guess effectively localises the threshold region to the valley containing the guessed value, figure (4.7).

2. Using the guessed threshold, a new value satisfying the optimal threshold criteria for a single solution, equation (4.10), is computed.
3. If the new value is identical to the guessed threshold the process terminates - if not step two is recalculated using the new threshold. Since the new value is clearly a better estimation than the initial guess, an iterative process will always converge to a solution, if it exists.
4. Finally, the new threshold value is compared to the actual intensities in the X-ray image and the closest value is taken as the final threshold.
This value is then used to produce the necessary segmentation, figure (4.9)
5. The background is now cropped to include only the objects of interest, figure (4.10).

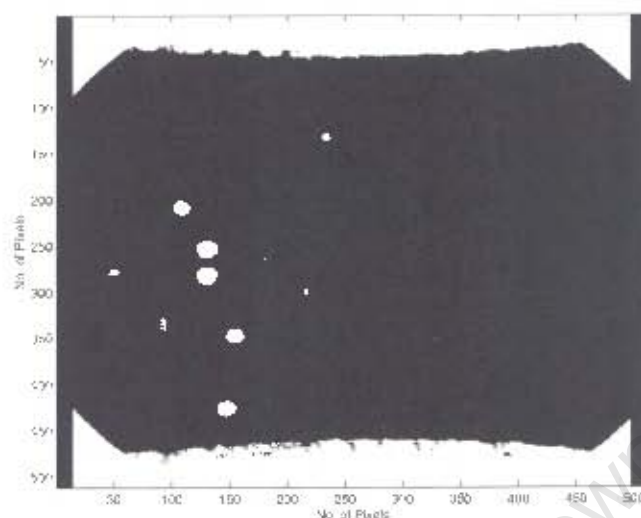


Figure 4.9: Segmentation using Optimal Thresholding Algorithm

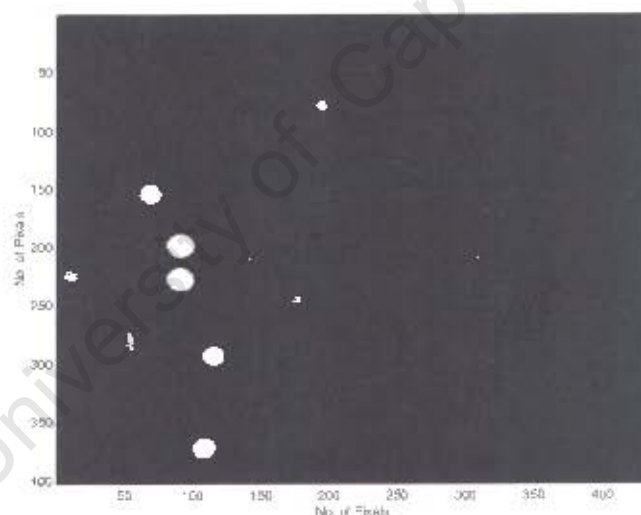


Figure 4.10: Cropped image after optimal thresholding

4.2.2.2 Discussion

The major advantage of the single threshold solution is the reduced number of iterations. The quickly converging solution (< 10 iterations) allows the process to be applied to all X-ray images, ensuring that the optimal result is always obtained separately for each image.

4.3 Edge Tracking

4.3.1 Motivation

The preprocessing stages encountered in the present work involve an averaging technique followed by Gaussian smoothing. The effects of such filtering techniques tend to blur detail in an image. These techniques may be broadly classified as integration processes. It is therefore natural to expect that differentiation will have the opposite effect and thus sharpen a given image, or at least cancel out the blurring effects.

4.3.2 Definition:

An edge is the boundary between two regions with relatively distinct grey-level properties. In general these regions must be sufficiently homogeneous so that the transition between the two regions can be established on the basis of grey-level discontinuities alone. The basic idea underlying most edge-detection techniques is the computation of a local derivative operator. The following diagrams illustrate this concept.

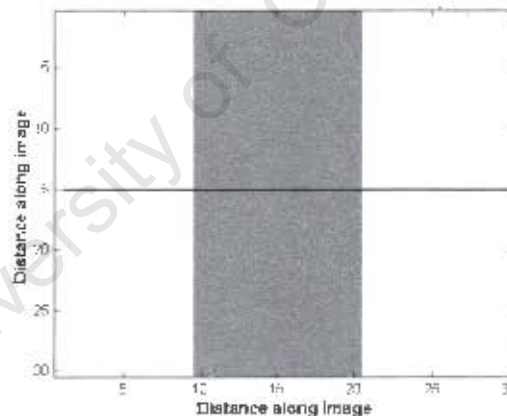


Figure 4.11: Grey strip with horizontal scan line

Figure (4.12) shows the intensity profile along the horizontal scan line of figure (4.11). The edge (transition from light to grey) is modelled as smooth rather than an abrupt change of intensity. This is indicative of the blurring effect at edges in digital images - a first indication of the uncertainty in locating true edges. Figure (4.13) shows that the first derivative of the grey level profile is negative at the leading edge, positive at the trailing edge and zero in areas of constant grey level. It is clear from this that a fluctuating grey level (a varying intensity profile) would make these edges less distinct, and hence more difficult to find. The second derivative, figure (4.14), is negative for that part of the transition associated with the light side of the edge, positive for that part of the transition associated with the dark side of the edge, and zero in areas of constant grey level. Accordingly, the magnitude

of the first derivative can be used to detect an edge in an image, and the sign of the second derivative indicates which side of the edge a particular edge pixel lies. It should also be noted that the second derivative has a zero crossing at the midpoint of a transition in a grey level. The second derivative is the Laplacian.

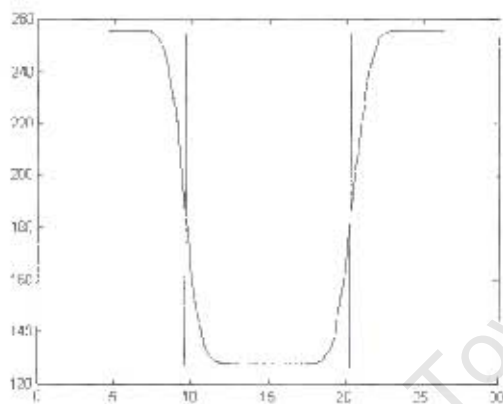


Figure 4.12: Intensity profile along scan line

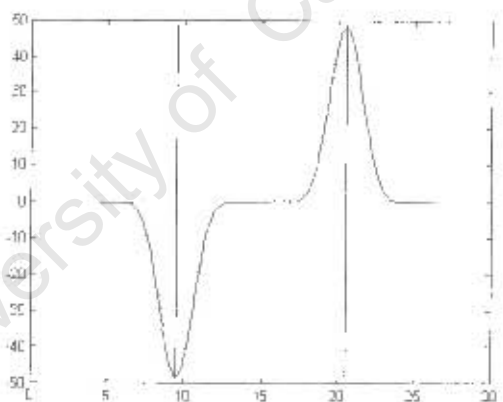


Figure 4.13: Derivative of intensity profile along scan line

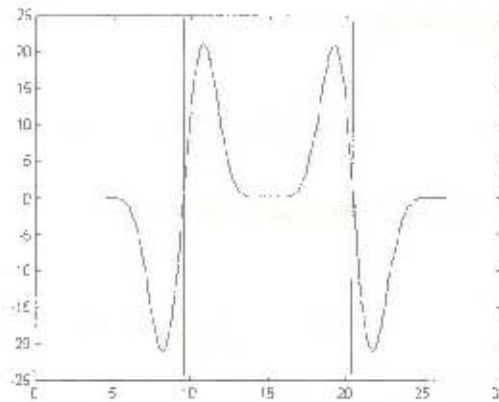


Figure 4.14: Laplacian of intensity profile along scan line

4.3.3 Implementation of edge detection

The general form of implementing edge detection is to find the appropriate derivative operator, essentially a matrix of dimension less than the image dimension, and apply it across the image in a particular manner. The most efficient way of applying this operator on the image is to convolve the image with it. Convolution utilises the powerful Fourier Transform to perform the operation, Appendix (A.1.1). The actual implementation employs the Fast Fourier Transform. Fourier analysis is well documented, and for this reason will be assumed in the following discussion. As discussed previously, the operator that very closely resembles the optimal filter is the first derivative of the Gaussian.

4.3.4 Canny Edge Detection

The Canny method [52] differs from most other edge detection methods in that it uses two different thresholds (to detect strong and weak edges), and includes the weak edges in the output image only if they are connected to strong edges. This method is therefore less likely than others to be "fooled" by noise, and more likely to detect true weak edges. The Canny operator uses the Gaussian for smoothing and the first derivative of the Gaussian as the edge detector.

Whenever a grey-level intensity change occurs, there will be a corresponding peak in the first directional derivative of intensity. In fact, we may define intensity change in this way, so that the task of detecting these changes can be reduced to finding the maxima of the first directional derivative of the grey-level intensity. Thus, we seek the maxima in

$$f(x, y) = D'[G(r) * I(x, y)] \quad (4.14)$$

where $I(x, y)$ is the image, $*$ is the convolution operator, and $G(r)$ is the Gaussian operator.

By the derivative rule for convolution,

$$f(x, y) = D'G * I(x, y) \quad (4.15)$$

We can write the operator $D'G$ as G' , so that in direction r

$$G'(r) = \frac{r}{\sigma^3 \sqrt{2\pi}} e^{-\frac{r^2}{2\sigma^2}} \quad (4.16)$$

where σ is the shaping parameters of the Gaussian function (standard deviation), and r is the polar coordinate equivalent for the cartesian (x, y) coordinates.

We now briefly illustrate the Canny process for an ideal step edge. The reader is referred to [91] and [52] for a more detailed description of the process, and variations thereof. The first step begins with Gaussian smoothing of the image to reduce the amount of noise in the image. The next stage, step two, involves the computation of the first derivative of the smoothed signal, where the x and y derivative components, dx and dy , are determined for each edge point. Magnitude and orientation images are computed in the third step using the following criteria:

$$(Magnitude)_i = \sqrt{dx_i^2 + dy_i^2} \quad (4.17)$$

$$(Orientation)_i = \tan^{-1} \left(\frac{dy_i}{dx_i} \right) \quad (4.18)$$

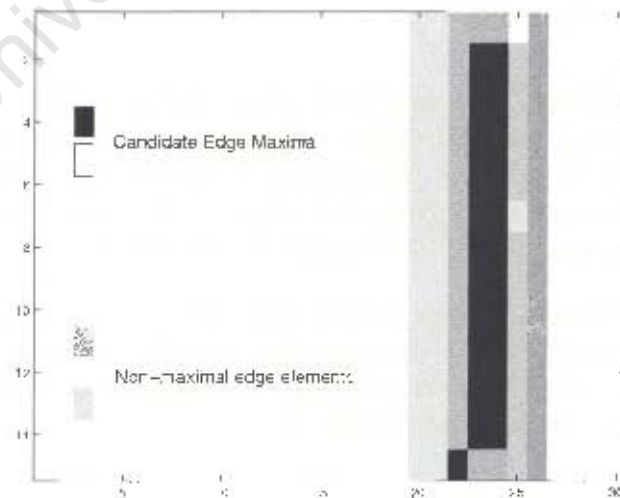


Figure 4.15: Results of the first derivative magnitude in the spatial domain

The axes of a digital system are defined with the rows as the y-axis, the columns as the x-axis and the origin at the top left corner of the screen. The orientations are computed in this coordinate system so that the angle is measured from the y-axis to the dark-to-bright edge transition [91].

The magnitude image is the first estimate of the edge map, figure (4.15). A local 3×3 operator is used to determine whether the central pixel is a non-maximal point by examining the magnitudes of the points parallel to the edge direction. If the central pixel magnitude is smaller than the examined points, then it is removed. Figure (4.16) illustrates the result of non-maximal suppression.

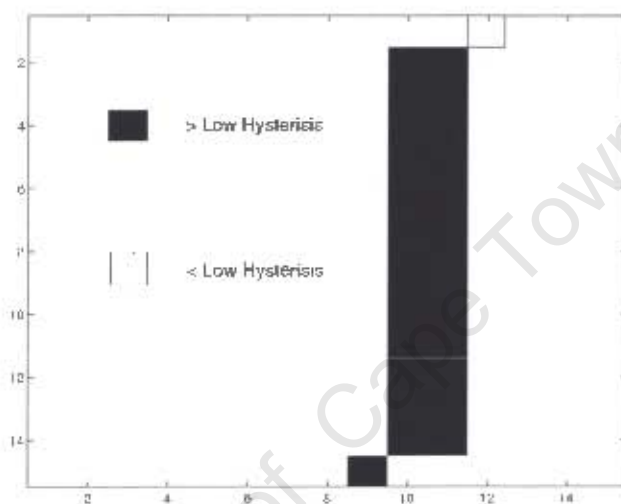


Figure 4.16: Edge after non-maximal suppression

Once the non-maximal points have been removed, edge tracking is performed to give the final robust edges. At this stage, to avoid edge streaking, Canny proposes the use of hysteresis. Hysteresis is a retardation of an effect from a high value to a lower value, such as slowing down of an object due to friction. In hysteresis-based edge tracking the process begins from a pixel with a high magnitude value and attaches points to the edge, as long as they do not fall below a lower threshold value. It is necessary to set this high and low thresholds before tracking begins. The low threshold is set above the probable noise level by use of a cumulative histogram on the magnitude image and a preset threshold percentage, say 20% - This is the low hysteresis value. The high hysteresis is normally set as a factor of the low threshold, say twice the threshold. Figure (4.17) illustrates the result of hysteresis edge tracking.

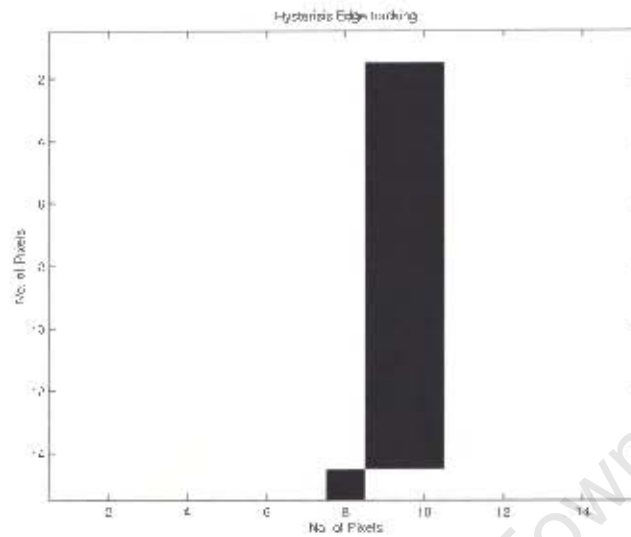


Figure 4.17: Resulting robust edge after hysteresis edge tracking

4.3.5 Edge Detection of X-ray images

The Canny process discussed in the previous section was applied to the X-ray images, in a fully automated program. A set of X-ray images are read-in as raw data. The images are then edge detected according to the procedure discussed above. A separation process, that distinguishes objects from noise is then applied to the images.

4.3.5.1 Canny edge detection and X-ray images

After applying the intensity flattening routine the Canny algorithm is applied to the averaged images. The convolution integral is applied from two different points. The first operation is from the top left pixel. The original image is then rotated through 180 degrees and convolved in the same manner. This is equivalent to convolving from the bottom right pixel. The edge-detected image is then rotated back through 180 degrees and added to the first edge-detected image. The two-fold convolution process ensures that no asymmetry is introduced due to directional bias. Figures (4.19, 4.20 and 4.21) illustrate the stages to symmetric edge detection of a theoretical circular area, figure (4.18).

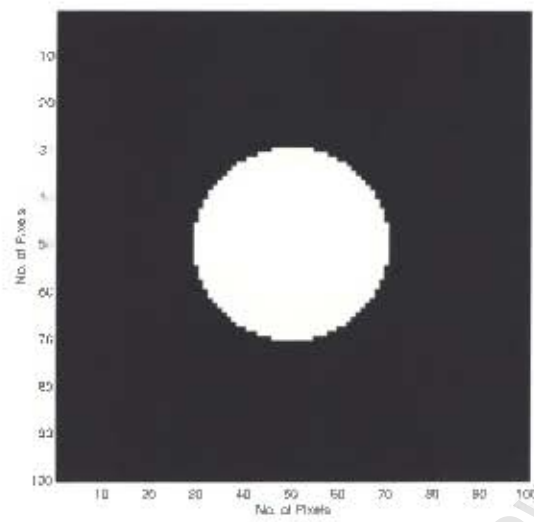


Figure 4.18: Theoretical circular area

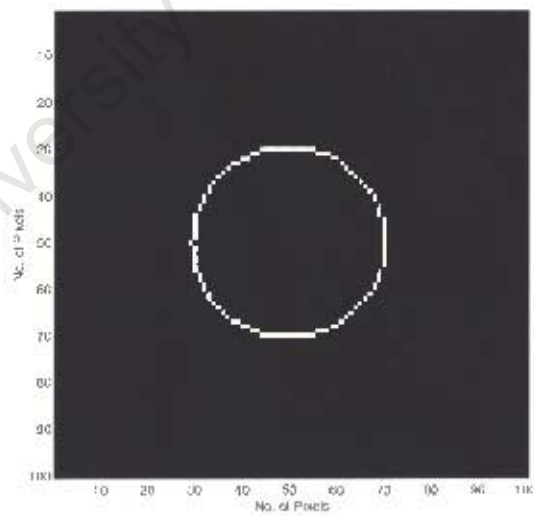


Figure 4.19: Left convolution using the Canny edge detector

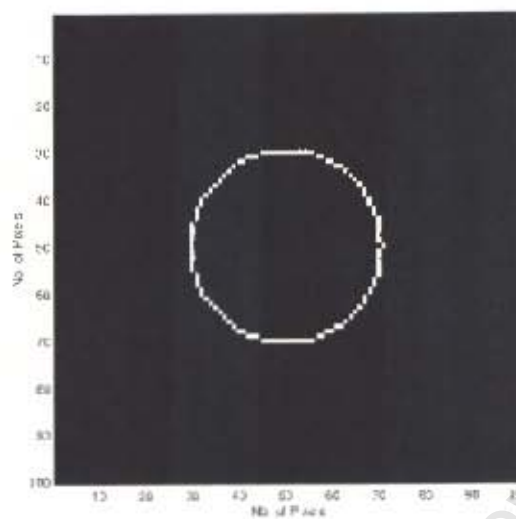


Figure 4.20: Right convolution using the Canny edge detector

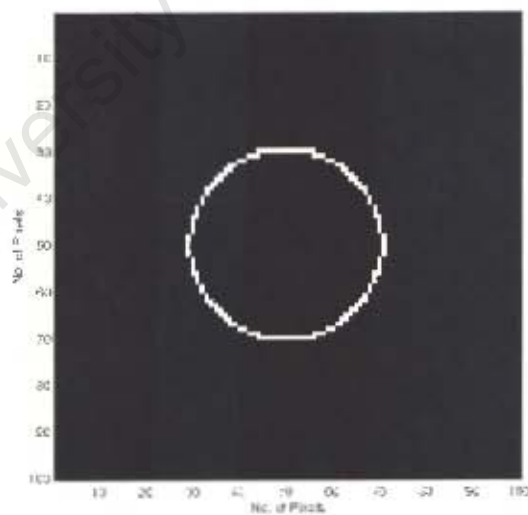


Figure 4.21: Two-fold Canny convolution on a theoretical circular area



Figure 4.22: Canny edge detection on X-ray image

Any noise still present along the borders of the image, see figure (4.22), is then filtered out using a simple masking technique. The resulting binary image is now composed of all the edges and some noisy artefacts within the images, figure (4.23). At this point it is observed that the edges belonging to the objects of interest resemble nearly closed circles, which are visually discernable from the other detected objects.

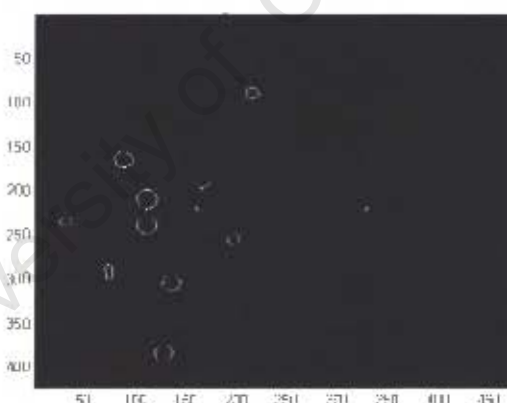


Figure 4.23: Masking of corners in edge detected image

The edge detected pixels are now dilated with a symmetric three-by-three structuring element (ones along the diagonals and zeros elsewhere), figure (4.24). The structuring element is convolved with the edge-detected image matrix using the two-fold process described above for the Canny convolution. The dilation process symmetrically pads the edge pixels. This effectively closes all the objects of interest without introducing any directional bias. The closed objects are then identified as distinct objects using an eight-connected neighbourhood search algorithm; A description of connectedness is given in Appendix (A.2). Each object of interest is then said to have two separate background elements (the part inside the closed loop and the part outside). These distinct backgrounds are labelled using a four-connected neighbourhood search algorithm. The four-connectedness ensures that the two background regions remain distinct from each other.

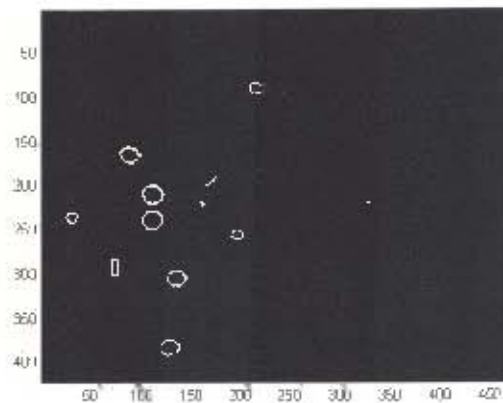


Figure 4.24: Pixel dilation using a 3-by-3 structuring element

Thereafter only the completely closed objects satisfying these criteria are filled with white using a flood-fill algorithm, figure (4.25). The flood-fill operation is applied only to the 'holes' (background pixels within the closed foreground). The filled pixels are also indexed. The neighbourhood search and flood-fill algorithms are well published in the literature ([87, 92]). All holes are subsequently mapped to two, as distinct to the numbers 1 and 0 referred to previously, and saved as a new image. The image consists of zeros (the background), ones (the dilated edge pixels) and twos (the flood filled holes).

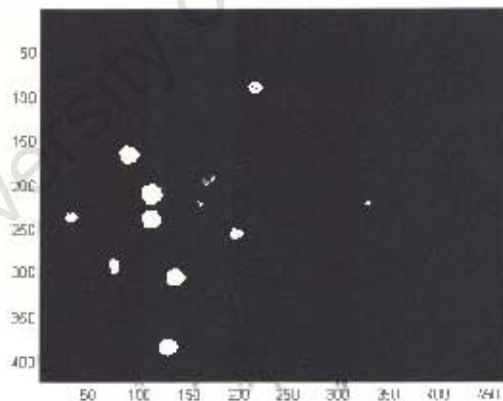


Figure 4.25: Filling of completely closed objects

Subtracting the filled binary image from the new image ensures that only the tracked balls remain. The resulting image is a binary image containing the tracked particles, figure (4.26).

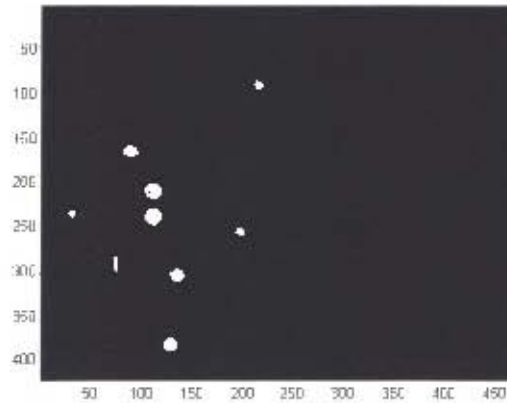


Figure 4.26: Subtraction process - only the tracked objects remain

The filled objects are labelled using an eight-connected region growing algorithm. The indexing numbers the object pixels from left to right and top to bottom, beginning with the number one for object one and so on. The labelled image thus appears with tonal variation, figure (4.27).

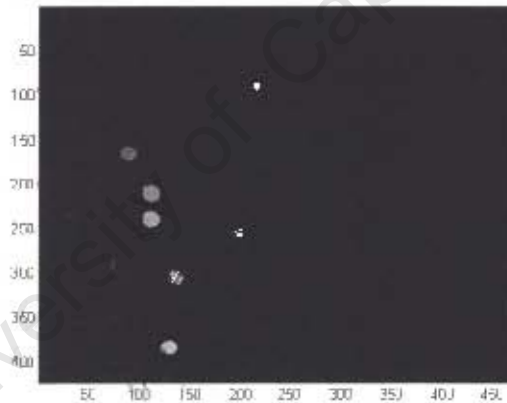


Figure 4.27: Eight-connected region growing algorithm

Provisional centroids, eccentricities, areas and pixel coordinates are computed. At this point the labelled images are ready for the target centering process.

4.4 Dynamic Template Matching (DTM)

4.4.1 Introduction

The tracked particles described in the preceding sections of this chapter have all been of higher densities than the bulk of the ball charge. These particles, though different from the bulk charge

(plastic beads), are still valid in the context of comminution as they represent the grinding ball charge (steel balls) within the rock charge of typical industrial mills. It is, however, desirable to track a typical bulk charge particle to account for any differences in motion due to particle density.

To this end a typical plastic bead coated with a thin layer of silver paint was tracked. A resultant X-ray image of the silver painted bead is shown in Figure (4.28); the above-mentioned bead is only vaguely discernable against the background charge. The tracked particle provided a new problem as far as the existing feature extraction routines were concerned. All of the routines described thus far could not meaningfully and consistently distinguish between the X-ray image of the tracked plastic bead and the background.

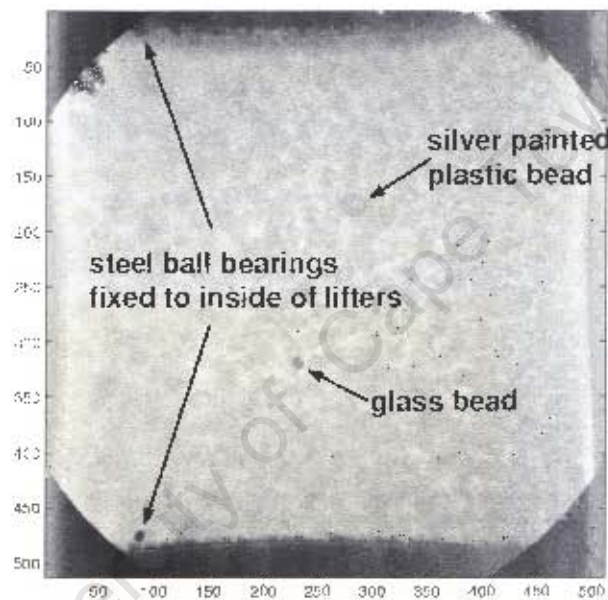


Figure 4.28: Typical X-ray image of silver painted bead after intensity flattening

4.4.2 Image processing requirements

The accurate verification of DEM place stringent demands on the processing requirements. In addition to automatically locating the particle, the centering accuracy must be equal to, if not better than, the accuracies achieved previously when tracking a steel ball bearing. The requirements of the feature extraction algorithm are:

1. Guarantee object localisation to within a 40×40 pixel window
2. Consistently resolve multiple matching cases into a single correct match
3. Be highly robust to noise
4. Adequately distinguish between object pixels and noise, and not withstanding

5. Return a meaningful (and accurate) centroid under all noise and occlusion conditions
6. Be computational efficient

In order to achieve localisation to within a 40×40 pixel window, some form of cross correlation, Appendix (A.1.2), or template matching is required. However, feature matching is further complicated if the images being searched are sufficiently different such that a single correlation filter will not suffice. These conditions occur in the X-ray image of the tracked bead. The relatively low X-ray attenuation of the bead results in noisy images – the projected shape and intensity of the bead changes from one image to the next. However, the differences are not appreciable between successive images. This implies that a correlation filter (a sub-image containing the tracked bead) extracted from image N most closely models a similar sub-image from image $N + 1$. Additionally, the matching is only valid for binary edge images (and sub-images). This is due to the close similarities between object and background intensities. The choice of binary image matching reduced the problem to shape matching which is less complicated if the general geometric properties are known (the shape of the tracked bead remains essentially elliptical).

4.4.3 DTM implementation

DTM is used to keep track of the marked particle's position and projected shape as it moves through different regions of the charge that have varying light intensities and surrounding contrasts. Using the 1st image, $F_1(x, y)$, of a given X-ray image scene, a provisional centroid location is obtained manually (using the mouse input). These coordinates form the centre of the first correlation filter, $W_1(x, y)$. The correlation filter is a 40×40 binary sub-image obtained from the edge image of $F_1(x, y)$ which we denote as $F_1^E(x, y)$; section (4.4.4) describes the generation of the edge image. The central part of the correlation between $W_1(x, y)$ and $F_2^E(x, y)$ contains the provisional centroid to $W_2(x, y)$. The central part of the correlation is simply that part of the correlation result that is the same size as $F_2^E(x, y)$ when the two images are viewed concentrically. A 40×40 sub-image centered on the provisional centroid is then cropped out from $F_2^E(x, y)$ to form $W_2(x, y)$. In general, correlation filter $W_N(x, y)$ is obtained from the cross correlation between $W_{N-1}(x, y)$ and $F_N^E(x, y)$. This process is fully automated and only requires the mouse input for the first image.

To illustrate this process, the analysis of two successive images are presented. The $(N)th$ image, denoted as $F_N(x, y)$, is edge detected, resulting in $F_N^E(x, y)$, figure (4.29). The correlation between $F_N^E(x, y)$ and $W_{N-1}(x, y)$ produces a maxima corresponding to a best match (dark spot) as illustrated in figure (4.30); $W_{N-1}(x, y)$ is shown in figure (4.31a). The coordinates of the dark spot in figure (4.30) denotes the provisional centroid to $W_N(x, y)$, figure (4.31b).

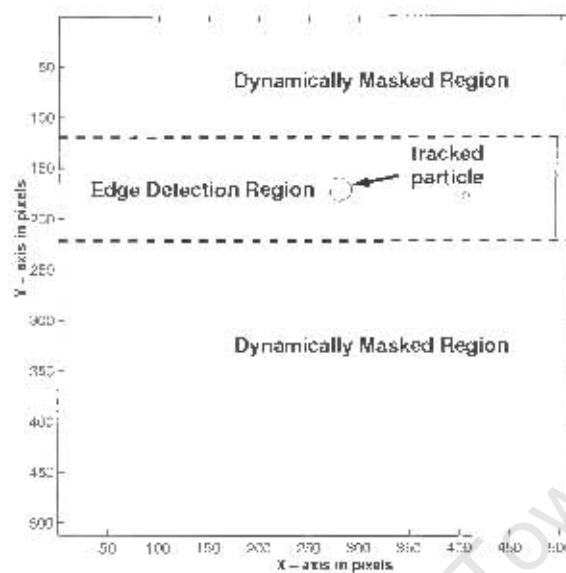
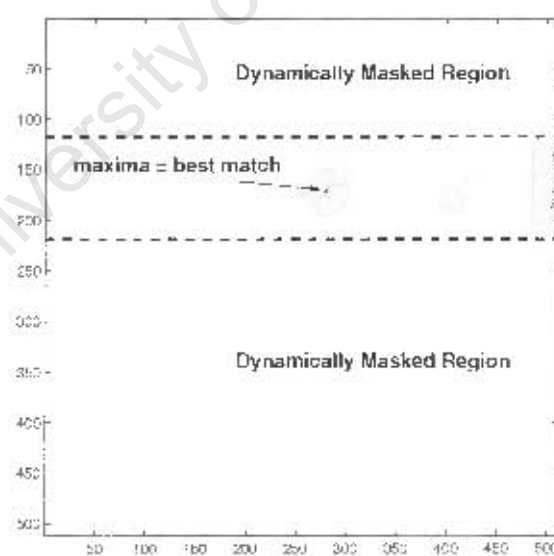


Figure 4.29: Nth image after edge detection

Figure 4.30: Correlation match on Nth image using the $(N-1)$ th correlation filter

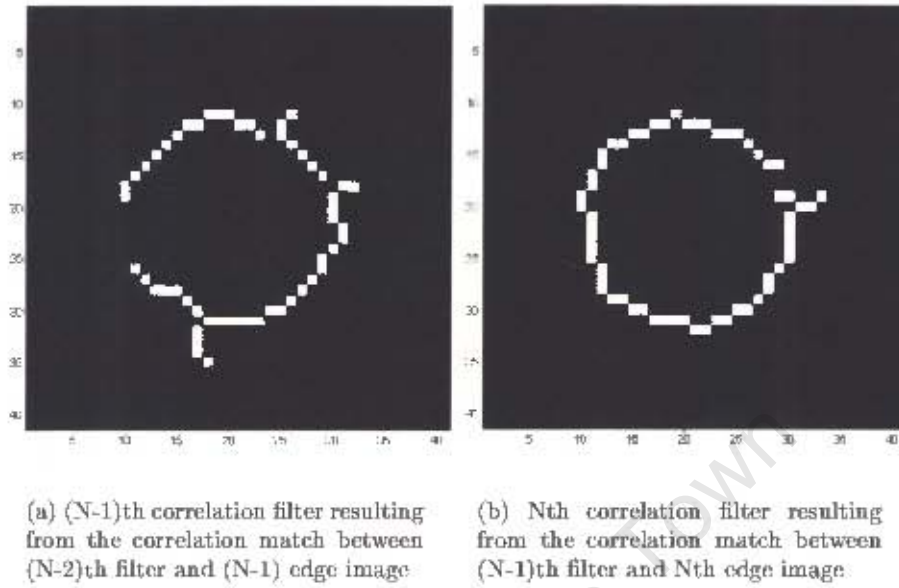


Figure 4.31: Successive correlation filters used in template matching

4.4.4 Generating the edge image

The edge detection scheme used is the Canny method, section (4.3.5.1). For a given X-ray image scene (a compressed set of Dicom compliant images generated from a single X-ray camera) there exists a set of parameters that govern the Gaussian filter design. These parameters should be modelled for both X-ray image planes within the given scene. The modelling entails finding the upper and lower intensity values to the edge pixels of the tracked object. The width of the Gaussian filter is taken to be one pixel for all images.

Three schemes for generating the required threshold inputs to the Canny edge detector were investigated. The basic idea was to formulate a discrete upper and lower threshold map for the X-ray image space. A set of images from a given scene was selected such that the tracked particle spanned most of the X-ray image space for that set. Each image was then processed according to the following schemes:

Scheme 1: A sub-image containing the tracked particle was manually selected from each of the images within the chosen set. The sub-images were then processed using the optimal thresholding algorithm described in section (4.2.2). The implementation of the algorithm produced only partial success because the tracked object was not sufficiently distinct from the background when its position was along the edges of the X-ray image.

Scheme 2: This involved the analyses of scan lines across the sub-images. The intensity profile along the scan lines facilitated the selection of the appropriate intensity values for input into the Canny edge detection algorithm. Figures (4.32 and 4.33) illustrate the scan lines and their corresponding

intensity profiles respectively. The intensity profile along the scan line is computed using a lowpass interpolation scheme described in section (7.7). The upper and lower threshold values are estimated visually from each scan line and an average computed for both. The threshold map resulting from this analysis showed very little variation within a 100 pixel radius from any given point. More importantly, the edge detection was relatively insensitive to these subtle differences.

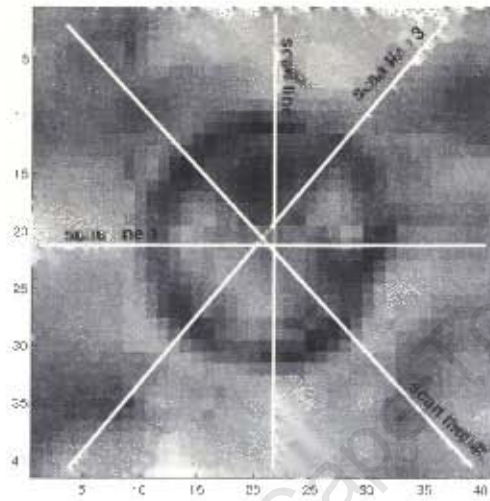


Figure 4.32: Scan lines used in scheme 2

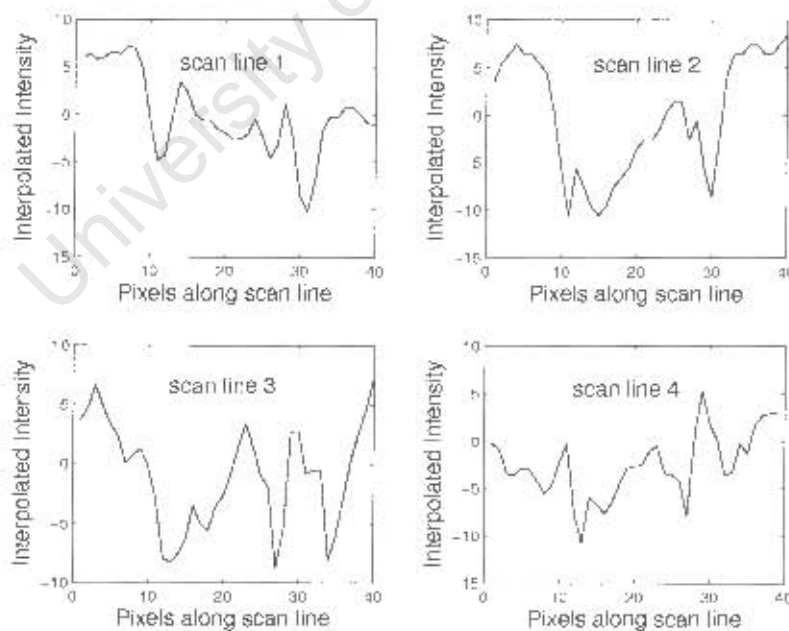


Figure 4.33: Intensity profiles across scan lines

Scheme 3: Based on the limitations of schemes 1 and 2, a more economical method was developed

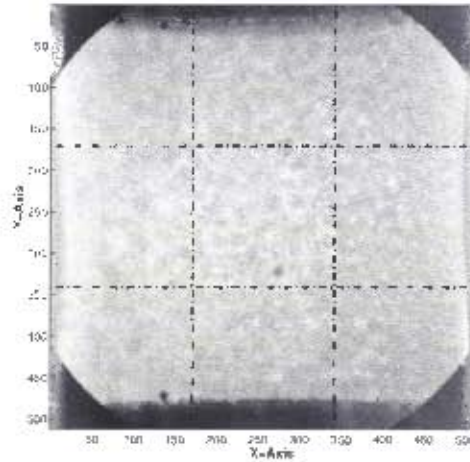


Figure 4.34: X-ray image of tumbling Perspex mill showing the 9 zones

to determine the parameters governing the Gaussian filter design. For the images belonging to a given X-ray plane, 9 random sets comprising approximately 30 images per set were selected. The 9 sets were chosen according to the 9 zones of an image in which the tracked particle could reside, Figure (4.34). The choice of 9 zones approximately agrees with the 100 pixel radius in which the threshold parameters show little variation. Scheme 2 was applied to each of the 9 zones and a single set of parameters computed for each zone.

Using the coordinates of the provisional centroid, (X_N, Y_N) , a dynamic masking technique was implemented - all pixels lying completely above or below the bounds of the subimage are set to zero. This simple masking structure can lead to problems when the tracked particle is along the top and bottom edges of the image. In such cases the mouse input is invoked to inform the masking bounds. This was deemed to be more economical than setting a generic mask that occasionally masks out the pixels of the tracked particle. The dynamically masked zones are illustrated in Figures (4.29 and 4.30).

After the provisional centroids have been accumulated for the entire scene, the images are quickly scanned to ensure that the centroids are those of the tracked particle. If an error is found it is corrected manually by locating the approximate centroid. The provisional centroids then form the basis for a more rigorous target centering algorithm. The centering algorithm is adapted after the conic fitting routine by Fitzgibbons [65]. The details of the conic fitting routine and its implementation in the current work is presented in section (5.3).

4.4.4.1 Dealing with multiple matches

Despite the dynamic masking which restricts the template matching to a narrow search space, multiple matches do occasionally occur. A multiple match simply implies that multiple maxima are found; maxima show up as bright spots in the central part of the correlation image, figure (4.30). A simple distance metric is used to exclude extraneous matches

$$\text{if } |D_N - C_{N-1}| > \text{distance metric} \quad \dots \text{ ignore maxima} \quad (4.19)$$

$$\text{if } |D_N - C_{N-1}| < \text{distance metric} \quad \dots \text{ maxima is valid} \quad (4.20)$$

where D_N is the coordinates of the maxima from the N^{th} image and C_{N-1} the provisional centroids of the $(N - 1)^{\text{th}}$ image. The distance metric is associated with the mean distance covered by the particle between two successive images. Using this method to remove extraneous matches can lead to no valid matches being found. In such cases the user is alerted and the error corrected manually by locating the approximate centroid using the mouse input.

4.5 Summary

In this chapter a description of the various techniques used to identify the objects of interest in the X-ray images of the tumbling Perspex mill are presented. The percentage threshold method and the optimal thresholding technique were computationally more efficient than the edge detection algorithms. The thresholding techniques worked well for distinct targets like steel, glass and lead balls, but failed to return meaningful results with the silver painted plastic bead. The edge detection implementations worked equally well for the distinct data, but required the pixels of interest to be localised for the silver painted bead. The tracking of the painted bead was achieved by first localising it with the dynamic template matching and then edge detecting. The template matching followed by the two-fold Canny implementation proved to be the most robust algorithm, returning meaningful and accurate edge data for all tracked particles. All algorithms were programmed and included as options in the automated image analyses of the X-ray image scenes. A comparison of the accuracies achieved by the various feature extraction routines is deferred to the next chapter due to the influence of the target centering algorithm.

Chapter 5

Target centering

5.1 Introduction

In close range photogrammetric applications, circular targets are generally favoured for use as control points, hence the precise location becomes a centering problem. However, in X-ray stereo photogrammetry, it has become common practice to use steel ball bearings. Ball bearings are preferred in X-ray imaging because they produce elliptical projections from any filming angle. These elliptical projections have the advantage of producing consistent images, but can complicate the target centering process. Various techniques have been investigated and programmed as options in the measurement system. These techniques may be broadly placed into two categories. The first category deals with distinct data (the steel ball bearings), where distinct implies that the pixels of interest can be located to the nearest pixel. The second category deals with data that is less distinct (particles with similar density to the background ball charge). In such cases, if we are to guarantee that all the imaged edge data of the object of interest is to be recovered in the edge detection process, then we will in general always recover noisy data as well. The reason for this is simply that the edge pixels of interest and the surrounding noise are very similar in intensity. Accordingly, the chosen centering algorithm must be highly robust to noise and provide useful results even under occlusion conditions. These techniques are formulated, discussed and analysed in this chapter.

Section two deals with the centering of distinct data using the grey values as weights, section three describes the use of conic fitting with particular reference to ellipse specificity in light of very scattered data, and the rest of the chapter focusses on various tests of these algorithms.

5.2 Weighted Centre of gravity (WCG)

The weighted centre of gravity technique using the grey value as a weight is a common area based centering routine (see for example Trinder [93], Wong & Ho [94], Gustafson & Handley [95] or Ruther & Parkyn [84]). This technique and a similar one using the square of the grey value as weight are covered in this section and have been programmed as options in the measurement system.

5.2.1 WCG with grey value as weight

This routine (and the grey value squared one) is formulated for a white target on a black background, however if the reverse situation exists then the grey values should be inverted for the equations to hold. The following equations are derived using moments:

$$y_0 = \frac{\sum yg}{\sum g} \quad (5.1)$$

$$x_0 = \frac{\sum xg}{\sum g} \quad (5.2)$$

in which (x, y) is the spatial coordinates of the pixel having a grey value of g and (x_0, y_0) is the centroid of the region under consideration.

5.2.2 WCG with grey value squared as weight

The equations for the centre of gravity using the grey value squared as weight are:

$$y_0 = \frac{\sum yg^2}{\sum g^2} \quad (5.3)$$

$$x_0 = \frac{\sum xg^2}{\sum g^2} \quad (5.4)$$

Using the grey value squared as weight, as opposed to grey value, reduces the effect caused by homogeneous intensity distributions. However it is also more sensitive to noisy pixel grey values.

For both methods, improvements in accuracy can be obtained by only including those pixels with grey values greater than the background level as observations.

5.3 Ellipse fitting to very scattered data

The discussions on target centering thus far have all assumed that the targets are well defined. Implicit to this assumption is that all the detected pixel data are accurate to within one pixel from the computed centroid. In other words, the edge pixels of the detected object have as their nearest neighbour pixels the true edges of the object of interest. However, in many situations the quality of the data is compromised either by inheritantly poor experimental data, or by contrived conditions that supersede data quality. The latter situation is encountered in the present work where:

1. The density of the tracked particle is required to be very close to that of the bulk charge, and
2. The 'chopping' of the X-ray beam results in partial occlusion of the tracked bead in certain regions along the mill periphery.

The first contrived condition implies that the intensities of object and background pixels are approximately equal, while the second guarantees that not all edges will be recovered during the image processing stage. It is also possible that combinations of both compromises will be encountered, especially for the large amounts of data expected in this thesis. The following section discusses a robust ellipse specific algorithm [65] that adequately deals with the conditions in the present work.

Although a wide range of literature on ellipse fitting routines is available (see for example [65, 66, 67, 72, 70] and other general methods described in the literature review), most are formulated for general conic fitting and biased towards good data. For the sake of uniformity we represent a general conic by an implicit second order polynomial:

$$F(\mathbf{a}, \mathbf{x}) = ax^2 + bxy + cy^2 + dx + ey + f = 0 \quad (5.5)$$

where $\mathbf{a} = [a \ b \ c \ d \ e \ f]^T$ and $\mathbf{x} = (x, y)^T$. $F(\mathbf{a}, \mathbf{x}_i)$ is called the algebraic distance of a point $\mathbf{x}_i$ to the conic $F(\mathbf{a}, \mathbf{x}) = 0$.

The general procedure to conic fitting is approached by defining a distance metric, similar to the algebraic distance (other metrics are described in the literature review), and minimising this distance. For example, using the definitions from equation (5.5) it is clear that the simplest approach is to minimise the algebraic distance in a least squares sense:

$$D(\mathbf{a}) = \sum_{i=1}^N F(\mathbf{a}, \mathbf{x}_i)^2 \quad (5.6)$$

where the N data points $\mathbf{x}_i$ are sampled about the curve. Since most of these distance metrics are not ellipse specific, non-ellipse fits are rejected using the discriminant test:

$$b^2 - 4ac < 0 \quad (5.7)$$

For this reason iterative methods must be employed. These methods increase the computational demand and do not guarantee convergence, especially when the data is noisy. The inclusion into an automated system is thus made all the more difficult, if at all possible. The automated detection and centering of tracked particles within the X-ray images placed very stringent demands on the ellipse fitting routines. The algorithm that most closely satisfied the imaging requirements was an ellipse specific, conic fitting routine by Fitzgibbons [65]. To illustrate this claim a simple comparison is conducted between the various conic fitting algorithms investigated (see section (2.3.5) for a description of the various conic fitting routines reviewed).

5.3.1 Evaluating the robustness of the various conic fitting routines under all noise and occlusion conditions

The evaluation of conic fitting algorithms is commonly achieved with synthetic data. This is usually done by computing the RMS error of the parameter residuals. The residuals are the differences between conic parameters before and after the introduction of simulated noise and occlusion. Similar comparisons can be achieved with real data provided that the underlying ('true') conic is known. The criteria used in discerning the underlying conic was motivated by Heath [50]. He showed that human visual assessment of edges provided a robust means of determining the true edge pixels in complex images. He also demonstrated that these visually determined edges can be used as the base case for testing the accuracy of edge detectors.

5.3.1.1 Evaluation conic fitting algorithms based on visually assessed parameters

The ellipse parameters used to evaluate the various conic fitting algorithms were based on the average elliptical properties of the plastic bead's projection onto the X-ray image plane¹. The mean values are calculated from a set of 300 images of the silver painted bead ($\varnothing 6.1$ mm) as it tumbles around in the Perspex mill. The 300 images were chosen such that their positions span the entire X-ray image. The images were processed manually, ensuring that only true edge pixels were used as input to the ellipse parameter estimation. To ensure that the test basis bore no dependency on the conic algorithms being tested, the parameters were computed by fitting an ellipse according to the explicit formulation. The analysis was done for both filming planes (denoted as plane A and plane B images from henceforth) and the average parameters computed from both. The mean parameters are given in table (5.1).

x_c	y_c	R_x	R_y	eccentricity (e)
20.0	20.0	10.8	11.6	0.36

Table 5.1: Mean ellipse parameters of the tracked particle sampled over 300 images, across both image planes. The parameters are determined from the least squares solution of the explicit ellipse equation

Table (5.2) and figure (5.1) illustrate the results of fitting to a highly occluded X-ray image while table (5.3) and figure (5.2) show the results for a noisy X-ray image. It should be noted that the final implementation of the conic fitting into an automated system includes additional metrics for optimum centering which are discussed later in this section and illustrated in figure (5.5). The tables give two eccentricity values for the conic fitting routine. The unbracketed values are eccentricities of the various conic fits using the original data while the bracketed values are calculated once extraneous pixels have been excluded through the use of additional metrics, see figure (5.5) for an illustration of the metrics used.

¹The rotation of the ellipse was not included in the evaluation

Algorithm/constraint	Cx	Cy	Rx	Ry	eccentricity (e)	ellipse rotation (θ) (degrees)
	(pixel units)					
Kanatani	29.0	14.3	13.7	21.1	0.76 (0.79)	51.6
Bookstein	19.2	20.9	8.5	12.1	0.71 (0.64)	-28.6
Linear algorithm	27.8	15.3	13.1	19.6	0.74 (0.78)	51.6
Geometric distance	24.5	17.8	12.4	15.2	0.58 (0.61)	51.6
Fitzgibbons	20.6	20.2	10.2	11.9	0.52 (0.42)	-22.9
$a + c = 1$	19.2	20.9	8.4	12.1	0.72 (0.65)	-28.6

Table 5.2: Ellipse parameters of highly occluded X-ray image of plastic bead

Algorithm/constraint	Cx	Cy	Rx	Ry	eccentricity (e)	ellipse rotation (θ) (degrees)
	(pixel units)					
Kanatani	hyperbolic fit					
Bookstein	24.7	13.3	10.3	33.5	0.95	22.9
Linear algorithm	24.3	18.1	10.2	26.2	0.92	28.6
Geometric distance	25.1	12.4	10.4	35.6	0.96	17.1
Fitzgibbons	21.0	21.8	16.4	12.7	0.63 (0.39)	-57.3
$a + c = 1$	24.7	13.3	10.3	33.5	0.95	-22.9

Table 5.3: Ellipse parameters of noisy X-ray image of plastic bead

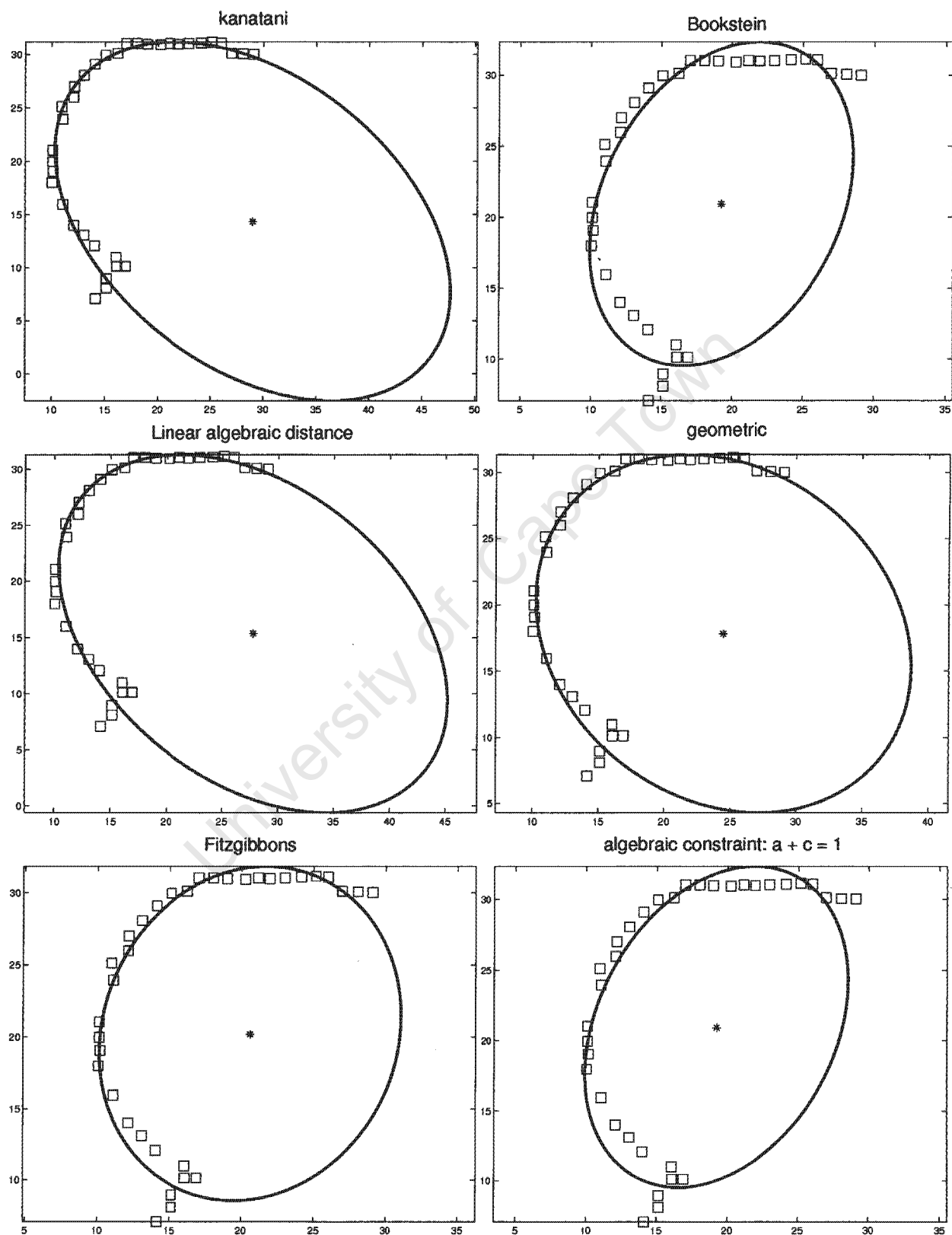


Figure 5.1: Conic fitting to highly occluded X-ray image of plastic bead. The squares denote the pixels after edge detection and the '*' denotes the centroid of the conic fitting.

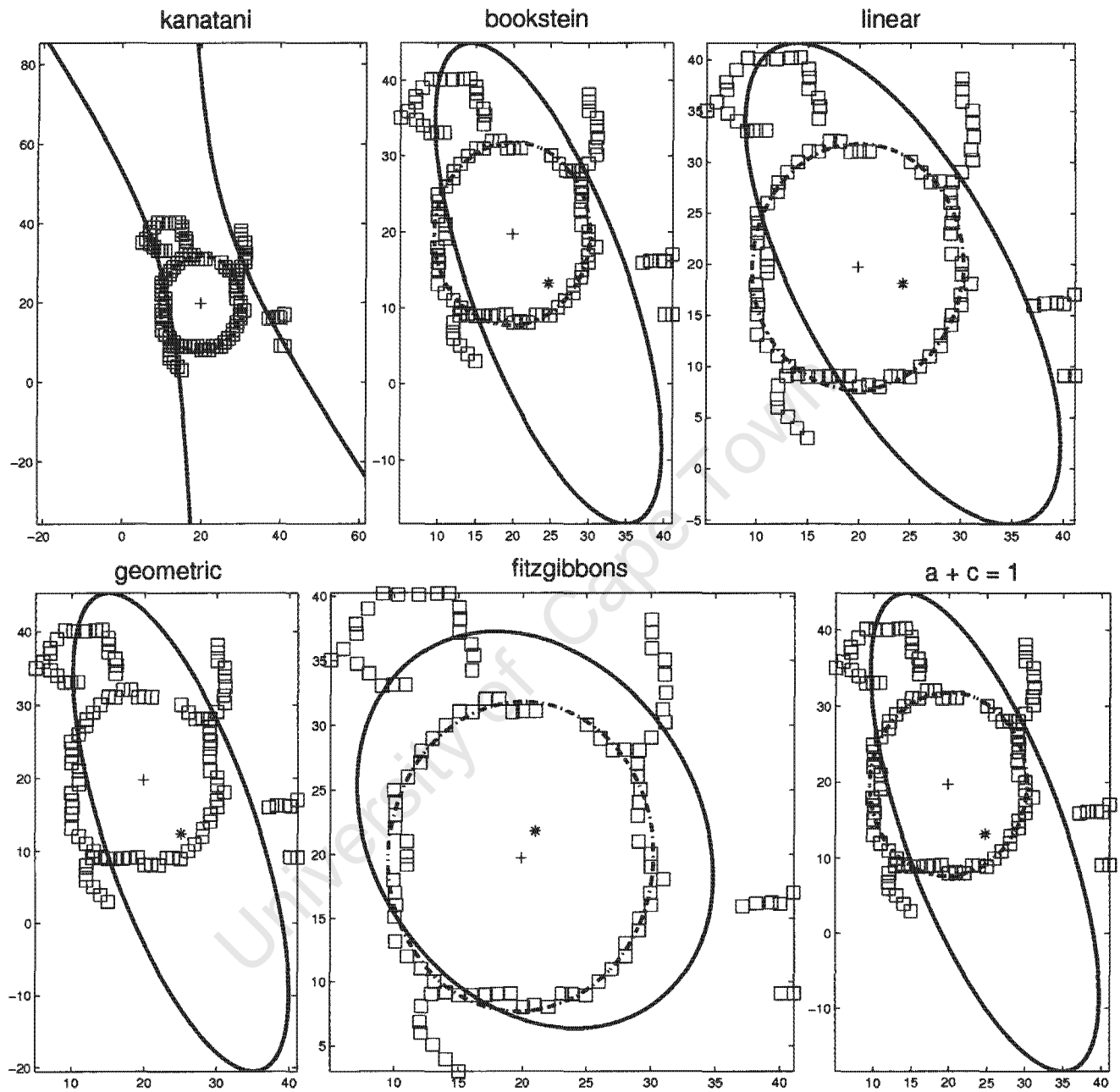


Figure 5.2: Conic fitting to noisy X-ray image of plastic bead. The '*' corresponds to the centroid of the conic fit. The 'dash-dot' line fit to the data illustrates the 'correct' fit. By simple visual inspection it is clear that the Fitzgibbons ellipse most closely resembles the 'correct' fit.

The parameter residuals are formed by subtracting the values in table (5.1) from those calculated in tables (5.2 and 5.3). The RMS error of the parameter residuals is calculated as:

$$\delta_i = \sqrt{\frac{\sum_{j=1}^4 (P_{ij} - P_j^{(m)})^2}{4}} \quad (5.8)$$

where

δ_i = RMS error of the i^{th} conic fitting algorithm

P_{ij} = j^{th} parameter of the i^{th} conic fitting algorithm

$P_j^{(m)}$ = j^{th} mean parameter from table (5.1)

and $j = 1, 2, 3, 4$ refers to the four parameters given in table (5.1).

5.3.2 Results from conic fitting routines investigated

Figures (5.3 and 5.4) illustrate histogram plots of the RMS error for images depicting occlusion and noise respectively. The height of the bars indicate the RMS error of the parameter residuals.

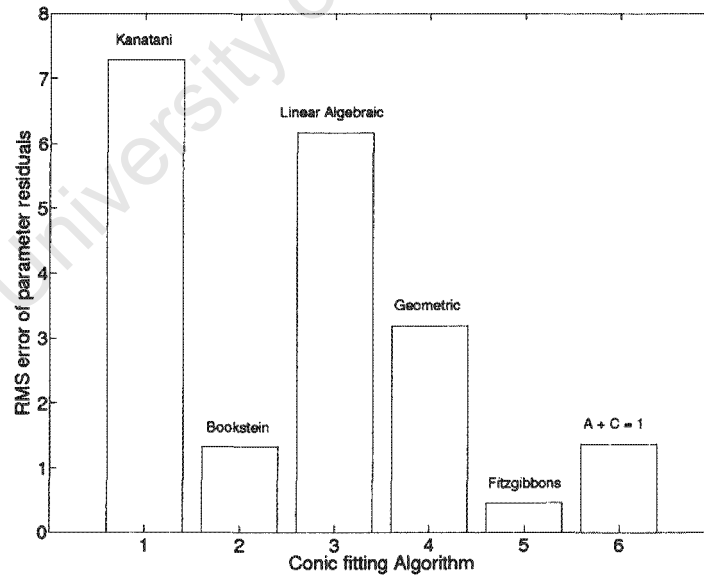


Figure 5.3: RMS error of parameter residuals for each of the conic fitting algorithms investigated. Analysis based on a highly occluded image

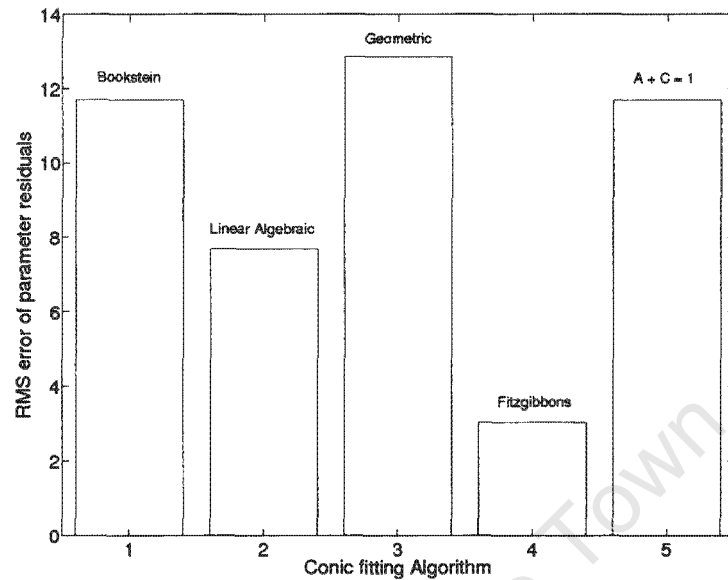


Figure 5.4: RMS error of parameter residuals for each of the conic fitting algorithms investigated. Analysis based on a noisy image

The Fitzgibbons algorithm produces the lowest RMS error of the parameter residuals for both the occlusion and noise tests. A somewhat disappointing observation was the performance of Kanatani's method. The reason for this might lie in the error assumption of his model - each pixel is a Gaussian distortion of the true value - which is not entirely valid for the X-ray image data. The complexity of implementing variations of Kanatani's algorithm prevented further investigations, especially since a viable solution was available.

These tests provided sufficient evidence for choosing the Fitzgibbons routine over the others. According to him, the algorithm satisfies the following conditions:

1. *Ellipse specificity*, providing meaningful results under all noise and occlusion conditions
 - (a) *Invariance* to Euclidean transformation of the data
 - (b) High robustness to noise
 - (c) High computational efficiency

5.3.3 Implementation of Fitzgibbons with additional metrics

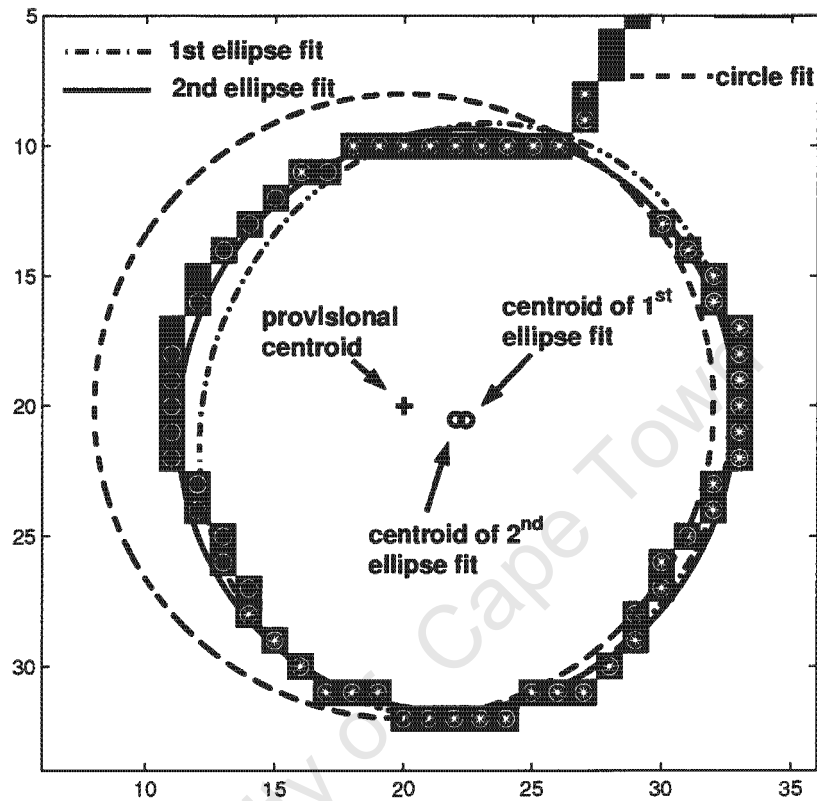


Figure 5.5: Ellipse fit to provisional centroid from template matching

The processing requirements of the X-ray data provided a rigorous test of these conditions. The binary sub-images obtained from the template matching provided the data for the target centering which is achieved in three stages. To ensure an accurate centering process it is important to isolate as many of the true pixels as possible while removing as many of the 'noisy' pixels as possible. Using the provisional centroid, denoted by the '+' in figure (5.5), a circle of radius 11 pixels centered on the '+' is fitted (denoted by the dashed lines). The choice of 11 pixels for the radius is based on the mean effective radius of the tracked plastic bead ($\varnothing 6.1$ mm) when projected onto the intensifier screen. The mean effective radius is a compromise between including all pixels of interest and excluding all extraneous pixels, and is calculated as the average of the maximum major and minor axes of all possible theoretical projections of the plastic bead onto the image plane. Strictly speaking, the pixels of interest form an elliptical shape. However the elliptical rotations of the pixel data vary considerably across the image scene making a mean representation impractical. The mean circle fit (stage 1) is the first attempt at isolating the pixels of interest. It can be observed that even though some of the edge pixels do not coincide with the perimeter of the circle, they are 'reasonably close'. A distance metric is then used to exclude bad pixels within the 40×40 pixel window.

Consider two circles centered on the '+' with radii of $(11 + 2.5)$ and $(11 - 2.5)$ pixels respectively

(these circles are not shown in figure 5.5). Then all pixels lying completely within the smaller circle or completely outside the larger circle are considered as extraneous pixels. All remaining pixels, denoted by the '*' in figure (5.5), are then used as input data to the ellipse fitting algorithm, stage 2, 1st ellipse fit. A geometric error of fit is then computed for the ellipse. The error of fit is similar to the distance metric used for the circle - all pixels outside a perpendicular distance of 1.6 pixels from the ellipse are considered as noise; the remaining pixels (denoted by the 'O' in figure (5.5)) are considered as true edge pixels. A final ellipse (stage 3, 2nd ellipse fit) is then fitted to the 'true' edge data.

It is clear from figure (5.5) that the simple distance metrics effectively isolate the true edge pixels from the noise. Of the 56 pixels used in the final ellipse fit, only one may be classified as noise. This translated to a 1.8% error which is reasonably good. All further use of the Fitzgibbons algorithm, in particular its implementation in the chapter (5), includes the additional metrics.

5.4 Testing with Ideal Targets

An ideal target is one in which the area within the ellipse has a grey value of 255 (saturation) and the area outside has a grey value equal to the background level. However in a discrete space, such as pixel space, the elliptical area is only an approximation to the ideal ellipse. A simple routine was developed for generating a circular area that very closely approximates the theoretical solution, figure (5.6). In terms of geometric properties, the area and eccentricity of the generated circle are exactly that obtained for an ideal circle.

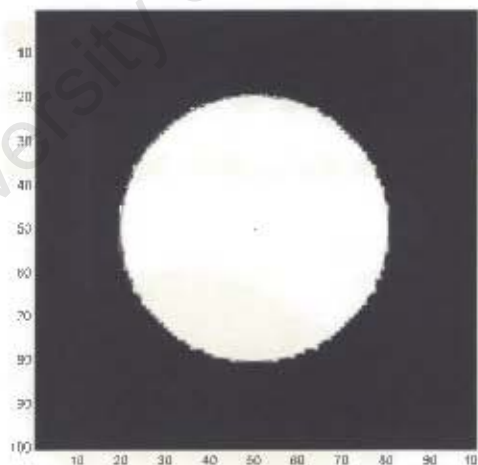


Figure 5.6: Generated circular area with theoretical circle superimposed

5.4.1 Fall-Off Functions and Random Generation of Ideal Targets

The effects of noise present in most digital images can be reasonably modelled by mathematical approximations. In particular fall-off functions model target blurring caused by de-focusing, over-

illumination and electronic effects. There are a number of methods which could be employed to generate and apply these Fall-off functions. For the series of tests conducted in the present work, the target fall-offs were obtained by convoluting the image containing the circle with a two-dimensional (7×7) discrete normal distribution with σ of 1 pixel. This is similar to blurring generated by Trinder [93], with the following variation: Each circular area was generated in a different position, within a separate (512×512) image, and individually convolved with a normal distribution. The normal distributions were generated independently for each image. This ensured that the blurring was purely random, and different for each circle. The resulting images were then combined into a single image, figures (5.7 and 5.9). A closer look at one of these synthetically generated targets with simulated noise, figure (5.8), reveals a 'halo' like edge which is characteristic of the penumbra effect encountered in X-ray imaging. This ensures that the tests bear some resemblance to the actual problem for which these centering routines were formulated.

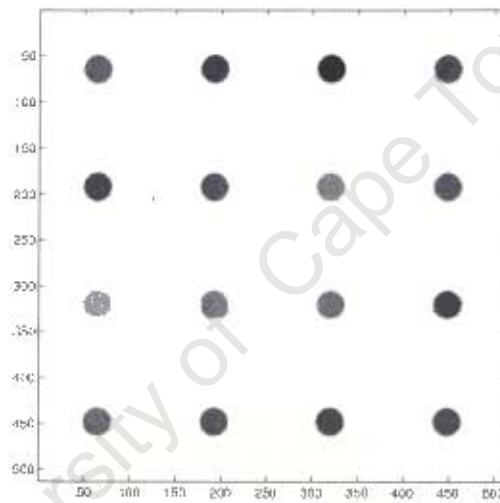


Figure 5.7: Blurred targets due to direct falloff: Equal diameters — 15 pixels

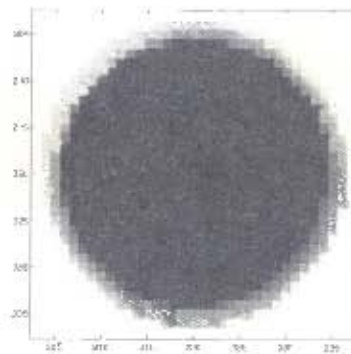


Figure 5.8: A closer look at one of the blurred targets: diameter = 15 pixels

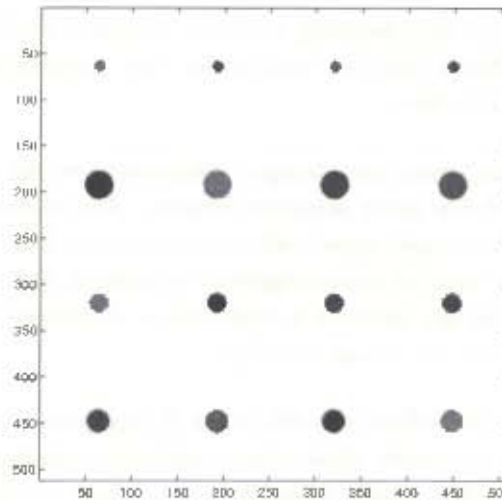


Figure 5.9: Blurred targets due to direct falloff: 4 diameters = (6,15,10,12) pixels

5.4.2 Centering results

The target centering methods were tested for all of the feature extraction techniques. Two target images, one with targets of equal diameter and one with four different diameters, were used in the tests. The varying diameter targets investigate the effects of scale within an image.

The first set of results with equal diameter circles are presented in table (5.4). Table (5.5) has 4 different diameters (6, 10, 12 and 15 pixels).

The methods are numbered as follows:

1. Center of gravity with uniform weight (UCG)
2. Weighted center of gravity (WCG)
3. Weighted center of gravity with grey value squared as weight (WCGSQ)
4. Ellipse specific conic fitting (ESCF)

The RMS Δx and Δy are the root mean squares of the differences between calculated and known coordinates of the circle centres. $\sqrt{(\Delta x)^2 + (\Delta y)^2}$ is the resultant RMS deviation for Δx and Δy . The max Δx and Δy are the maximum differences for each method. It should be noted that the percentage and optimal threshold methods produce areas (not edges) which are inappropriate for the ESCF centering method and are therefore not included in tables (5.4 and 5.5).

The results show that the one-fold Canny method is the least accurate for all centering routines. This may be attributed to the directional bias introduced in the convolution. The two-fold Canny method,

although less accurate than the thresholding routines, produce improved centering when compared to the one-fold method. This shows that the directional bias introduced in the convolution is overcome with a globally symmetric procedure.

The two thresholding routines, quite surprisingly, outperformed the edge detection. In particular, the percentage method produced the most accurate results. The accuracy of the percentage method is greatly influenced by the percentage input value. In real data, like the X-ray images, the percentage occupied by object pixels can only be approximately calculated, but with ideal targets, this percentage is known exactly. The percentage method is well suited to images with clearly defined objects, but can become less accurate when the image is noisy.

As regards target centering, the ellipse specific conic fitting produced the most accurate results. The algorithm proves to be more accurate than other centering techniques with 'good' data and equally important, it returns meaningful results when the noise levels are high.

Method	RMS Δx	RMS Δy	$\sqrt{(\Delta x)^2 + (\Delta y)^2}$	max Δx	max Δy
Percentage Method (% = 4.5)					
1	0.2685	0.2075	0.3393	0.4345	-0.4300
2	0.1681	0.1342	0.2151	0.4438	0.2507
3	0.1432	0.1022	0.1759	0.3591	0.2199
Optimal Thresholding Method					
1	0.2325	0.2292	0.3262	0.5270	0.4716
2	0.1748	0.1590	0.2363	0.3629	-0.2765
3	0.1651	0.1303	0.2103	0.2750	-0.2904
One-Fold Canny Convolution : ($\sigma = 1$ pixel), [Upper, Lower] = [0.1, 0.13]					
1	0.4958	0.5675	0.7536	0.6768	0.8352
2	0.3415	0.4187	0.5403	0.7426	0.6179
3	0.2410	0.3161	0.3975	0.4973	0.4694
4	0.1799	0.1635	0.2431	1.1327	1.2464
Two-Fold Canny Convolution : ($\sigma = 1$ pixel), [Upper, Lower] = [0.1, 0.18]					
1	0.3759	0.4551	0.5903	-0.8459	-0.9162
2	0.2873	0.3547	0.4564	-0.6987	-0.6561
3	0.2279	0.2818	0.3624	-0.5835	-0.5304
4	0.1620	0.1509	0.2214	-1.2689	-1.2362

Table 5.4: Centering results using ideal targets with same diameter, units : pixels

Method	RMS Δx	RMS Δy	$\sqrt{(\Delta x)^2 + (\Delta y)^2}$	max Δx	max Δy
Percentage Method (% = 2.42)					
1	0.2079	0.2073	0.2936	-0.4185	0.4338
2	0.1522	0.1520	0.2151	-0.4185	-0.2827
3	0.1547	0.1395	0.2083	-0.4185	-0.3017
Optimal Thresholding Method					
1	0.1930	0.2266	0.2976	0.4498	0.4956
2	0.1741	0.1976	0.2634	0.3614	0.4956
3	0.1777	0.1933	0.2626	0.4055	0.4956
One-Fold Canny Convolution: ($\sigma = 1$ pixel), [Upper, Lower] = [0.1, 0.15]					
1	0.6088	0.5699	0.8339	0.8485	0.6661
2	0.4613	0.3334	0.5692	0.8485	0.8470
3	0.4362	0.2817	0.5193	0.8485	0.5748
4	0.1629	0.1948	0.2539	1.0427	1.4503
Two-Fold Canny Convolution: ($\sigma = 1$ pixel), [Upper, Lower] = [0.1, 0.18]					
1	0.4744	0.4633	0.6631	-0.8458	0.8377
2	0.3064	0.3629	0.4749	-0.7075	-0.8765
3	0.2601	0.3408	0.4287	-0.5903	0.8377
4	0.1543	0.1755	0.2337	-1.1878	-1.4411

Table 5.5: Centering results using ideal targets with four diameters, units: pixels

5.5 Testing with Real Targets

A technique for testing the relative accuracy of the imaging and centering algorithms with real targets is discussed in this section. The method involves transforming the image data of control points to 3D real space coordinates. The residuals from comparing calculated to measured 3D coordinates then form the basis for the analysis. The control frame used in this analysis comprised 21 points, listed in table (5.6) and illustrated in figure (5.10). The control points are measured using a coordinate measuring machine (CMM), the functions and accuracy of which are described in Appendix (B.5). The frame in question is the original frame constructed for calibrating X-ray space and is used to illustrate the relative accuracy of the various imaging and centering processes. The control frame used in the 3D reconstruction of the final experimental data, and required for the desired accuracy reported in table (6.2), contained an effective 110 points and is given in table (B1).

Point No.	X (mm)	Y (mm)	Z (mm)
1	0.00	150.48	0.00
2	12.06	150.78	0.07
3	32.50	150.69	1.32
4	92.01	150.68	0.14
5	0.60	64.47	-27.97
6	12.31	64.39	-26.54
7	39.95	64.38	-27.26
8	71.05	64.41	-27.54
9	92.33	64.36	-26.85
10	-3.84	92.93	-56.63
11	12.15	93.18	-56.51
12	51.30	93.08	-56.66
13	92.10	93.25	-56.64
14	-3.79	121.86	-84.08
15	11.73	121.93	-85.27
16	69.37	121.86	-85.26
17	91.88	121.86	-85.70
18	-0.09	35.61	-113.15
19	11.94	35.67	-113.65
20	32.51	35.69	-114.21
21	91.88	35.73	-113.94

Table 5.6: 3D coordinates of the 21 control points used to assess the various image processing and centering algorithms



Figure 5.10: Control frame used to assess the various image processing and centering algorithms. The orientation of the axes are shown in the figure with ball position number 1 denoting the origin.

The real targets are made up of 3mm ball bearings positioned on a rigid frame such that they are clearly visible from orthogonal perspectives. The frame is immersed in the intersecting X-ray beams, and the resulting digital images provide the data for testing the target centering routines for

each of the feature extraction methods. Real targets do not have a priori target centers in image space, however the spatial coordinates are known, Appendix (B.5). The testing therefore involves calculating the image target centers and then transforming them to real space. The RMS values of residuals between measured and calculated spatial target centers form the basis for the analysis. Like the ideal target tests, the RMS values are computed for each of the three cartesian coordinates. The standard DLT equations were used to transform image data back to real space coordinates. The resulting errors produced from the residual analysis includes the transformation errors, however, given the same image, it is expected that these errors are equivalent. It should be noted that the magnitudes of the residual analysis do not quantify the error associated with unknown points and are therefore not a true reflection of the error. However, this was not the intention of this exercise. The relative accuracy of the various techniques is what's important at this stage. The error associated with unknown points is deferred to chapter (6).

5.5.1 Centering results

Method	RMS Δx	RMS Δy	RMS Δz	Resultant	max Δx	max Δy	max Δz
<i>Percentage Method : (Plane A = 19 %), (Plane B = 17 %)</i>							
1	0.5021	0.3331	0.3080	0.6767	0.9947	0.5817	0.5511
2	0.5012	0.3312	0.3094	0.6758	0.9937	0.5805	0.5511
3	0.5005	0.3298	0.3111	0.6753	0.9928	0.5793	0.5509
<i>Optimal Thresholding Method</i>							
1	0.5069	0.3328	0.3140	0.6829	0.9698	0.5766	0.5966
2	0.5060	0.3309	0.3158	0.6821	0.9687	0.5754	0.5966
3	0.5053	0.3295	0.3179	0.6819	0.9675	0.5742	0.5964
<i>One-Fold Canny Convolution, ($\sigma = 1$ pixel), [upper, lower] = [0.1, 0.18]</i>							
1	0.5023	0.3334	0.3168	0.6811	0.9922	0.5772	0.6149
2	0.5023	0.3317	0.3117	0.6806	0.9925	0.5759	0.6146
3	0.5023	0.3303	0.3187	0.6804	0.9928	0.5745	0.6142
4	0.1802	0.1795	0.1723	0.3072	0.3316	0.2987	0.3101
<i>Two-Fold Canny Convolution, ($\sigma = 1$ pixel), [upper, lower] = [0.1, 0.18]</i>							
1	0.5001	0.3375	0.3078	0.6773	0.9963	0.5664	0.5942
2	0.5003	0.3363	0.3079	0.6769	0.9969	0.5607	0.5942
3	0.4998	0.3354	0.3088	0.6765	0.9968	0.5547	0.5940
4	0.1769	0.1710	0.1603	0.2951	0.3011	0.2914	0.3135

Table 5.7: Centering Results for Real Targets, units : millimeters

The RMS Δx , Δy and Δz are the root mean squares of the differences between calculated and known coordinates of the spatial target centre coordinates. *Resultant* is the resultant vector RMS deviation

for Δx , Δy and Δz . The max Δx , Δy and Δz are the maximum differences for each centering method.

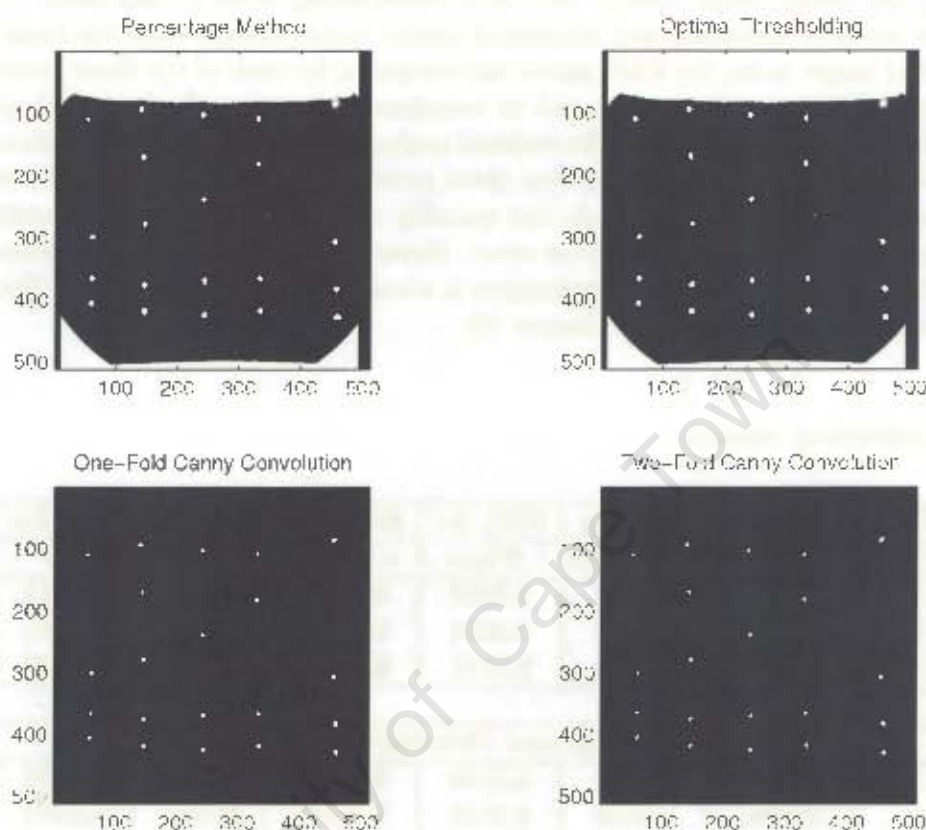


Figure 5.11: Feature extraction based on the four processing techniques

From table (5.7), the ESCF routine clearly out-performs the essentially equivalent results of the center-of-gravity based techniques. It is also evident that much of the resultant centering error is due to the x -coordinate. Comparing this to the original CMM coordinates, i.e. before rotating the CMM data into the coordinate system of the X-ray beam, the error corresponds to the depth measurement (Z -direction) of the CMM. Figure (5.11) illustrates the resultant images of the feature extraction algorithms.

The Percentage and Optimal Thresholding Methods do not remove extraneous objects during the processing, white sections in figure (5.11). This can potentially complicate target centering, especially when the noise is in the region of the targets, and more importantly in the path of the tracked particles. This problem is adequately dealt with in the Canny convolution routines.

The weighted routines (WCG and WCGSQ) were not appreciably different from the uniform centering method (UCG). This may be attributed to the relatively homogeneous intensity distribution within the targets.

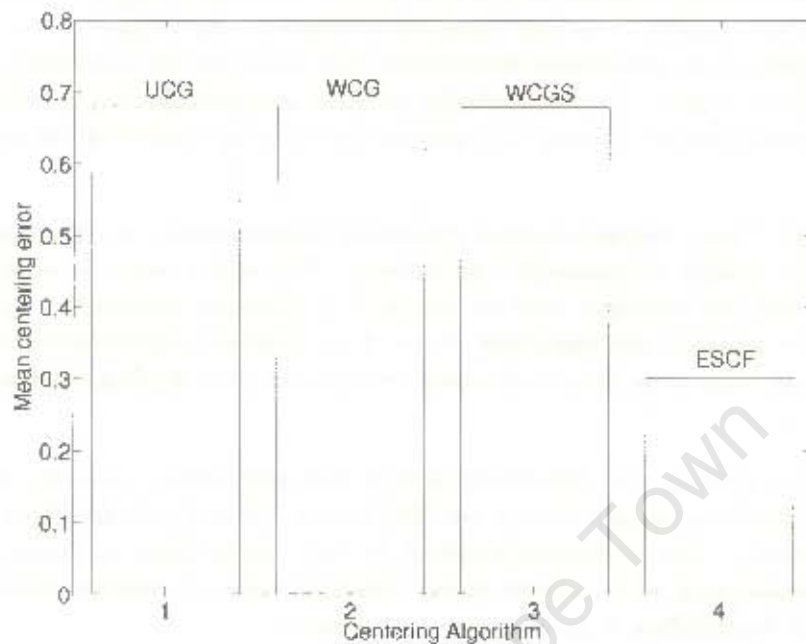


Figure 5.12: Mean resultant error of various target centering algorithms applied to real data

5.6 Conclusion

The testing of the centering algorithms for each of the feature extraction routines were conducted using synthetic and real data. The center of gravity based techniques are essentially equivalent, with the WCGS routine producing subpixel improvements on the WCG and UCG algorithms. The ESCF technique is, on average, twice as accurate as the other centering routines with an overall improvement of one pixel when tested against real data.

The relative robustness of the centering algorithms on the X-ray data were assessed by combining the resultant residuals of the feature extraction routines in an average sense. Figure (5.12) illustrates a histogram plot of the combination for each of the centering routines. The ESCF centering, though not directly applicable to area based, feature extraction routines, produces the best centering results.

The following discussion summarises the findings pertaining to the feature extraction routines.

1. The area based, percentage threshold method is very accurate with a priori knowledge of the percentage occupancy in image space. Unfortunately, such information is not readily available with real data, and even with good estimates, the results can be spurious due to noisy pixels of similar intensities occupying equivalent areas. The technique can be successfully implemented with distinct targets and relatively low noise levels, though such conditions are usually associated with trivial problems.

2. Optimal thresholding provides an efficient means of distinguishing a meaningful threshold when the choice is not obvious. The final threshold is however only as good as the initial estimate. The bimodal nature of the probability distribution that underlies the technique, limits its application to fairly distinct objects. The applicability to X-ray images depicting the typical penumbra effect has been demonstrated, making the optimal thresholding algorithm the preferred thresholding technique.
3. The two-fold Canny implementation produced improvements on the standard form, thereby affirming the validity of symmetric procedures. The improvement is computationally twice as expensive, and can therefore only be justified in problems demanding subpixel improvement. Such gains in accuracy are important to the X-ray tracking experiments, though the additional computational time is an issue that cannot be ignored when dealing with large amounts of data.

Based on the above findings, the feature extraction and subsequent centering algorithm selected for the processing of the X-ray images are the two-fold Canny method and the ellipse specific conic fitting algorithm respectively. The accuracies achieved for the control data are based on the standard, 11 parameter, implementation of the direct linear transformation. A further improvement in accuracy based on full DLT formulation is given in the next chapter.

Chapter 6

X-ray Photogrammetry

Most of photogrammetry can be based on a pair of equations known variously as the perspective equations, the imaging equations, the projective equations or the collinearity equations. The exterior orientation problem, as used in photogrammetry, is well described by the collinearity equations, equations (6.1 and 6.2). These equations relate the two-dimensional coordinates x and y of image points to the three dimensional coordinates ξ , η and τ of points in object space, figure (6.1). Of the various formulations of the analytical orientation problem, the Direct Linear Transformation, or DLT ([60], [59]), was found to best fit the requirements of the BAE. The projective transformation as referred to by Adams [60], was derived using the geometry of the X-ray beam and is used in the present work. Unlike the traditional bundle adjustment solution [58], this method does not require a calibrated camera, or fiducial markings on the scintillating screen, and can be solved without supplying initial approximations for the transformation parameters.

In section (6.1) we develop the projective transformation equations using homogeneous coordinates. The link to X-ray geometry is then used to develop a linear least squares solution for mapping X-ray image coordinates to object space. Section (6.2) deals with the conventional description of collinearity encountered in photogrammetry. This section allows the reader to explore an alternate solution scheme if needed, but more importantly it introduces some of the mathematical detail that facilitate the descriptions in subsequent sections. The conventional description uses nine linearly independent parameters to geometrically describe the image capturing mechanism. The link between the linearly dependent DLT parameters and the linearly independent camera parameters are elucidated here. A geometric interpretation of the X-ray imaging in terms of conventional cameras is also described.

The mathematical descriptions developed in section (6.2) facilitate the incorporation of optical errors (lens distortion effects) and random deviations into the DLT model which is the focus of section (6.3). A technique for modelling the variances of image and object space coordinates is illustrated in section (6.4).

Section (6.5), discusses the merits of the techniques developed in this thesis by comparing them to the standard solution schemes. In particular the proposed variance model is assessed according to accuracy and stability.

6.1 Projective transformations in X-ray photogrammetry

The projective transformation is given by:

$$\chi_i + \Delta\chi_i = \frac{b_{11}\xi_i + b_{12}\eta_i + b_{13}\tau_i + b_{14}}{b_{31}\xi_i + b_{32}\eta_i + b_{33}\tau_i + 1} \quad (6.1)$$

$$\gamma_i + \Delta\gamma_i = \frac{b_{21}\xi_i + b_{22}\eta_i + b_{23}\tau_i + b_{24}}{b_{31}\xi_i + b_{32}\eta_i + b_{33}\tau_i + 1} \quad (6.2)$$

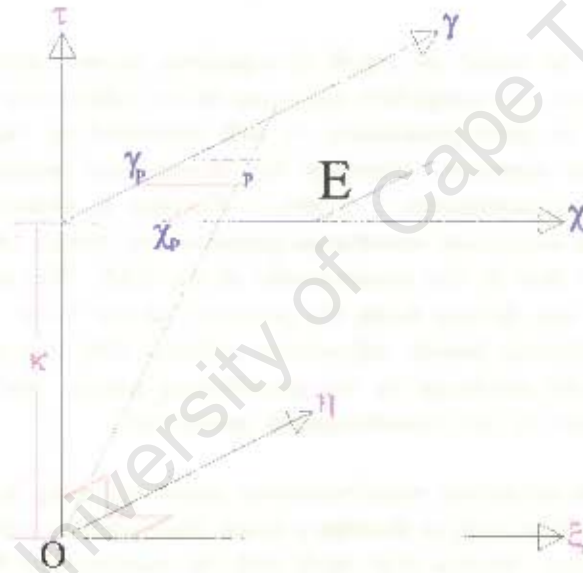


Figure 6.1: Homogeneous co-ordinates in two dimensional space

where χ_i, γ_i are the image coordinates of point i , $\Delta\chi_i, \Delta\gamma_i$ are the systematic errors of the image coordinates of point i (caused by lens distortion, etc.), ξ_i, η_i, τ_i are the object space coordinates of point i and the b_{ij} terms are the unknown transformation parameters from which the elements of interior and exterior orientation can be calculated. This linearly dependent equation set involves eleven transformation parameters instead of nine linearly independent camera parameters. An approximation lies in ignoring the linear dependence between parameters. However, the DLT solution does have some advantages in that it converges quickly even for poor provisional values and it appears to converge in spite of the high correlations, which would affect a bundle adjustment solution. Karara & Abdel-Aziz [59] point out "the linear components of lens distortion and film deformation are taken into account by the eleven transformation coefficients". This is particularly useful for an apparatus

A pair of observation equations can be written for each of the m cameras (in the case of the BAE, $m = 2$). The normal equation matrices $\mathbb{A}$, $\mathbb{X}$ and $\mathbb{B}$ are then given by

$$\mathbb{A} = \begin{bmatrix} \frac{u^{(1)}L_9^{(1)} - L_1^{(1)}}{R^{(1)}} & \frac{u^{(1)}L_{10}^{(1)} - L_2^{(1)}}{R^{(1)}} & \frac{u^{(1)}L_{11}^{(1)} - L_3^{(1)}}{R^{(1)}} \\ \frac{w^{(1)}L_9^{(1)} - L_5^{(1)}}{R^{(1)}} & \frac{w^{(1)}L_{10}^{(1)} - L_6^{(1)}}{R^{(1)}} & \frac{w^{(1)}L_{11}^{(1)} - L_7^{(1)}}{R^{(1)}} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \frac{u^{(m)}L_9^{(m)} - L_1^{(m)}}{R^{(m)}} & \frac{u^{(m)}L_{10}^{(m)} - L_2^{(m)}}{R^{(m)}} & \frac{u^{(m)}L_{11}^{(m)} - L_3^{(m)}}{R^{(m)}} \\ \frac{w^{(m)}L_9^{(m)} - L_5^{(m)}}{R^{(m)}} & \frac{w^{(m)}L_{10}^{(m)} - L_6^{(m)}}{R^{(m)}} & \frac{w^{(m)}L_{11}^{(m)} - L_7^{(m)}}{R^{(m)}} \end{bmatrix} \quad (6.123)$$

$$\mathbb{X} = [X \ Y \ Z]^t \quad (6.124)$$

$$\mathbb{B} = \left[\frac{L_4^{(1)} - u^{(1)}}{R^{(1)}} \quad \frac{L_8^{(1)} - w^{(1)}}{R^{(1)}} \quad \cdot \quad \cdot \quad \frac{L_4^{(m)} - u^{(m)}}{R^{(m)}} \quad \frac{L_8^{(m)} - w^{(m)}}{R^{(m)}} \right]^t \quad (6.125)$$

from which the least squares solution is

$$\mathbb{X} = (\mathbb{A}^t \mathbb{W}_r \mathbb{A})^{-1} (\mathbb{A}^t \mathbb{W}_r \mathbb{B}) \quad (6.126)$$

The following steps can be used in incorporating the reconstruction solution into a computer program

1. The first iteration is solved with the identity matrix used for the weighting. Note that the R terms cancel out for this condition.
2. From the second iteration on, compute the weight vectors according to section (6.3.1.2). Use the object space coordinates solved in the previous iteration to generate the weights and the R terms.
3. Repeat the procedure until the object space coordinates of new points have converged sufficiently. A convergence criteria can be set to achieve this.

a control volume. More details on control frame design is deferred to Appendix (B.2). On careful examination of the normal equations it becomes apparent that there is an implicit non-linearity - the coefficient matrix (design matrix) requires R which is a function of (L_9 to L_{11}). This system cannot be solved directly. An iterative scheme that solves (L_9 to L_{11}) in the previous iteration must be employed to calculate R in the current iteration. The following steps must be taken when incorporating the calibration solution into a computer program:

1. Solve the standard 11 parameter DLT equations using the identity matrix for the weighting. Note that columns 12 to 16 of the coefficient matrix, equation (6.116), and rows 12 to 16 of matrix $\mathbb{X}$, equation (6.117) vanish in this part of the solution. With this in place the R_i terms cancel out of the solution. This will be the first iteration.
2. From the second iteration on, use the previously computed parameters (from step 1) to solve for the R_i terms, the principal point coordinates (x_p, y_p), equations (6.87 and 6.88) and the variances, equations (6.99 and 6.100). Use these parameters to generate the normal matrices ($\mathbb{A}$, $\mathbb{X}$ and $\mathbb{B}$), equations (6.116, 6.117 and 6.118 respectively), and hence form the least squares solution given by equation (6.119).
3. Repeat the procedure until the 16 DLT parameters have converged sufficiently. This is normally achieved by setting a convergence criteria.
4. At this point we need to compute the variance-covariance matrix, equation (6.108), which will be needed in the reconstruction phase.

6.3.3 Reconstruction Stage using the SFDLT

This phase deals with the solution of new object space coordinates with the assumption that the 16 parameters are known and optimised. Again, a set of normal equations must be formed. Equations (6.87) and (6.88) can be rearranged for the $[X, Y, Z]$ to give (in matrix form)

$$\frac{1}{R} \begin{bmatrix} uL_9 - L_1 & uL_{10} - L_2 & uL_{11} - L_3 \\ wL_9 - L_5 & wL_{10} - L_6 & wL_{11} - L_7 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \frac{1}{R} \begin{bmatrix} L_4 - u \\ L_8 - w \end{bmatrix} \quad (6.120)$$

where

$$[u, w] = [x - \Delta x, y - \Delta y] \quad (6.121)$$

$$R = L_9X + L_{10}Y + L_{11}Z + 1 \quad (6.122)$$

and $(\Delta x, \Delta y)$ is given by equations (6.83 and 6.84) respectively.

Equation (6.115) can be generated for each of the n control points resulting in $2n$ observation equations. The normal equation matrices $\mathbf{A}$, $\mathbf{X}$ and $\mathbf{B}$ are then given by

$$\mathbf{A} = \begin{bmatrix} \frac{X_1}{R_1} & \frac{Y_1}{R_1} & \frac{Z_1}{R_1} & \frac{1}{R_1} & 0 & 0 & 0 & 0 & \frac{-x_1 X_1}{R_1} & \frac{-x_1 Y_1}{R_1} & \frac{-x_1 Z_1}{R_1} \\ 0 & 0 & 0 & 0 & \frac{X_1}{R_1} & \frac{Y_1}{R_1} & \frac{Z_1}{R_1} & \frac{1}{R_1} & \frac{-y_1 X_1}{R_1} & \frac{-y_1 Y_1}{R_1} & \frac{-y_1 Z_1}{R_1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{X_n}{R_n} & \frac{Y_n}{R_n} & \frac{Z_n}{R_n} & \frac{1}{R_n} & 0 & 0 & 0 & 0 & \frac{-x_n X_n}{R_n} & \frac{-x_n Y_n}{R_n} & \frac{-x_n Z_n}{R_n} \\ 0 & 0 & 0 & 0 & \frac{X_n}{R_n} & \frac{Y_n}{R_n} & \frac{Z_n}{R_n} & \frac{1}{R_n} & \frac{-y_n X_n}{R_n} & \frac{-y_n Y_n}{R_n} & \frac{-y_n Z_n}{R_n} \end{bmatrix}$$

$$\begin{bmatrix} \xi_1 r_1^2 & \xi_1 r_1^4 & \xi_1 r_1^6 & \xi_1 \eta & (r_1^2 + 2\xi_1^2) \\ \eta_1 r_1^2 & \eta_1 r_1^4 & \eta_1 r_1^6 & \eta_1 \xi_1 & (r_1^2 + 2\eta_1^2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \xi_n r_n^2 & \xi_n r_n^4 & \xi_n r_n^6 & \xi_n \eta_n & (r_n^2 + 2\xi_n^2) \\ \eta_n r_n^2 & \eta_n r_n^4 & \eta_n r_n^6 & \eta_n \xi_n & (r_n^2 + 2\eta_n^2) \end{bmatrix} \quad (6.116)$$

$$\mathbf{X} = [L_1 \quad L_2 \quad \cdot \quad \cdot \quad L_{15} \quad L_{16}]^t \quad (6.117)$$

$$\mathbf{B} = [\frac{x_1}{R_1} \quad \frac{y_1}{R_1} \quad \cdot \quad \cdot \quad \frac{x_n}{R_n} \quad \frac{y_n}{R_n}]^t \quad (6.118)$$

from which the least squares solution is

$$\begin{aligned} \mathbf{AX} &= \mathbf{B} \\ \mathbf{X} &= (\mathbf{A}^t \mathbf{W}_c \mathbf{A})^{-1} (\mathbf{A}^t \mathbf{W}_c \mathbf{B}) \end{aligned} \quad (6.119)$$

Implicit to this formulation is that the object space coordinates $[X, Y, Z]$ are the measured control points and are thus known. Again, the control points must not be coplanar - that is they must form

The weight matrix for the reconstruction is then formed as

$$\mathbb{W}_r = \begin{bmatrix} \frac{1}{\sigma^2(e_{x1})} & 0 & 0 & . & . & 0 & 0 \\ 0 & \frac{1}{\sigma^2(e_{y1})} & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & \frac{1}{\sigma^2(e_{xm})} & 0 \\ 0 & 0 & . & . & . & 0 & \frac{1}{\sigma^2(e_{ym})} \end{bmatrix} \quad (6.111)$$

where m represents the number of cameras. In the X-ray imaging context, $m = 2$.

6.3.2 Calibration Stage

The calibration phase seeks to optimise the 11 DLT parameters and the additional lens distortion terms. The normal equations for the calibration is obtained as follows:

Rearranging equations (6.87) and (6.88) yields

$$\frac{1}{R}x = \frac{1}{R}(L_1X + L_2Y + L_3Z + L_4 - L_9xX - L_{10}xY - L_{11}xZ) + \Delta x \quad (6.112)$$

$$\frac{1}{R}y = \frac{1}{R}(L_5X + L_6Y + L_7Z + L_8 - L_9yX - L_{10}yY - L_{11}yZ) + \Delta y \quad (6.113)$$

where

$$R = L_9X + L_{10}Y + L_{11}Z + 1 \quad (6.114)$$

In matrix form the two observation equations (for a single point) become

$$\frac{1}{R} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{R} \begin{bmatrix} X & Y & Z & 1 & 0 & 0 & 0 & 0 & -xX & -xY & -xZ \\ 0 & 0 & 0 & 0 & X & Y & Z & 1 & -yX & -yY & -yZ \end{bmatrix} \begin{bmatrix} L_1 \\ L_2 \\ . \\ . \\ L_{15} \\ L_{16} \end{bmatrix} \quad (6.115)$$

where

$$\begin{aligned} \mathbb{U} &= [U_1 \ U_2 \ . \ . \ . \ U_{10} \ U_{11}] \\ &= \left[\begin{array}{ccccccccccc} \frac{-X}{R} & \frac{-Y}{R} & \frac{-Z}{R} & \frac{-1}{R} & 0 & 0 & 0 & 0 & \frac{xX}{R} & \frac{yY}{R} & \frac{zZ}{R} \end{array} \right] \end{aligned} \quad (6.106)$$

and

$$\begin{aligned} \mathbb{V} &= [V_1 \ V_2 \ . \ . \ . \ V_{10} \ V_{11}] \\ &= \left[\begin{array}{ccccccccccc} 0 & 0 & 0 & 0 & \frac{-X}{R} & \frac{-Y}{R} & \frac{-Z}{R} & \frac{-1}{R} & \frac{yX}{R} & \frac{yY}{R} & \frac{yZ}{R} \end{array} \right] \end{aligned} \quad (6.107)$$

$\sigma(L_i, L_j)$ = variances and covariances between the 11 DLT parameters and is represented by

$$\sigma^2(L) = \begin{bmatrix} \sigma(L_1, L_1) & \sigma(L_1, L_2) & . & . & \sigma(L_1, L_{10}) & \sigma(L_1, L_{11}) \\ \sigma(L_1, L_1) & \sigma(L_1, L_2) & . & . & \sigma(L_1, L_{10}) & \sigma(L_1, L_{11}) \\ \sigma(L_2, L_1) & \sigma(L_2, L_2) & . & . & \sigma(L_2, L_{10}) & \sigma(L_2, L_{11}) \\ . & . & . & . & . & . \\ \sigma(L_{10}, L_1) & \sigma(L_{10}, L_2) & . & . & \sigma(L_{10}, L_{10}) & \sigma(L_{10}, L_{11}) \\ \sigma(L_{11}, L_1) & \sigma(L_{11}, L_2) & . & . & \sigma(L_{11}, L_{10}) & \sigma(L_{11}, L_{11}) \end{bmatrix} \quad (6.108)$$

and calculated as

$$\sigma^2(L) = MSE \cdot (\mathbf{A}^t \mathbf{W}_c \mathbf{A}) \quad (6.109)$$

where $\mathbf{A}$ is the $(2n \times 16)$ coefficient matrix of the calibration phase, equation (6.116), and $\mathbf{W}_c$ the weight matrix of the calibration stage, equation (6.101). The terms along the main diagonal of equation (6.108) are the variances and the other terms the covariances. MSE (a scalar quantity) is the mean square error from the least squares estimates and is computed as

$$MSE = \frac{\mathbf{B}^t \mathbf{W}_c \mathbf{B} - \mathbf{X}^t \mathbf{A}^t \mathbf{W}_c \mathbf{B}}{2n - 16} \quad (6.110)$$

$$\sigma^2(e_x) = \left(\frac{xL_9 - L_1}{R}\right) \sigma^2(X) + \left(\frac{xL_{10} - L_2}{R}\right) \sigma^2(Y) + \left(\frac{xL_{11} - L_3}{R}\right) \sigma^2(Z) + \sigma^2(x) \quad (6.99)$$

$$\sigma^2(e_y) = \left(\frac{yL_9 - L_5}{R}\right) \sigma^2(X) + \left(\frac{yL_{10} - L_6}{R}\right) \sigma^2(Y) + \left(\frac{yL_{11} - L_7}{R}\right) \sigma^2(Z) + \sigma^2(y) \quad (6.100)$$

with $[\sigma^2(x), \sigma^2(y)]$ = variances of image coordinates and $[\sigma^2(X), \sigma^2(Y), \sigma^2(Z)]$ = variances of the object space coordinates. The variances to the observations must be obtained independently and is described in sections (6.4 and 7.4). The weight matrix is then formed as the reciprocal of the equation variances.

$$W_c = \begin{bmatrix} \frac{1}{\sigma^2(e_{x_1})} & 0 & 0 & \dots & 0 & 0 \\ 0 & \frac{1}{\sigma^2(e_{y_1})} & \dots & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & \frac{1}{\sigma^2(e_{x_n})} & 0 \\ 0 & 0 & \dots & \dots & 0 & \frac{1}{\sigma^2(e_{y_n})} \end{bmatrix} \quad (6.101)$$

where n is the number of control points.

6.3.1.2 Weight matrix for the reconstruction stage using the SFDLT

Again, if we Taylor expand equations (6.92 and 6.93) with respect to the image coordinates of the new points and the 11 DLT parameters,

$$\partial e_x = -\frac{X}{R} \partial L_1 - \frac{Y}{R} \partial L_2 - \frac{Z}{R} \partial L_3 - \frac{1}{R} \partial L_4 + \frac{xX}{R} \partial L_9 + \frac{xY}{R} \partial L_{10} + \frac{xZ}{R} \partial L_{11} + \partial x \quad (6.102)$$

$$\partial e_y = -\frac{X}{R} \partial L_5 - \frac{Y}{R} \partial L_6 - \frac{Z}{R} \partial L_7 - \frac{1}{R} \partial L_8 + \frac{yX}{R} \partial L_9 + \frac{yY}{R} \partial L_{10} + \frac{yZ}{R} \partial L_{11} + \partial y \quad (6.103)$$

where (X, Y, Z) and R were kept constant in the Taylor expansion. The equation variances are then given by

$$\sigma^2(e_x) = \sum_{i=1}^{11} \sum_{j=1}^{11} U_i U_j \sigma(L_i, L_j) + \sigma^2(x) \quad (6.104)$$

$$\sigma^2(e_y) = \sum_{i=1}^{11} \sum_{j=1}^{11} V_i V_j \sigma(L_i, L_j) + \sigma^2(y) \quad (6.105)$$

We will now formulate the equation variances for both the calibration and reconstruction stages based on the standard implementation (SFDLT). The calibration stage requires variances to the image and object space coordinates of the control points while in the reconstruction stage, the variances to the image coordinates as well as the variances and covariances to the 11 DLT parameters are needed. A detailed description of the variance model for the observations is given in section (6.4).

The random errors of the observation equations, equations (6.81 and 6.82), can be expressed simply as:

$$e_x = x - \Delta x - \frac{L_1X + L_2Y + L_3Z + L_4}{L_9X + L_{10}Y + L_{11}Z + 1} \quad (6.92)$$

$$e_y = y - \Delta y - \frac{L_5X + L_6Y + L_7Z + L_8}{L_9X + L_{10}Y + L_{11}Z + 1} \quad (6.93)$$

6.3.1.1 Weight matrix for the calibration stage using the SFDLT

Rearranging equations (6.92 and 6.93):

$$e_x = \frac{1}{R} (-L_1X - L_2Y - L_3Z - L_4 + L_9xX + L_{10}xY + L_{11}xZ) - \Delta x + \frac{1}{R}x \quad (6.94)$$

$$e_y = \frac{1}{R} (-L_5X - L_6Y - L_7Z - L_8 + L_9yX + L_{10}yY + L_{11}yZ) - \Delta y + \frac{1}{R}y \quad (6.95)$$

The equation residuals can then be computed by applying a Taylor series expansion to first order.

$$\partial e_x = \left(\frac{xL_9 - L_1}{R} \right) \partial X + \left(\frac{xL_{10} - L_2}{R} \right) \partial Y + \left(\frac{xL_{11} - L_3}{R} \right) \partial Z + \partial x \quad (6.96)$$

$$\partial e_y = \left(\frac{yL_9 - L_5}{R} \right) \partial X + \left(\frac{yL_{10} - L_6}{R} \right) \partial Y + \left(\frac{yL_{11} - L_7}{R} \right) \partial Z + \partial y \quad (6.97)$$

where we assume L_1 to L_{11} and R are constants and that

$$\partial(\Delta x) = \partial(\Delta y) = 0 \quad (6.98)$$

The corresponding variances are

Parameters	Description of error
L_1 to L_{11}	Standard DLT parameters
L_{12} to L_{14}	3rd, 5th and 7th order optical distortions
L_{15} and L_{16}	de-centering distortion terms

Table 6.1: Description of the lens distortion terms in the DLT equations

(x_p, y_p) is the principle point $\mathbf{P}$ and is defined by the perpendicular intersection of a line from the X-ray source to the scintillation screen of the angiographic equipment, see figure (6.2).

$$x_p = \frac{L_1 L_9 + L_2 L_{10} + L_3 L_{11}}{L_9^2 + L_{10}^2 + L_{11}^2} \quad (6.87)$$

$$y_p = \frac{L_5 L_9 + L_6 L_{10} + L_7 L_{11}}{L_9^2 + L_{10}^2 + L_{11}^2} \quad (6.88)$$

The solution to the full DLT equations may be divided into two stages namely calibration and reconstruction. In both cases the DLT equations are arranged into a set of normal equations and an iterative least squares solution is implemented in the usual manner

$$\mathbf{A}\mathbf{X} = \mathbf{B} \quad (6.89)$$

$$\mathbf{W}\mathbf{A}\mathbf{X} = \mathbf{W}\mathbf{B} \quad (6.90)$$

$$\mathbf{X} = (\mathbf{A}^t \mathbf{W} \mathbf{A})^{-1} (\mathbf{A}^t \mathbf{W} \mathbf{B}) \quad (6.91)$$

where $\mathbf{A}$ is the design matrix, $\mathbf{W}$ the weight matrix to ensure a stable set of zeros and $\mathbf{X}$ is the column vector containing either the DLT parameters (calibration stage) or the object space coordinates of the tracked particle (reconstruction stage). It should be noted that the elements of the weight matrix are different for the calibration and reconstruction stages. In section (6.3.1) we formulate the weight matrices.

6.3.1 Determining the weighting factors to the observation equations

The weighting factors are associated with the variances of the image coordinates, object space coordinates and the 11 DLT parameters. These variances are integral to the stability of the solution and must be carefully modelled. Such stability can only be achieved if the modelling is based on physically correct principles. This implies that artificially constructed variances produce quickly diverging solutions for a linearly dependent system like the DLT. For this reason it is normally safer to assume uniform weighting if no other information regarding the observations are available.

Additionally, meaningful and accurate DEM verification demands that the highest quality experimental data be used for the comparisons. In the context of statistically valid data this translates to large amounts of accurate 3D coordinates of object space points. In this section we present the standard technique for incorporating the higher order terms into the full DLT solution. To facilitate the ensuing discussion the following acronyms will be used:

- SDLT: standard 11 parameter DLT solution (coordinates assumed to be error free)
 SFDLT: standard, full DLT solution (solution based on the standard implementation)
 VFDLT: variance model, full DLT solution (solution uses a physical model for the variances)

To facilitate the inclusion of the errors into the solution, the change of notation used in section (6.2.3) will be adopted.

$$x - \Delta x = \frac{L_1X + L_2Y + L_3Z + L_4}{L_9X + L_{10}Y + L_{11}Z + 1} \quad (6.81)$$

$$y - \Delta y = \frac{L_5X + L_6Y + L_7Z + L_8}{L_9X + L_{10}Y + L_{11}Z + 1} \quad (6.82)$$

where the L terms represent the standard 11 DLT parameters, $[x, y]$ the image coordinates of the observations, $[X, Y, Z]$ are the object coordinates of the observations and $[\Delta x, \Delta y]$ are the errors caused by lens distortion.

The full DLT solution has the optical errors expressed as:

$$\Delta x = \xi (L_{12}r^2 + L_{13}r^4 + L_{14}r^6) + L_{15}\xi\eta + L_{16}(r^2 + 2\xi^2) \quad (6.83)$$

$$\Delta y = \eta (L_{12}r^2 + L_{13}r^4 + L_{14}r^6) + L_{15}\xi\eta + L_{16}(r^2 + 2\eta^2) \quad (6.84)$$

where

$$[\xi, \eta] = [x - x_p, y - y_p] \quad (6.85)$$

$$r^2 = \xi^2 + \eta^2 \quad (6.86)$$

The five additional parameters are related to the optical and de-centering distortions (see table 6.1 for details)

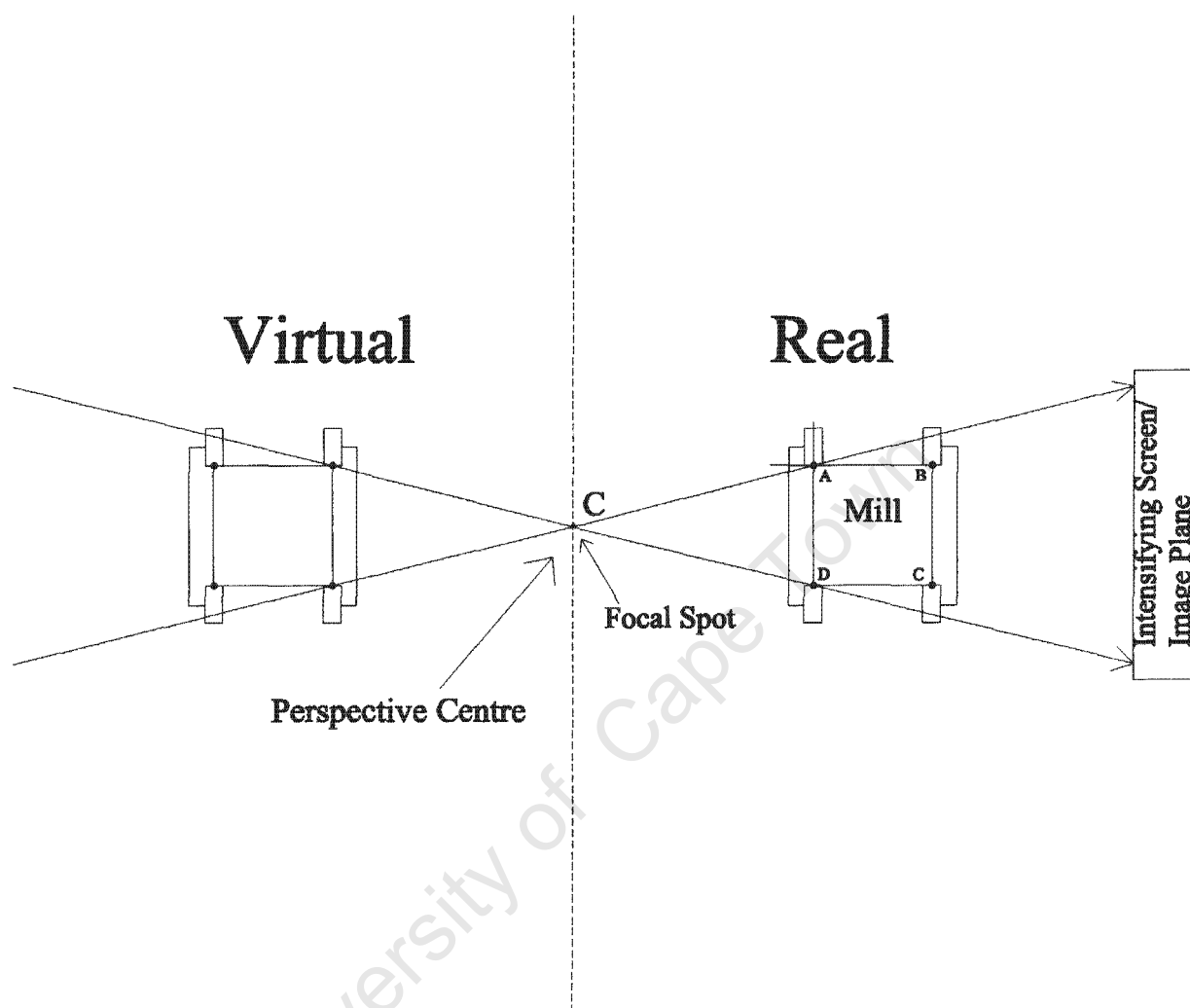


Figure 6.3: Interpreting the BAE configuration as a large camera with a virtual object

6.3 The full DLT solution: Incorporating optical errors and random deviations

The common approach employed for the DLT equations is to generate a set of normal equations and simply ignore the effects of lens distortion and random errors in the final solution (equations 6.24 and 6.32), as in section (6.1). The motivation for ignoring these errors is largely due to the difficulty associated with modelling these effects, especially when prior knowledge of camera calibration and coordinate variances are unavailable. In addition the improvements in accuracy are not always sufficient to justify the added effort and in some cases the solution may still diverge. The divergence typifies the instability of the parametrisation and is largely due to an unphysical characterisation of the de-centering and optical distortions.

$$D = \frac{-1}{\sqrt{(L_9^2 + L_{10}^2 + L_{11}^2)}} \quad (6.72)$$

$$x_p = (L_1 L_9 + L_2 L_{10} + L_3 L_{11}) D^2 \quad (6.73)$$

$$y_p = (L_5 L_9 + L_6 L_{10} + L_7 L_{11}) D^2 \quad (6.74)$$

$$c = \sqrt{(L_1^2 + L_2^2 + L_3^2 + L_5^2 + L_6^2 + L_7^2) D^2 - x_p^2 - y_p^2} \quad (6.75)$$

$$\varphi = \sin^{-1}(L_9 D) \quad (6.76)$$

$$\omega = \tan^{-1}\left(\frac{-L_{10}}{L_{11}}\right) \quad (6.77)$$

$$m_{11} = \frac{D(x_p L_9 - L_1)}{c} \quad (6.78)$$

$$\kappa = \frac{\cos^{-1}(m_{11})}{\cos \varphi} \quad (6.79)$$

$$\begin{bmatrix} X_c \\ Y_c \\ Z_c \end{bmatrix} = \begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix}^{-1} \begin{bmatrix} L_4 \\ L_5 \\ 1 \end{bmatrix} \quad (6.80)$$

6.2.4 Interpreting the BAE as a pinhole camera

Due to the proprietary nature of the BAE the actual camera mechanism, and hence the nine parameters, could not be directly investigated. However it is known that the scintillations produced by the intensifying screen is the first record of the 2D image data. In fact, the final digital images are simply scaled recordings of the intensifying screen³. The BAE can thus be compared to a pinhole camera, with the perspective centre C or pinhole at the focal spot (anode), the principal distance equal to the FFD (see chapter 3) of the X-ray beam and the objects being analysed are analogous to a virtual object, figure (6.3). With this analogy in place all the usual mathematical descriptions of photogrammetry, as applied to conventional cameras, can be applied to the BAE.

³We assume that the camera axis passes through the center of the scintillation screen

$$L_1 = \frac{(x_p m_{31} - c m_{11})}{D} \quad (6.60)$$

$$L_2 = \frac{(x_p m_{32} - c m_{12})}{D} \quad (6.61)$$

$$L_3 = \frac{(x_p m_{33} - c m_{13})}{D} \quad (6.62)$$

$$L_4 = x_p + \frac{c(m_{11}X_c + m_{12}Y_c + m_{13}Z_c)}{D} \quad (6.63)$$

$$L_5 = \frac{(y_p m_{31} - c m_{21})}{D} \quad (6.64)$$

$$L_6 = \frac{(y_p m_{32} - c m_{22})}{D} \quad (6.65)$$

$$L_7 = \frac{(y_p m_{33} - c m_{23})}{D} \quad (6.66)$$

$$L_8 = y_p + \frac{c(m_{21}X_c + m_{22}Y_c + m_{23}Z_c)}{D} \quad (6.67)$$

$$L_9 = \frac{m_{31}}{D} \quad (6.68)$$

$$L_{10} = \frac{m_{32}}{D} \quad (6.69)$$

$$L_{11} = \frac{m_{33}}{D} \quad (6.70)$$

$$D = -(m_{31}X_c + m_{32}Y_c + m_{33}Z_c) \quad (6.71)$$

In particular the camera parameters can be obtained from the DLT parameters, if desired, by algebraic solution of the DLT equations:

$$a = kRA \quad (6.54)$$

$$\begin{bmatrix} x_i - x_p \\ y_i - y_p \\ c \end{bmatrix} = kR \begin{bmatrix} X_i - X_c \\ Y_i - Y_c \\ Z_i - Z_c \end{bmatrix} \quad (6.55)$$

This produces three equations that describe the physical situation. Since the scale factor, k , is of no interest in itself, it can be eliminated by dividing the 1st and 2nd equations by the 3rd and rearranging to yield a functional form. Including distortion parameters, δx and δy , this gives the most commonly used form of the *collinearity equations*:

$$x_i - \delta x_i = x_p + c \frac{m_{11}(X_i - X_c) + m_{12}(Y_i - Y_c) + m_{13}(Z_i - Z_c)}{m_{31}(X_i - X_c) + m_{32}(Y_i - Y_c) + m_{33}(Z_i - Z_c)} \quad (6.56)$$

$$y_i - \delta y_i = y_p + c \frac{m_{21}(X_i - X_c) + m_{22}(Y_i - Y_c) + m_{23}(Z_i - Z_c)}{m_{31}(X_i - X_c) + m_{32}(Y_i - Y_c) + m_{33}(Z_i - Z_c)} \quad (6.57)$$

The collinearity equations can now be written for each image of each interest point, and the equations combined into a simultaneous solution for the unknown parameters and is commonly referred to as the *bundle adjustment*, after Brown [58]. The implicit non-linearity of these equations necessitate the use of Taylor expansion w.r.t. all parameters in order to obtain a linearised set of equations that can then be solved in an iterative least squares adjustment. A detailed description of this solution scheme is elegantly presented by Brown [58] and is therefore not included in this thesis.

6.2.3 Relating the camera parameters to the DLT parameters

One derivation of the DLT equations, Abdel-Aziz & Karara [59], is to begin with the collinearity equations. After some algebraic manipulation, the following equations are obtained²:

$$x - \Delta x = \frac{L_1 X + L_2 Y + L_3 Z + L_4}{L_9 X + L_{10} Y + L_{11} Z + 1} \quad (6.58)$$

$$y - \Delta y = \frac{L_5 X + L_6 Y + L_7 Z + L_8}{L_9 X + L_{10} Y + L_{11} Z + 1} \quad (6.59)$$

where

²These equations are identical to equations (6.1 and 6.2) except for the difference in notation

$$m_{11} = \cos \varphi \cos \kappa \quad (6.43)$$

$$m_{12} = \sin \omega \sin \varphi \cos \kappa + \cos \omega \sin \kappa \quad (6.44)$$

$$m_{13} = -\cos \omega \sin \varphi \cos \kappa + \sin \omega \cos \kappa \quad (6.45)$$

$$m_{21} = -\cos \varphi \sin \kappa \quad (6.46)$$

$$m_{22} = -\sin \omega \sin \varphi \sin \kappa + \cos \omega \cos \kappa \quad (6.47)$$

$$m_{23} = \cos \omega \sin \varphi \sin \kappa + \sin \omega \cos \kappa \quad (6.48)$$

$$m_{31} = \sin \varphi \quad (6.49)$$

$$m_{32} = -\sin \omega \cos \varphi \quad (6.50)$$

$$m_{33} = \cos \omega \cos \varphi \quad (6.51)$$

6.2.2 The Collinearity Equations

Having defined the exterior orientation of the image, by establishing the coordinate systems and the relationship between them, we can now describe the imaging process. The image vector, expressed in the image coordinate system, is

$$a = \begin{bmatrix} x_i - x_p \\ y_i - y_p \\ c \end{bmatrix} \quad (6.52)$$

where

x_i, y_i are the image coordinates of the point i , and x_p, y_p are defined in figure (6.2).

The vector from the perspective centre to the interest point, expressed in the object space coordinate system, is

$$A = \begin{bmatrix} X_i - X_c \\ Y_i - Y_c \\ Z_i - Z_c \end{bmatrix} \quad (6.53)$$

where

X_i, Y_i, Z_i are the object space coordinates of the interest point I , and

X_c, Y_c, Z_c are the object space coordinates of the perspective centre.

The imaging process requires that the image and object rays be collinear, thus we multiply the object space vector by the rotation matrix, R , to bring it into the same coordinate system. An unknown scale factor, k , is then included, giving

In figure (6.2), the camera is represented by two components, namely the perspective centre of the lens C , and the image plane, shown here as a photographic negative. An image coordinate system x and y of arbitrary origin and orientation is defined in the image plane. An intermediate system x', y', z' , is also defined with its origin at the perspective centre C , with the x', y' axes parallel to x, y , and the positive z' axis directed from the perspective centre C , towards the image plane. The z' axis is perpendicular to the image plane. The perpendicular distance between the image plane and the perspective centre C , is known as the principal distance or camera constant, and will be denoted here by the lower case letter, c . All image points have $z' = c$ as it is assumed they all lie on the image plane. The point P with image coordinates (x_p, y_p) , in figure (6.2), is the principal point and it is defined by the intersection of the z' axis and the image plane.

6.2.1 Exterior Orientation

"The term *exterior orientation* refers to the position of the camera (more precisely, the position of the front nodal point of the lens) when the image was taken, and to the angular relationship between the image and the object space coordinate system." [59]

The first step in determining exterior orientation, is to establish the coordinate system, see figure (6.2). The "*object space*" coordinate system is a 3D Cartesian coordinate system whose orientation is defined by the given coordinates. The "*image*" coordinate system is a 3D right-handed Cartesian system, with the x and y axes being in the image plane and the z -axis being towards the perspective centre. The angular relationship between the image and object space coordinate systems is described by a 3×3 orthogonal rotation matrix, the elements of which are the cosines of the angles between each pair of coordinate axes. One commonly used system is the definition of the orientation matrix by sequential rotations around the X , Y and Z axes by angles ω , φ and κ respectively, in that order. These matrices are multiplied together in the proper sequential order to give:

$$M = M_{\kappa} M_{\varphi} M_{\omega} \quad (6.41)$$

or

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \quad (6.42)$$

with

$$C_{21} = (b_{21} - \gamma.b_{31}) \quad (6.38)$$

$$C_{22} = (b_{22} - \gamma.b_{32}); \quad L_2 = \gamma - b_{24} \quad (6.39)$$

$$C_{23} = (b_{23} - \gamma.b_{33}) \quad (6.40)$$

The last two rows are identical in form except for the barred terms.

6.2 Collinearity in conventional photogrammetry

In this section we develop the collinearity equations from the perspective of the nine camera parameters. The relationship between the DLT equations and the camera parameters are also presented. The notational changes introduced in this section serve only to facilitate the discussion in section (6.3).

An alternate, and more traditional, formulation of the exterior orientation problem is to determine the collinearity relationship according to the nine camera parameters. The following discussion briefly describes the collinearity equations for the nine parameters and provides a geometric interpretation for relating X-ray space to conventional photogrammetric descriptions.

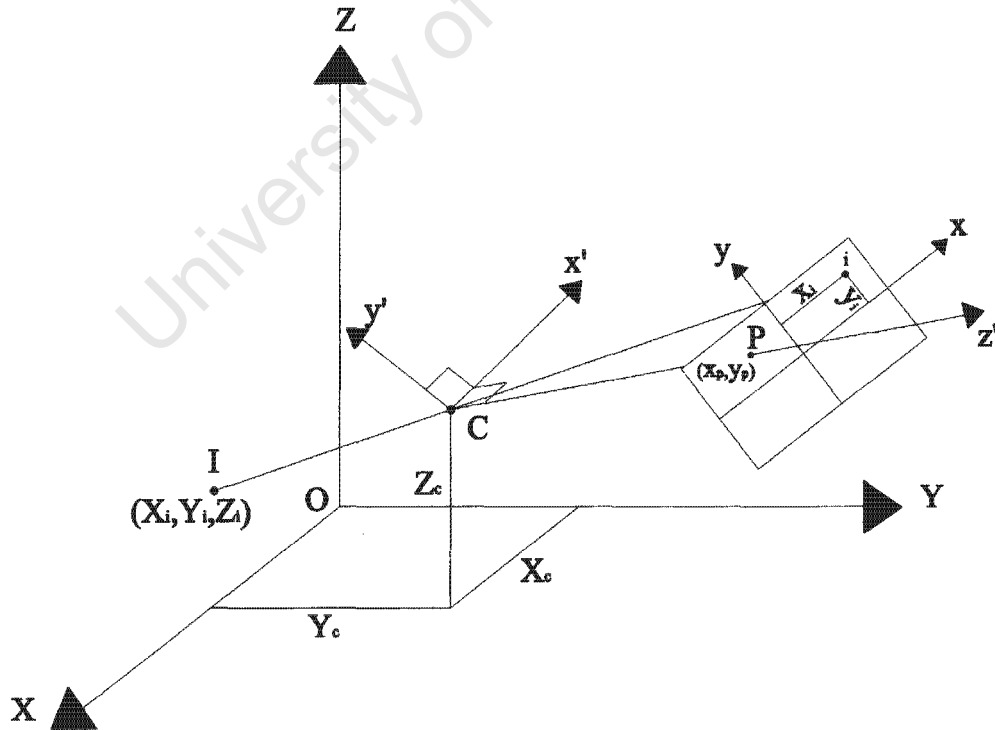


Figure 6.2: Geometric view of a conventional camera imaging system

If we substitute the image coordinates (obtained from planes A and B) of the new point into equations (6.22 and 6.23), we have four equations in three unknowns. This is sufficient to solve for the real space coordinates of the new point, in a least squares sense.

The four equations, after some rearranging of equations (6.22 and 6.23) are:

$$\chi = (b_{11} - \chi.b_{31})\xi + (b_{12} - \chi.b_{32})\eta + (b_{13} - \chi.b_{33})\tau + b_{14} \quad (6.28)$$

$$\gamma = (b_{21} - \gamma.b_{31})\xi + (b_{22} - \gamma.b_{32})\eta + (b_{23} - \gamma.b_{33})\tau + b_{24} \quad (6.29)$$

$$\bar{\chi} = (\bar{b}_{11} - \bar{\chi}.\bar{b}_{31})\xi + (\bar{b}_{12} - \bar{\chi}.\bar{b}_{32})\eta + (\bar{b}_{13} - \bar{\chi}.\bar{b}_{33})\tau + \bar{b}_{14} \quad (6.30)$$

$$\bar{\gamma} = (\bar{b}_{21} - \bar{\gamma}.\bar{b}_{31})\xi + (\bar{b}_{22} - \bar{\gamma}.\bar{b}_{32})\eta + (\bar{b}_{23} - \bar{\gamma}.\bar{b}_{33})\tau + \bar{b}_{24} \quad (6.31)$$

In matrix form we have:

$$\begin{pmatrix} \xi \\ \eta \\ \tau \end{pmatrix} = (\mathbf{C}^T \mathbf{C})^{-1} \mathbf{C}^T \mathbf{L} \quad (6.32)$$

where

$$\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \\ C_{41} & C_{42} & C_{43} \end{bmatrix} \quad (6.33)$$

$$\mathbf{L} = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{bmatrix} \quad (6.34)$$

The C -matrix is formed from the coefficients of equations (6.28, 6.29, 6.30 and 6.31), where

$$C_{11} = (b_{11} - \chi.b_{31}) \quad (6.35)$$

$$C_{12} = (b_{12} - \chi.b_{32}); \quad L_1 = \chi - b_{14} \quad (6.36)$$

$$C_{13} = (b_{13} - \chi.b_{33}) \quad (6.37)$$

$$\mathbf{A} = \begin{bmatrix}
 \xi_2 & \eta_2 & \tau_2 & 1 & 0 & 0 & 0 & 0 & -\chi_2\xi_2 & -\chi_2\eta_2 & -\chi_2\tau_2 \\
 \xi_3 & \eta_3 & \tau_3 & 1 & 0 & 0 & 0 & 0 & -\chi_3\xi_3 & -\chi_3\eta_3 & -\chi_3\tau_3 \\
 \xi_5 & \eta_5 & \tau_5 & 1 & 0 & 0 & 0 & 0 & -\chi_5\xi_5 & -\chi_5\eta_5 & -\chi_5\tau_5 \\
 \xi_6 & \eta_6 & \tau_6 & 1 & 0 & 0 & 0 & 0 & -\chi_6\xi_6 & -\chi_6\eta_6 & -\chi_6\tau_6 \\
 \xi_8 & \eta_8 & \tau_8 & 1 & 0 & 0 & 0 & 0 & -\chi_8\xi_8 & -\chi_8\eta_8 & -\chi_8\tau_8 \\
 \xi_9 & \eta_9 & \tau_9 & 1 & 0 & 0 & 0 & 0 & -\chi_9\xi_9 & -\chi_9\eta_9 & -\chi_9\tau_9 \\
 \xi_{11} & \eta_{11} & \tau_{11} & 1 & 0 & 0 & 0 & 0 & -\chi_{11}\xi_{11} & -\chi_{11}\eta_{11} & -\chi_{11}\tau_{11} \\
 \xi_{12} & \eta_{12} & \tau_{12} & 1 & 0 & 0 & 0 & 0 & -\chi_{12}\xi_{12} & -\chi_{12}\eta_{12} & -\chi_{12}\tau_{12} \\
 0 & 0 & 0 & 0 & \xi_2 & \eta_2 & \tau_2 & 1 & -\gamma_2\xi_2 & -\gamma_2\eta_2 & -\gamma_2\tau_2 \\
 0 & 0 & 0 & 0 & \xi_3 & \eta_3 & \tau_3 & 1 & -\gamma_3\xi_3 & -\gamma_3\eta_3 & -\gamma_3\tau_3 \\
 0 & 0 & 0 & 0 & \xi_5 & \eta_5 & \tau_5 & 1 & -\gamma_5\xi_5 & -\gamma_5\eta_5 & -\gamma_5\tau_5 \\
 0 & 0 & 0 & 0 & \xi_6 & \eta_6 & \tau_6 & 1 & -\gamma_6\xi_6 & -\gamma_6\eta_6 & -\gamma_6\tau_6 \\
 0 & 0 & 0 & 0 & \xi_8 & \eta_8 & \tau_8 & 1 & -\gamma_8\xi_8 & -\gamma_8\eta_8 & -\gamma_8\tau_8 \\
 0 & 0 & 0 & 0 & \xi_9 & \eta_9 & \tau_9 & 1 & -\gamma_9\xi_9 & -\gamma_9\eta_9 & -\gamma_9\tau_9 \\
 0 & 0 & 0 & 0 & \xi_{11} & \eta_{11} & \tau_{11} & 1 & -\gamma_{11}\xi_{11} & -\gamma_{11}\eta_{11} & -\gamma_{11}\tau_{11} \\
 0 & 0 & 0 & 0 & \xi_{12} & \eta_{12} & \tau_{12} & 1 & -\gamma_{12}\xi_{12} & -\gamma_{12}\eta_{12} & -\gamma_{12}\tau_{12}
 \end{bmatrix} \quad (6.25)$$

$$B = [b_{11} \ b_{12} \ b_{13} \ b_{14} \ b_{21} \ b_{22} \ b_{23} \ b_{24} \ b_{31} \ b_{32} \ b_{33}]^T \quad (6.26)$$

$$L = [\chi_2 \ \chi_3 \ \chi_5 \ \chi_6 \ \chi_8 \ \chi_9 \ \chi_{11} \ \chi_{12} \ \gamma_2 \ \gamma_3 \ \gamma_5 \ \gamma_6 \ \gamma_8 \ \gamma_9 \ \gamma_{11} \ \gamma_{12}]^T \quad (6.27)$$

Since the b_{ij} terms are averaged values, the more control points used the better the mean value, i.e. the closer we are to the true values. The above matrix equation incorporates a least squares fitting to the overconstrained system. The danger of using too many control points is that some of them might be outlying points. This results in a undue weighting to these points according to the least squares fit. To obviate this one should systematically omit certain points and check the accuracy and conditioning of the parameters. This allows you to identify the outlying points. On the other extreme, if you choose to use a minimum of six control points it is essential that:

1. No more than four control points be coplanar, and
2. No more than three images be colinear

Now since we have two pictures, from different view points, of the same object we can solve for the b_{ij} and $\bar{b}_{ij}$ terms, where the barred terms are for plane A images and the unbarred terms for plane B¹. Substituting these parameters back into equations (6.22 and 6.23), we can solve for the real space coordinates of a new point that has been imaged in both planes A and B. It should be noted that the parameters are only valid provided that the orientation of, and distance between the X-ray target and image intensifier are exactly the same for filming of the control frame and the new points.

¹The change of description from left and right image planes to planes A and B is used to coincide with the jargon of the BAE

where the χ, γ plane is parallel to the ξ', η' plane and τ' is arbitrary and constant.

Substituting the collinearity relations into equations (6.14, 6.15 and 6.16), we have:

$$\chi = \frac{a_{11}\xi + a_{12}\eta + a_{13}\tau + a_{14}}{a_{31}\xi + a_{32}\eta + a_{33}\tau + a_{34}} \quad (6.18)$$

$$\gamma = \frac{a_{21}\xi + a_{22}\eta + a_{23}\tau + a_{24}}{a_{31}\xi + a_{32}\eta + a_{33}\tau + a_{34}} \quad (6.19)$$

Eliminating one of the transformation parameters (a_{34}) by dividing all terms on the right by a_{34} and introducing new transformation parameters $b_{ij} = \frac{a_{ij}}{a_{34}}$ so that we may write:

$$\chi = \frac{b_{11}\xi + b_{12}\eta + b_{13}\tau + b_{14}}{b_{31}\xi + b_{32}\eta + b_{33}\tau + 1} \quad (6.20)$$

$$\gamma = \frac{b_{21}\xi + b_{22}\eta + b_{23}\tau + b_{24}}{b_{31}\xi + b_{32}\eta + b_{33}\tau + 1} \quad (6.21)$$

The need for the change of notation mentioned earlier is made clear in this step.

Rewriting equations (6.20 and 6.21) and gathering terms

$$\chi = b_{11}\xi + b_{12}\eta + b_{13}\tau + b_{14} - b_{31}\chi\xi - b_{32}\chi\eta - b_{33}\chi\tau \quad (6.22)$$

$$\gamma = b_{21}\xi + b_{22}\eta + b_{23}\tau + b_{24} - b_{31}\gamma\xi - b_{32}\gamma\eta - b_{33}\gamma\tau \quad (6.23)$$

In terms of the X-ray imaging as applied to the BAE, equations (6.22 and 6.23) is the relationship between the two dimensional coordinates (χ, γ) on the scintillating screen, and the three dimensional space coordinates (ξ, η, τ) of points on the line joining the focal spot to the image point. In addition, there are eleven unknown parameters. However, if we had sufficient (> 6) known space points and their corresponding image points this would provide us with enough equations to solve for the b_{ij} terms. Then for a given X-ray image with known (χ, γ) data we have three unknowns (ξ, η, τ) corresponding to a particular space point. But two images of the same space point provide us with sufficient information to uniquely locate it. Therefore we must first solve for the b_{ij} and $\bar{b}_{ij}$ parameters corresponding to the right and left X-ray images of the same space point at the given time.

One method to solving these equations is a least squares iterative solution. This method is common practice in photogrammetry where the space control points are assumed to be error free and the image points subject to errors of observation. A mathematical description of these errors is presented in section (6.3), however by assuming both co-ordinates to be error free a solution for the mean ' b_{ij} ' terms can be obtained in a least squares sense as follows:

$$\mathbf{B} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{L} \quad (6.24)$$

where

Equation (6.7) becomes:

$$\rho' \xi' = a_{11}\xi + a_{12}\eta + a_{13}\zeta + a_{14}\tau \quad (6.10)$$

$$\rho' \eta' = a_{21}\xi + a_{22}\eta + a_{23}\zeta + a_{24}\tau \quad (6.11)$$

$$\rho' \zeta' = a_{31}\xi + a_{32}\eta + a_{33}\zeta + a_{34}\tau \quad (6.12)$$

We may thus write (to within the arbitrary common factor ρ')

$$\begin{bmatrix} \xi' \\ \eta' \\ \zeta' \\ \tau' \end{bmatrix} = A \begin{bmatrix} \xi \\ \eta \\ \zeta \\ \tau \end{bmatrix} \quad (6.13)$$

where A is non-singular. (i.e. the determinant of A is non-zero). The case of A being singular is however of particular interest as it represents the mapping of space upon a plane, as in every central projection, eg., in photography and X-ray imaging. For this reason the general case, A is non-singular, will not be discussed further. In the central projection of the X-ray image a singular transformation implies that to distinct planes in space there corresponds one, and only one, plane, the picture plane. Such a transformation cannot be uniquely inverted for to every point in the picture plane, there corresponds an infinite number of space points.

With regard to the singular transformation, the practically useful approach to introducing homogeneous coordinates is as follows:

If ξ, η, τ are the rectangular coordinates of 3D space, mapped upon itself via the projective transformation equations (this is known as collineation), then the projective transformation equations, now called the collinearity equations, may be written as:

$$\xi' = \frac{a_{11}\xi + a_{12}\eta + a_{13}\tau + a_{14}}{a_{41}\xi + a_{42}\eta + a_{43}\tau + a_{44}} \quad (6.14)$$

$$\eta' = \frac{a_{21}\xi + a_{22}\eta + a_{23}\tau + a_{24}}{a_{41}\xi + a_{42}\eta + a_{43}\tau + a_{44}} \quad (6.15)$$

$$\tau' = \frac{a_{31}\xi + a_{32}\eta + a_{33}\tau + a_{34}}{a_{41}\xi + a_{42}\eta + a_{43}\tau + a_{44}} \quad (6.16)$$

Thus for the two dimensional coordinates χ, γ say and using homogeneous coordinates ξ', η', τ' we have

$$\chi = \frac{\xi'}{\tau'} \quad \gamma = \frac{\eta'}{\tau'} \quad (6.17)$$

Consider now a two dimensional space - a plane E say, figure (6.1). In this plane, point p has coordinates (χ, γ) . We now interpret the homogeneous coordinates ξ, η, τ as rectangular coordinates in space and, in this space, we arbitrarily choose the plane $\tau = \kappa$ parallel to the ξ, η plane, as the plane E. In this plane E, $\chi = \xi, \gamma = \eta$. Joining O, the origin of the rectangular coordinates, to (χ, γ) of plane E by a straight line, we find that $\frac{\xi}{\tau}$ and $\frac{\eta}{\tau}$ are constant. Now since κ is arbitrary, setting $\kappa = 1$, we may write

$$\frac{\xi}{\tau} = \chi \quad \frac{\eta}{\tau} = \gamma \quad (6.6)$$

since for $\tau = 1$, $\xi = \chi, \eta = \gamma$. Accordingly, the introduction of homogeneous coordinates signifies simply the representation of the plane E into that space cone of rays with the origin O as center, of which E is a section. That is, the homogeneous coordinates of a point are the space coordinates of the points of the projecting ray of that point. With this analogy, the indefiniteness of the homogeneous coordinates is made clear. The exclusion of the system of values $\xi = \eta = \tau = 0$ has its geometric basis in the fact that the point O determines no ray, and hence no point at plane E. Accordingly the condition on κ is that it is arbitrary, and $\kappa \neq 0$. Although this analogy has a physical interpretation for two dimensional space, it is by no means restricted to this space. We can easily extend this to three dimensional space where we think of space as a section of $\tau = 1$ of a four dimensional auxiliary space and we relate it to the space cone which projects it from the origin of the auxiliary space. In this, the use of four dimensional space is only a convenient means of expression. This example used to illustrate homogeneous coordinates has obvious similarities to X-ray imaging where plane E is the X-ray picture plate (or scintillating screen of the BAE), O is the focal spot, and κ is the focal length.

If we now introduce homogeneous coordinates into the projective transformation equations, equation (6.5) may be written as:

$$\begin{aligned} \frac{\xi'}{\tau'} &= \frac{a_{11}(\frac{\xi}{\tau}) + a_{12}(\frac{\eta}{\tau}) + a_{13}(\frac{\zeta}{\tau}) + a_{14}}{a_{41}(\frac{\xi}{\tau}) + a_{42}(\frac{\eta}{\tau}) + a_{43}(\frac{\zeta}{\tau}) + a_{44}} \\ \frac{\eta'}{\tau'} &= \frac{a_{21}(\frac{\xi}{\tau}) + a_{22}(\frac{\eta}{\tau}) + a_{23}(\frac{\zeta}{\tau}) + a_{24}}{a_{41}(\frac{\xi}{\tau}) + a_{42}(\frac{\eta}{\tau}) + a_{43}(\frac{\zeta}{\tau}) + a_{44}} \\ \frac{\zeta'}{\tau'} &= \frac{a_{31}(\frac{\xi}{\tau}) + a_{32}(\frac{\eta}{\tau}) + a_{33}(\frac{\zeta}{\tau}) + a_{34}}{a_{41}(\frac{\xi}{\tau}) + a_{42}(\frac{\eta}{\tau}) + a_{43}(\frac{\zeta}{\tau}) + a_{44}} \end{aligned} \quad (6.7)$$

Setting

$$\rho' = \frac{d\tau}{\tau'} \quad \rho' > 0 \quad (6.8)$$

where

$$d = a_{41}(\frac{\xi}{\tau}) + a_{42}(\frac{\eta}{\tau}) + a_{43}(\frac{\zeta}{\tau}) + a_{44} \quad (6.9)$$

that precludes any monitoring of the filming equipment, as is the case with the BAE. In this section we derive the 11 parameter collinearity solution according to Klein [61], and include the application, as proposed by Adams [60], to X-ray photogrammetry.

According to Klein [61] the projective transformation is the general case of the Affine Transformation, and may be stated as:

$$\begin{aligned} x' &= \frac{a_1x + b_1y + c_1z + d_1}{a_4x + b_4y + c_4z + d_4} \\ y' &= \frac{a_2x + b_2y + c_2z + d_2}{a_4x + b_4y + c_4z + d_4} \\ z' &= \frac{a_3x + b_3y + c_3z + d_3}{a_4x + b_4y + c_4z + d_4} \end{aligned} \quad (6.3)$$

That is:

To every point x, y, z there corresponds, accordingly, a definite finite point x', y', z' , provided only that the common denominator is not zero.

In the treatment of problems arising from a projection it is very convenient to use *homogeneous coordinates*, i.e., in place of the three point coordinates x, y, z , to use four quantities ξ, η, ζ, τ , defined by the equations

$$x = \frac{\xi}{\tau}, \quad y = \frac{\eta}{\tau}, \quad z = \frac{\zeta}{\tau} \quad (6.4)$$

These four quantities are to vary independently of each other, but not all four are to vanish simultaneously, and none of them is to become infinite. We thus have a one-to-many mapping, i.e., To every point x, y, z there will then correspond infinitely many systems of values $\rho\xi, \rho\eta, \rho\zeta, \rho\tau$, where ρ is an arbitrary factor, $\rho \neq 0$. The value of homogeneous coordinates is that it makes complicated relations in space of n -dimensions appear simpler and easier to understand, through auxiliary considerations in space of $n + 1$ dimensions.

The particular value of the homogeneous coordinates is in projecting space onto a plane. To facilitate the discussion that ensues, a slight change of notation to the transformation equations, that will become clearer as the discussion develops, is needed. The projective transformation equations can thus be written, without any loss of generality, as follows:

$$\begin{aligned} x' &= \frac{a_{11}x + a_{12}y + a_{13}z + a_{14}}{a_{41}x + a_{42}y + a_{43}z + a_{44}} \\ y' &= \frac{a_{21}x + a_{22}y + a_{23}z + a_{24}}{a_{41}x + a_{42}y + a_{43}z + a_{44}} \\ z' &= \frac{a_{31}x + a_{32}y + a_{33}z + a_{34}}{a_{41}x + a_{42}y + a_{43}z + a_{44}} \end{aligned} \quad (6.5)$$

6.4 Modelling the variances to the observations

In most photogrammetric applications, the variances to image coordinates is obtained by repeating the digitisation process. Similarly the variances to the object space coordinates of the control frame are computed from repeated measurement. Implicit to these practices is that only static objects can have physically meaningful (measured) variances. The approach taken in section (6.3) was based on the assumption that the variance of new points in image and object space cannot be determined without direct measurement, and more so if the new objects are moving. This assumption is largely responsible for the linearly dependent formulation used in determining the variances in the reconstruction phase of the DLT solution. Such systems are typically unstable and very often lead to quickly diverging solutions. These problems can be tempered with robust solvers like the Levenberg-Marquardt method [63, 96] or Singular Value Decomposition [96]. However, such schemes do not address the problem directly and in many cases the solution obtained is still not much better than the 11 parameter DLT solution.

In this section we describe a technique for modelling the variances of new image and object space coordinates. The aim is to circumvent the variance calculation in section (6.3.1.2) with a physical model.

There are a number of techniques that can be used to estimate the variance at a point given the variances of nearby sample points. Some of the common techniques are natural neighbour interpolation (NNI), polynomial interpolation, spline interpolation and interpolation within Delaunay simplexes (IDS). An implementation of IDS and NNI employing Voronoi cells are given here. Both methods employ a weighted average technique for computing the variance of a new point from nearby sampled data.

6.4.1 Triangulation

Triangulation involves creating from the input sample points a set of triangularly bounded facets that do not overlap. Of the many triangulation algorithms available, the more popular ones are the radial sweep method and Watson's algorithm which implement Delaunay triangulation.

Delaunay triangulation (tessellation in n -dimensional space) produces a set of triangles (simplexes in n -dimensional space) such that none of the sample points are contained in any of the triangles circumcircles (tetrahedrons circumspheres in 3D). Figure (6.4) illustrates the triangulation by the thick lines with the circles indicating the sampled data (X-ray image coordinates of the control points). Delaunay triangulation is geometrically related to the Voronoi tessellation (also known as Dirichlet or Thiessen tessellations). These tessellations split the plane (formed by the sample points) into a number of polygonal regions called tiles. Each tile has one sample point (node) in its interior called a generating point. Figure (6.4) illustrates the Voronoi tiles as thin lined polygons. All other points inside the polygonal tile are closer to the tile's generating point than any other generating point. In other words, each Voronoi cell defines a locus of points that are closer to its generating point than any other node. The Delaunay triangulation is created by connecting all generating points that share a common tile edge. Thus formed, the triangle edges (sides) are perpendicular bisectors

of the tile edges (sides). The vertices to the Voronoi tiles are the circumcenters to the Delaunay triangles.

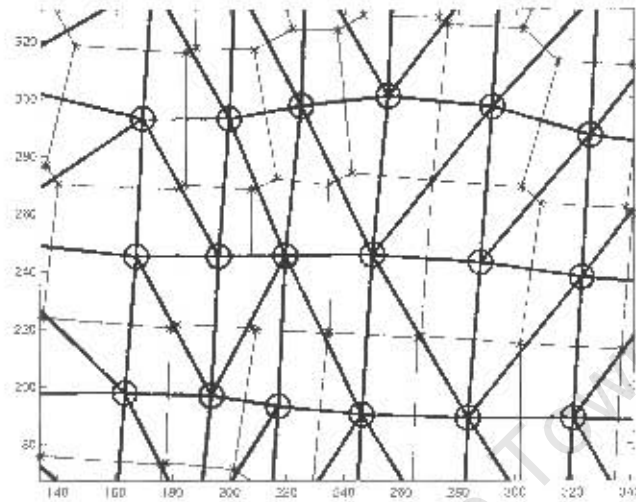


Figure 6.4: Tessellation of X-ray image coordinates: Delaunay triangles (thick lines) with circles at the vertices and Voronoi tiles (thin lines)

In the above description we considered two-dimensional Euclidean space $\mathbb{R}^2$. The theory, however, is applicable in a general k - dimensional framework. In this multidimensional framework the Delaunay tessellation is a set of simplexes such that no data points are contained in any simplex's circumsphere. In three-dimensional space a simplex is a tetrahedron. Figure (6.5) illustrates the 3D Delaunay tessellation for the measured control point data.

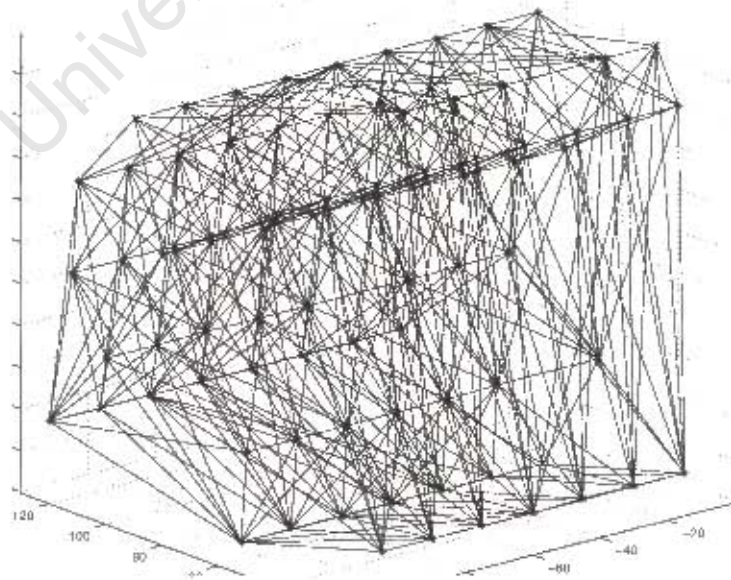


Figure 6.5: 3D Delaunay tessellation of the measured control points

6.4.2 Natural Neighbour Interpolation employing Voronoi cells and simplex integration

The natural neighbours of any sample point (node) are those in the neighbouring Voronoi polygons, or equivalently, those to which the node is connected by the sides of the Delaunay triangles. For example in figure (6.6) all the nodes are indicated by the circles. In particular the natural neighbours of node 1 (denoted by the number 1) are p, q, r, s, t and u. It is also important to note that the natural neighbours are a function of position and can vary depending on the density of the surrounding nodes.

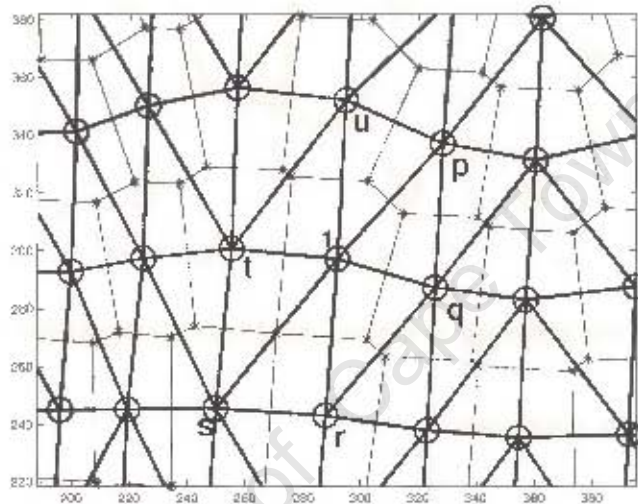


Figure 6.6: Illustrating the natural neighbours to a given node with Voronoi cells and Delaunay triangles

Using the above description we can formulate a technique for determining the properties of a new sample point - not from the original sample - using the information contained at its natural neighbours.

Let X be the new sample point (illustrated by the diamond shape) introduced into the Voronoi diagram of the control frame image coordinates, figure (6.7). The figure shown is a zoomed-in image of 14 control points (denoted by the circles) with Voronoi polygons drawn in (shown as thin lined polygons). The inclusion of point X results in a localised restructuring of the Voronoi cells. However, the only cells impacted by this restructuring are the cells containing:

- the natural neighbours of the new point; points 1, 2, 4, 5 and 6 in figure (6.7), and
- the new point (this cell also contains point 3)

The introduction of the new point X and creation of a new cell (denoted by the thick lined polygon) is confined locally to six cells. It should be noted that once the Voronoi diagram is redrawn to include

the new point, the thin lines forming the six secondary polygons within the new polygon do not exist, but are used to facilitate the calculation of the weighting factors used in the interpolation.

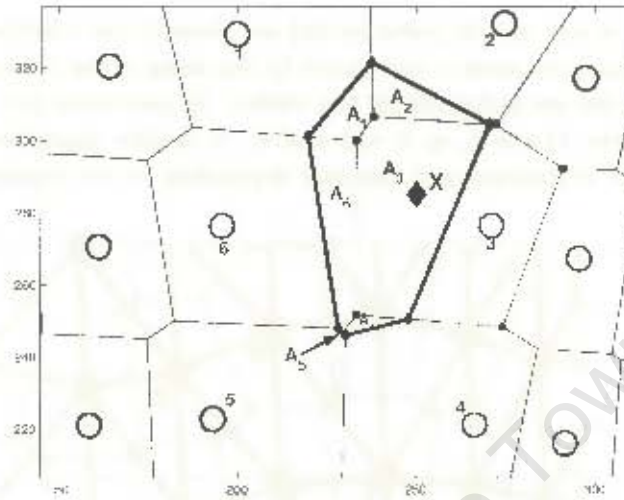


Figure 6.7: Illustrating the principle of natural neighbour interpolation for X-ray image coordinates

The variance at X is then calculated as:

$$\begin{aligned}\sigma(\mathbf{x}) &= \sum \phi_i(\mathbf{x})\sigma_i \\ \phi_i(\mathbf{x}) &= \frac{A_i}{A_T}, \quad i = 1, 2, \dots, 6 \\ A_T &= \sum A_i\end{aligned}\tag{6.127}$$

where σ_i is the variance at node i , A_i is the area of the secondary polygon formed in the cell containing node i and the $\phi_i(\mathbf{x})$ are the natural neighbour coordinates (weights associated with each node).

In general the enclosing polygon of some new point $X(\mathbf{x})$ can overlap with n Voronoi cells resulting in an interpolated variance of

$$\sigma(\mathbf{x}) = \sum_{i=1}^n \phi_i(\mathbf{x})\sigma_i\tag{6.128}$$

Using this definition, the following properties are self evident:

$$\sum_{i=1}^n \phi_i(\mathbf{x}) = 1\tag{6.129}$$

$$0 \leq \phi_i(\mathbf{x}) \leq 1\tag{6.130}$$

Now if X were to coincide with, say node 3 for instance, then it is readily seen that

$$\begin{aligned}\phi_3(\mathbf{x}) &= 1 & \text{and} \\ \phi_i &= 0, & i \neq 3\end{aligned}\tag{6.131}$$

Therefore the natural neighbour coordinates satisfy the Kronecker-delta property

$$\phi_i(\mathbf{x}_j) = \delta_{ij}\tag{6.132}$$

The areas forming the weighting factors are computed using the 2D simplex integration method and given as:

$$A = \frac{1}{2} \sum_{i=1}^n (x_i y_{i+1} - x_{i+1} y_i)\tag{6.133}$$

where (x_i, y_i) are the coordinates to vertex i of an n -sided polygon.

The extension to 3D is completely analogous to 2D with the areas replaced by volumes. The polygonal volumes are computed in a two stage process: First the surfaces bounding the Voronoi volumes are split into triangular facets using the Qhull routines from the Geometry Center, University of Minnesota (<http://www.geom.umn.edu/software>). Figure (6.8) illustrates the voronoi cell of control point No. 2 after the faceting process.

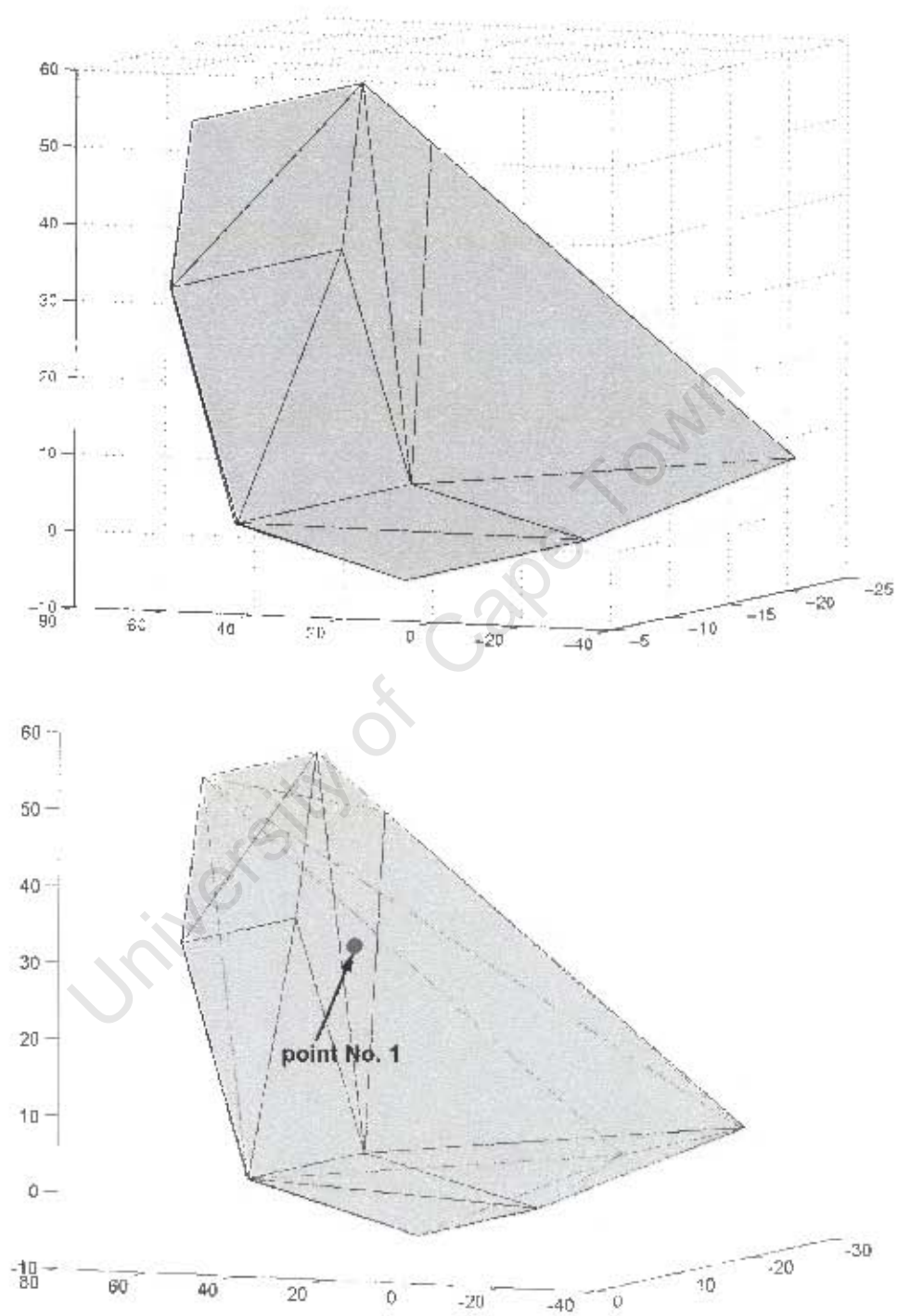


Figure 6.8: 3D Voronoi cell for control point number 2 used in X-ray space parametrisation

The volumes are then calculated using 3D simplex integration between each triangle and the generating point.

$$V_i = \frac{1}{6} \det \begin{vmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{vmatrix} \quad (6.134)$$

$$V = \sum_{i=1}^n V_i \quad (6.135)$$

where V_i is the volume of the i^{th} simplex and n is the number of simplexes constituting the Voronoi volume.

6.4.3 Interpolation using Delaunay tessellation and simplex integration

Let X be the new sample point introduced into the Delaunay tessellation diagram, figure (6.9), splitting the enclosing Delaunay simplex into three secondary triangles (A_1 , A_2 and A_3) in 2D and four secondary tetrahedrons (formed between each of the four faces of the primary tetrahedrons and point X) in 3D. Using the known variances at the nodes of the enclosing simplex we can apply an interpolation scheme based on the natural neighbours. The variance at sample point X is then computed as:

2D interpolation using Delaunay triangulation	3D interpolation using Delaunay tessellation
$\sigma(\mathbf{x}) = \sum_{i,j} \phi_i(\mathbf{x}) \sigma_j \quad i, j = 1, 2, 3 \quad (i \neq j)$	$\sigma(\mathbf{x}) = \sum_{i,j} \phi_i(\mathbf{x}) \sigma_j \quad i, j = 1, 2, 3, 4 \quad (i \neq j)$
$\phi_i(\mathbf{x}) = \frac{A_i}{A_T}$	$\phi_i(\mathbf{x}) = \frac{V_i}{V_T}$
$A_T = \sum A$	$V_T = \sum V_i$

where i, j form a permuted cyclic pair, σ_j is the variance at node j , the A_i are the areas of the secondary triangles, the V_i are the volumes of the secondary simplexes and $\phi_i(\mathbf{x})$ is the weight associated with node j . Again if X were to coincide with, say node 3 for instance, then it is readily seen that

$$\begin{aligned} \phi_3(\mathbf{x}) &= 1 & \text{and} \\ \phi_i &= 0, & i \neq 3 \end{aligned} \quad (6.136)$$

Therefore the coordinates of the connecting nodes satisfy the Kroneka-delta property

$$\phi_i(\mathbf{x}_k) = \delta_{ik} \quad (6.137)$$

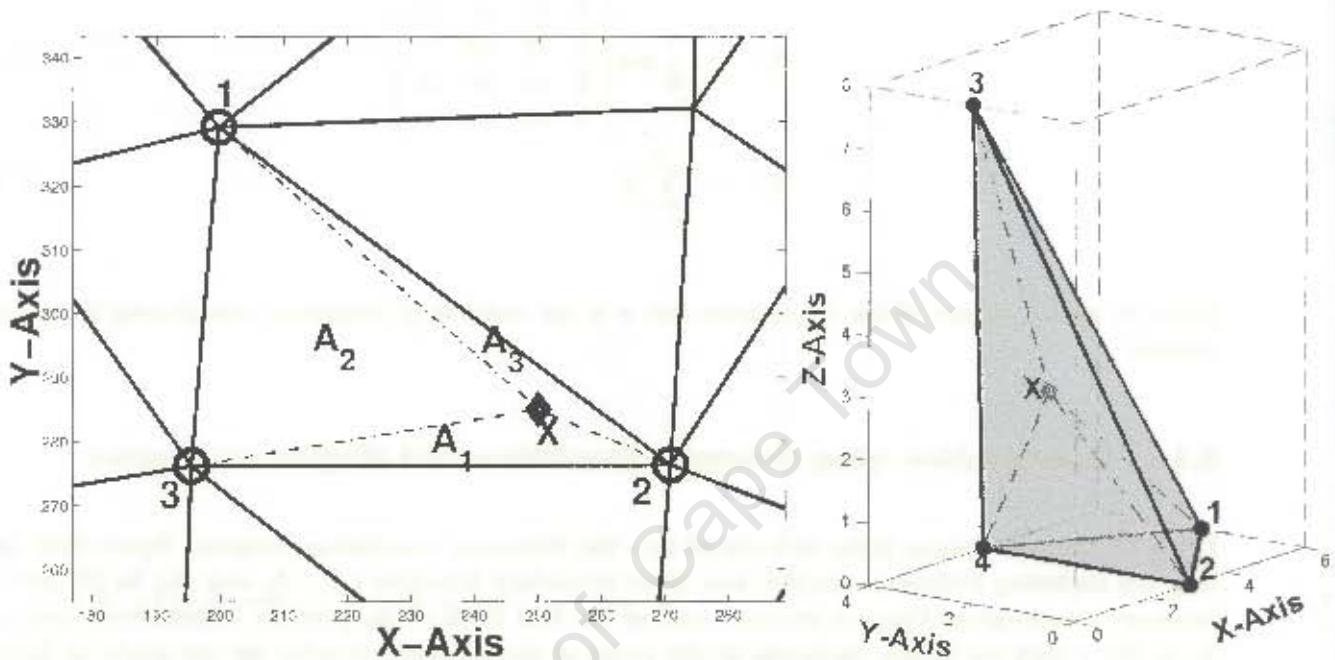


Figure 6.9: Delaunay tessellation

6.4.4 Discussion on interpolation schemes

Despite the similarities Delaunay schemes are often preferred over its dual structures the Voronoi cells as Voronoi cells tend to be unbounded along the periphery of the data set while Delaunay structures always produce simplexes. However in the context of photogrammetry, a well parameterised object space will ensure that the interpolation occurs well within the boundaries of the control space thus obviating the need of having to deal with unbounded Voronoi cells. In the present work the tracked particle does occasionally venture into the unbounded Voronoi cells of the X-ray images. Depending on the position of the tracked particle, two possible scenarios could arise as illustrated in figure (6.10) (The thick lines indicate the secondary Voronoi cell). Figure (6.10a) produces a bounded secondary cell which is interpolated in the usual manner, while figure (6.10b) fails due to the unbounded secondary Voronoi cell. In the latter case the variance for the new point is simply the average of the nearest neighbour variances.

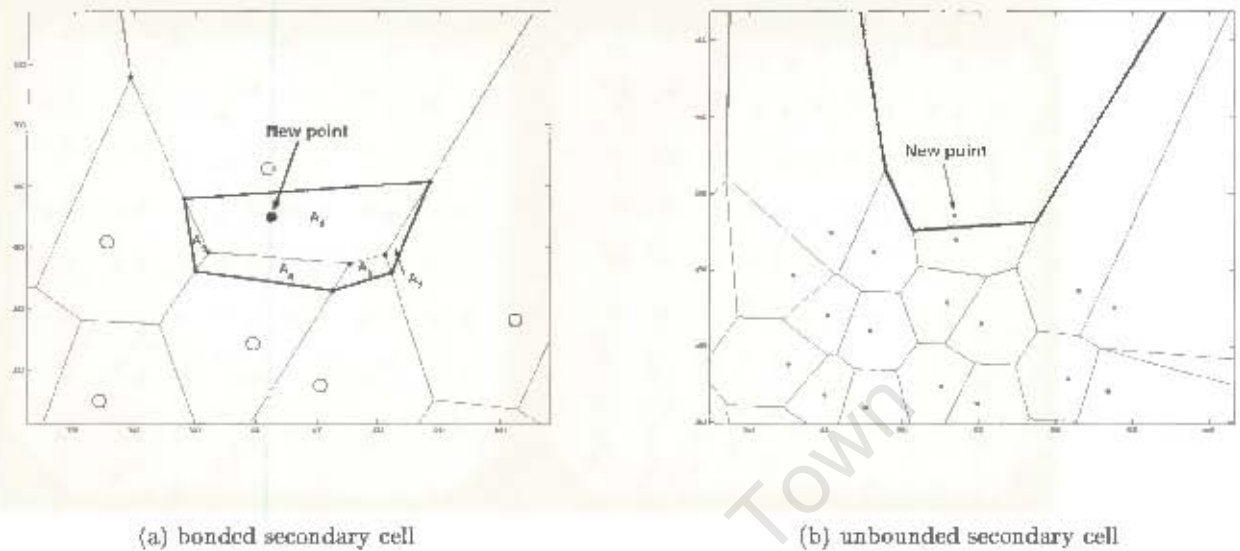


Figure 6.10: Tracked particles position introduced into an unbounded Voronoi cell

6.5 Evaluation of variance model

The variance model was tested against the standard implementations of the full DLT solution, section (6.3), and the eleven parameter DLT solution according to the following criteria:

1. Accuracy
2. Stability

The data used to evaluate the variance model comprise image and object space coordinates of a control frame. Figures (6.11 and 6.12) illustrates a typical X-ray images of the control frame. A total of 99 control points (frame comprised 110 points) were simultaneously discernable in both orthogonal planes. The 3D coordinates of entire control frame, with standard deviations, are given in Appendix (B.5). The image coordinates of the test data were determined using the dynamic template matching scheme coupled with the two-fold Canny convolution for the provisional edge data, and the ellipse specific conic fitting routine (ESCF) with the additional metrics for the final centering (see sections (4.4 and 5.3.3) for details of the image processing and target centering respectively). The 3D object coordinates of the control frame were obtained with a coordinate measuring machine (CMM), the details of which are described in Appendix (B.5).

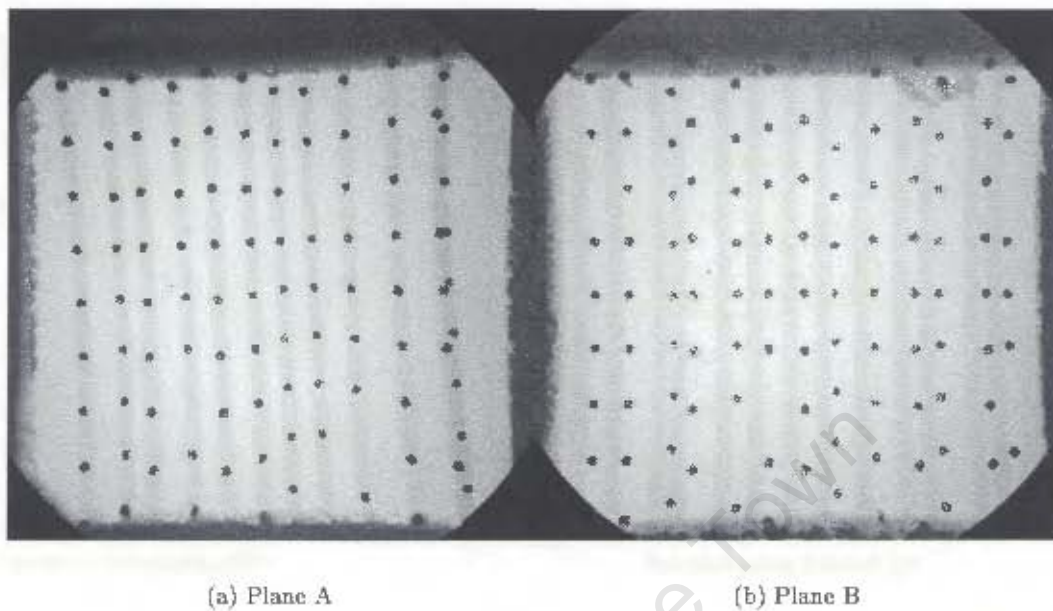


Figure 6.11: X-ray images of control frame used to evaluate variance model



Figure 6.12: Control frame in the fully inter-linked position

6.5.1 Accuracy

The standard method for evaluating accuracy is to estimate the errors of calibration and reconstruction using the image and object space coordinates of a control frame. The accuracy of the calibration process for camera planes A and B of the X-ray system was obtained as follows:

$$\varepsilon_{x_i} = x_i + \Delta x_i - \frac{L_1 X_i + L_2 Y_i + L_3 Z_i + L_4}{L_9 X_i + L_{10} Y_i + L_{11} Z_i + 1} \quad (6.138)$$

$$\varepsilon_{y_i} = y_i + \Delta y_i - \frac{L_5 X_i + L_6 Y_i + L_7 Z_i + L_8}{L_9 X_i + L_{10} Y_i + L_{11} Z_i + 1} \quad (6.139)$$

$$\varepsilon_x = \frac{1}{n} \sum_{i=1}^n \sqrt{\varepsilon_{x_i}^2} \quad (6.140)$$

$$\varepsilon_y = \frac{1}{n} \sum_{i=1}^n \sqrt{\varepsilon_{y_i}^2} \quad (6.141)$$

$$\varepsilon_c = \frac{1}{n} \sum_{i=1}^n \sqrt{\varepsilon_{x_i}^2 + \varepsilon_{y_i}^2} \quad (6.142)$$

The reconstruction error is the deviation of the reconstructed coordinates of the control frame from the measured

$$\varepsilon_{X_i} = X_{R_i} - X_i, \quad \varepsilon_{Y_i} = Y_{R_i} - Y_i, \quad \varepsilon_{Z_i} = Z_{R_i} - Z_i, \quad (6.143)$$

$$\varepsilon_X = \frac{1}{n} \sum_{i=1}^n \sqrt{\varepsilon_{X_i}^2} \quad (6.144)$$

$$\varepsilon_Y = \frac{1}{n} \sum_{i=1}^n \sqrt{\varepsilon_{Y_i}^2} \quad (6.145)$$

$$\varepsilon_Z = \frac{1}{n} \sum_{i=1}^n \sqrt{\varepsilon_{Z_i}^2} \quad (6.146)$$

$$\varepsilon_R = \frac{1}{n} \sum_{i=1}^n \sqrt{\varepsilon_{X_i}^2 + \varepsilon_{Y_i}^2 + \varepsilon_{Z_i}^2} \quad (6.147)$$

where $(X_{R_i}, Y_{R_i}, Z_{R_i})$ are the reconstructed coordinates for the i^{th} control point. The errors of calibration and reconstruction are presented in table (6.2) for the various solution schemes investigated. ε_{AC} and ε_{BC} are the calibration errors in planes A and B respectively and ε_R is the overall reconstruction error.

According to Hatze [79], the transformation matrix produced in the DLT formulation is known to violate the orthogonality requirement, making it perturbatively unstable. An iterative solution scheme is thus likely to fail if the search direction is incorrect, even slightly. A consequence of this condition is also to bring the transformation matrix close to singular. The latter condition is known to be alleviated with singular value decomposition [96] of the transformation matrix, while the optimal search direction is best determined by the Levenberg-Marquardt [63, 96] algorithm. Accordingly,

the numerical solvers used are the linear least squares (LSQ), Levenberg-Marquardt (Lever-Mar) and singular value decomposition (SVD). Implementations for these routines are well documented, with the current use being credited to the matrix manipulation software OCTAVE.

To incorporate the interpolation schemes into the reconstruction error calculation, 18 control points were chosen from the 9 zones described in section (4.4.4) and treated as unknown points. Essentially two points per zone (one from each X-ray image plane) was selected for the error analysis. The calibration error calculation incorporated all but the 18 control points while the reconstruction error was based entirely on the 18 omitted control points. The results, indicated by ε_R in table (6.2), show very little differences between the various combinations of interpolation schemes indicating that the Voronoi and/or Delaunay combinations are essentially equivalent. The improvement on the standard 11 parameter solution and the standard implementation of the full DLT solution is apparent. The added effort of incorporating physically valid variances into the photogrammetric model is therefore justified.

Weighting scheme	ε_X	ε_Y	ε_Z	ε_R	ε_A	ε_B	iterations	solver
2D & 3D Delaunay tessellation	0.07	0.11	0.10	0.16	0.24	0.22	0	LSQ
2D & 3D NNI	0.08	0.11	0.11	0.17	0.24	0.22	0	LSQ
2D Delaunay tessellation & 3D NNI	0.07	0.10	0.10	0.16	0.24	0.22	0	LSQ
2D NNI & 3D Delaunay tessellation	0.06	0.10	0.09	0.15	0.24	0.22	0	LSQ
Full DLT: standard implementation	Solution diverges						0	LSQ
Full DLT: standard implementation	0.13	0.18	0.18	0.29	0.44	0.32	3	Lev-Mar
Full DLT: standard implementation	0.15	0.20	0.20	0.31	0.38	0.33	3	SVD
Standard 11 parameter DLT solution	0.17	0.20	0.19	0.32	1.15	0.85	0	LSQ

Table 6.2: Reconstruction error for 18 control points; Calibration error based on remaining control points.

NNI – Natural Neighbour Interpolation

6.5.2 Stability

The building blocks of the variance model are the control points. Therefore the greater the concentration of points per unit volume, the more reliable the interpolated variance within the given volume. An obvious test of stability will therefore be to reduce the number of control points and investigate the effect on the reconstruction of the omitted points. The steps undertaken to assess stability are as follows:

1. step 1: Remove 18 points from within the control space volume and treat them as unknowns
2. step 2: Remove 36 more points (i.e. the 18 points from step 1 and 18 more points) and treat them as unknowns
3. step 3, 4, ... : Continue the above process until the solution diverges or all points are used up

The unknowns are then solved for in each step and the errors of calibration and reconstruction computed. Based on the results of the accuracy test in section (6.5.1), only the (2D NNI & 3D Delaunay tessellation) combination is included from the interpolation combinations in the stability test. The methods used to evaluate stability are:

1. *method 1*: 2D NNI & 3D Delaunay tessellation (LSQ)
2. *method 2*: Full DLT: standard implementation (LSQ)
3. *method 3*: Full DLT: standard implementation (Lev-Mar)
4. *method 4*: Full DLT: standard implementation (SVD)
5. *method 5*: Standard 11 parameter DLT solution (LSQ)

Figure (6.13) illustrates the overall reconstruction error (ε_R) after removing consecutively, 18, 36, 54 and 72 points. The full DLT with the LSQ solution scheme (method 2) diverges at the zeroth iteration and is thus not included in the figure. The number of iterations to reach convergence is illustrated in table (6.3) at each step of the removals. The removal of the control points was conducted such that at each step, all 9 zones of the X-ray image contained at least 1 point.

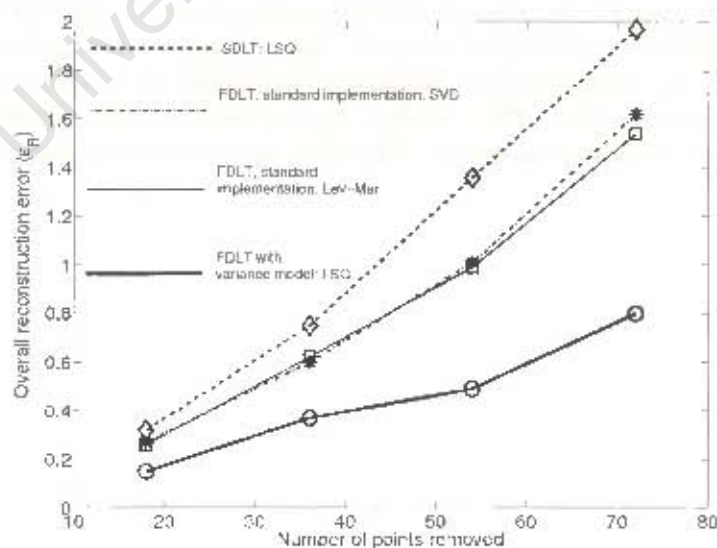


Figure 6.13: Stability test of the DLT algorithms

	Number of points removed			
	18	36	54	72
method 1	0	0	0	0
method 2	Solution diverges			
method 3	3	7	13	28
method 4	3	9	17	35
method 5	0	0	0	0

Table 6.3: Number of iterations taken by the various methods at each step of the removals

After 54 points were removed, only the standard DLT scheme (method 5) produced an overall error greater than 1 mm. The full DLT method produced similar errors for both the Lev-Mar (method 3) and SVD (method 4) solution schemes. Beyond the removal of 54 points, only the full DLT method employing the variance model (method 1) produced an overall error less than 1 mm. Method 1 also produced the smallest overall error for all removals up 72 points. It is therefore concluded that method 1 is more stable than the other DLT variations for the X-ray images of the control points. Table (6.3) also shows method 1 to be computationally most efficient.

6.6 Conclusion

The development of the DLT solution was reported in this chapter. The full DLT solution was shown to be an improvement on the standard 11 parameter formulation provided that the appropriate solver was used. In this regard, the Levenberg-Marquard solution scheme [63, 96] proved to be more robust than the standard least squares and the singular value decomposition [96] solvers. Further improvement in accuracy and stability was shown when a physical model of the variances was employed. Unlike the standard implementation of the full DLT solution, the variance model implementation did not require a sophisticated solver to compensate for the non-orthogonality of the transformation matrix, known to afflict the DLT method, Hatzel [79]. In fact, the variance model converged with zero iterations, even with the standard least squares solver. The other solvers required more iterations with increased point removals.

Chapter 7

Errors encountered in the BAE

7.1 Introduction

The errors encountered in digital X-ray photogrammetry can be divided into two categories: Distortions of X-ray image arising from systematic errors and distortions of reconstruction. The systematic errors are normally due to unflatness of the image intensifier screen and non-synchronised X-ray cameras which should be corrected prior to the 3D reconstruction phase. The latter is pertinent to moving objects while the former may be treated by addressing its consequences which usually manifests itself in reconstruction distortions. The assumption is therefore that image unflatness is not an explicit issue, although a well characterised reconstruction phase does account for it, if only partially. The reconstruction accuracy of digitised data is affected by a number of factors including digitisation errors, lens distortion, penumbra effects caused by finite sized focal spots and vibrations in the imaging system. However, many X-ray photogrammetrists still ignore these errors arguing that the required accuracies are achieved with first order approximations. Adams ([60]), for example, implemented a standard DLT solution, ignoring all of the above-mentioned errors, in his application of 3D X-ray imaging of brain tumors and still obtained a reconstruction standard error of 0.4 mm. Another reason for using approximate photogrammetric descriptions is probably due to the difficulty associated with finding a physically representative error model and an efficient numerical scheme for solving it. This is particularly true for X-ray imaging. The level of detail included in the chosen photogrammetric model ultimately depends on the objectives of the investigator. An implicit objective of this thesis was to provide high accuracy trajectory data of the tumbling mill for the purpose of DEM validation. For this reason an inclusion of error models into the existing mapping equations was essential.

In terms of the DLT method the reconstruction error is formulated as higher order optical distortions and decentering, the details of which are given in section (6.3). The reconstruction error formulation depends on a mathematically consistent and robust variance model which is described in section (6.4). The variance model must however be based on physically measurable deviations in the image and object coordinates in order for the variance model to be valid. The reconstruction errors addressed in this thesis are the effects of penumbra (section 7.2), distortions arising from X-ray projection

systems (section 7.3) and oscillations in the image capture mechanism (section 7.4), all of which can be resolved with a robust description of the variance.

Another source of error in the BAE is the non-synchronised cameras, commonly referred to as phase lag. Phase lag between the X-ray cameras seems to be built into the BAE, however no investigators to date have attempted to quantify its effect on the 3D reconstruction accuracy of moving objects. The factors affecting phase lag error are:

1. the velocities of the moving objects,
2. the frequency of the image capture mechanism (shutter speed and/or sampling rates), and
3. the initial lag time between the two cameras.

Phase lag in general is poorly dealt with in the photogrammetric literature with the exception of Yeadon & King [97] and Pourcelot et al [98]. Most improvements in X-ray photogrammetry over the past decade have been towards self-calibration systems [62] because they introduce minimal variation to the existing surgical procedures and are therefore preferred by the clinician. Unfortunately, none of them account for the errors introduced by phase lag even when studying motion anomalies of the heart. A quantitative analysis of the phase lag between the two X-ray cameras of the BAE is presented in section (7.7).

The next three sections attempt to quantify errors that are collectively addressed by variance model. Their purpose is therefore to address the physical significance of the variance model described in the previous chapter. The last section, section (7.7), describes a scheme for minimising the error introduced by phase lag.

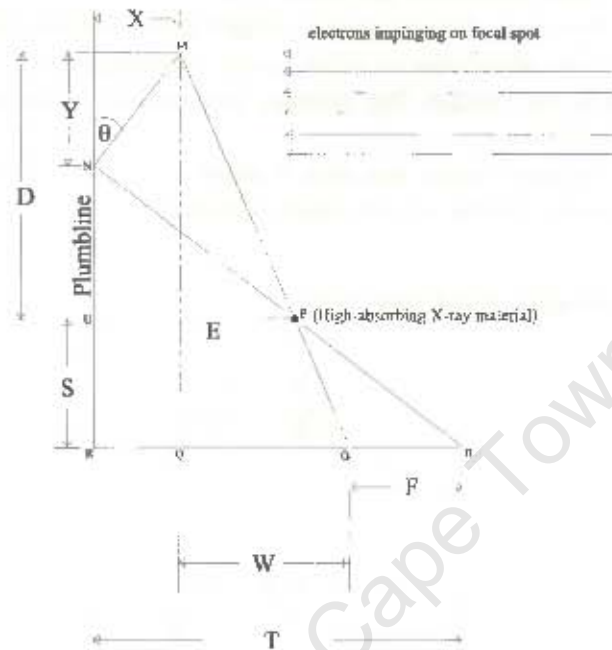


Figure 7.1: Penumbra effect due to a finite sized focal spot

The effective size of the focal spot used in medical X-ray work has a practical limit according to Moffit [99]. The limited size focal spot results in a penumbra effect throughout the X-ray image [90]. A penumbra effect tends to spread out the edge gradient and results in a soft image on the film. Consider the cross sectional view of the focal spot as it is bombarded by X-rays, figure (7.1). The edge of the object at point P is composed of radio opaque material. Only X-rays that pass through the high absorbing material are considered; the boundaries of which are indicated by lines MO and NB . The area enclosed by triangle POB illustrates the penumbra or area of partial non-uniform radiation (partial shadow). The area to the right of the penumbra is the area of minimum radiation (full shadow). It is clear from the diagram that if the radiation emanated from a point-source, there would be no penumbra effect. The following discussion includes the derivation of the magnitude of the penumbra distance (distance F), and its effect on quantitative and qualitative analysis of the X-ray system.

The notation associated with figure (7.1) is as follows:

- X is the horizontal projection of focal spot or effective length of focal spot
 Y is the vertical projection of focal spot
 θ is the angle between plumbline and focal spot
 D is the distance from top of focal spot to object plane (object distance)
 E is the distance from plumbline to point under consideration
 (the plumb line is the vertical line through the bottom of the focal spot)
 F is the penumbra distance
 S is the distance between image and object plane.
 ($S + D$ is commonly known as the image distance)

Considering similar triangles NBR and NPU ,

$$\frac{NR}{BR} = \frac{NU}{UP} \quad (7.1)$$

this implies

$$\frac{D - Y + S}{T} = \frac{D - Y}{E} \quad (7.2)$$

therefore

$$T = \frac{(D - Y + S)E}{D - Y} \quad (7.3)$$

$$= \frac{[D - X \cot(\theta) + S]E}{D - X \cot(\theta)} \quad (7.4)$$

Similarly in triangles MOQ and MPV ,

$$\frac{MQ}{QO} = \frac{MV}{VP} \quad (7.5)$$

which implies

$$\frac{D + S}{W} = \frac{D}{E - X} \quad (7.6)$$

and rearranging, we have

$$W = \frac{(D + S)(E - X)}{D} \quad (7.7)$$

Using these relations, we see that the penumbra distance F is simply

$$F = T - W - X \quad (7.8)$$

Substituting and simplifying results in equation (7.8) yields:

$$F = \frac{SX}{D} + \frac{SEX \cot(\theta)}{D^2 - DX \cot(\theta)} \quad (7.9)$$

The conditions arising from this relationship is:

1. The penumbra distance increases as E increases positively (i.e. to the right of the plumbline)
2. The penumbra distance decreases as E increases negatively (i.e. to the left of the plumbline)
3. The penumbra distance is zero when,

$$E = -(D \tan(\theta) - X) \quad (7.10)$$

4. The penumbra distance decreases as the focal spot decreases, the object-film distance (S) decreases, and the object distance (D) increases.

To illustrate the effect of the penumbra we consider a typical case scenario when employing the use of the BAE: The focal spot used with the BAE is a square target of dimensions 0.4 mm, oriented at an angle of 8 degrees to the plumbline. Using the diagonal across the target as the cross sectional view, the focal spot is approximately 0.6 mm. The maximum value of E encountered in the experiments is: $E_{\max} = 71.079$ mm. This corresponds to the radius of the mill plus half the effective length of the focal spot. Geometrically, we are looking at a small particle along the perimeter of the mill, with the assumption that the mill centre and midpoint of the effective focal length lie along the mill axis. The minimum object distance is $D = 634.4$ mm and object-film distance is $S = 173.6$ mm.

Substituting these values into equation (7.9), we have:

$$F = \frac{(173.6)(0.079)}{634.4} + \frac{(173.6)(71.079)(0.079) \cot(8)}{(634.4)^2 - (634.4)(0.079) \cot(8)} \quad (7.11)$$

$$= 0.04 \text{ mm} \quad (7.12)$$

The error caused by the finite sized focal spot is consistent with the deviations required to model higher order optical distortions and decentering, see section (6.4) for details on the variance model.

7.3 Projection of a sphere onto the film plane using analytical X-rays

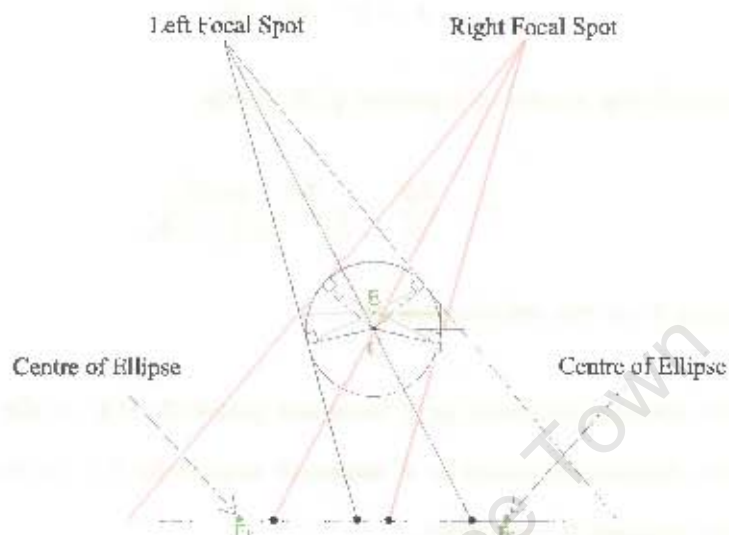


Figure 7.2: Stereo radiography of a sphere

A stereoscopic view of a radio opaque sphere with centre C produces an interesting geometrical condition. When viewed in cross section, the situation illustrated in figure (7.2) is observed. The images are those of ellipses centered on E_R and E_L . In particular, if the sphere is replaced by an ellipsoid centered on E the images produced are exactly that obtained for the sphere. However, the centre of the sphere and the ellipsoid, in real space, are clearly different. This implies that a transformation from the centres of the elliptical images to real space will find the centre of the space ellipse and not the centre of the sphere. This however, need not be a concern, provided that the difference between the image centres of the sphere and the ellipse is not large, in relation to other errors that cannot be overcome. To these ends we determine the deviation Δx between the projection of the sphere and the ellipse. For simplicity, we include only one focal spot, figure (7.3).

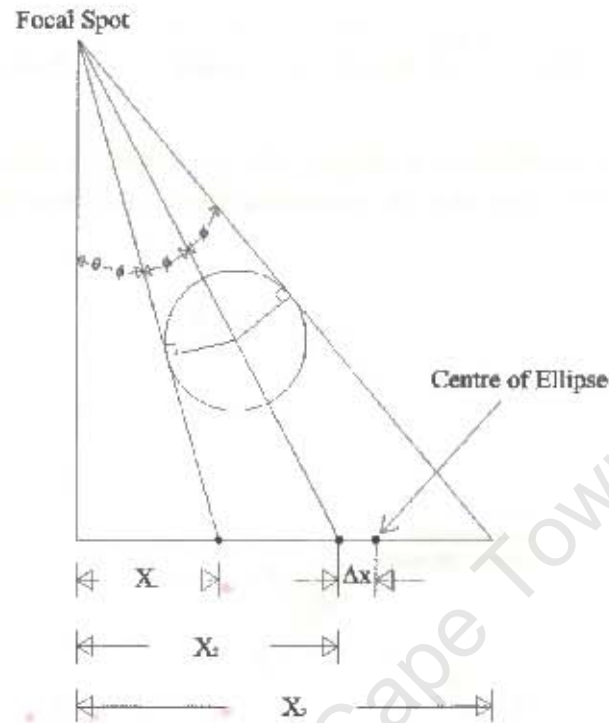


Figure 7.3: Centre of ellipse relative to centre of sphere

From figure (7.3), it follows that

$$X_1 = d \tan(\theta - \varphi) \quad (7.13)$$

$$X_2 = d \tan(\theta) \quad (7.14)$$

$$X_3 = d \tan(\theta + \varphi) \quad (7.15)$$

and half the length of major axis of the elliptical image is given by

$$\frac{(X_3 - X_2) + (X_2 - X_1)}{2} = X_2 - X_1 + \Delta x \quad (7.16)$$

Rearranging and simplifying,

$$\Delta x = \frac{1}{2}[(X_1 + X_3) - 2X_2] \quad (7.17)$$

Therefore

$$\Delta x = \frac{FFD}{2} [\tan(\theta - \varphi) + \tan(\theta + \varphi) - 2 \tan(\theta)] \quad (7.18)$$

Based on the parametric descriptions in chapter (3), the condition that would make Δx a maximum is illustrated in figure (7.4). Note that by comparing figure (7.3) with figure (7.4), it is clear that

$$\varphi = \frac{\beta - \theta}{2} \quad (7.19)$$

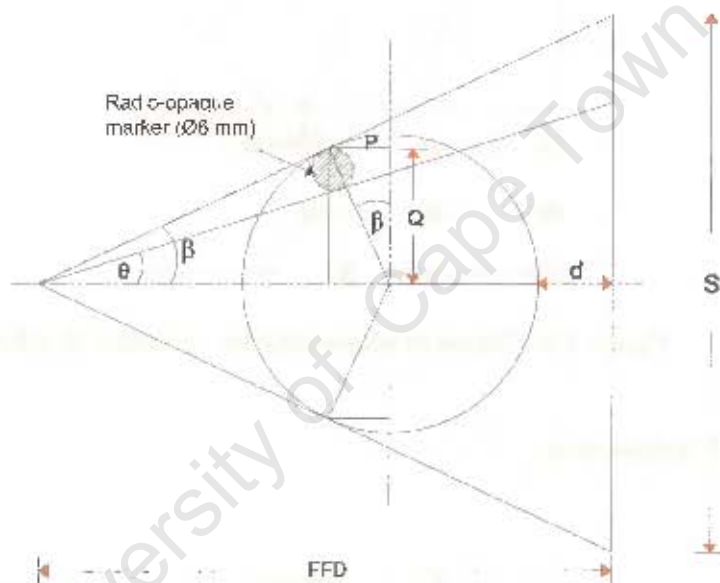


Figure 7.4: Position of radio-opaque object that maximises the projection error (Δx)

Using $FFD = 950$ mm, $\theta \approx 5.29^\circ$, $\varphi \approx 0.16^\circ$, the corresponding deviation is:

$$\Delta x \approx \frac{950}{2} [\tan(5.29 - 0.16) + \tan(5.29 + 0.16) - 2 \tan(5.29)] \quad (7.20)$$

$$= 6.9 \times 10^{-4} \text{ mm} \quad (7.21)$$

The projection error is clearly negligible when compared to the errors due to penumbra and vibration. We will therefore ignore the effects caused by projection on the final centering accuracy of the projected objects.

7.4 Errors due to oscillations in BAE filming system

Errors due to higher order optical distortions can be caused by perturbations in the imaging system. Variations of this nature lead to decentering errors that ultimately affect the reconstruction accuracy of unknown object points. The BAE system is a complex of mechanical and electronic equipment and as such is susceptible to vibrations. The vibrations appear to be random and vary from one scene to the next.

The control frame is filmed for 16.7 seconds yielding 835 static X-ray images of the control points. Each image is analysed separately, resulting in 835 centroids for each control points projected shape. A mean centroid is then computed followed by a residual error analysis.

For a X-ray image scene comprising 835 images, the mean centroid for any given point is:

$$[\bar{x}_c, \bar{y}_c] = \left[\frac{\sum_{i=1}^{835} x_i}{835}, \frac{\sum_{i=1}^{835} y_i}{835} \right] \quad (7.22)$$

where $\bar{x}$, $\bar{y}$ are mean centroid coordinates for a given point and x_i , y_i are the centroid coordinates of the given point in the i^{th} image.

The standard deviation for the given point is then computed as:

$$[\varepsilon_x, \varepsilon_y] = \left[\frac{\sqrt{\sum_{i=1}^{835} (x_{ci} - \bar{x}_c)^2}}{835 - 1}, \frac{\sqrt{\sum_{i=1}^{835} (y_{ci} - \bar{y}_c)^2}}{835 - 1} \right] \quad (7.23)$$

$$\varepsilon_R = \sqrt{(\varepsilon_x)^2 + (\varepsilon_y)^2} \quad (7.24)$$

where $[\varepsilon_x, \varepsilon_y]$ is the standard deviation of the residuals $[(x_{ci} - \bar{x}_c), (y_{ci} - \bar{y}_c)]$ and ε_R is the resultant error.

To illustrate the error, the centroid of an arbitrary control point (the projected shape of the control point) is tracked through an entire dicom scene (comprising 835 frames) in biplanar mode. The analysis is presented in two graphical forms (figures 7.6 and 7.7). The upper plot shows the variation of a control points centroid (x and y pixel coordinates) through a dicom image scene while the lower plot represents the standard deviation of the control points centroid as the radius of a circle centered on the mean centroid. Table (7.1) shows the results of the calculations.

The results shown in table (7.1) is a typical illustration of the control frame data obtained in this thesis. The point being considered in this particular example is randomly selected and shown in figure (7.5).

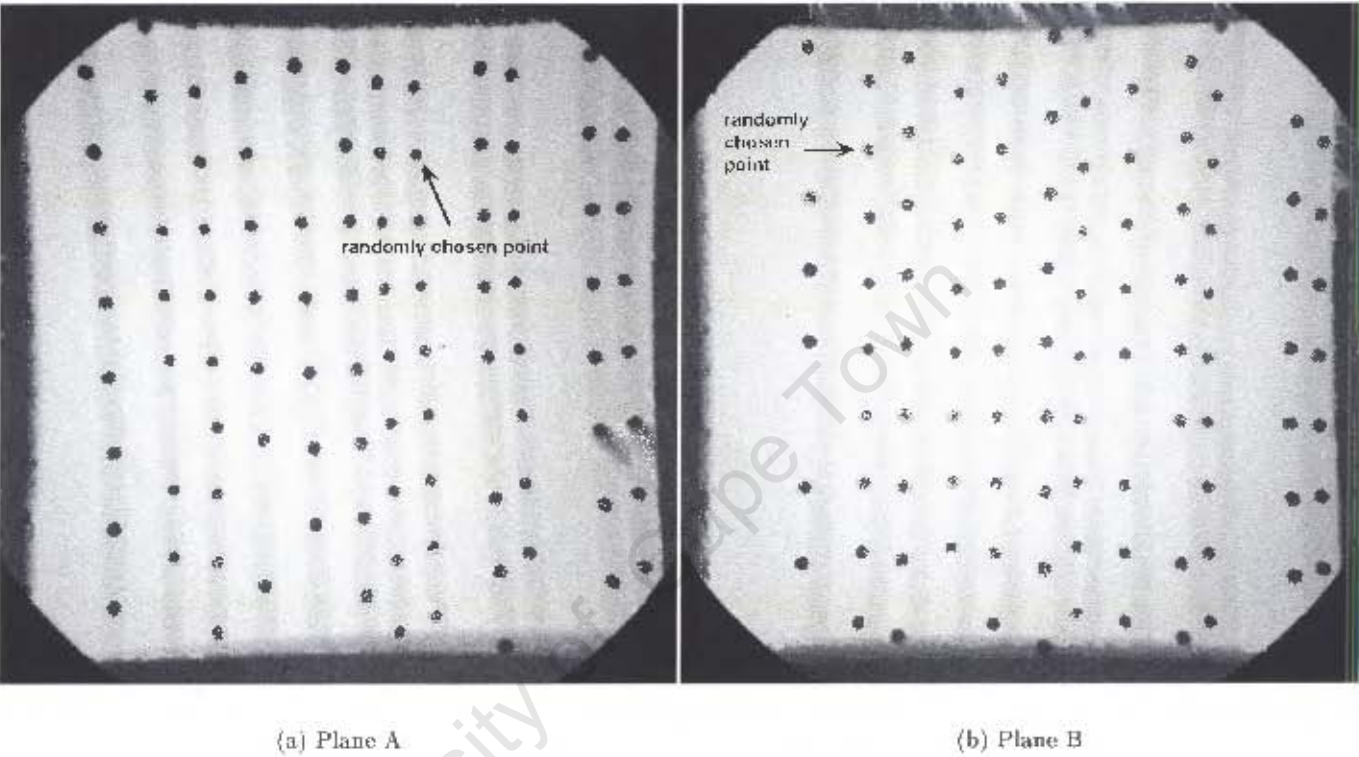


Figure 7.5: Randomly selected point used to illustrate standard deviation in centroid due to vibrations in the BAE

error	plane A: (pixel units)	plane A: (mm)	plane B: (pixel units)	plane B: (mm)
$\overline{x_c}$	15.73	5.66	16.87	6.07
$\overline{y_c}$	16.62	5.98	16.56	5.96
ε_x	0.04	0.01	0.05	0.02
ε_y	0.06	0.02	0.05	0.02
ε_R	0.08	0.03	0.07	0.03
ε_{max}	0.26	0.09	0.20	0.07

Table 7.1: Standard deviation for the centroid of a control points projected shape through a dicom scene

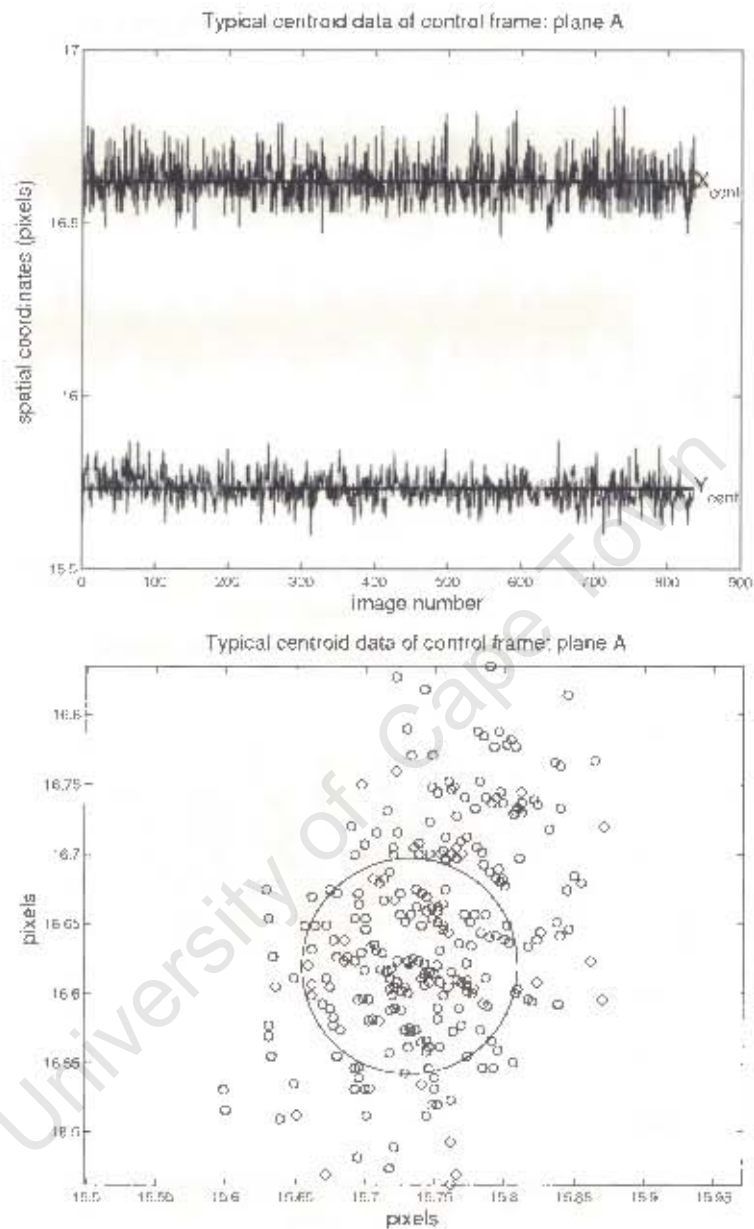


Figure 7.6: Variation of centroid about the mean for the projected shape of the control point: plane A

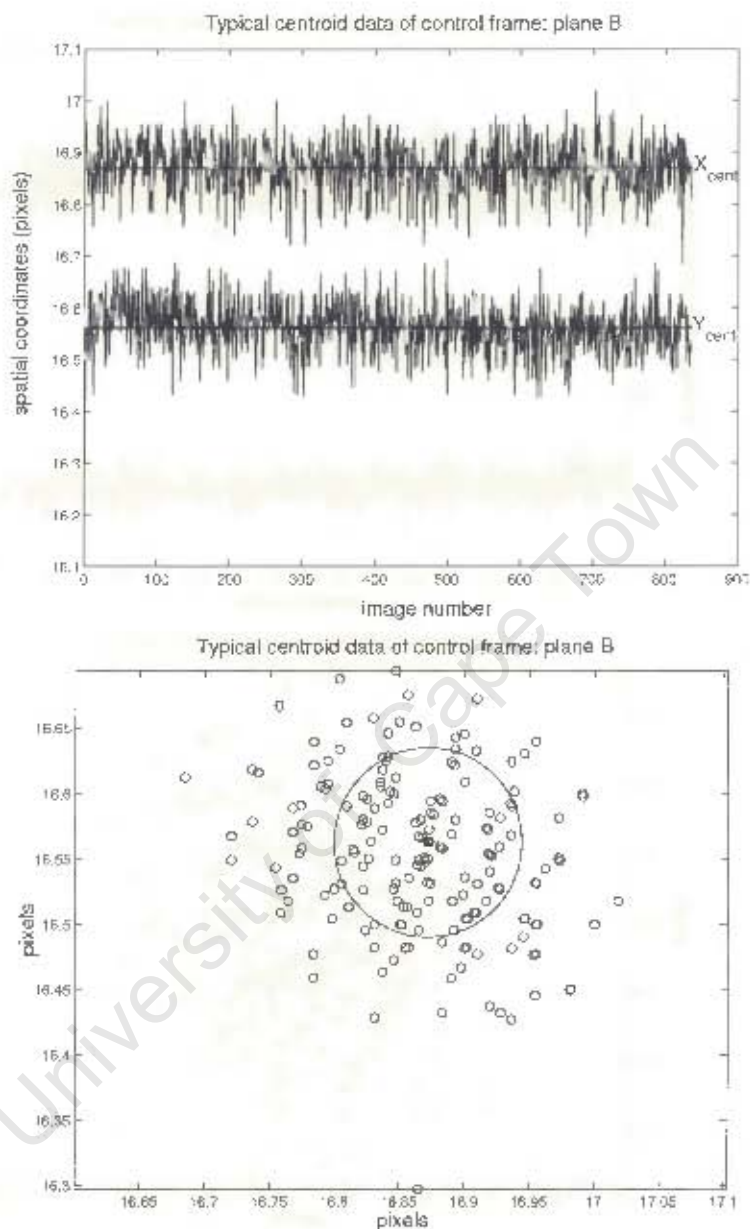


Figure 7.7: Variation of centroid about the mean for the projected shape of the control point: plane B

7.5 Conclusions on errors due to penumbra and BAE vibrations

By treating the errors of penumbra, section (7.2), and BAE vibrations, section (7.4), as perturbations in the imaging system that result in target decentering, a basis for a variance model can be established with these errors. The control frame is designed such that the projected shapes of the ball bearings

span (discretely) most of the X-ray image. The error due to BAE vibrations can be obtained according to section (7.4) for each of the control points. The error due to penumbra can also be estimated according to section (7.2) for each of the control points. Thus for a given control point with known spatial coordinates (X, Y, Z) , relative to the focal spot, the combined error of penumbra and vibration can be estimated as

$$\Delta R = \varepsilon_R + F \quad (7.25)$$

where ΔR is the combined perturbation error in locating the centroid of the control points projection in the image plane, ε_R is the error due to vibration and F is the error caused by penumbra. The perturbation error in centering the projections of the control points is illustrated in table (7.2). The calculation of the maximum values (ε_R^{max} and F_{max}) are described in sections (7.4 and 7.2). The RMS error values are computed as follows:

$$\varepsilon_{RMS} = \sqrt{\frac{\sum_{i=1}^n (\varepsilon_R^{(i)})^2}{n}} \quad (7.26)$$

$$F_{RMS} = \sqrt{\frac{\sum_{i=1}^n (F^{(i)})^2}{n}} \quad (7.27)$$

where i refers to the i^{th} control point's projection and n is the number of control points that are projected onto the image plane and simultaneously discernable in both planes. In most experiments $n = 100$, though this can vary depending on the positioning of the frame.

Vibration Error		Penumbra error		Combined error	
ε_R^{max} (mm)	ε_{RMS} (mm)	F_{max} (mm)	F_{RMS} (mm)	ΔR_{RMS}	ΔR_{max} (mm)
0.09	0.05	0.04	0.02	0.07	0.13

Table 7.2: Centering error for image coordinates of control point

7.6 Overall expected error of the X-ray vision system

A reasonable estimate of the overall reconstruction error can be obtained by combining the physical errors of the imaging system and that of the control frame. The former was discussed in the previous section, while the actual measured deviations of the control frame was determined by calculating the standard deviation from repeated measurements of the control points. The mean control point

coordinates and the associated deviations are given in table (B1). The resultant deviation of these measured values was calculated as:

$$\begin{aligned}\varepsilon_{RO} &= \sqrt{(\bar{\varepsilon})_X^2 + (\bar{\varepsilon})_Y^2 + (\bar{\varepsilon})_Z^2} \\ &= \sqrt{0.036^2 + 0.038^2 + 0.033^2} \\ &= 0.06 \text{ mm}\end{aligned}\quad (7.28)$$

where

$[(\bar{\varepsilon})_X, (\bar{\varepsilon})_Y, (\bar{\varepsilon})_Z]$ = average of the control point standard deviations given in table (B1)

ε_{RO} = average resultant error of the control points

The maximum average resultant error of the control points can be determined by:

$$\begin{aligned}\varepsilon_{RO}^{max} &= \sqrt{(\varepsilon_X^{max})^2 + (\varepsilon_Y^{max})^2 + (\varepsilon_Z^{max})^2} \\ &= \sqrt{0.132^2 + 0.1226^2 + 0.089^2} \\ &= 0.20 \text{ mm}\end{aligned}\quad (7.29)$$

where

$[\varepsilon_X^{max}, \varepsilon_Y^{max}, \varepsilon_Z^{max}]$ = the maximum standard deviation of the control points in each of the three cartesian directions.

The expected, overall reconstruction error, based on the physically quantified errors should then be:

$$\begin{aligned}E &= \Delta R_{RMS} + \varepsilon_{RO} \\ &= 0.07 + 0.06 \\ &= 0.13 \text{ mm}\end{aligned}\quad (7.30)$$

with the expected, maximum overall reconstruction error given by:

$$E^{max} = \Delta R_{max} + \varepsilon_{RO}^{max} \quad (7.31)$$

$$= 0.13 + 0.20 \quad (7.32)$$

$$= 0.33 \text{ mm} \quad (7.33)$$

The 11 parameter DLT solution was shown to produce an overall reconstruction error of 0.32 mm in table (6.2), suggesting that it does not account for any of the errors. In fact, it yields approximately the maximum error that can be expected within a systematically consistent experiment. The full DLT solution, which claims to account for distortions in the imaging and calibration processes, should be achieving reconstruction errors of 0.13 mm, however, the results from the accuracy assessment of the DLT formulation, table (6.2), suggest that the full DLT formulation, without a physical model of the variances, is inadequate, producing overall reconstructions errors that are 0.16 mm (Lev-Mar solver) and 0.18 mm (SVD solver) greater than the expected mean error of equation (7.30). The full DLT solution based on the variance model produces an overall reconstruction error of 0.15 mm, which is 0.02 mm larger than is expected, but approximately consistent with the physically quantified error.

It should be noted that the accuracy does diminish as the number of control points are uniformly removed, see section (6.5.2). The results from the stability test suggests that the removal of 36 points or more (i.e. using 63 points or less to calibrate the X-ray space) produces errors that are greater than those quantified in this thesis. This implies that the number of control points is an important criteria in determining the final reconstruction accuracy.

The following guidelines may be used to design an X-ray vision system based on accuracy requirements.

1. If the accuracy requirements of the X-ray imaging system are approximately 1.0 mm, the 11 parameter DLT equation with at least 6 control points are needed. The target centering requirements for this accuracy would be no more than using the mouse input on an enlarged image, and simply clicking the 'eye-balled' centroid.
2. If the accuracy requirements of the X-ray imaging system are approximately 0.3 mm, the 11 parameter DLT equation will suffice, but at least 20 control points are needed. The target centering would need to be fairly sophisticated, and must be preceded by a robust area or edge based feature extraction routine
3. If the accuracy requirements of the X-ray imaging system are approximately 0.15 mm, the full DLT solution scheme with approximately 70 control points are needed in conjunction with a physically valid weighting scheme. The image processing and subsequent target centering would also needed to be tailored to the specific nature of the data.
4. Any X-ray vision system that employs the DLT equations must have an accurate and independent means of measuring the control frame. The coordinate measuring machine is amongst the best tools for this purpose, though usage comes at a relatively high cost.

7.7 Phase lag in BAE

In keeping with the chosen photogrammetric formulation used in this thesis, the synchronisation of the two X-ray cameras was achieved by exploiting aspects of the standard DLT method. The technique employed has been adapted from the work by Yeadon & King [97] and Pourcelot et al [98] for the

X-ray imaging system. Using the formulation given in equations (6.81 and 6.82), the DLT equations can be written without the error terms as:

$$x = \frac{L_1X + L_2Y + L_3Z + L_4}{L_9X + L_{10}Y + L_{11}Z + 1} \quad (7.34)$$

$$y = \frac{L_5X + L_6Y + L_7Z + L_8}{L_9X + L_{10}Y + L_{11}Z + 1} \quad (7.35)$$

Following the methodology of Stewart [100], the observation equations of a tracked object can be rearranged into the familiar form of a plane, equation (7.36), and written for both X-ray fields as in equation (7.37).

$$a_iX + b_iY + c_iZ = d_i \quad i = 1, 2, 3, 4 \quad (7.36)$$

$$\begin{bmatrix} L_1 - L_9x & L_2 - L_{10}x & L_3 - L_{11}x \\ L_5 - L_9y & L_6 - L_{10}y & L_7 - L_{11}y \\ L'_1 - L'_9x' & L'_2 - L'_{10}x' & L'_3 - L'_{11}x' \\ L'_5 - L'_9y' & L'_6 - L'_{10}y' & L'_7 - L'_{11}y' \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} x - L_4 \\ y - L_8 \\ x' - L'_4 \\ y' - L'_8 \end{bmatrix} \quad (7.37)$$

All symbols have their usual meaning as in section (6.3) and the primed symbols are associated with the second camera. After solving for the eleven DLT parameters, a least squares estimate (X_0, Y_0, Z_0) of a moving object's 3D position can be obtained. Section (6.1) describes the numerical scheme required for a least squares estimate of the DLT parameters and 3D coordinates of unknown objects. The residuals of the least squares estimates are the errors associated with the coordinates. By normalising the equation of the plane, equation (7.36), the residuals r_i of the the least squares solution can be expressed in the form of equation (7.38).

$$r_i = \frac{|a_iX_0 + b_iY_0 + c_iZ_0 - d_i|}{\sqrt{a_i^2 + b_i^2 + c_i^2}} \quad i = 1, 2, 3, 4 \quad (7.38)$$

According to Fuller & Tarwater [101] the residual, r_i , is the geometric distance of (X_0, Y_0, Z_0) from the plane. It should be noted that the normalisation is not mandatory although it does improve convergence when the system is ill-conditioned.

The RMS distance between each least squares estimate, (X_0, Y_0, Z_0), and the four planes is then calculated using:

$$r = \frac{\sqrt{r_1^2 + r_2^2 + r_3^2 + r_4^2}}{4} \quad (7.39)$$

An overall RMS error estimate for all moving points that appear in both X-ray images is then computed according to equation (7.40)

$$R = \sqrt{\frac{\sum_{j=1}^2 \sum_{i=1}^m r_{ij}^2}{2m}} \quad (7.40)$$

where r_{ij} is the RMS distance r of the i^{th} moving point in image j , m is the number of moving points that appears in both images, 2 denotes the number of intensifier screens (image planes) and R represents the global RMS distance.

The global RMS distance is a 'lumped' error estimate of the reconstruction accuracy that includes errors of digitisation, lens distortion and synchronising errors. By combining systematic and reconstruction errors into a global error estimate we can investigate the influence of phase lag on the global RMS distance. In particular R will tend to be smallest (a minimum) when the image coordinates from the two X-ray image planes are correctly synchronised since the digitisation and lens distortion errors will be similar for different phase offsets. The problem therefore becomes one of varying the offsets between the two X-ray images until the global RMS distance is minimised.

The procedure used was to keep plane B image data fixed and to adjust plane A. Different phase offset values are investigated using lowpass interpolation by Oetken et al [102]. The algorithm designs a symmetric filter which increases the sampling rate by an integer factor of N . The resulting filter allows the original input samples (image coordinates) to pass through the filter unchanged and interpolates $N - 1$ sample values between each pair of original samples in such a way that the mean square error between the interpolated values and their theoretically ideal values is minimised. The design of the filter is achieved through the formulation of an $L \times L$ matrix inversion where $2L$ is the number of original sample values used to interpolate each new value. According to Oetken et al [102], L should be limited to 10 or less.

The data used in the analysis was the image coordinates of a steel ball bearing fixed to the inner surface of the lifter bars. The particle or 'mill tracer' therefore traces out a circular path equivalent to the mill outline along that cross-section of the mill. Figure (7.8) illustrates the interpolation scheme used for twenty image points of the mill tracer (the axes are not to scale for illustration purposes). The corresponding filter used to interpolate this data is shown in figure (7.9).

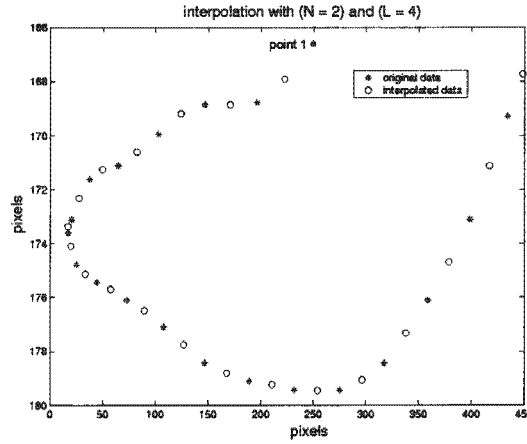


Figure 7.8: Illustration of the interpolation scheme for 20 image points of the mill tracer

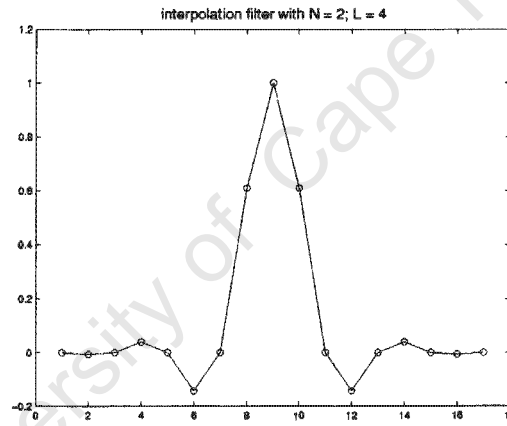


Figure 7.9: Interpolation filter used to resample 20 image coordinates of mill tracer.

For each interpolated image coordinate the phase offset was calculated as follows:

$$T_j = \left(\frac{P_j - P_i}{P_{i+1} - P_i} \right) \quad (7.41)$$

where

- P_i = position in the i^{th} image
- P_{i+1} = position in the $(i + 1)^{th}$ image
- P_j = the j^{th} interpolated position
- the i^{th} and $(i + 1)^{th}$ position; and $j = 1, 2, 3, \dots, N - 1$
- T_j = phase offset of the j^{th} interpolated position

The RMS phase offset for the j^{th} interpolated position is calculated as:

$$T_{RMS}^{(j)} = \sqrt{\frac{\sum_{k=1}^{n-1} T_{jk}^2}{n}} \quad (7.42)$$

where n is the number of images in the dicom scene.

An example of a typical minimisation is shown in figure (7.10).

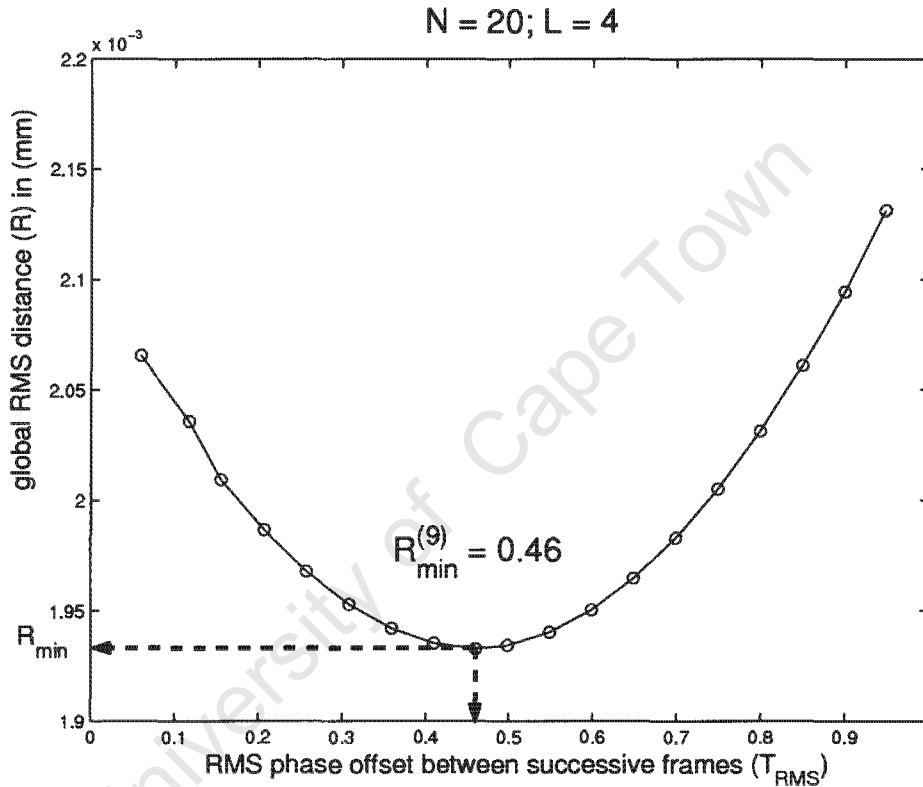


Figure 7.10: Global RMS error (R) versus RMS phase offset (T_{RMS}) for the mill tracer

The RMS phase difference value that minimises the global RMS distance was then used to select the appropriate interpolated position, P_j , of plane A image data. In the example shown in figure (7.10), $j = 9$ and $T_{RMS}^{(9)} = 0.54$, corresponds to the minimisation of the global RMS distance. The global RMS error was shown to be less than 2×10^{-3} mm for all dicom scenes. This error is also considered negligible in relation to the other errors contributing to the overall reconstruction accuracy of object points.

7.7.1 Testing the accuracy of the phase lag correction

The evaluation of the phase lag correction theory requires tests against some accurate measurement of objects in the different fields. For example, Yeadon & King [97] used common events in genlocked

video fields to determine the actual time offset. The difference between the actual and predicted offset then formed the basis for evaluating the accuracy of their method. Pourcelot et al [98] rotated a $\varnothing 64$ cm wheel at 1.16 rps fitted with one marker and recorded with four video cameras at shutter speeds of $1/1000$ s. While the wheel was recorded, a light bulb, placed in the field of the four cameras, was switched on every 10 s. For each camera, the frame on which the first light of the bulb appeared was selected to synchronise approximately the four recordings. The synchronised data was then used to reconstruct the 3D coordinates of the marker and thus evaluate their method.

The accuracy of the implementation in this thesis was tested against measurements of the tracer particle. The mill's inner diameter, T_a in figure (7.11), was measured for deviations along the cross section occupied by the tracer. The diameter measurements were then adjusted to match up with the path followed by the actual tracer. The path traced out by the centroid of the mill tracer is illustrated in figure (7.11) by the dashed lines and labelled (T_c). The adjusted diameter data then forms the basis for testing the accuracy of the phase lag correction. The test is conducted in four steps:

1. Reconstruct the 3D path of the mill tracer for the phase corrected image coordinates and the original data.
2. Fit an ellipse to each of the reconstructed 3D data sets.
3. Extract diameters around the fitted ellipse.
4. Statistically compare the measured and calculated diameters.

The measured diameter data is presented in table (7.3).

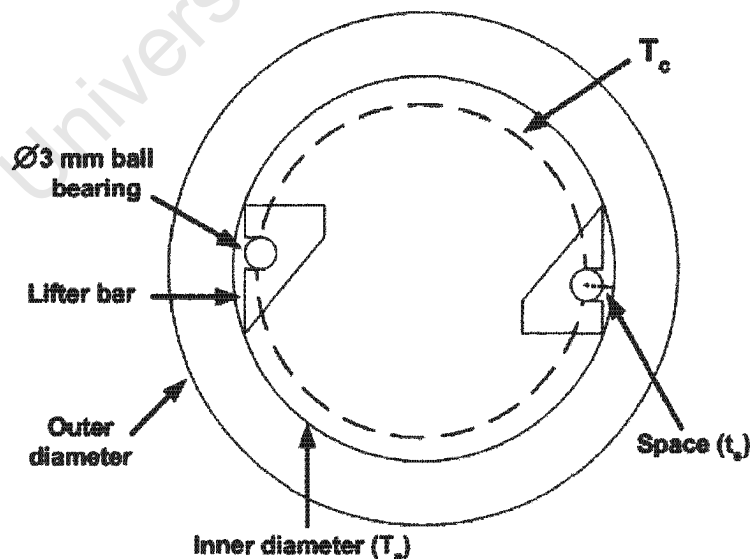


Figure 7.11: Cross sectional view of path traced out by centroid of ball bearing: Not to scale

T_a (mm)	$T_c = T_a - (3 + 2t_s)$ (mm)
142.32	138.72
142.24	138.64
141.84	138.24
141.60	138.00
141.78	138.18
141.52	137.92
141.56	137.96
141.48	137.88
140.40	137.80
140.96	137.36
140.52	136.92
140.58	136.98
140.50	136.90
140.4	136.80
140.36	136.76

Table 7.3: Measurement of mill diameter: $t_s = 0.3$ mm

The results shown in table (7.4) indicate that the phase corrected (PC) and uncorrected (O) $\bar{D}$ agrees with the measured (M) $\bar{D}$ and are within the uncertainties. An important observation from table (7.4) is that the maximum diameter deviations are more accurately described (in terms of agreement with the mean and standard deviation of the measured data) for the (PC) diameters than the uncorrected (O) values. These deviations from the measured mill outline can lead to incorrect interpretation of the tracked particles motion and interaction with the mill shell and should therefore be accurately characterised and corrected for if necessary.

PC or U or M	Method used	$\bar{D}$ (mm)	D_{max} (mm)	D_{min} (mm)	σ_D (mm)
M	Measured	137.67	138.72	136.76	0.67
PC	Fitzgibbons	137.96	138.91	136.89	0.76
PC	Circle fit	137.78	137.78	137.78	0.96
O	Fitzgibbons	137.56	145.51	129.62	0.62

Table 7.4: Measured (M), Phase Corrected (PC) and Original (O) diameters of mill tracer's trajectory

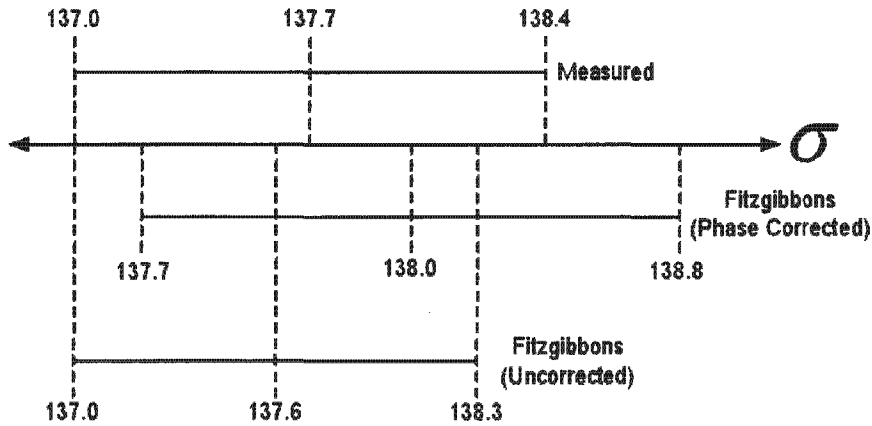


Figure 7.12: Standard deviation in T_c : Experimental, Phase Corrected and Uncorrected

7.8 Conclusion

The errors associated with the penumbra effect, vibration of the BAE unit and to a lesser extent projection errors of a sphere onto a plane have been physically quantified for the X-ray imaging system. The combined error of these quantities was shown to be 0.07 mm. The error associated with the control frame was also estimated and given as 0.06 mm. The overall reconstruction error, which is composed of systematic errors and distortions of reconstruction, is shown to be equivalent to the additional terms that constitute the full DLT formulation. However, achieving this accuracy with the full DLT solution is only possible if the initial estimates to the variances are physically related to the actual systematics and reconstruction distortions. A model for these initial estimates was developed in chapter (6), and shown to converge without iteration, producing reconstructed coordinates that are more accurate and stable than the standard implementation, even when robust numerical solvers were used for the latter. The model implementation was also shown to be computationally more efficient than its standard counterparts, and again, irrespective of numerical solvers.

The phase lag between the two X-ray cameras of the BAE unit was minimised using the validated procedures of Yeadon & King [97] and Pourcelot et al [98]. Notwithstanding, the evaluation of the phase lag correction was conducted against the path traced out by a ball bearing fitted to the inside of a lifter bar. The results indicate that both the corrected and uncorrected phase offset paths are in statistical agreement with the actual path of the tracer. However, the maximum deviations from the actual path were statistically significant for the uncorrected phase lag with notable improvement, with regard to statistical agreement in relation to the actual path, for the phase corrected data. The assessment of the phase lag minimisation is not conclusive, however, the validated formulation of Yeadon & King [97] and Pourcelot et al [98] reassures the implementation.

The accuracy achieved in the current work is limited by the pixel resolution of 512×512 . Further improvements could be achieved with the newer BAE units that produce images at resolutions of 1024×1024 . It is anticipated that at this higher resolution, the current implementations of image

processing, target centering and photogrammetric reconstruction should be able to reach reconstruction accuracies < 0.1 mm.

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Chapter 8

Validation tools for verification of DEM

The ability of DEM to accurately simulate the behaviour of experimental tumbling mills has been investigated by several researchers, with promising results. Cleary & Hoyer [103] demonstrated good agreement (in terms of power draw and visual snapshot comparisons of the charge motion) between the experimental and 2D DEM simulations of a centrifugal mill. Agrawala et al [104] and Rajamani et al [105] presented similar comparisons between the experimental tumbling mills and DEM simulations thereof. The ability of 2D DEM to model the dynamics of an instrumented experimental mill at start-up was established by Monama & Moys [106] by comparing the power drawn of the experimental and simulated mill. The position of the centre of circulation (CoC), as defined by Powell & Nurick [19] was used by Govender et al [107] to compare 2D DEM simulations with experimental particle trajectory data from a scale tumbling mill. More recent comparisons of 2D and 3D DEM with an experimental SAG mill using the position of the shoulder and toe of the charge as well as the CoC were presented by Cleary et al [108]. While the power drawn by a mill provides a consistent measure of the mill behaviour, visual comparisons of charge motion will always introduce some measure of subjectivity.

None of the the validation techniques presented in the literature are sufficiently rigorous to claim that DEM can simulate all aspects of the system correctly. Validating DEM against one aspect of a complex system, such as a mill, does not imply that the model adequately describes the full system. The hypothesis for this chapter is that a series of validation techniques (a validation toolbox) and objective comparisons are required to ensure that all features of the system are adequately simulated. This work contributes several techniques towards such a validation toolbox and is based on work by McBride et al [15] and Govender et al [89].

8.1 Validation algorithms

A series of validation algorithms were developed to allow meaningful comparisons to be made between the trajectory history of the tracked particle within the experimental mill and the trajectory histories of ten randomly selected particles within the DEM simulations. The validation algorithms allow the behaviour of a system of near identical particles to be inferred from that of a single particle or

multiple particles. The duration over which the marked particle is tracked must be sufficient to allow the particle to pass through all regions of the charge and thereby provide a representation of the bulk charge behaviour.

8.1.1 Bin algorithms

To provide a statistically meaningful means of comparison between the experiment and simulation, a probability distribution function of particle position, or any measured variable, within the mill is generated using a recursive binning algorithm as described by Govender et al [107].

The cross section of the mill is uniformly divided into a fine grid (50×50 cells) where each cell represents a bin. All data points within the individual bins are grouped together. The normalised count of experimental points falling within each bin represents the probability function of particle position within the mill. The bin algorithm allows the frequency of any measured variable to be expressed as a function of position within the mill. The velocity and acceleration of the tracked particle is determined from the experimental trajectory data using a second order Lagrange interpolation polynomial, finite differencing scheme according to Chapra & Canale [109]. The same finite differencing scheme is used to determine the velocities and accelerations of the tracked particle from their simulated values in the DEM simulation. The reader is referred to Jeffreys & Jeffreys [110] for a more detailed account of interpolation schemes.

Bin plots of a selected variable in both experiment and DEM simulations can be subtracted from one another to give an indication of the relative error; a bin difference matrix containing zero values would indicate a perfect match.

8.1.2 Locating key features of the bulk charge

The identification of unique features of the charge allows direct comparisons to be made between the experimental and DEM data. Two such features are the CoC and the equilibrium surface, Powell & Nurick [19] and McBride et al [15]. Powell defined the CoC as the point about which all the charge in the mill circulates and the equilibrium surface as the surface dividing the ascending, *en masse* charge from the descending charge. The process of identifying the CoC and the equilibrium surface was via visual inspection of photographs of the mill, figure (8.1), and X-ray trajectory plots of a single tracked particle, and therefore introduced operator bias. Cleary et al [108] used the concept of the CoC (termed the vortex centre or centre of recirculation) and the positions of the shoulder and toe of the charge to compare the charge motion in a scale mill with DEM simulations. The position of the CoC (determined via visual inspection of streak images) was shown to provide the most sensitive measure of the accuracy of their DEM simulation.

Definitions

Head: Apex of particle trajectory.

Departure shoulder: Region where particles depart from shell and enter free fall.

Centre of circulation: Point about which all charge in mill circulates.

Equilibrium surface: Curve differentiating the ascending en-masse charge from the descending.

Bulk toe: Point of intersection of tumbling (cascading) charge with mill shell.

Impact toe: Region where cataracting charge impacts shell or bulk charge.

K: Angle of repose of the charge.

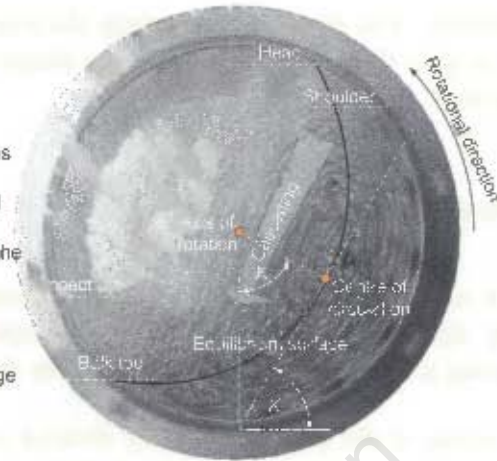


Figure 8.1: Photograph of an experimental mill showing the CoC and the equilibrium surface, McBride et al [15]

The following section provides rigorous definitions for the CoC and equilibrium surface using basic conservation principles, allowing their positions to be evaluated objectively using automated numerical techniques.

The principle of conservation of mass flow within a closed system is used to identify the CoC and equilibrium surface. Consider the cross section of the mill shown in figure (8.2) with the tracked particle's trajectory data superimposed. The cross section of the mill is sectioned using horizontal, vertical and radial planes, termed the control surfaces, whose normals lie in the plane of the cross section. The data in 3D is projected onto a vertical cross section to produce 2D planar data for this analysis. End effects due to the interaction of the particles with the front and back ends of the mill and other possible longitudinal position dependent effects are not quantified in this work. The number of particle trajectories crossing each control surface, the location of the intersection point (defined as the point where a particle's trajectory path intersects the control surface) and the relative sense of the crossings are recorded for the duration of the analysis. The relative sense of the crossing is defined as follows:

$$s = \text{sgn}(\tilde{d} \cdot \tilde{n}) \quad (8.1)$$

$$\tilde{n} = \tilde{n}_z \times \overrightarrow{AB} \quad (8.2)$$

where $\tilde{d}$ is the particle trajectory vector, $\tilde{n}$ the normal vector to the control surface, $\tilde{n}_z$ a vector defining the mills axis of rotation, and $\overrightarrow{AB}$ a vector orthogonal to $\tilde{n}$ in the XY plane, figure (8.2). The definition of the normal vector to the control surface $\tilde{n}$ is dependent on whether the mass flux count is performed from A to B or from B to A .

The cumulative count of particles crossing the control surface (representing the mass flux ϕ along the control surface) is performed for each control surface from both A to B and from B to A . The point

of maximum mass flux along a control surface, termed the mass flux equilibrium point or the balance point, occurs at the intersection of the mass flux count performed from A to B and from B to A . The point represents the position along a control surface where the amount of charge moving in a positive sense relative to the control surface is in equilibrium with the amount of charge moving in a negative sense, i.e. where

$$\frac{d\phi}{dl} = 0 \quad (8.3)$$

and l is the distance along the control surface.

The surface formed by linking successive mass flux equilibrium points within a family of control surfaces is termed the mass flux equilibrium surface. The position and physical interpretation of the mass flux equilibrium surface is governed by the orientation of the control surfaces over which the mass flux summation was performed and is apparent when the equilibrium surfaces are superimposed on selected velocity component bin plots, as shown in figure (8.3). The equilibrium surface generated by sectioning the mill with a family of control surfaces whose normals are parallel to the vertical Y axis is termed the vertical mass flux equilibrium surface. The vertical mass flux equilibrium surface separates the ascending, i.e. travelling in the positive Y direction, *en masse* charge from the descending charge. The horizontal mass flux equilibrium surface, formed by sectioning the mill using a family of control surfaces whose normals are parallel to the horizontal X axis, separates the *en masse* charge moving in a positive sense in the X direction from that moving in a negative sense. The radial mass flux equilibrium surface is formed by sectioning the mill using a family of control surfaces passing through the centre of the mill's cross section and whose normals are orthogonal to the mill's axis of rotation. The radial equilibrium surface separates *en masse* charge possessing a positive radial velocity from that possessing a negative radial velocity.

As a direct consequence of the principle of mass conservation, the mass flux along any control surface passing through the CoC will be maximum and independent of the control surface's orientation. This allows the CoC to be identified as follows, figure (8.4): the mass flux equilibrium point (the point of maximum mass flux along a single control surface within the family) on the control surface with the greatest maximum mass flux is identified and represents the CoC for that specific equilibrium surface.

It is possible for multiple CoCs to be identified using the aforementioned process. These multiple maxima bound the region in which the instantaneous CoC moves as a natural consequence of fluctuations in the motion of the tumbling charge, figure (8.5). The CoC along the respective control surface is assumed to be that of the position of the mean of the multiple maxima. The procedure described to determine the CoC was then applied to all three families of control surfaces (horizontal, vertical and radial) and their CoCs are calculated. The averaged position of the CoC, used to compare the experimental and simulated data, is defined as the centroid of the triangle whose vertices are the CoCs of the three different equilibrium surfaces.

An alternate approach to determine the position of the CoC would be to calculate the intersection of the mass flow equilibrium surfaces generated from the three families of control surfaces.

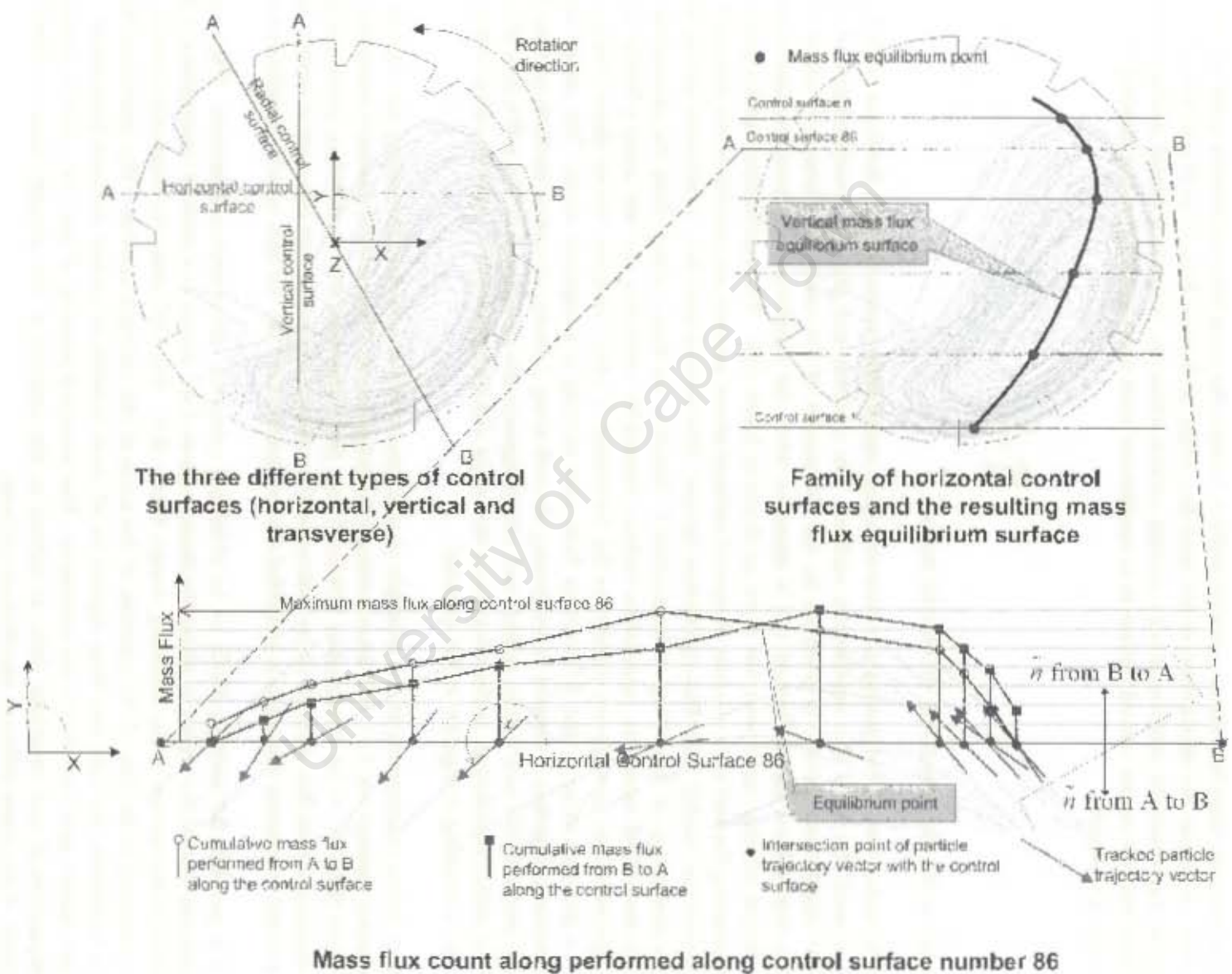


Figure 8.2: Schematic sectioning of the mill using various control surfaces and the mass flux count performed along a selected control surface to determine the position of the flux equilibrium point, McBride et al [15]

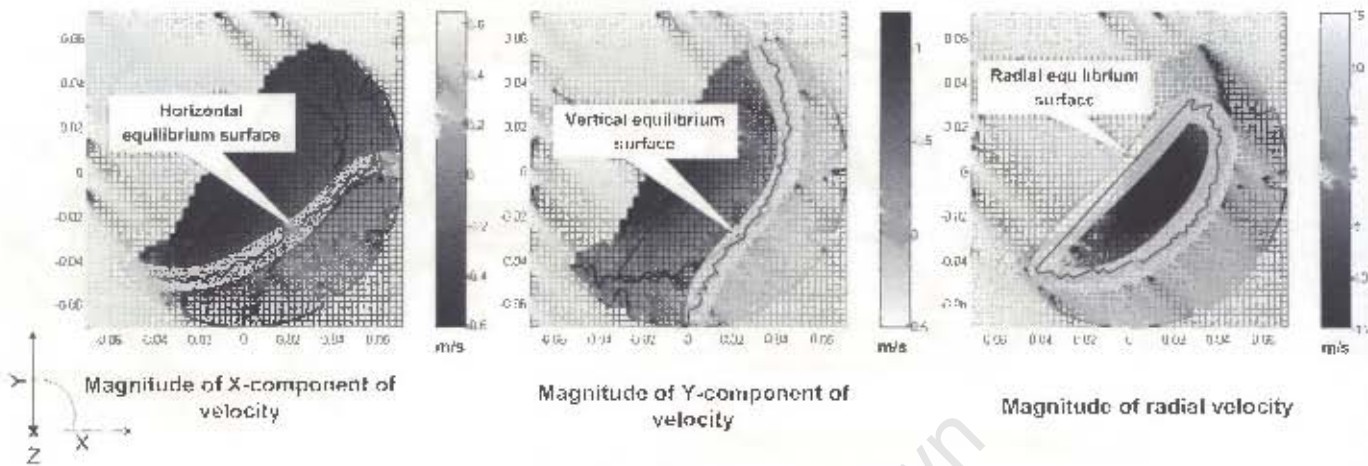


Figure 8.3: Physical interpretation of the mass flux equilibrium surfaces, McBride et al [15]

Powell and Nurick's [19] original definition of the equilibrium surface as the surface separating the ascending en masse charge from the descending charge, as shown in figure (8.1), is the combination of the horizontal equilibrium surface below the CoC and the vertical equilibrium surface above the CoC.

Previous attempts to locate the CoC and the equilibrium surfaces using numerical techniques by identifying the particle trajectory turning points (the points where the horizontal or vertical component of velocity of a particle changes sign) and fitting either an ellipse or a least squares cubic spline, Govender et al [107], to the data have several disadvantages when compared to the mass flux method presented. The trajectory path of a particle within the bulk of the charge is not smooth. Deviations from a smoothed trajectory path occur due to the numerous particle interactions within the charge. These deviations cause significant scatter in the particle turning point data and necessitated the development of techniques requiring operator input to identify the actual turning points, thereby introducing some level of potential user bias. Spurious turning points are dealt with naturally using the mass flux balance method as deviations from the smoothed trajectory path are cancelled upon integration. Inherent error estimates, based on parameters such as the mass flux across any control surface passing through the CoC being a maximum and independent of the orientation of the control surface and the net mass flux across an equilibrium surface being zero, add to the robustness of the method.

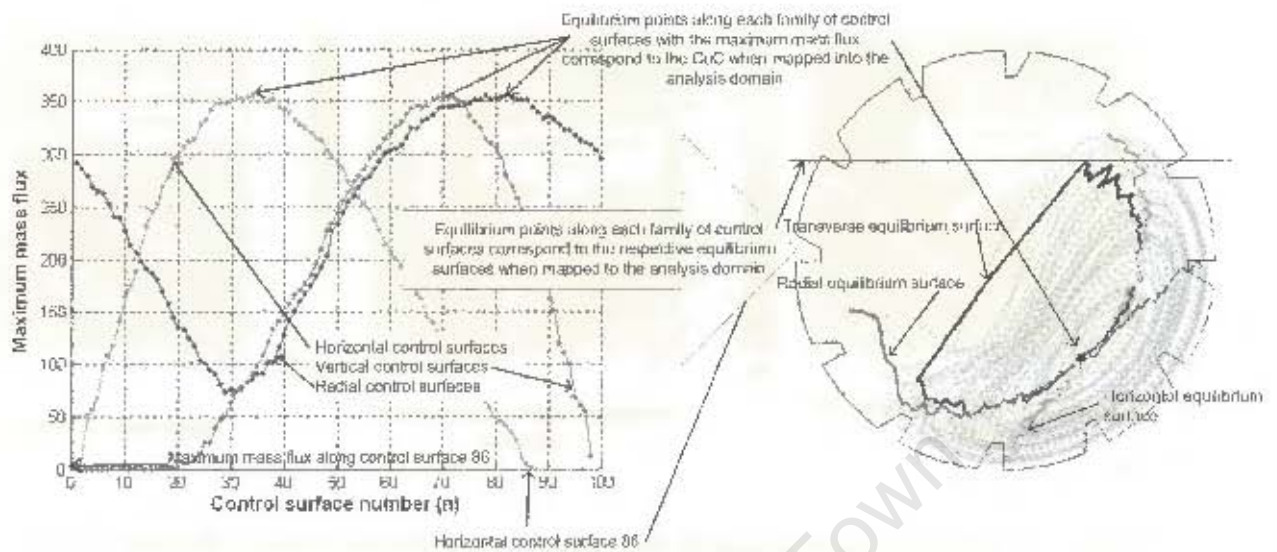


Figure 8.4: Relating the points of maximum mass flux along each equilibrium surface to the position of the CoC, McBride et al [15]

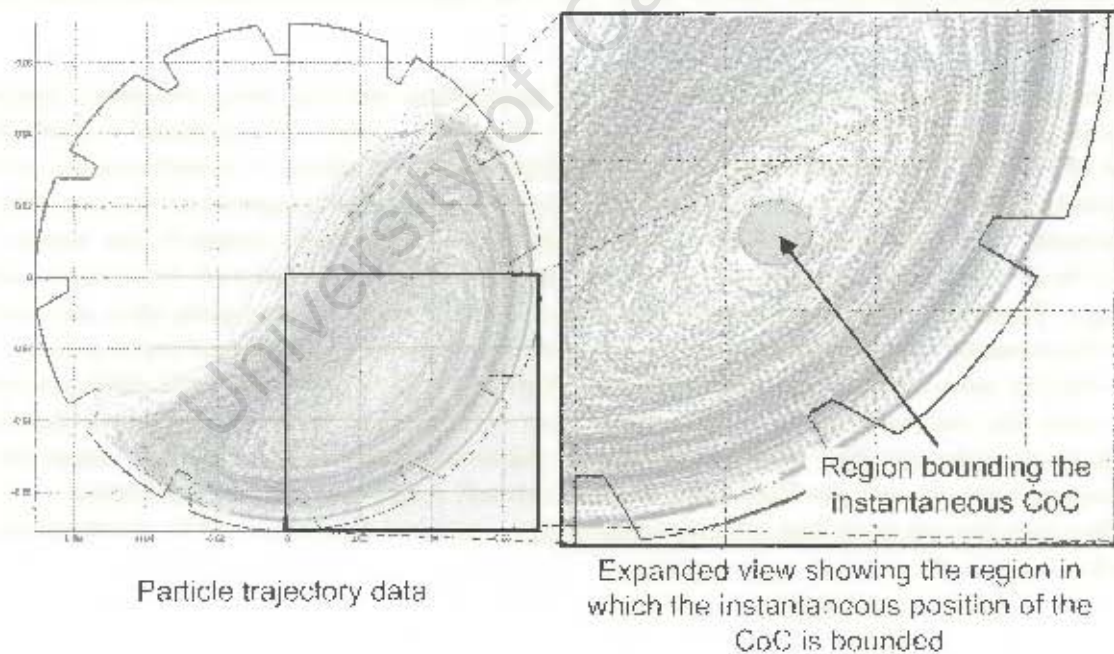


Figure 8.5: DEM trajectory data showing the region in which the instantaneous CoC moves, McBride et al [15]

Fraction of critical damping in the normal direction $C_n^*(-)$	0.2
fraction of critical damping in the shear direction $C_s^*(-)$	0.1
Normal contact spring stiffness (N.m^{-1})	1×10^6
Shear contact spring stiffness (N.m^{-1})	1×10^6
Coefficient of friction $(-)$	0.23
Maximum permissible time-step duration (s)**	1×10^{-5}

Table 8.1: Parameters used in the DEM simulations. The particles and the mill boundaries were assumed to have the same properties.

Notes: *The coefficient of resitution is related to the fraction of critical damping via an expression described by Mishra [28]; **the actual time-step duration could be below this value to satisfy stability criteria, Itasca [112]

8.2 DEM simulations

8.2.1 DEM model

The DEM simulations of the experimental mill were performed using both the 2D and 3D software package particle flow code (PFC) from Itasca. Contact between particles is modelled using the soft contact approach, whereby contacting objects are allowed to overlap. The contact force law, used to relate the overlap between contacting particles to a contact force, is the standard linear spring and dashpot model. An overview of the DEM and the contact model used in the PFC code is described in section (2.1).

The selection of the input parameters for the DEM simulation can have a significant influence on the behaviour of the simulation (Cleary et al [108]; Dong & Moys [23]). The experimental measurement of these input parameters (in particular, the coefficient of normal and tangential restitution, and the coefficient of friction) needs to be performed under conditions that reproduce those found in the experimental system, as they may be dependent on the relative contact velocity, Lorenz et al [111]. An experimental device to measure the particle interaction properties and the coefficient of friction under conditions similar to those experienced in the experimental mill has been constructed as part of a masters program that ran concurrently with this project. Unfortunately the measurement of the DEM properties for the plastic beads used in this work was not completed within the time frame of this thesis. Particle interaction properties from similar tests were used as estimates, Lorenz et al [111], and a series of 2D DEM simulations performed to adjust their values to account for the specific conditions found in the experimental mill, Govender et al [107]. The interaction properties were adjusted within a realistic range to find the best agreement between the simulated and experimental position of the CoC. Bin difference plots of particle position, velocity and acceleration were also used to assess the agreement between the experimental and simulated systems. The parameters used in the DEM simulations are listed in table (8.1). Discrepancies between the experimental beads and the numerical representation thereof as perfect spheres or disks arise as the plastic beads are not perfectly spherical and contain a hole through their centre, see chapter (3) for a quantification of the sphericity.

The simulations conducted by McBride (co-worker) represents the mill shell as a cylindrical geometric primitive in the 3D DEM simulation and as a series of line segments in the 2D simulation. In the 2D

DEM simulation ten equal length line segments are used to approximate the shell between consecutive lifters. Discrepancies between the experimental mill shell and the DEM representation therefore exist and contribute to differences in the experimental and simulated data.

The DEM simulation was broken into two stages; namely, settlement and analysis. The particles were generated within the mill and allowed to settle under gravity during the settlement stage. A representative sample of the plastic beads that constitute the bulk charge diameters were measured and the standard deviation thereof is included in the numerical model. An estimate for the number of particles to be generated was obtained by assuming an 85 % mill filling by volume and a void ratio of 0.3. By overfilling the mill, subsequent analyses can be performed without needing to regenerate the particle system. The “random” filling of an arbitrary region in space with particles conforming to a size distribution is an area of immense importance for the development of DEM and (the seminal work of Feng et al [113], and more recently Feng et al [114], deserve mention in this regard. The initial particle positions within the simulated mill were obtained using the random generation of particle positions, subject to the criteria that generated particles could not overlap. The initial radii of the generated particles were reduced by a factor of 30% to expedite this process. The particles’ radii were restored to their initial values once all particles had been regenerated. The process of expanding the particle radii can cause considerable overlap to occur between particles, resulting in excessive contact forces, and in turn excessive particle velocities. The particles’ velocities were therefore set to zero every 100 time-steps during the first quarter of the settlement phase.

A horizontal line is drawn across the mill to give the desired volumetric filling. All particles protruding above this line are removed at the beginning of the analysis phase. The mill was then rotated at full speed for two revolutions and the power draw was monitored to assess the system state. It was found that approximately one mill revolution was required to stabilise the power draw, indicating that a steady state had been achieved. The trajectory histories of ten randomly selected particles within the bulk charge were then monitored for a specified number of revolutions (15 revolutions in the 3D simulation and 45 in the 2D).

8.2.2 Comparison of DEM and experimental data

The trajectories of the monitored particles in the experimental and simulated systems are superimposed upon one another in figure (8.6). The position of the toe, the region where the outermost layer of cataracting or cascading particles impact the mill liner or other particles, prior to being drawn into the bulk of the charge, is exaggerated in the 2D simulation (see figure (8.1) for a schematic definition of terminology). This exaggeration is due to the void ratio of a 2D system of disks being greater than a similar 3D system of spheres, i.e. the packing efficiency is greater in a 3D structure than in a 2D one. This limitation of 2D systems was also noted by Cleary et al [108] when comparing 2D DEM simulations with experimental mill trajectory data. The position of the shoulder of the charge, the region where particles leave the liner and enter into projectile motion or cascade down the surface of the charge, is exaggerated in both 2D and 3D DEM simulations. The simulated trajectory paths cross the experimental ones in this region, indicating that the motion is not identical. The reason for this discrepancy is not verified at present, but it is most likely due to the estimated coefficient of friction being incorrect, and using only one coefficient of friction as opposed to different static and

kinetic coefficients of friction, which was identified by Powell [115] as being critical in obtaining the correct particle trajectory as it is projected off a lifter bar.

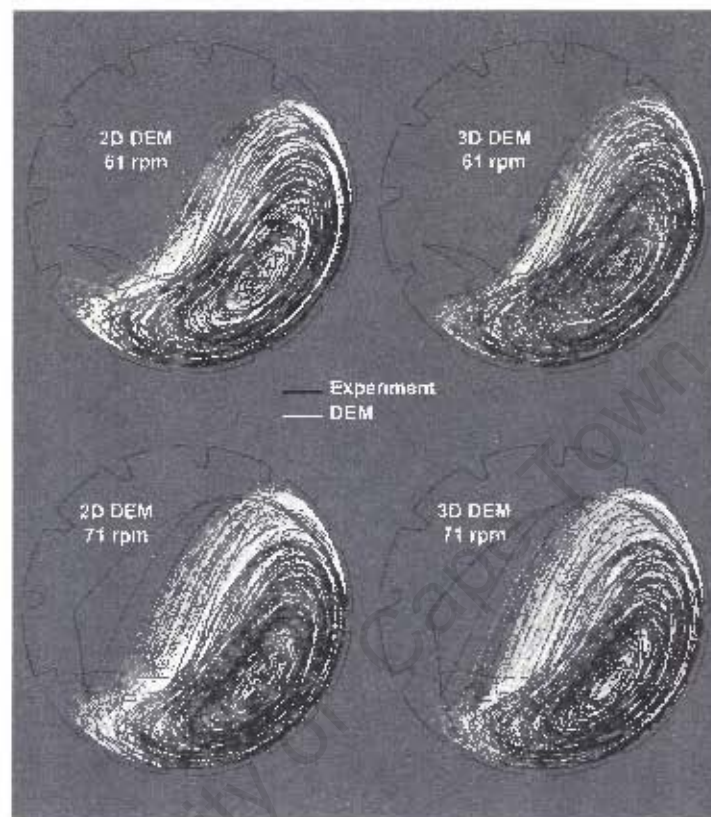


Figure 8.6: Comparison of the DEM and experimental trajectory data, McBride et al [15]

Experimental data appears to be lacking in certain regions, most notably in the region bounded by the cataracting and cascading trajectory paths. If no experimental data occurs within a certain region it becomes rather subjective as to whether the experimental sampling duration was insufficient to capture behaviour in this region or if instead it is highly improbable that a particle will ever enter the region. Only additional experiments will resolve this issue. One major component of the charge motion that obviously cannot be simulated in 2D DEM is the longitudinal particle drift in the mill. Figure (8.7) shows the trajectory path of a single tracked particle in both the experimental and simulated mills. The effect of the longitudinal charge motion becomes significant when flow through the mill, classification by a grate, and end effects are to be considered. The amount of longitudinal drift is significant in both the experimental mill and DEM simulation (note, the duration over which the experimental data was sampled is longer than the DEM simulation period and thus more longitudinal drift is present in the experimental system). No attempt to quantify the longitudinal drift is made in the current work.

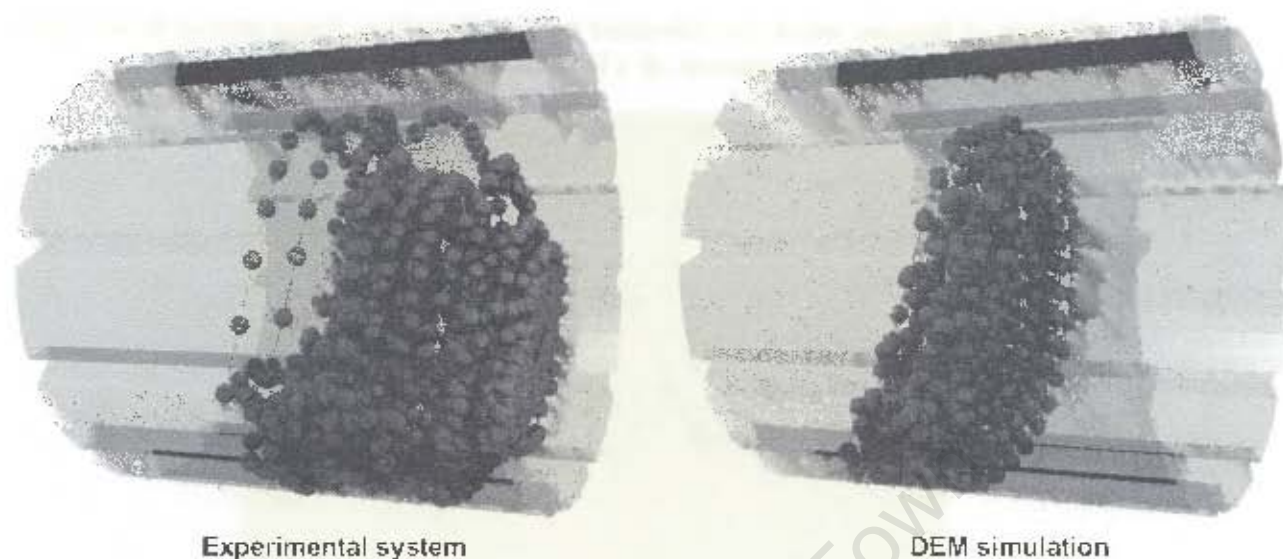


Figure 8.7: Comparison of the DEM and experimental trajectory data along the length of the mill

The experimental and simulated bin plots of the X, Y and radial components of velocity for test 1 (mill speed of 61rpm) are shown in figure (8.8) with the mass flux equilibrium surfaces superimposed. The smoothness of the bin plots for the simulated data relative to the experimental data highlights the need to perform additional experiments. The upper bound of the Y and radial components of velocity in the experimental and simulated systems are in good agreement. The upper bound of the X-component of velocity in the experimental data is approximately twice that of the simulated systems. Closer analysis revealed that this was probably due to linking particle trajectory data from separate experimental runs or due to missing experimental data. The absolute velocity of the particles in contact with the liner within the bulk of the charge corresponds exactly, as expected, with the rotational velocity of the mill (0.46 m/s). It is recommended that future work in this area should be to objectively define other key regions of the charge, such as the shoulder and toe, using the bin plots, thereby providing physically correct definitions that limit possible user biasing.

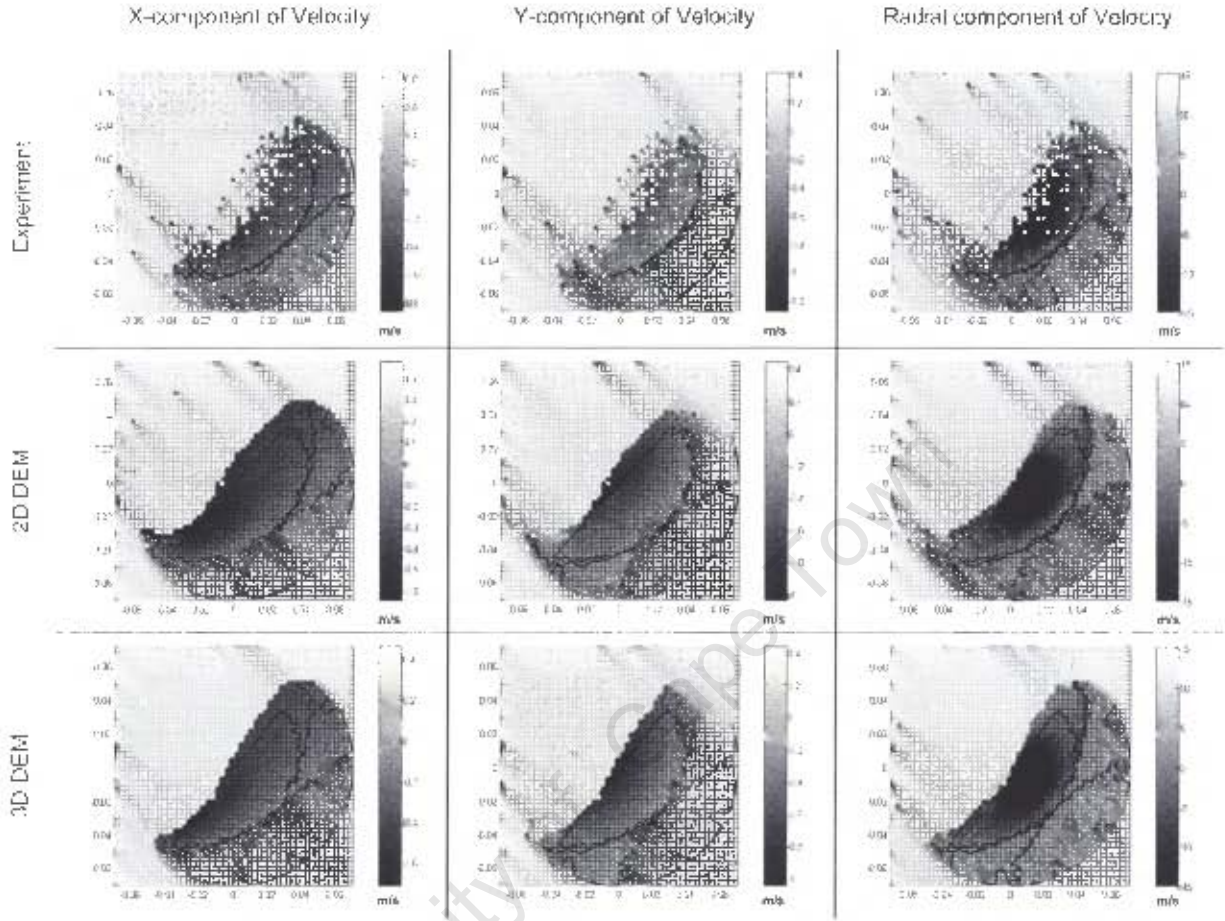


Figure 8.8: Experimental and DEM bin plots of selected components of particle velocity with the mass flow equilibrium surfaces superimposed

The bin difference plots of normalised frequency (i.e. the normalised probability distribution plots of particle position) and absolute velocity between the experimental and simulated mill system for the first test series (mill speed of 61rpm) are shown in figure (8.9); the scale indicates the percentage error between experimental and simulated data sets with a zero bin value being a perfect correlation. The normalised probability distribution plots of particle position were calculated by dividing the value in each cell of the bin plot of particle position by the cumulative bin count and then normalising to unity. The error E is calculated for each cell in the bin plot as follows:

$$E = \left(\frac{Y_E - Y_{DEM}}{Y_E} \right) \times 100\% \quad (8.4)$$

where Y_E is the bin value of the measured parameter in the experimental system and Y_{DEM} the bin value of the parameter in the simulated system. The error is undefined in regions where the experimental data is not present, and by definition is set to zero in these regions.

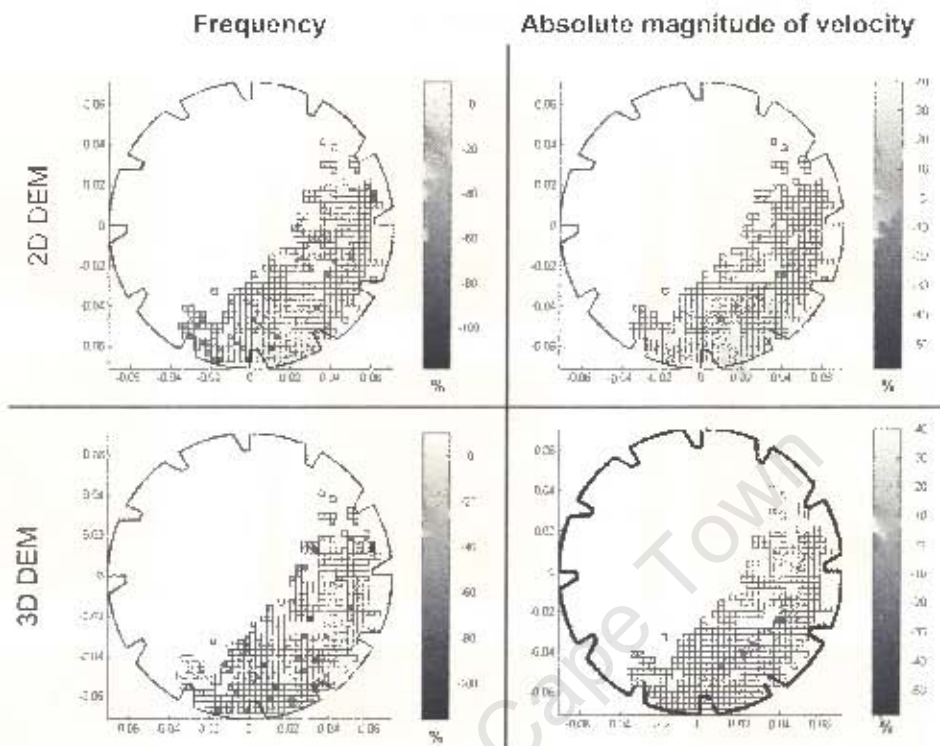


Figure 8.9: Bin difference plots (expressed as % difference) of frequency and absolute magnitude of velocity between the experimental and simulated systems

The position of the CoC can be used to compare experimental and simulated data. The position of the CoC would be invariant under stable equilibrium conditions, but the location of the charge in a mill fluctuates with time and a detailed investigation showed that the CoC moves within a bounded region between successive mill revolutions, as shown for the simulated 3D trajectory data in figure (8.10) (the analysis duration of 15 revolutions was divided into 5 sub-sets of three revolutions and the CoC calculated for each sub-set). This motion can be attributed to the surging and slumping of the charge, a phenomena observed in large scale mills (Vermeulen et al [116]) and DEM simulations of an experimental tumbling mill, Agrawala et al [104]. The CoC is therefore the centroid of the region of maximum mass flux that bounds the instantaneous CoC. The position of the centre of mass of the charge also moves in a similar manner to the CoC causing fluctuations in the power drawn by the mill, as evident in the simulated system. The time averaged position of the CoC, summarised in table (8.2), provides a unique means to characterise the charge, provided that the duration of the experiment is sufficient. As the mill speed increases from test 1 to test 2, the CoC moves upwards towards the shoulder and outwards towards the mill shell.

The 2D simulations appear to give better correlation to the experimental data than the 3D, but this is misleading. The exaggerated position of the toe in the 2D simulation draws the position of the CoC towards the toe and in turn towards the position of the experimental CoC, thus improving the simulated results. This emphasises the fact that the validation of DEM needs to be performed using a range of different comparative techniques. The simulated results differ by no more than approximately

	COC test 1		COC test 2	
	Speed 61.34 rpm		Speed 70.88 rpm	
	Position (mm)	Deviation (mm)	Position (mm)	Deviation (mm)
2D DEM	X = 26	3	X = 28	4
	Y = -30		Y = -27	
3D DEM	X = 29	10	X = 29	6
	Y = -22		Y = -23	
Experimental	X = 27		X = 32	
	Y = -32		Y = -28	

Table 8.2: Experimental and simulated position of the CoC

one and a half times the mean particle diameter from the experimental CoC.

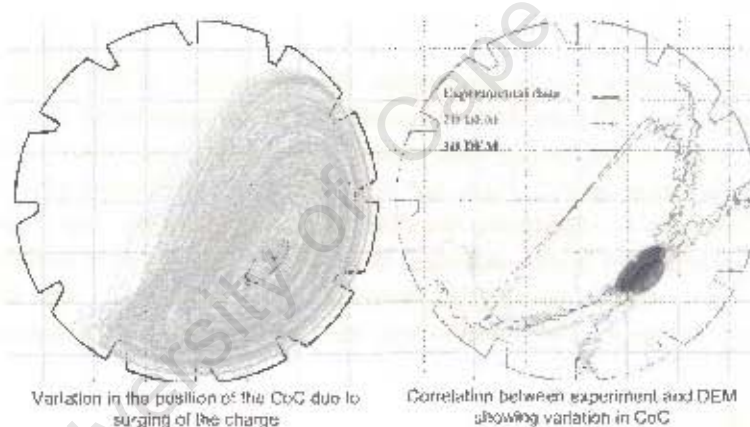


Figure 8.10: Variation in the position of the CoC and a comparison of simulated and experimental equilibrium surfaces (mill speed of 61 rpm)

8.3 Conclusion

The validation techniques presented characterise the mass behaviour of the charge based on observations of a limited number of particles. They are robust and noise free due to the inherent insensitivity to minor fluctuations in charge motion (i.e. spurious turning points) and the noise suppressing nature of the integration of experimental data. These methods have scope beyond experimental tumbling mills and can be used to characterise the motion of many closed systems.

Chapter 9

Analysis of trajectory data

9.1 Introduction

The objective of this chapter was to investigate the kinematics of the tracked particle for the range of experiments conducted. The analysis is based on the 3D positional data of the tracked particles. Each configuration (mill speed, charge filling and lifter profile) was filmed in 6 sets with a mandatory 30 second interval between sets¹. Each set was filmed at 50 frames per second for 16.7 seconds, resulting in 100.2 seconds of trajectory data for each configuration. As a consequence of correcting for the phase lag between the X-ray cameras, the effective filming rate was increased by a factor of 2 using the lowpass interpolation filter [102] described in section (7.7). The final image data are then effectively 0.01 seconds apart. The conventions used to describe a configuration is:

H: height of the lifter

W: width of the lifter

A: leading lifter face angle

S: speed of mill (% of critical)

Thus **H5-W9-A45-S90** refers to a mill run at 90% of critical speed and with lifters that are 5 mm high, 9 mm wide and with a 45° face angle.

9.1.1 Heuristic assessment of the filming duration

The X-ray experiments are based on the assumption that a single particle's trajectory can be used to represent the entire charge motion provided the filming duration is sufficient. The question of sufficiency is further complicated by logistical issues such as limited access to the angiographic equipment

¹ The BAE cuts off after 16.7 s of filming and takes approximately 30 s to startup.

and storage constraints. These factors necessitated a heuristic assessment of the filming duration for each configuration in order to ensure *a priori* validity of the experimental data.

The assessment of the filming duration was conducted by McBride et al [117]. The milling dimensions was that of the Perspex mill charged with 4596 balls and run at 70% of critical speed. Ten lifters, 9 mm high and wide, with a face angle of 60° was used. The procedure was as follows:

1. Track all particles in the mill for two revolutions to establish a base case.
2. Bin the positions of the base case data.
3. Track five particles for 38 seconds at the sampling rate of the angiographic equipment (50 frames per second). These represent the DEM experimental mill.
4. Bin the positions of the five particles.
5. Combine the bin data of the five particles and compute the percentage of base case bins occupied by each combination, see figure (9.3) and table (9.1) for details of combinations.

The combinations used are illustrated in figure (9.1) and listed in table (9.1). In terms of statistical agreement, one standard deviation (68% of the base case bins occupied) requires 45s of filming. The storage constraints of the angiographic equipment allow 100.2s of filming before the data must be downloaded.

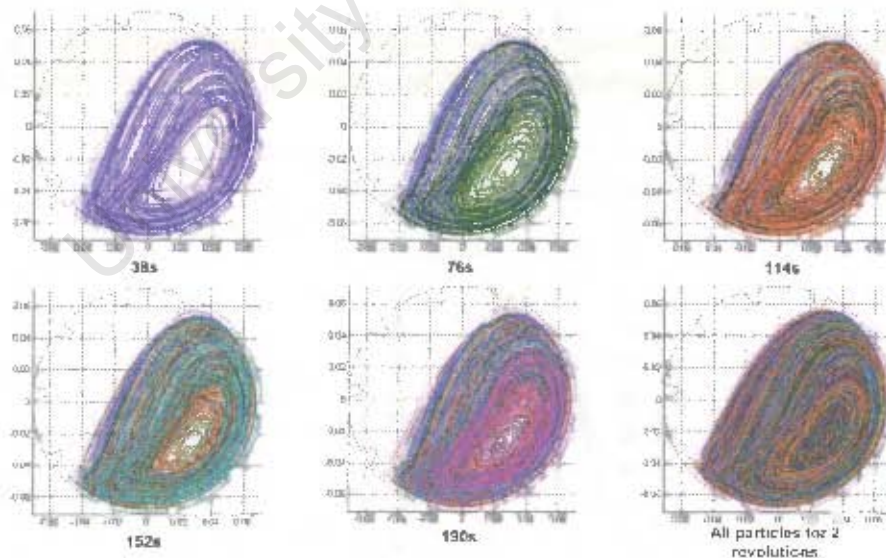


Figure 9.1: Combinations of trajectory plots of the five particles. Combinations and the effective sampling time is listed in table (9.1).

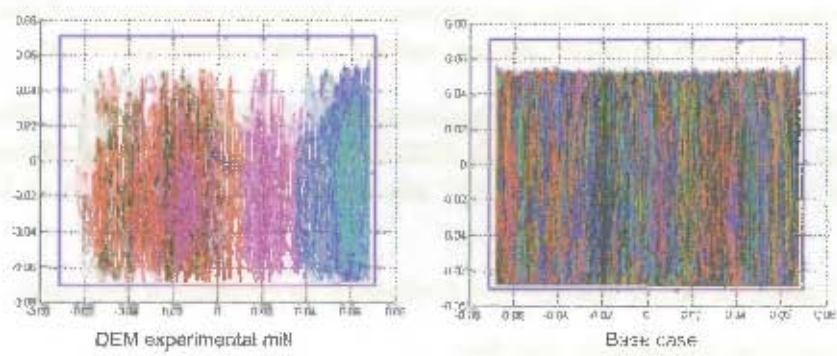


Figure 9.2: Side view of trajectory plots of the five particles demonstrating longitudinal drift.

particle combinations	combined sampling time	% of base case bins occupied
1	38	60
1 & 2	76	79
1, 2 & 3	114	85
1, 2, 3 & 4	152	88
1, 2, 3, 4 & 5	190	89

Table 9.1: The effect on the percentage of base case bins occupied by increasing the effective sampling time. The sampling time is effectively increased by combining bin data of identical particles.

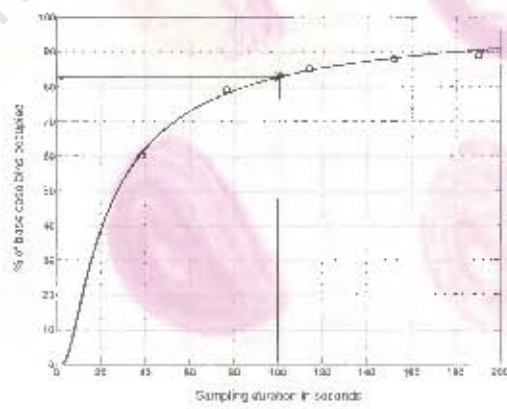


Figure 9.3: Heuristic assessment of filming duration. The percentage of base case bins occupied by the sampled DEM data plotted for a range of sample times. The solid line through the data follows an exponential trend.

Findings

1. The maximum allowed filming time is greater than one standard deviation and is therefore chosen as the filming time for each experimental configuration.
2. The percentage of base bins occupied in 100.2 s is 83%, which is statistically acceptable.
3. Longitudinal drift of the DEM experimental mill is apparent, figure (9.2).

9.1.2 Key zones in a circulation

In order to facilitate the analysis that ensues, the circulation of the tracked particle is divided into a number of distinct zones according to Powell [1], figure (9.4).

1. The circular path in the *en masse* region, where the ball is lifted up by the rotary motion of the mill - from A to B. This zone is used to calculate the slip between layers of charge, section (9.5).
2. The shoulder zone - B to C - where the ball falls away from the circular path, passes through the maximum height S, and then begins to fall down towards the toe region.
3. Cascading/cataracting zone - from C to T - where the ball tumbles down the other descending balls (cascading) or falls freely (cataracting), until it impacts in the toe region.
4. The toe region - T to A - where the ball impacts on the charge and is drawn into the rotary motion of the mill. This is an active region of the charge, and where the interactive forces are greatest.

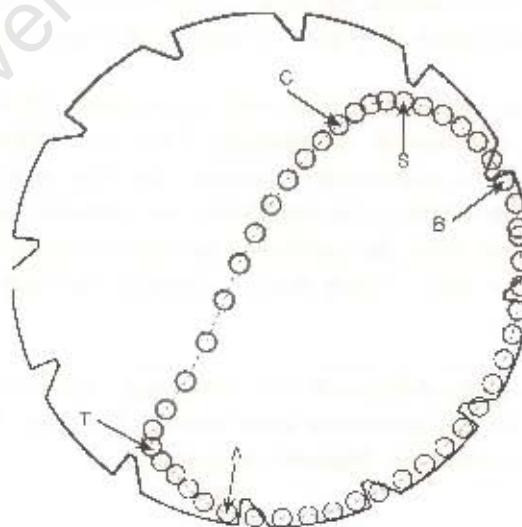


Figure 9.4: A typical circulation path followed by the ball, showing the various distinct zones

The analysis of the reconstructed trajectory data is conducted under the following categories:

1. Approximations of velocity and acceleration from positional data
2. An investigation into useful comminution based on the average, cumulative power draw
3. Toe and shoulder estimates
4. Slip within the charge
5. Charge surging phenomenon
6. A heuristic power model based on CoC and CoM trends

9.2 Approximations of velocity and acceleration from positional data

The experimental data may be viewed as a function $f(x)$ with a known set of discrete points $x_1, x_2, \dots, x_N$, however, we do not have an analytic expression for $f(x)$ that lets us calculate its value at an arbitrary point or determine its derivatives. If the calculations are confined to $x, x', x'' \in [x_1, x_N]$, the typical approach would be to use piecewise interpolation, where the primes denote differentiation.

The most widely used interpolation scheme is Lagrange interpolation polynomials [110]. Additionally, if the points are evenly spaced in time, the finite difference method can be employed for the differentiation. Another useful technique for determining the instantaneous velocity at the midpoint between positions x_i and x_{i+1} is the weighted average method used by Parker et al [44]. This technique is commonly used in PEPT experiments for slow to medium rotational speeds, see for example Cox et al [45]. The formulation described by Parker is said to give good approximations for the velocity.

The Lagrange interpolation polynomial was used to compare DEM and experimental kinematics in cases where the DEM time step varied. Essentially, if the user defined time step in PFC is not small enough to ensure stability at a particular iteration, the code automatically halves it. It should be noted that even though the DEM code calculates the velocity and acceleration directly from the contact, they are recalculated from the positional information in order to keep the analysis equivalent to that of the experimental data. This step is essential for meaningful comparison of DEM and experimental data.

This section describes the finite difference and Lagrange interpolation polynomial methods as possible schemes to be used on the positional data arising from the X-ray experiments. The weighted average method is also presented for illustration purposes.

9.2.1 Finite Difference Method

A key step in the finite difference method is to replace the continuous derivatives with appropriate approximations in terms of the dependent variable evaluated at different points in the region of interest. The basis for doing this is related to the Taylor Series expansion for a function in the vicinity of some discrete point, t_i . If the domain of interest is broken into evenly spaced increments, then

$$t_i = t_0 + i \Delta t \quad (9.1)$$

$$t_i = t_0 \pm ih \quad \text{where } h = |\Delta t| \quad (9.2)$$

In the case of the tracked particle, t_0 denotes the starting time of the filming, t_i is the time of the i^{th} frame and $\Delta t = 0.01 \text{ s}$ is the constant time increment between frames.

For the dependent variable, $x(t)$, the discrete representation becomes

$$x(t_i) = x_i \quad (9.3)$$

where $x(t_i)$ is the position of the particle in the i^{th} frame.

The Taylor Series expansion of x_{i+1} and x_{i-1} yields

$$x_{i+1} = x_i + x'_i h + \frac{x''_i h^2}{2!} + \frac{x'''_i h^3}{3!} + \dots \quad (9.4)$$

$$x_{i-1} = x_i - x'_i h + \frac{x''_i h^2}{2!} - \frac{x'''_i h^3}{3!} + \dots \quad (9.5)$$

where the primes denote differentiation.

These two equations are used to derive many of the common derivative approximations for linear systems. The first derivative approximations (the velocity) are:

Forward Difference Approximation

$$x'_i = \frac{x_{i+1} - x_i}{h} + O(h) \quad (9.6)$$

Backward Difference Approximation

$$x'_i = \frac{x_i - x_{i-1}}{h} + O(h) \quad (9.7)$$

Central Difference Approximation

By subtracting equation (9.5) from equation (9.4), we get

$$x_{i+1} - x_{i-1} = 2x'_i h + \frac{2x''_i h^3}{3!} - \dots \quad (9.8)$$

or

$$x'_i = \frac{x_{i+1} - x_{i-1}}{2h} + O(h^2) \quad (9.9)$$

By comparing the three difference approximations, equations (9.6, 9.7 and 9.9), it is clear that the central difference approximation is more accurate, with truncation errors on the order of h^2 (denoted as $O(h^2)$) while the forward and backward difference approximations have truncation errors on the order of h (denoted as $O(h)$).

The central difference approximation of the second derivative (the acceleration) is calculated by adding equations (9.5) and (9.4):

$$x''_i = \frac{x_{i-1} - 2x_i + x_{i+1}}{h^2} + O(h^2) \quad (9.10)$$

These approximations are the most commonly used representations of the first and second derivative. Investigations show that using approximations with higher order truncation errors do not improve the smoothness of the data. These derivatives tend to be large, particularly near sharp gradients like the toe region of the mill.

9.2.2 Polynomial interpolation

Another method for generating discrete derivatives is the Lagrange interpolation polynomial [110]. Here we assume that the solution to the discrete particle trajectory data can be approximated by a polynomial, and that the values at these points are exact. We thus differentiate the polynomial to obtain expressions for various derivatives.

Assume that $x(t) = at^2 + bt + c$:

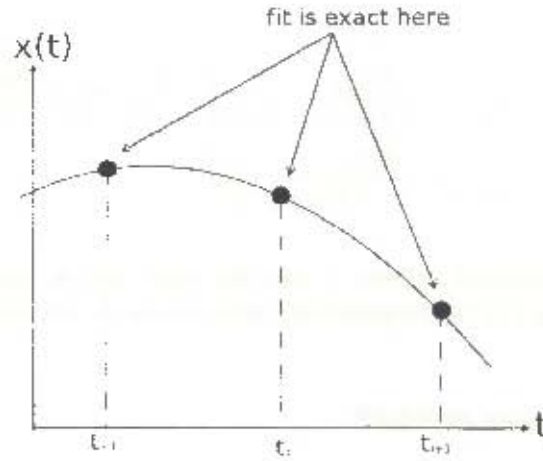


Figure 9.5: Illustration of polynomial fit to discrete data

In order to solve for the coefficients a , b and c , we apply polynomials to the three points illustrated in figure (9.5). Note that the time spacing between points need not be equal.

$$x_{i-1} = at_{i-1}^2 + bt_{i-1} + c \quad (9.11)$$

$$x_i = at_i^2 + bt_i + c \quad (9.12)$$

$$x_{i+1} = at_{i+1}^2 + bt_{i+1} + c \quad (9.13)$$

After solving for a , b and c , we obtain

$$x = x_{i-1} \left[\frac{(t - t_i)(t - t_{i+1})}{(t_{i-1} - t_i)(t_{i-1} - t_{i+1})} \right] + x_i \left[\frac{(t - t_{i-1})(t - t_{i+1})}{(t_i - t_{i-1})(t_i - t_{i+1})} \right] + x_{i+1} \left[\frac{(t - t_i)(t - t_{i-1})}{(t_{i+1} - t_i)(t_{i+1} - t_{i-1})} \right] \quad (9.14)$$

Differentiating equation (9.14) yields

$$x' = x_{i-1} \left[\frac{2t - t_{i+1} - t_i}{(t_{i-1} - t_i)(t_{i-1} - t_{i+1})} \right] + x_i \left[\frac{2t - t_{i+1} - t_{i-1}}{(t_i - t_{i-1})(t_i - t_{i+1})} \right] + x_{i+1} \left[\frac{2t - t_{i-1} - t_i}{(t_{i+1} - t_{i-1})(t_{i+1} - t_i)} \right] \quad (9.15)$$

and

$$x'' = \left[\frac{2x_{i-1}}{(t_{i-1} - t_i)(t_{i-1} - t_{i+1})} \right] + \left[\frac{2x_i}{(t_i - t_{i-1})(t_i - t_{i+1})} \right] + \left[\frac{2x_{i+1}}{(t_{i+1} - t_i)(t_{i+1} - t_{i-1})} \right] \quad (9.16)$$

If these derivatives are evaluated at time t_i and the time step is uniform, equations (9.15 and 9.16) reduce to equations (9.9 and 9.10) respectively with errors on the order of h^2 .

9.2.3 Weighted average method

The instantaneous velocity of the tracked particle at the midpoint of the i^{th} and $(i+1)^{th}$ positions, denoted, $\bar{v}_{i,i+1}$, is:

$$\bar{v}_{i,i+1} = \frac{\sum_{j=i-2}^{i+2} b_j v_j}{\sum b_j} \quad (9.17)$$

where v_j is the velocity at the j^{th} position and b_j is the weighting factor associated with the v_j . The velocities, v_j , are usually approximated using the Forward difference scheme, equation (9.6).

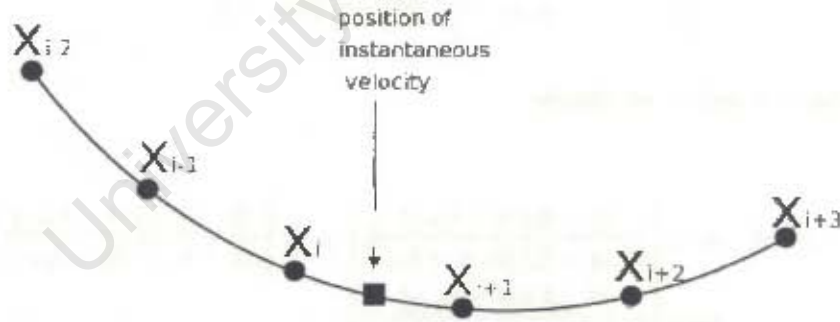


Figure 9.6: Illustration of points used in estimating the instantaneous velocity at the midpoint of the i^{th} and $(i+1)^{th}$ positions.

The weighted average scheme used by Parker et al [44] is based on this formulation and is given as:

$$\bar{v}_{i,i+1} = \frac{1}{9} \left[1 \times \left(\frac{x_{i-2} - x_{i-1}}{t_{i-2} - t_{i-1}} \right) + 2 \times \left(\frac{x_i - x_{i-1}}{t_i - t_{i-1}} \right) + 3 \times \left(\frac{x_{i+1} - x_i}{t_{i+1} - t_i} \right) + 2 \times \left(\frac{x_{i+2} - x_{i+1}}{t_{i+2} - t_{i+1}} \right) + 1 \times \left(\frac{x_{i+3} - x_{i+2}}{t_{i+3} - t_{i+2}} \right) \right] \quad (9.18)$$

In light of the discussion on difference approximations, it is apparent that the central difference approximation, equation (9.9), would be a better choice for calculating the v_j terms. Hence, a more accurate and robust estimate of the instantaneous velocity at the midpoint of the i^{th} and $(i+1)^{th}$ positions is:

$$\bar{v}_{i,i+1} = \frac{\sum_{j=i-1}^{i+2} b_j v_j}{\sum b_j} \quad (9.19)$$

where the v_j terms are now estimated using equation (9.9).

The symmetric distribution of the v_j terms in estimating $\bar{v}_{i,i+1}$, figure (9.6), enforces the robustness of equation (9.19). The instantaneous acceleration at the midpoint of the i^{th} and $(i+1)^{th}$ positions can be determined in a similar manner and is calculated as:

$$\bar{a}_{i,i+1} = \frac{\sum_{j=i-1}^{i+2} b_j a_j}{\sum b_j} \quad (9.20)$$

where a_j is the central difference approximation of the acceleration, equation (9.10), and the b_j are the associated weights.

The weighting scheme used in investigating the WACD in the present work is:

$$\bar{v}_{i,i+1} = \frac{1}{8} \left[1 \times \left(\frac{x_{i-2} - x_i}{2 \Delta t} \right) + 3 \times \left(\frac{x_{i+1} - x_{i-1}}{2 \Delta t} \right) + 3 \times \left(\frac{x_{i+2} - x_i}{2 \Delta t} \right) + 1 \times \left(\frac{x_{i+3} - 2x_{i+2} + x_{i+1}}{2 \Delta t} \right) \right] \quad (9.21)$$

and

$$\bar{a}_{i,i+1} = \frac{1}{8} \left[1 \times \left(\frac{x_{i-2} - 2x_{i-1} + x_i}{(\Delta t)^2} \right) + 3 \times \left(\frac{x_{i+1} - 2x_i + x_{i-1}}{(\Delta t)^2} \right) + 3 \times \left(\frac{x_{i+2} - 2x_{i+1} + x_i}{(\Delta t)^2} \right) + 1 \times \left(\frac{x_{i+3} - 2x_{i+2} + x_{i+1}}{(\Delta t)^2} \right) \right] \quad (9.22)$$

9.2.4 Comparison of interpolation schemes

The various interpolation schemes were compared in order to establish the method most suited for the experimental trajectory data. The comparisons are based on expected outcomes from the tracked particles kinematics. Although the particle trajectory may be classified as non-linear, there are regions of predictable motion like the outermost layer of the *en masse* region, region A to B in figure (9.4), and the cataracting zone, region C to T in figure (9.4). If we assume negligible vibration and a constant mill speed, it is expected that the angular velocity of particles, ω_p , moving contiguous to the mill shell in the *en masse* region will be equal to that of the mill's angular velocity, ω_m . The acceleration along this layer (the centripetal acceleration) is also expected to be directed radially inwards with a magnitude of $m\omega_m^2 R$, where m is the mass of the tracked particle and R the mill radius. The cataracting zone should produce particle accelerations, a_{cat} , of 9.81 m.s^{-2} if there are no mid air collisions between the particles while the x -component of velocity, v_x , should be constant in this zone.

The comparison was done for the mill run at 70% of critical speed with 12 rectangular lifters of height 7 mm and width 9 mm. This configuration was selected because the *en masse* layer along the mill shell and the cataracting zone are distinct. The uniform time step introduced by the X-ray filming result in the central difference approximations being equivalent to the Lagrange interpolation polynomial method and which are thus not presented explicitly in the comparisons. Table (9.2) presents the comparisons as the average value $\pm$ standard deviation. The comparisons are illustrated in figures (9.7 - 9.10).

	WACD	WAFD	LIP	FD	Expected/Measured
outermost layer of <i>en masse</i> zone					
$\bar{\omega}_p \pm \delta(\omega_p)$ (rad.s ⁻¹)	8.05 ± 0.22	8.03 ± 0.21	8.07 ± 0.27	8.15 ± 0.26	8.29 ± 0.01 (Measured)
$\Delta\theta \pm \delta(\Delta\theta)$ (°)	0.12 ± 0.02	0.14 ± 0.03	0.5 ± 0.2	0.5 ± 0.2	0 (Expected)
cataracting zone					
$\bar{a}_{cat} \pm \delta(a_{cat})$ (m.s ⁻²)	500 ± 38	500 ± 35	9.67 ± 1.66	9.61 ± 1.72	9.81 (Expected)
$\pm\delta(v_x)$ (m.s ⁻¹)	± 0.015	± 0.016	± 0.015	± 0.016	0 (Expected)

Table 9.2: Comparison of the various techniques used to estimate particle velocity and acceleration from positional data. The techniques are compared to expected *en masse* kinematics along the mill periphery, and to theoretical free-flight acceleration and *x*-component velocity of the cataracting particle.

WACD: weighted average central difference

WAFD: weighted average forward difference (Parker et al.)

LIP: Lagrange interpolation polynomial

FD: forward difference

Expected: expected values assuming constant mill speed and negligible vibration

$\Delta\theta$: average angular deviation between the radial direction and the direction of the acceleration vector for trajectories along the outermost layer of the *en masse* zone

$\delta(v_x)$: uncertainty in the average *x*-component of velocity (v_x) in the cataracting zone

$\bar{a}_{cat}$: average freefall acceleration in the cataracting zone

$\bar{\omega}_p$: average angular velocity for outermost trajectories in *en masse* zone

9.2.4.1 WACD and WAFD

The weighted average method produces very smooth velocity vectors at the interpolated midpoints (figures 9.7a and 9.9a). This conclusion is reinforced by the relatively low standard deviation of the *x*-components of velocity in the free fall zone. Both schemes produced unphysical trajectory profiles in magnitude and direction (figures 9.7b and 9.9b). Interestingly, the acceleration direction produced by the weighted methods along the mill periphery in the *en masse* region showed very slight angular deviations, $\Delta\theta$, from the theoretically expected radial direction. There are no noticeable differences in average angular deviation between the averaging used by Parker et al. and that suggested in this work.

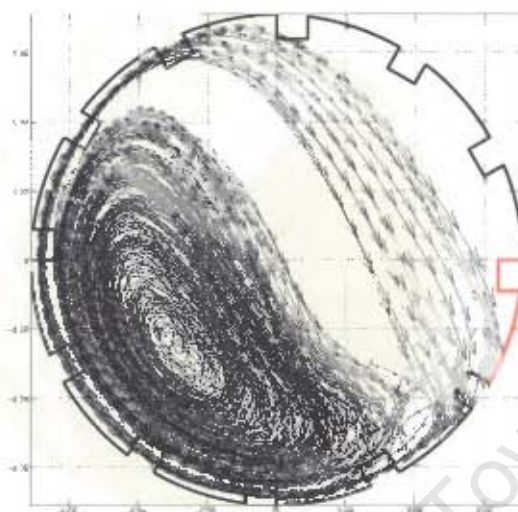
9.2.4.2 LIP and FD

The Lagrange interpolation polynomial scheme and the forward difference approximation produced realistic velocity and acceleration profiles as shown in figures (9.8 and 9.10). The deviations in the angular velocity are in statistical agreement with the measured values for the mill. The calculated acceleration in the cataracting zone is also statistically consistent with the known value of free-fall. The low standard deviation of horizontal velocity in the cataracting zone is acceptable in comparison to the other errors.

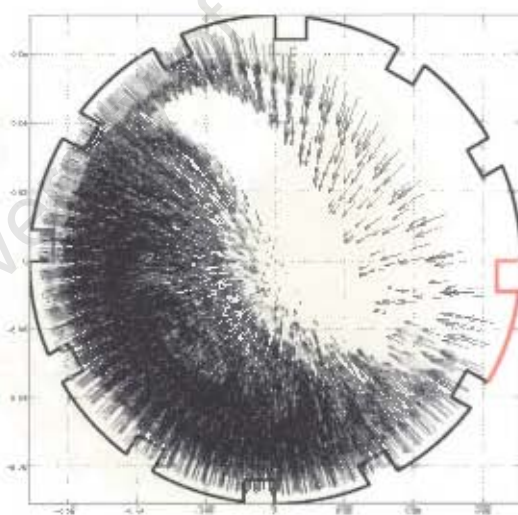
9.2.4.3 Conclusion

Despite the errors the LIP and FD methods produce angular velocities and free-fall accelerations that agree with the measured and expected values respectively within the uncertainties while the WACD and WAFD methods do not. None of the methods provide statistically acceptable centripetal acceleration directions, even though the weighted averaging methods come close.

The weighted averaging scheme was not used in the final analysis because of the unrealistic acceleration profile produced. The LIP method was statistically the most accurate with regard to kinematic predictions. All techniques performed poorly in the impact zone. The LIP method was used in the final analyses of the data. Plots of the velocity vectors, superimposed onto the particle positions, are shown in section (9.4).

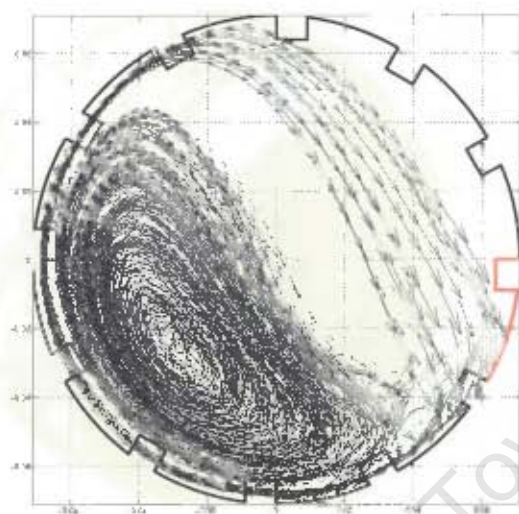


(a) Velocity

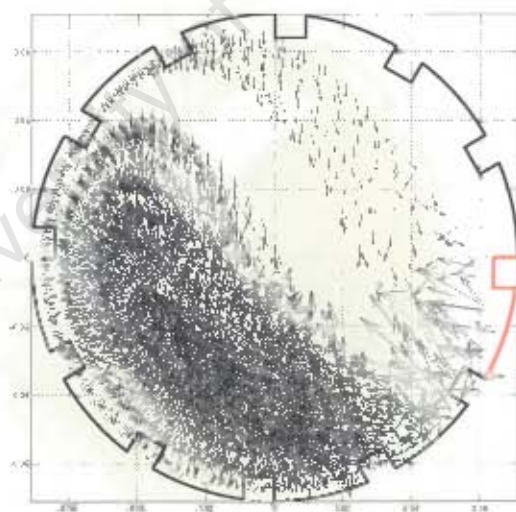


(b) Acceleration

Figure 9.7: Weighted average method using the central difference scheme

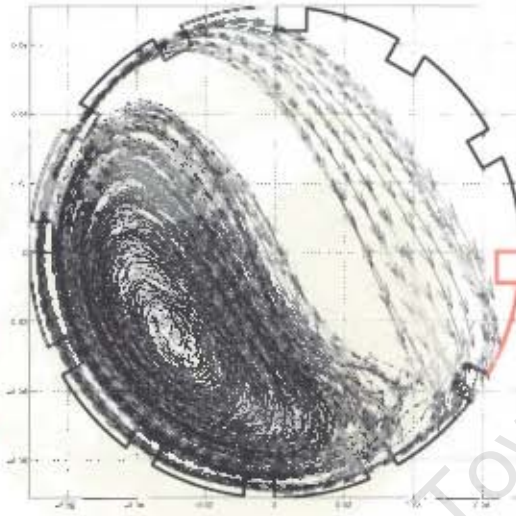


(a) Velocity

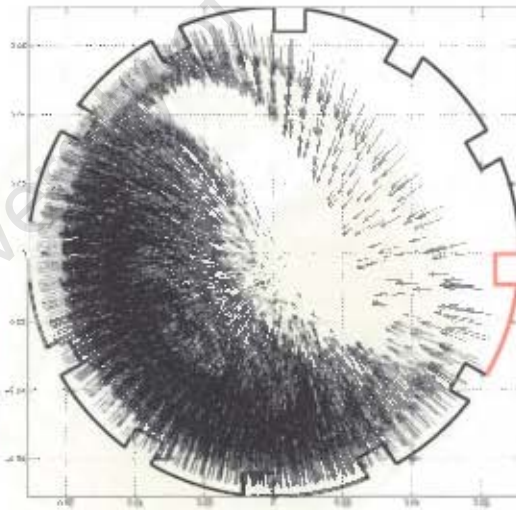


(b) Acceleration

Figure 9.8: 2nd order, Lagrange interpolation polynomial method

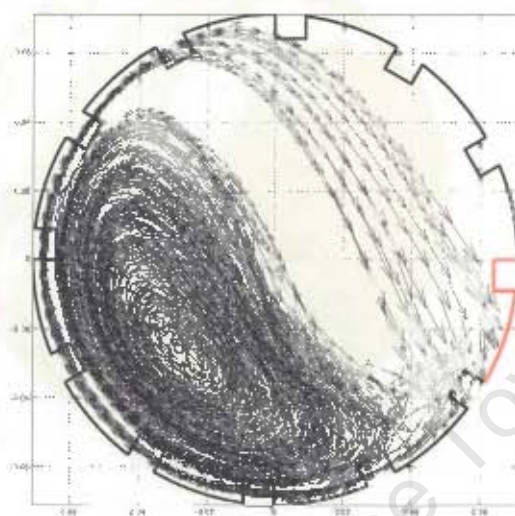


(a) Velocity

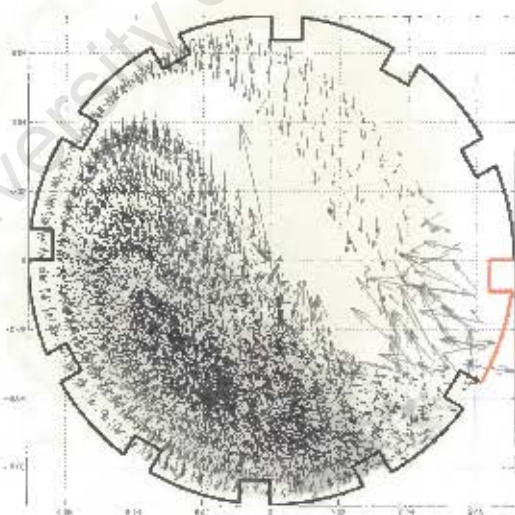


(b) Acceleration

Figure 9.9: Weighted average method using the forward difference scheme (weighting according to Parker et al [44])



(a) Velocity



(b) Acceleration

Figure 9.10: 1^{st} order, forward difference approximation

9.3 Circulation trends

The circulation of a particle is an important quantity in comminution due to its intimate relationship with SAG modelling. In particular the current SAG models assume that the energy associated with the particle is imparted once per mill revolution to the charge body and mill. This assumption has not been measured experimentally. The number of times a particle circulates as it passes down the length of the mill is key to determining the likelihood of it being broken to the required discharge size. Such information is traditionally difficult, if not impossible, to measure due to the harsh conditions in industrial mills. Parameters like the circulation rate are usually defined to make the mathematical manipulation of the models tractable, which is probably why the value of one is popular. The trajectory data of the tracked particle is well suited to investigated circulation trends. In fact, this is preferred due to the relatively short filming time afforded to the particle tracking experiments. These trends are now described.

9.3.1 Longitudinal motion

A typical industrial mill is characterised by a pressure gradient that occurs primarily due to particles flowing through the mill. The gradient is sustained by the continuous 'feeding' of rocks at the inlet and preferential removal of the smaller particles at the opposite end. The rate at which the particles are discharged is a complicated balance with the grinding requirements, and ultimately describes the residence time, defined to be the time taken for a particle to pass through the mill. For example, high speed mills produce the required outgoing sizes relatively faster, due to the increased impact breakage, leading to a higher throughput, coarser grind and consequently a shorter residence time, while slow speed mills tend to favour abrasion and attrition, leading to a finer grind, reduced throughput, and ultimately a longer residence time for the outgoing particles. Most mature grinding models indirectly incorporate residence time into their structure via the discharge rate. However, the description is still empirical, requiring further study to bring it into the realm of fundamentals.

A key ingredient to the understanding, and ultimately predicting, the discharge rate is intimately related to the longitudinal motion of the particles. Unfortunately, the batch nature of the current experimental work prevents the flow gradient from being formed. It was therefore decided to look for natural flow patterns, typified by changes in the tracked particle's axial motion.

Movement along the mill length was clearly observed when looking at the particle's trajectory from the side of the mill, figures (9.11, 9.12, 9.13, 9.14, 9.15, 9.16 and 9.17). A possible reason for such motion may be attributed to the end-window effect, which tends to set up a flow away from the window. The lack of a statistical basis for this conclusion, necessitated that the analysis be confined to a relatively shorter axial regime. To this end, the longitudinal displacement per circulation was investigated. Figure (9.18) illustrates the average axial displacement per circulation with mill speed, with the actual values displayed in table (9.3).

Configuration	Axial displacement (mm)
H5-W9-A45-S90	1.05
117-W9-A45-S90	1.53
H7-W9-A90-S70	0.26
H7-W9-A90-S90	0.24
H9-W9-A60-S55	1.30
119-W9-Acurved-S70	0.48
H9-W9-A60-S63	1.15

Table 9.3: Average axial displacement per circulation for each experimental configuration investigated

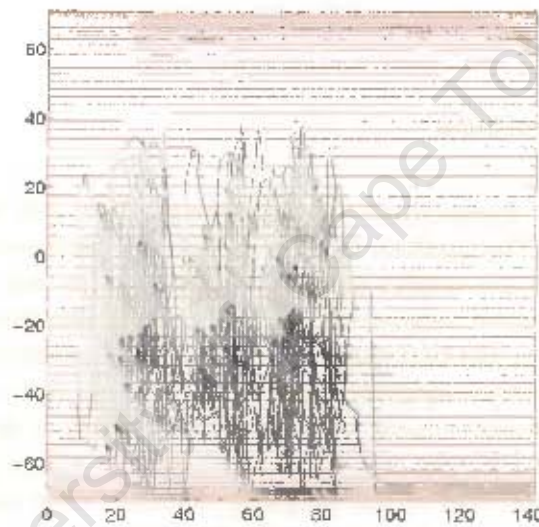


Figure 9.11: Longitudinal motion of particle: H5-W9-A45-S50

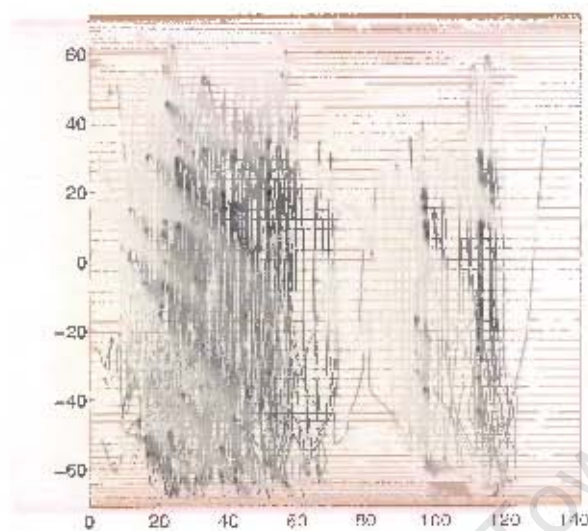


Figure 9.12: Longitudinal motion of particle: H7-W9-A45-S90

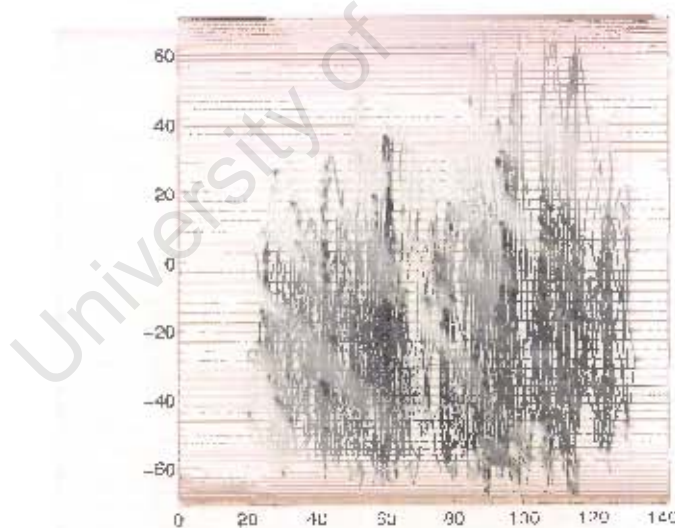


Figure 9.13: Longitudinal motion of particle: H7-W9-A90 S70

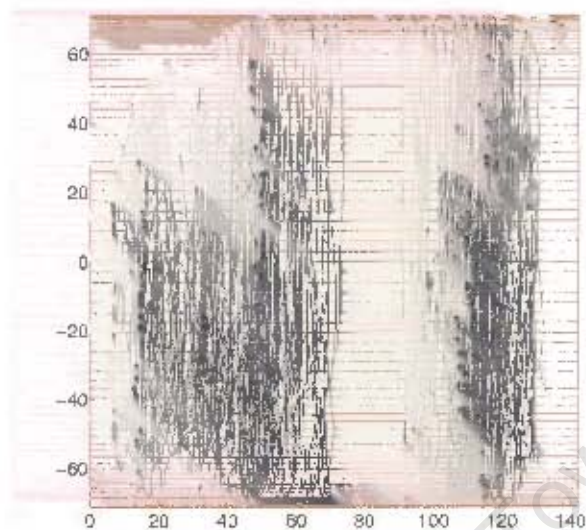


Figure 9.14: Longitudinal motion of particle: H7-W9-A90-S90

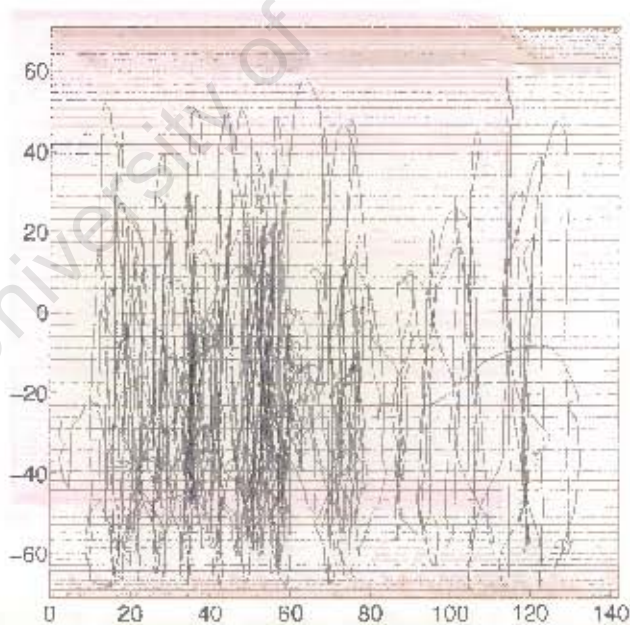


Figure 9.15: Longitudinal motion of particle: H9-W9-A60-S55

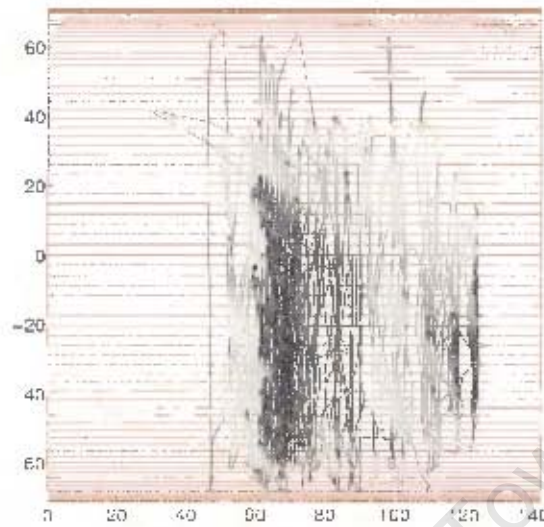


Figure 9.16: Longitudinal motion of particle: II9-W9-A60-S63

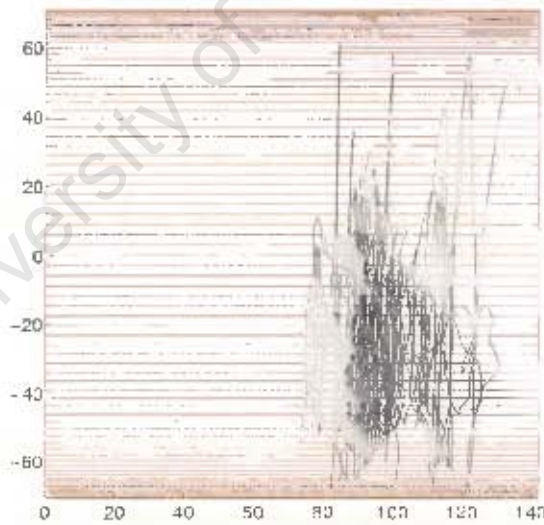


Figure 9.17: Longitudinal motion of particle: H9-W9-Acurved-S70

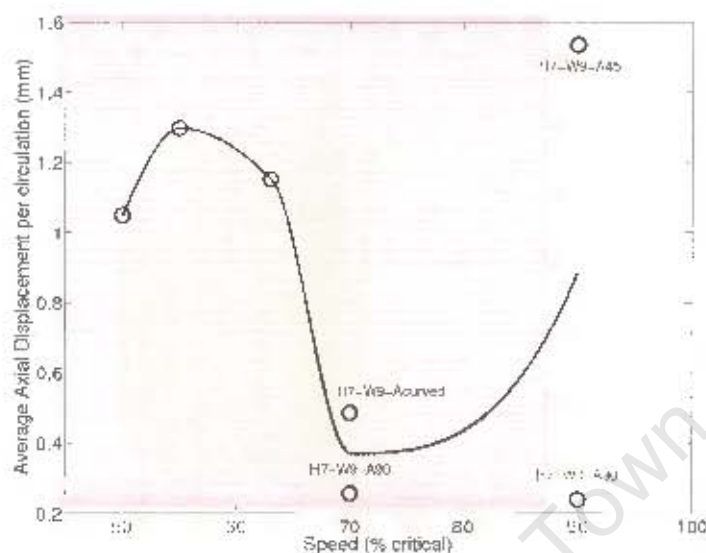


Figure 9.18: Average axial displacement per circulation with mill speed.

The following observations were made:

1. Longitudinal motion is clearly evident, even in the absence of the flow gradient
2. Axial displacement decreases with speed up to 70% of critical speed
3. The aggressive lifters produce distinctly smaller axial displacements per circulation than their less aggressive counterparts.

Comments

The duration of the filming prevented a rigorous and statistical assessment of longitudinal motion. The choice of investigating the average longitudinal displacement per circulation was deemed the only sensible approach to discerning statistically meaningful trends. This being said, it is interesting to observe the aggressive lifters produce conservative longitudinal displacement with the opposite being true for their less aggressive counterparts. A possible explanation could be due to the near centrifuging trajectory path followed by the particle at the H7-W9-A90-S90 milling configuration. The motion becomes more predictable, almost circular, with a reduced probability of being knocked off the axial slice around which it circulates. This observation is reaffirmed by the circulation rate described in the next section. The sharp leading face of the 45° lifter at the same speed produces trajectory paths that are significantly shorter than those of the aggressive lifters, and consequently the probability of being knocked off the axial slice path is increased.

9.3.2 Circulation rates

On the subject of a particle's behaviour per circulation, a useful quantity is the circulation rate, defined as the average number of circulation per revolution of the mill. It is known that most grinding models take this value to be one, implying that the particles impart energy once per revolution of the mill, irrespective of the mill speed. A plot of the circulation rate against mill speed is given in figure (9.19). The trend is distinctly linear with a rate of approximately 1.7 at 50% of critical speed, dropping linearly to a value of 1 at an extrapolated 116% of critical speed.

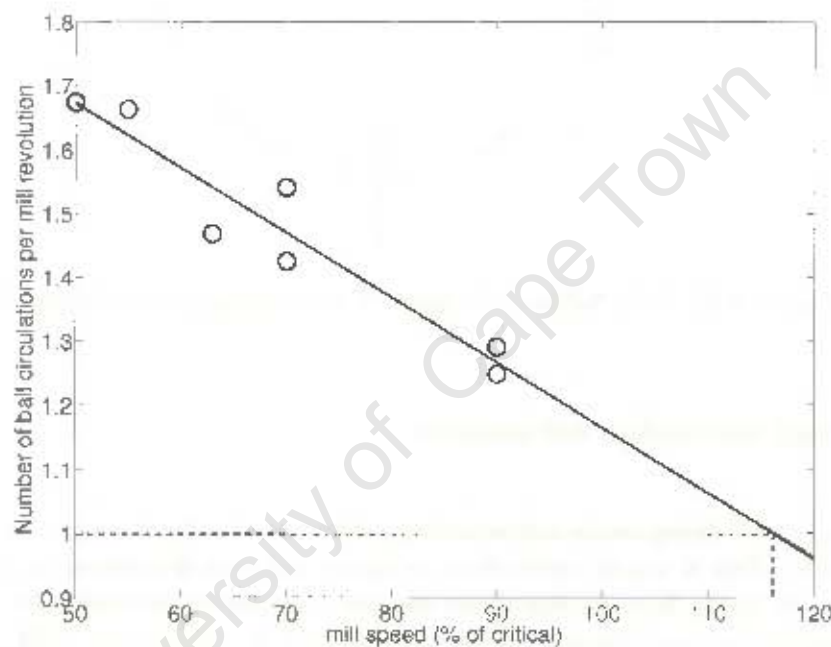


Figure 9.19: Variation of circulation rate with mill speed

Comments

At a speed conducive to cataracting, a particle is projected into freefall, following the path of the dashed line in figure (9.20). This path is clearly shorter than the path along the shell from A and B. In addition the tangential speed of the shell is lower than that of the freefalling particle between A and B. These differences culminate in the particle circulating faster than that of the mill and consequently the circulation rate of the particle is greater than one. Circulation rates of one can only be achieved when the particle is completely centrifuging. These observations are pertinent to current mill models that assume a circulation rate of one irrespective of mill speed.

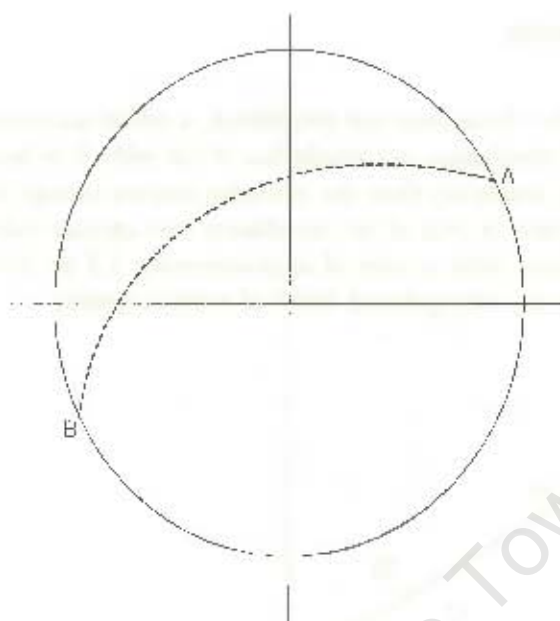


Figure 9.20: Path followed by particle after being projected into freefall

9.4 Toe and shoulder estimates

The characterisation of charge motion in tumbling mills is seldom done without including the notion of a toe and shoulder. This is usually qualitative in nature and very dependent on the perspective of the researcher, many of whom have employed the concept to facilitate the definition of an angle of repose - the line joining the toe and shoulder of the charge, Roif & Vongluekiet [118], Yashima et al [119], Liddell & Moys [37], Fuerstenau et al [39]. Powell [1], and more recently Powell & McBride [46], note that these “eye-balled” shoulder identifications seem reasonable at low mill speeds, but as the speed is increased above 80% of the critical speed, the definition becomes meaningless. The location of the toe suffers from similar shortcomings, becoming unclear, and author-dependent, when there is direct impacting of material on the shell of the mill.

With the advent of DEM, accuracy and exactness of charge descriptors have become important tools for the purpose of validating the numerical models. Cleary et al [108] and Powell & McBride [46] deserve mention in this regard for their improved definitions of the toe and shoulder region of the mill. These definitions are presented in figure (9.21) using trajectory data from an experimental run depicting clear cascading and cataracting: H7-W9-A90-S70.

Cleary et al [108] used the concept of a shoulder and toe in order to compare DEM simulations to photographs of a scale model SAG mill taken by Ward [120]. The definition of the shoulder and toe angles proposed by Ward [120] are measured counter-clockwise from the 3 o'clock location. The shoulder is defined by the angle associated with the highest point reached by the charge in the shoulder region of the mill while the toe is given by the angle to the highest point reached by the charge in the

toe region of the mill. The authors note that both definitions are straight forward at low mill speeds with the shoulder location becoming quite arbitrary at speeds $> 95\%$ of critical speed. In order to accomodate these difficulties the angles were bounded by an upper and lower limit.

Shoulder

- The lower limit of the shoulder angle is chosen to be the point where the trajectories of the particles near the lifters begin to diverge from the mill shell.
- The upper limit of the shoulder angle is chosen to be the point at which the trajectories of the particles deemed to be in the bulk of the charge (as opposed to being in the cataracting stream) have reached their highest point. This is identified by the horizontal streaks in the image.

Toe

- The lower limit of the toe angle, provided the mill speed is sufficiently high, is that point of the cataracting stream which has the highest concentration of cataracting particles above it.
- The upper limit of the toe position is identified by determining the point at which the curve of the free surface of the charge intersects the mill liner.

The definition of the toe and shoulder proposed by Powell & McBride [46] refine the ideas of Powell & Nurick [19] and is based on the analytically derived equilibrium surface obtained by McBride et al [15]. They argue that although the toe and shoulder concepts are useful in characterising the charge in the mill, they are poorly defined and don't allow for an objective charge description.

According to Powell & McBride [46] the shoulder region is defined using the concepts of a head and a departure shoulder of the charge, figure (9.21). The departure shoulder is defined as the uppermost point at which the charge departs from the shell of the mill. The media may still travel along the surface of a lifter bar, and may still be travelling in an upwards direction, prior to entering cascading motion or free flight. The head is defined as the apex position obtained by the charge, so it is the highest vertical position that the charge attains.

The toe region of the charge is defined using the concepts of an impact toe and a bulk toe, figure (9.21). The bulk toe is located by the point where the horizontal extension of the horizontal equilibrium surface from the inflection point (defined as the point at which the horizontal equilibrium surface moves in an upwards direction towards the impact toe) intersects the shell. The impact toe is where the cataracting charge impacts the shell in the toe region when the speed is sufficiently high. This is given by the point where the horizontal equilibrium surface intersects the shell of the mill in the toe region.

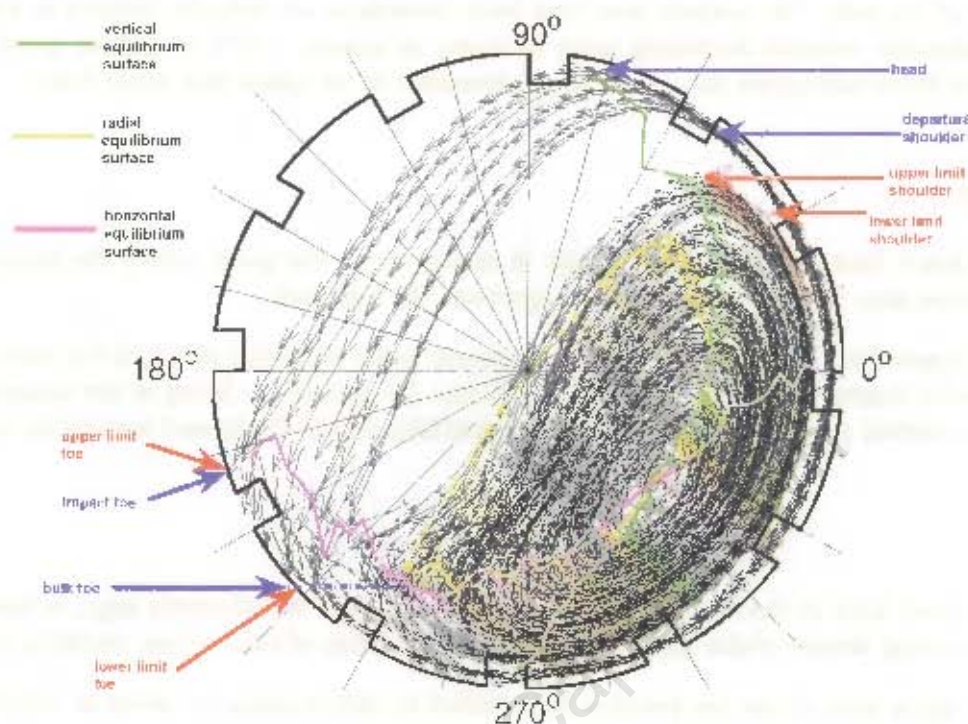


Figure 9.21: Experimental data of X-ray mill used to illustrate shoulder and toe estimates according to Powell & McBride [46] and Cleary et al [108]: H7-W9-A90-S70. The radial lines are intended to facilitate the estimation of the toe and shoulder angles by the reader.

red arrows: Cleary et al.

blue arrows: Powell & McBride

9.4.1 Discussion

The Powell & McBride definitions allow the departure shoulder and impact toe to be numerically obtained by counting the contact events along the shell of the mill, where a contact event is defined to occur when the particle's trajectory enters within a specified distance from the shell (usually one ball radius). The authors suggest searching along histogram plots (radial or conventional) of the contact events against angular position, see figure (9.22) for an implementation of the conventional histogram plot based on the charge profile shown in figure (9.21). The angle corresponding to the first zero count locates the departure shoulder while the impact toe position is located by the next non-zero count. The head of the shoulder region is simply the highest point reached by the charge while the bulk toe can be located by the upward turning point of the horizontal equilibrium surface.

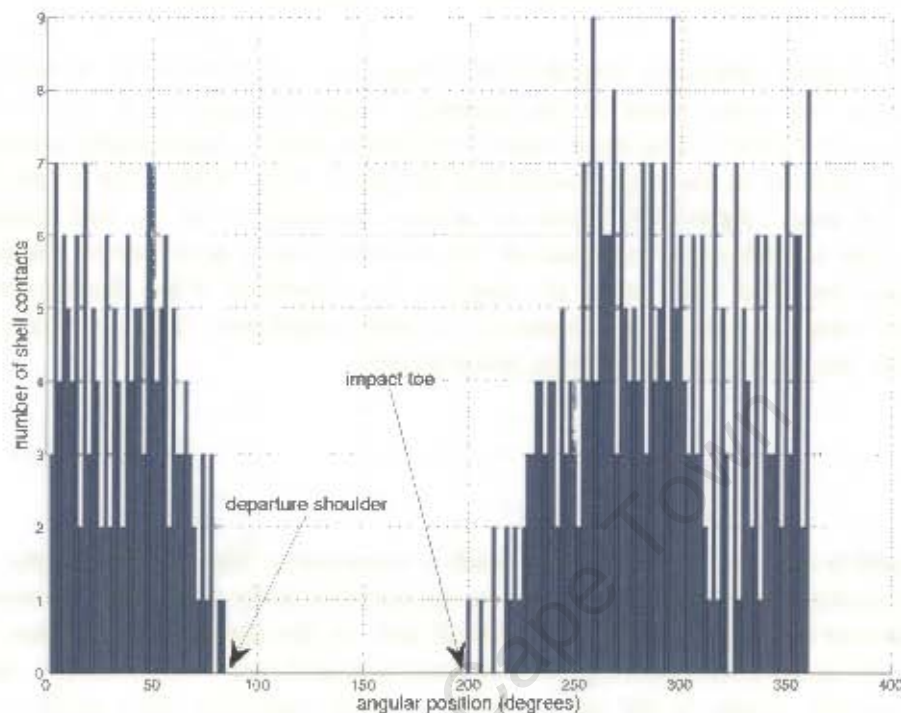


Figure 9.22: Histogram plot of contact events with shell in order to determine departure shoulder and bulk toe: H7-W9-A90-S70

The shoulder and impact toe can be numerically obtained according to the methodology suggested by the authors. The bulk toe location is not easily implemented in a numerical sense due to the noisy path followed by the horizontal equilibrium surface. It does however follow the trend of a third order polynomial for all charge shapes investigated in this thesis. It is therefore suggested that a third order polynomial be fitted to the data in the least squares sense and the lower inflection point used to determine the vertical position of the horizontal line that locates the bulk toe. Appendix (A.6) describes the fitting of polynomials in the least squares sense.

The Cleary et al. definition of the shoulder is not easily implemented in a numerical sense due to non-specific descriptions like “... *trajectories of particles near the lifters* ...”. The exclusion of the cataracting stream also complicates the location of the shoulder for charge profiles that are dominated by cataracting. In order to incorporate their definition into a numerical routine, the following assumption was made:

- The lower limit of the shoulder angle is chosen to be the point where the trajectories of the particles located approximately *one lifter height away from the shell* begin to diverge from the mill shell. Note that “*near the lifters*” was replaced by “*one lifter height away from the shell*”

9.4.2 Toe and shoulder estimates for the experimental data

The toe and shoulder estimates described by Cleary et al. and Powell & McBride were numerically implemented in this thesis based on the numerical schemes suggested by the authors and the modifications described above. The third order polynomial used to numerically implement the bulk toe estimation is indicated by the blue, dashed lines in figures (9.23 - 9.26). The inflection point is denoted by the big, red point. Table (9.4) shows the angular positions of the toe and shoulder relative to the mill center. Due to differences in definition, the shoulder region described by Powell & McBride is on average higher than that of Cleary et al., however, the movement of the shoulder with changes in mill configuration seems to follow similar trends for both definitions. The toe region depicts negligible differences for the milling configurations investigated.

Findings

When the mill is operated at a low speed with a conservative lifter profile, figures (9.23a and 9.25a), no cataracting occurs and the two toe regions are approximately coincident. Comparing this to figure (9.23b) indicates that even a dramatic upward shift in the shoulder region due to a relatively big change in mill speed (from 50% to 90% of critical speed) and maintaining a similar lifter profile causes a negligible change in the toe position. A more aggressive lifter profile even at a moderate mill speed (70% of critical speed) dilates the shoulder region and moves the upper limit of the toe relatively higher due to the induced cataracting of the charge while the lower limit remains relatively unchanged, figure (9.24a). Similar charge behaviour is observed with a slightly less aggressive lifter profile, figure (9.25b). Keeping the aggressive lifter profile and increasing the speed to 90% of critical brings the most dramatic changes in the shoulder and upper limit of the toe, and still no appreciable changes in the lower limit, figure (9.24b).

Conclusion

1. A conservative lifter profile has a greater influence on charge shape than high mill speeds, figure (9.23).
2. Aggressive lifter profiles seem to induce cataracting effects even if the speed is maintained at a moderate level, figures (9.24a and 9.25b).
3. Aggressive lifters combined with moderate or high mill speeds cause distinct cataracting effects. The cataracting stream is maintained even when the lifter begins to wear², figure (9.25b).

²a worn lifter profile is simulated by using a curved surface to the leading lifter face

Mill configuration	shoulder (°)		toe (°)		shoulder (°)		toe (°)	
	Cleary et al.				Powell & McBride			
	SU	SL	TU	TL	head	dept. shoulder	impact toe	bulk toe
H5-W9-A45-S50	34	19	210	210	35	27	210	210
H7-W9-A45-S90	55	55	210	222	64	64	210	222
H7-W9-A90-S70	50	33	199	223	77	52	199	223
H7-W9-A90-S90	56	31	178	220	90	90	178	220
H9-W9-A60-S55	41	7	225	225	41	5	225	225
H9-W9-Acurved-S70	47	26	214	228	68	52	214	222
H9-W9-A60-S63	30	15	225	225	63	35	225	225

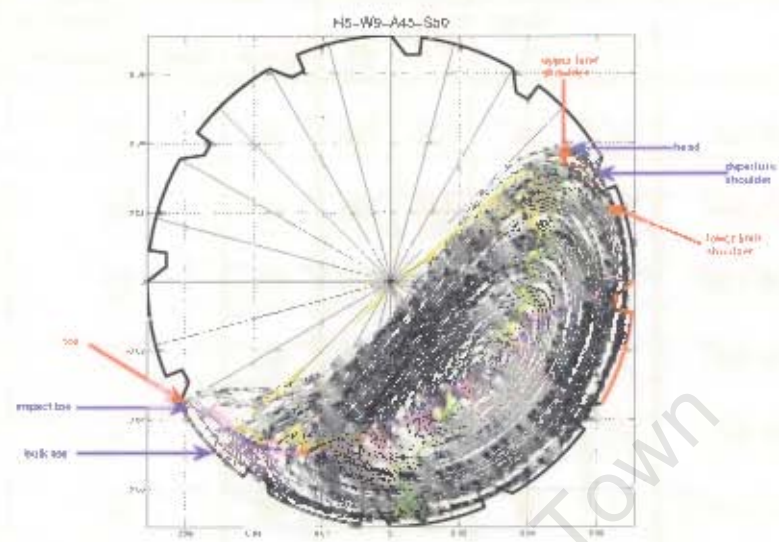
Table 9.4: Angular position of toe and shoulder using analytic methods based on the definitions by Cleary et al [108] and Powell & McBride [46]

SU: shoulder upper limit

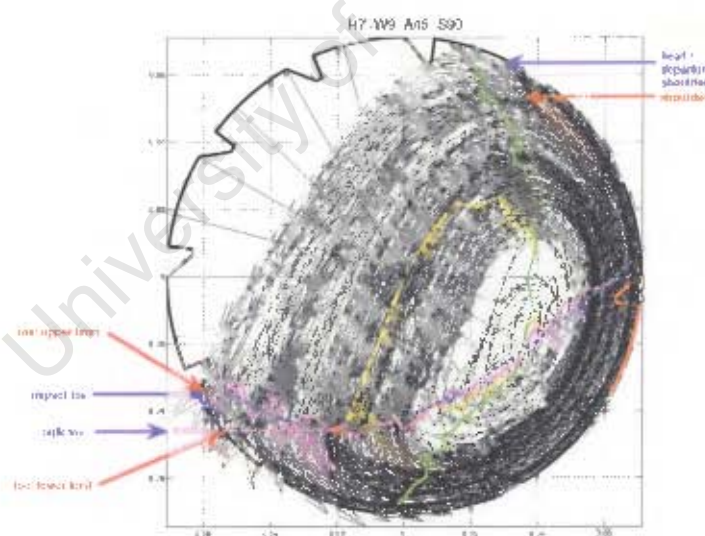
SL: shoulder lower limit

TU: toe upper limit

TL: toe lower limit

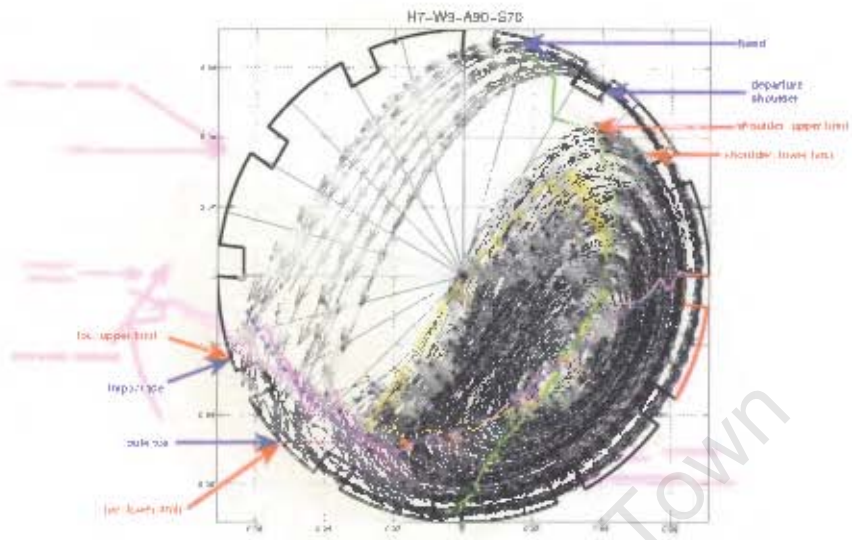


(a)

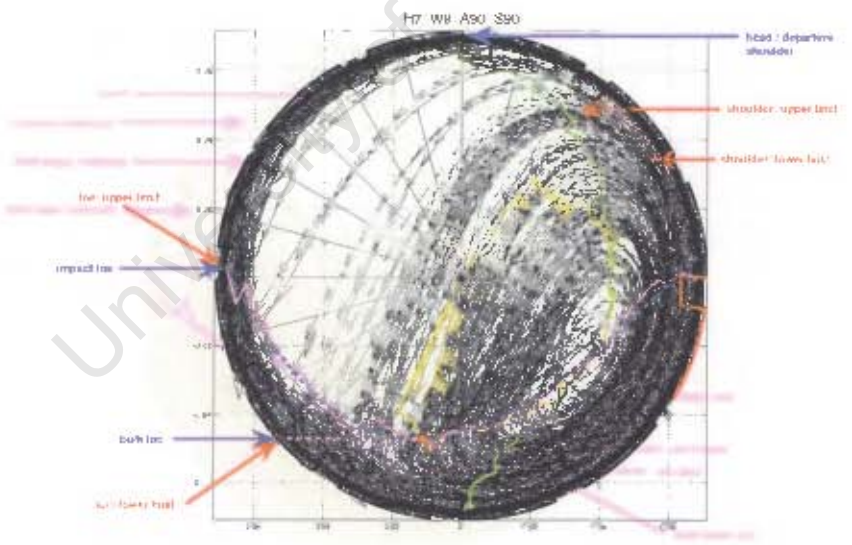


(b)

Figure 9.23: toe and shoulder estimates

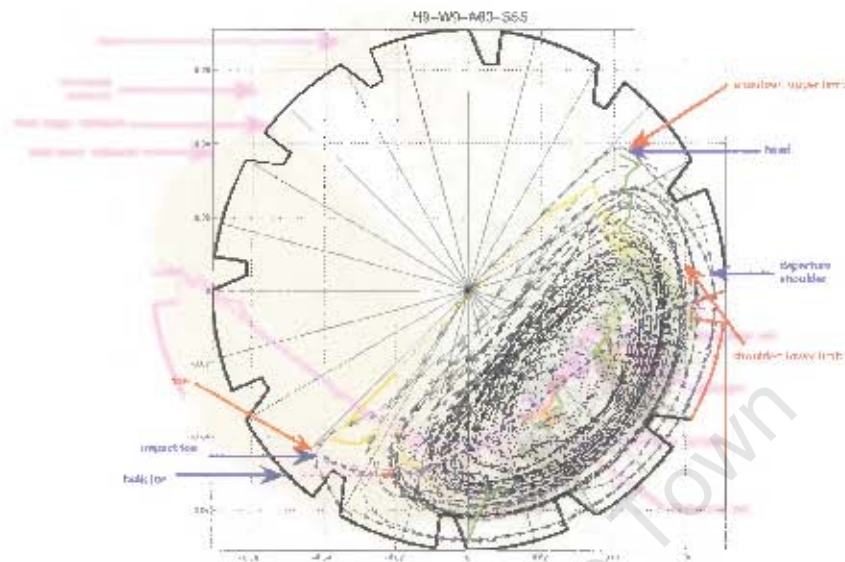


(a)

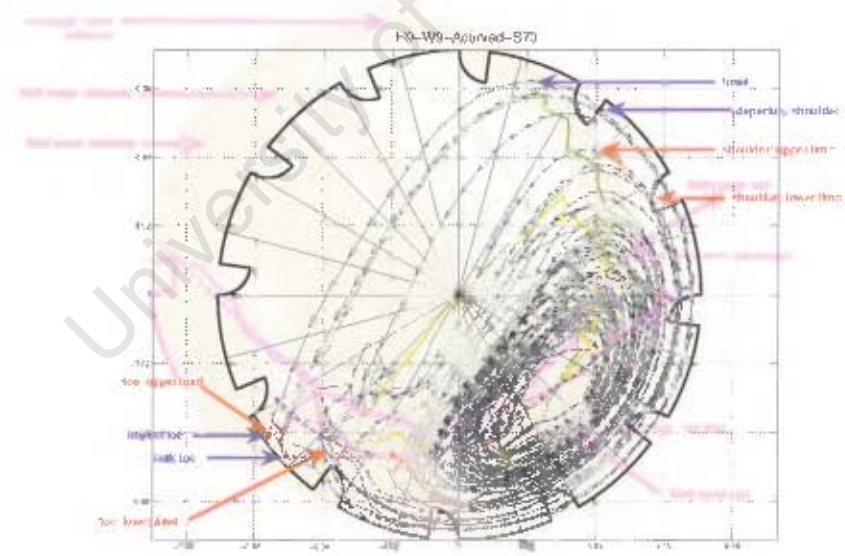


(b)

Figure 9.24: toe and shoulder estimates



(a)



(b)

Figure 9.25: toe and shoulder estimates

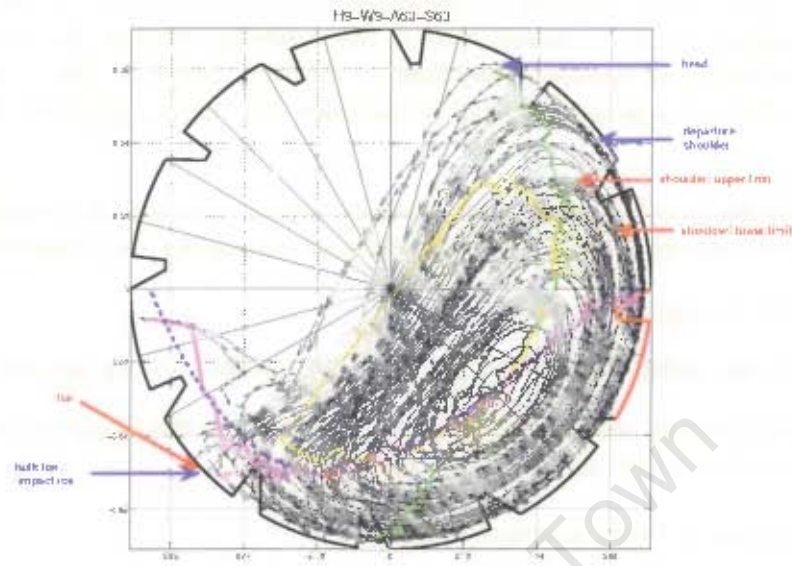


Figure 9.26: toe and shoulder estimates

9.5 Slip within charge

A significant amount of breakage in tumbling mills is due to abrasion, where fine fragments break loose from the particle surface due to shear force interactions. These particles constitute the fines in a typical mill charge. The production of fines is caused by combinations of:

1. Slip between layers of charge due to velocity gradients across layers.
2. The effective pressure of the charge acting on the particle.

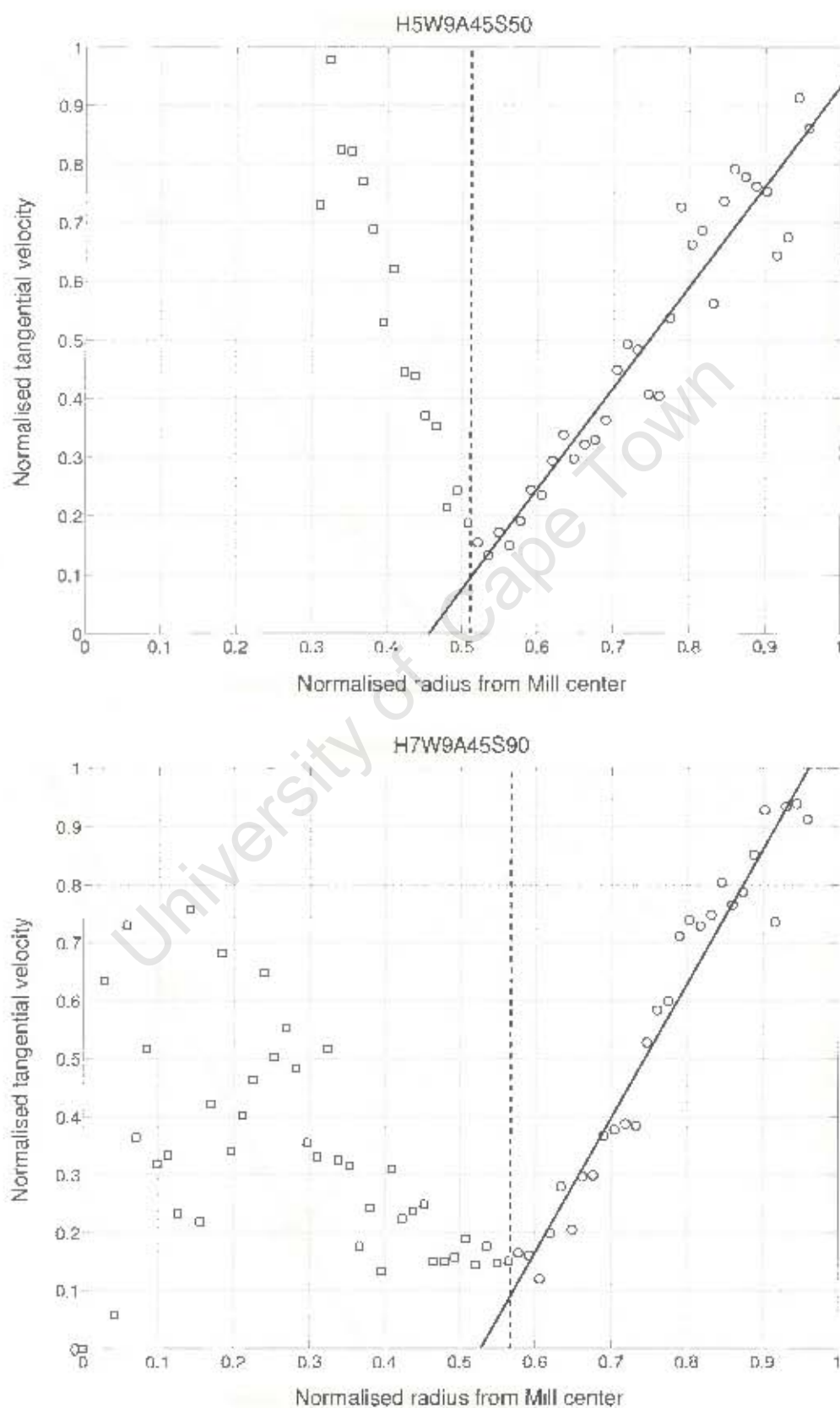
Velocity gradients across circulation layers are typically used to demonstrate slip within the charge. Powell [1] noted that balls in the *en masse* region and travelling upwards, driven by the mill shell and the balls surrounding them, are not necessarily moving with the same angular velocity as the mill. The relative difference between the angular velocity of the mill and the ball was used to indicate the degree of slip taking place. Vermeulen & Howat [10] also showed that the angular velocity of the concentric layers decreases as the CoC is approached, becoming zero at the CoC. Morrell [42] used laboratory experiments to derive an expression for the variation of a particles rotational rate with radial position. This gave an effective measure of slip within the charge as a function of radial position, which he used in his power model.

The nature of the X-ray data prevented a direct analysis of charge pressure and its influence on slip. However, the velocity gradients across ball layers is a natural consequence of the velocity fields calculated in section (9.2). The slip region is defined to include all trajectories within angular positions

from A to B, figure (9.4). Position A corresponds to the first point at which the particle becomes completely keyed-in to the *en-masse* motion after passing through the toe region. Point B corresponded to the departure shoulder described by Powell & McBride [46]. The shoulder positions are given in table (9.4). The procedure used to calculate the velocity gradients is as follows:

1. At each radial position within the *en-masse* region compute the average tangential velocity of the particle; see Appendix (A.7) for details on calculating the tangential velocity
2. Normalise the tangential velocity
3. Normalise the radial positions of the particle by dividing it by the mill radius
4. Using the technique outlined in section (8.1.2), identify the radius to the CoC, R_{CoC} , and divide the tangential velocity data into two sets:
 - (a) velocities at radial positions $> R_{CoC}$
 - (b) velocities at radial positions $< R_{CoC}$
5. Using the values in 4(a), fit a least squares straight line to the normalised tangential velocities versus normalised radial positions. The slope of the straight line gives a measure of the degree of slip

The data, with least squares fit superimposed, is shown in figures (9.27 - 9.30). The dashed ("-") vertical line denotes the radial position of the CoC. The graphs are cropped in favour of the least squares fit thereby omitting those high velocities in the turbulent region of the CoC. The lack of discernable trends in and around the CoC region reinforces the choice to exclude this region.

Figure 9.27: Quantification of the degree of slip between charge layers in the *en-masse* region

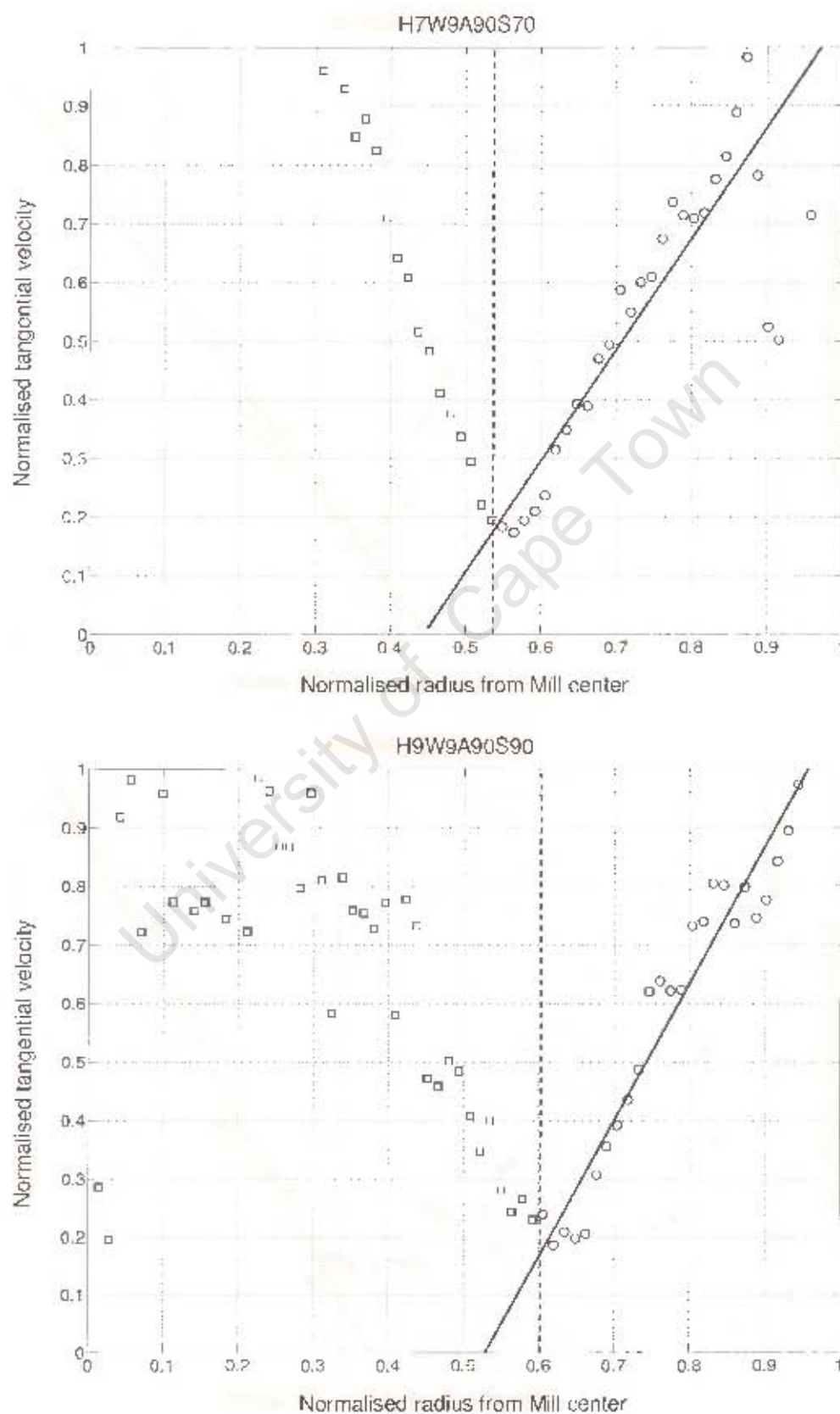


Figure 9.28: Quantification of the degree of slip between charge layers in the *en masse* region

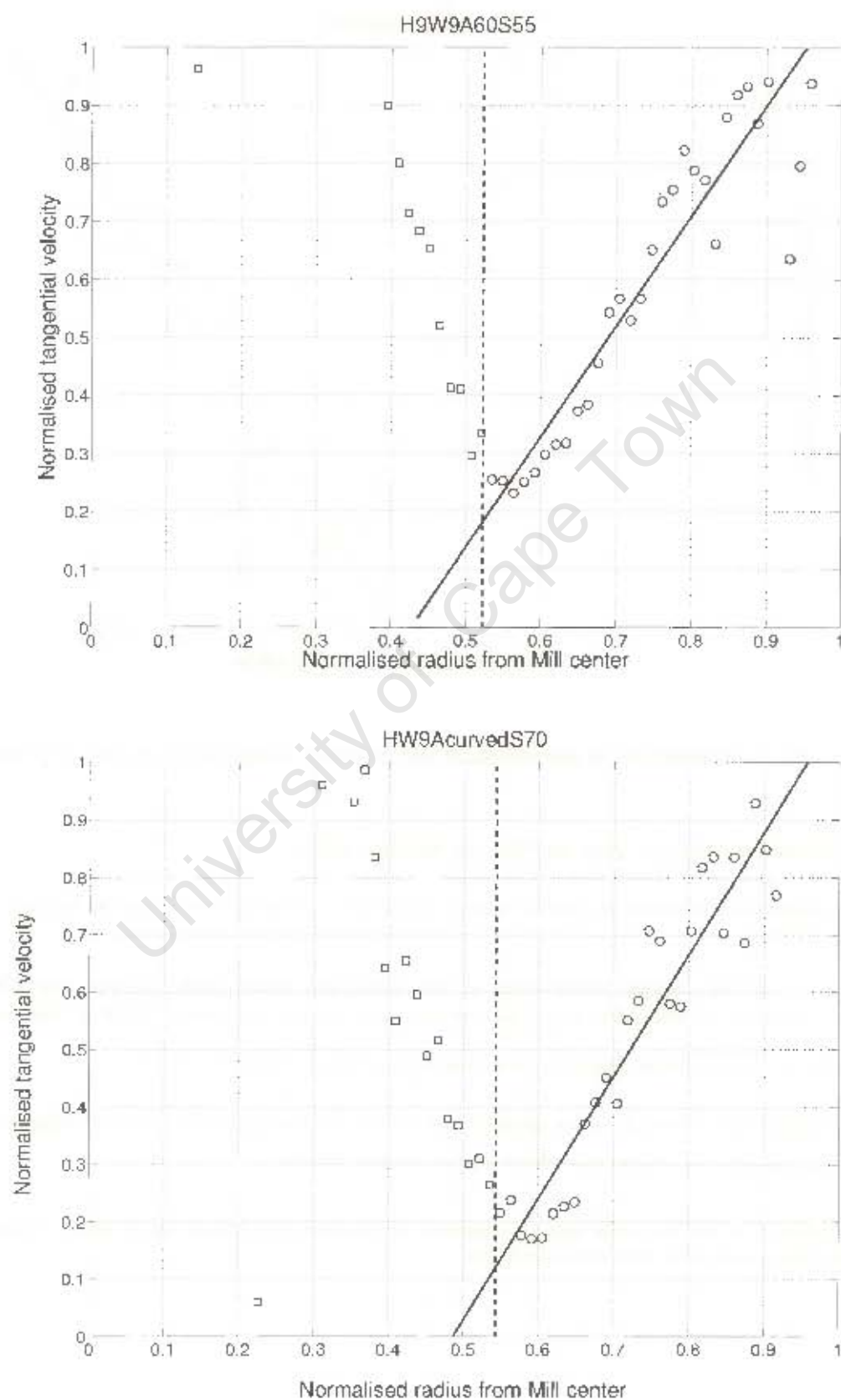


Figure 9.29: Quantification of the degree of slip between charge layers in the *en-masse* region

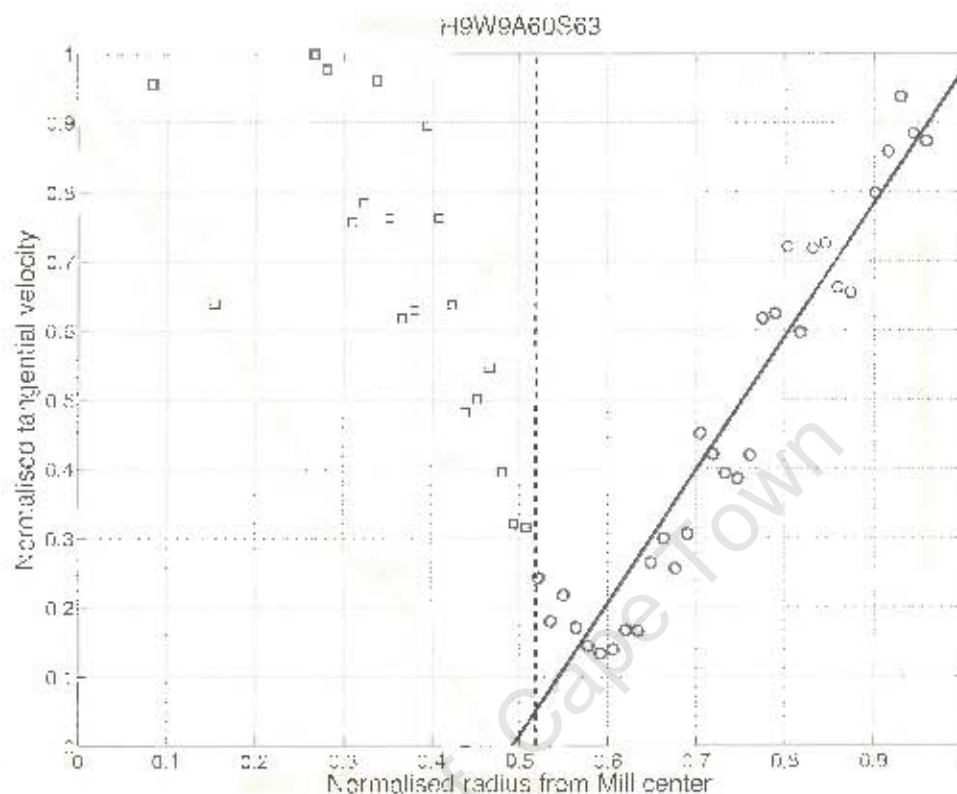


Figure 9.30: Quantification of the degree of slip between charge layers in the *en masse* region

9.5.1 Observations on slip in the *en-masse* zone

1. The tangential velocities in the *en-masse* zone follow a linear trend, similar to that reported by Morrell [42].
2. All layers of the charge, irrespective of the speed and lifter configuration, slip relative to the mill - There is no perfectly keyed-in motion, even for the outermost layer of the charge³
3. Degree of slip increases linearly with mill speed, figure (9.31).
 - (a) Aggressive lifters produce greater slip of the tracked particle than less aggressive lifters
 - (b) Speed has the dominant effect on the degree of slip

³Perfect keying-in of the two outer layers, as observed by Vermeulen & Howat [10], is more a function of the 2D type packing that is associated with mono-sized rods.

"The tangent to the equilibrium surface at the CoC is perpendicular to the radial line passing through the CoC. This condition is uniquely defined because the equilibrium surface has a different curvature to the mill shell, so only one point on the surface can have a tangent perpendicular to a radial line."

Closer inspection of the geometric implications of the Powell & McBride definition reveal that the angle of repose is simply the angular position of the CoC, figure (9.32); which is probably what motivated the definition.

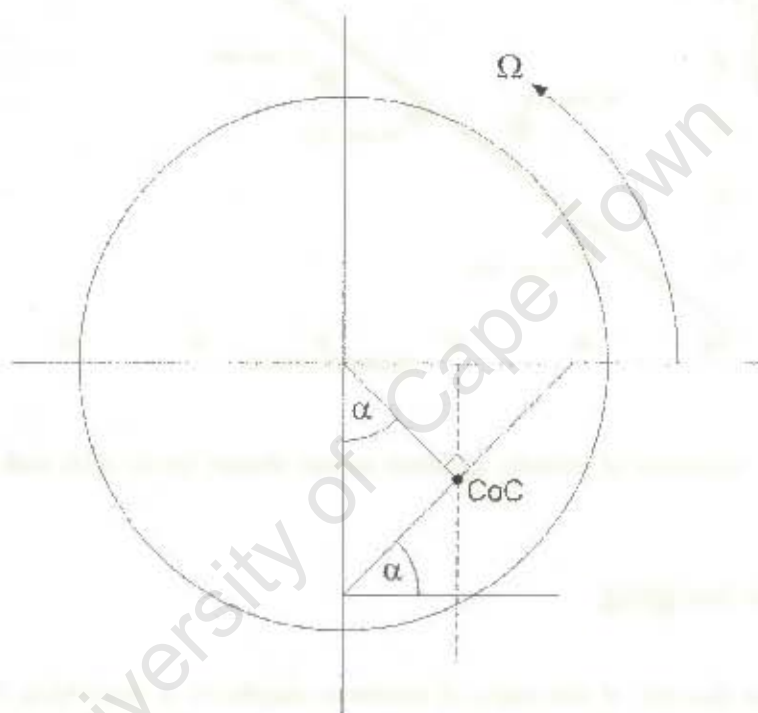


Figure 9.32: Angle of repose defined by Powell & McBride [46]

9.6.2 A surging mechanism

The surging mechanism is analogous to a block of mass M placed on an inclined plane that oscillates according to the rotation of the mill, figure (9.33). The block represents the charge body and the slope characterises the angle of repose of the charge.

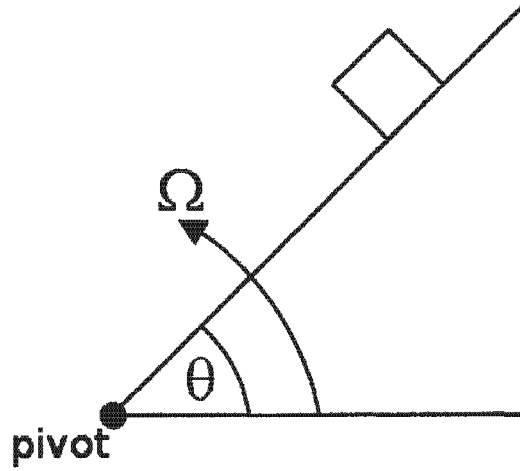


Figure 9.33: Surge model

At the lowest angular position, $\theta = \theta_{min}$, of the oscillation, the block 'sticks' to the plane as it rotates clockwise (upwards) at a constant rate

$$\frac{d\theta(t)}{dt} = \Omega \quad (9.23)$$

The static frictional force F_s during the upward rotation is given by

$$F_s = Mg \sin \{\theta(t)\} \quad (9.24)$$

where g is the acceleration due to gravity and θ is the angle of repose. F_s maintains the block in static equilibrium, relative to the incline, up to its limiting friction:

$$F_{s, max} = \mu_s Mg \cos \{\theta_{max}\} \quad (9.25)$$

where μ_s is the coefficient of static friction and θ_{max} denotes the highest angular position reached by the slope. At this point the maximum static friction is said to be exceeded and simultaneously drops to the lower kinetic friction, F_k :

$$F_k = \mu_k Mg \cos \{\theta(t)\} \quad (9.26)$$

where μ_k is the coefficient of kinetic friction. The equilibrium of the block, in the reference frame of the incline plane, is now compromised and it begins sliding down the slope. The initial stage of

slip is characterised by a net acceleration down the slope. Choosing down the slope as positive, the acceleration is calculated as:

$$a(t) = g [\sin \theta(t) - \mu_k \cos \{\theta(t)\}] \quad (9.27)$$

The movement of the block down the slope is accompanied by a step change in the rotation of the incline plane as it rotates clockwise in favour of a lower repose angle. The counter rotation seeks to restore the equilibrium of the block relative to the slope. The decrease in the angle of repose causes the acceleration of the block to decrease. The decrease continues, through zero and into deceleration, until the block eventually comes to a momentary rest. The rest position signifies the resumption of the 'sticking' phase and the cycle repeats itself.

The case of zero acceleration in equation (9.27) corresponds to the mean angle of repose, given by

$$\theta_{mean} = \arctan(\mu_k) \quad (9.28)$$

Figure (9.34) shows the variation of μ_k with θ_{mean} . If the angular changes caused by the surging is small relative to the mean angle of repose, figure (9.34) becomes useful in establishing the coefficient of kinetic friction between the charge body and the shell, provided that the repose angle is known.

The amplitude of the oscillation $\Delta\theta$ can be established if θ_{max} or μ_s is known. Starting with knowledge of μ_s , which is physically quantifiable by experiment, the relationship between θ_{max} and μ_s is easily shown to be

$$\mu_s = \tan(\theta_{max}) \quad (9.29)$$

The amplitude of the oscillation is then simply computed as

$$\Delta\theta = \theta_{max} - \theta_{mean} \quad (9.30)$$

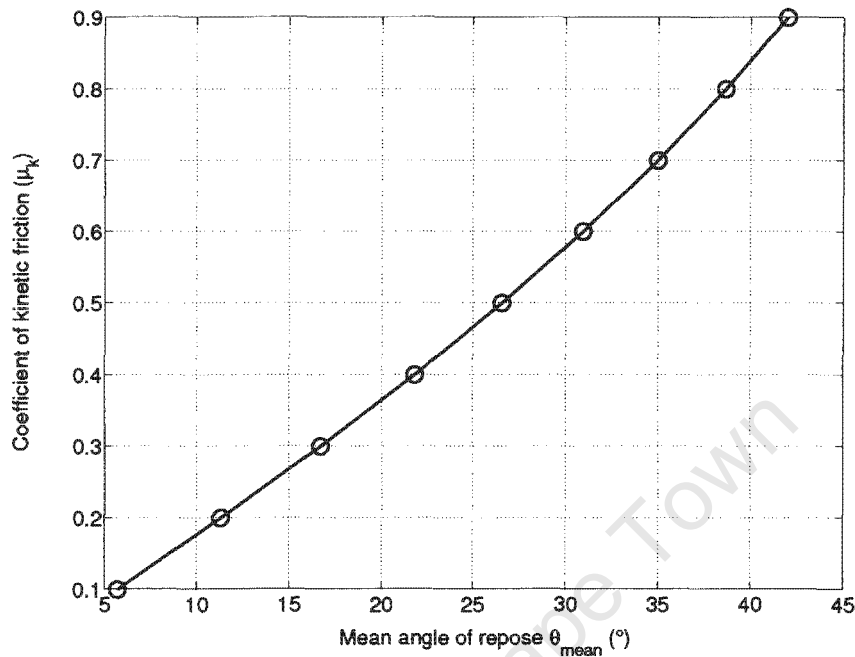


Figure 9.34: Mean angle of repose for different sliding friction coefficients

For a mill moving with a constant angular velocity of Ω , the steady state force balance at the CoM of the charge body is given according to figure (9.35), where Mg is the weight of the charge acting through the CoM, N is the net normal reaction force on the charge, assumed to be in the radial direction, F is the frictional force between the charge mass and the inner wall of the mill shell and α is complimentary to the angle of repose. The frictional force is not constant but varies periodically in response to the surging action of the charge, figure (9.36). Starting at the lowest point of the surging action, the charge 'sticks' to the liner and is lifted upwards with the rotating shell. During this upward movement of the charge body, α decreases and consequently $Mg \cos(\alpha)$ increases. Now provided limiting friction is not exceeded, $F \cos(\theta)$ must increase in order to maintain the charge body in static equilibrium. In particular, the now static frictional force F_s also increases until it reaches a maximum $F_{s, \max}$. At this point the frictional requirements are exceeded and the charge 'slips'. The frictional response is a sudden drop to minimum kinetic friction $F_{k, \min}$. The slipping part of the surging motion is characterised by a downwards movement of the charge body. During the slipping phase, kinetic friction increases as the angle of repose of the charge decreases. The sliding comes to an end when the initial sticking point, the point of minimum static friction $F_{s, \min}$, is reached. At this point the cycle repeats itself.

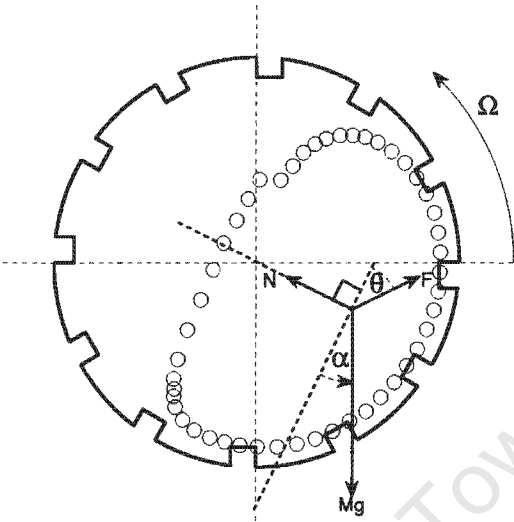


Figure 9.35: Forces acting through the CoM

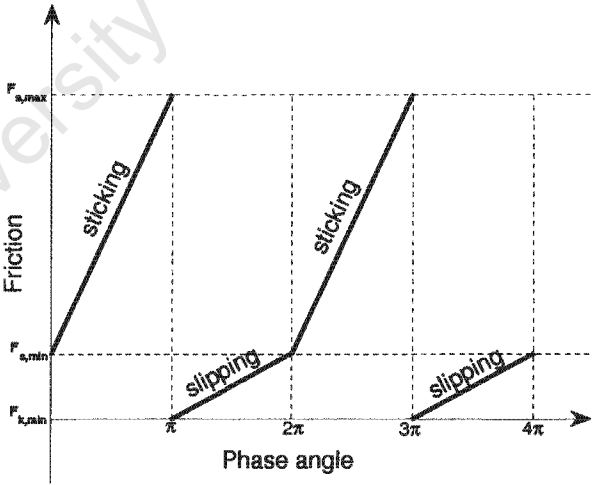


Figure 9.36: Variation of frictional force in response to the surging action of the charge

9.6.3 Quantification of power surge

The oscillation of the charge body along the repose slope manifests itself as a power surge, commonly observed on industrial mills. The power surge can be directly computed for the tracked particle using the CoM definition proposed by Powell & Nurick [19].

For a series of particle positions of the tracked ball with mass m , the CoM is given by:

$$X_{CoM} = \frac{\sum_{j=1}^q \sum_{i=1}^{N_j} x_{ij}}{\sum_{j=1}^q N_j}, \quad Y_{CoM} = \frac{\sum_{j=1}^q \sum_{i=1}^{N_j} y_{ij}}{\sum_{j=1}^q N_j}, \quad Z_{CoM} = \frac{\sum_{j=1}^q \sum_{i=1}^{N_j} z_{ij}}{\sum_{j=1}^q N_j} \quad (9.31)$$

where N_j is the number images of the tracked particle in the j^{th} circulation and q is the number of complete circulations made by the tracked particle. The CoM coordinates (X_{CoM} , Y_{CoM} , Z_{CoM}) represent the time-averaged center of mass. The procedure used to calculate the power fluctuation, for each milling configuration, is as follows:

1. Compute the CoM per circulation and corresponding power:

$$P_i = [Mg \cos(\theta_i)] R_i \Omega \quad (9.32)$$

where M is the total mass of the charge, g is the acceleration due to gravity, (R_i, θ_i) are the polar coordinates to the CoM of the i^{th} circulation and Ω is the average angular velocity of the mill. The relevant quantities are indicated in figure (9.35)

2. Compute the standard deviation of the power, ΔP

The fluctuation is dependent on various factors including lifter profile, repose angle and mill speed, with the latter being the dominant influence. The variation of ΔP with mill speed, for a range of lifter profiles, is shown in figure (9.37). The calculated surge amplitudes for each of the milling configurations is shown in table (9.5).

Mill configuration	Surge amplitude (mW)
H5-W9-A45-S50	20.5
H7-W9-A45-S90	63.5
H7-W9-A90-S70	64.4
H7-W9-A90-S90	81.1
H9-W9-A60-S55	35.3
H9-W9-Acurved-S70	72.7
H9-W9-A60-S63	61.9

Table 9.5: Power Surge amplitudes

Findings

- 1. The surge amplitude increases rapidly with mill speed up to 60% of critical, with further increases in speed having less of an influence on the amplitude.
- 2. The more aggressive the lifter profile, the greater the surge amplitude.
- 3. The influence of the lifter on the surge amplitude increases with increase in speed.

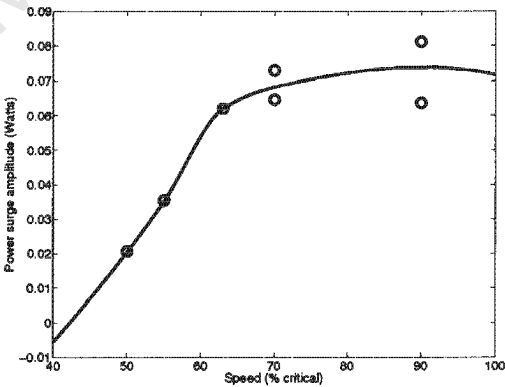


Figure 9.37: Variation of power surge amplitude with mill speed. The trend line fitted to the data is a shape preserving spline.

9.7 Heuristic power modelling

The simple torque-arm principle, which is the basis for many power equations [29, 30, 38, 37, 40, 42, 21], is employed in formulating a heuristic power model. The model is based on the true CoM coordinates which are shown to be linearly correlated with that of the CoC. The heuristics pertain to the best fit curves that model these correlations. An analysis of the error, including the error of fit, is then conducted on the model. Finally, the resultant power curve is compared to published power models, discussed in the literature review, section (2.2), of this thesis.

The objective of this section is to formulate a power equation that can predict power draw, given the mill speed, mass of the charge and the position of the CoC. Based on the simplified conditions of the mill, the resultant heuristic curves are not generally applicable to real charge, comprising slurry, balls and a size distribution of rocks. However, the correlation of the CoC with the CoM in order to calculate the torque-arm power draw, and based on physically quantifiable parameters, is a generally valid modelling methodology. The model structure therefore demonstrates, in principle, what is needed to formulate a generally valid power equation.

The unique data resulting from the X-ray experiments permits the quantification of fundamental quantities underlying most published power models and therefore facilitated a critical analysis of them. It should be noted that the angles used to characterise the CoC and CoM are measured counter-clockwise from the positive x-axis for a mill moving in the same direction.

The following empirical relationships form the basis to the model:

1. The variation of radial distance to the CoM and CoC with mill speed, figure (9.38).
2. The angular location of the CoM and CoC against mill speed, figure (9.39).
3. The polar coordinates to the CoC and CoM plotted against each other for the same range of speeds, figures (9.40 and 9.41)

Table (9.6) presents the polar coordinates of the CoC and CoM for the seven configurations investigated.

Mill configuration	R_{CoC}	θ_{CoC}	R_{CoM}	θ_{CoM}
H5-W9-A45-S50	0.0364	313.22	0.0351	314.42
H7-W9-A45-S90	0.0402	326.35	0.0262	329.52
H7-W9-A90-S70	0.0381	318.41	0.0328	321.31
H7-W9-A90-S90	0.0427	321.17	0.0219	328.27
H9-W9-A60-S55	0.0371	314.67	0.0369	318.41
H9-W9-Acurved-S70	0.0385	316.16	0.0331	315.73
H9-W9-A60-S63	0.0368	317.20	0.0353	320.17

Table 9.6: CoC and CoM coordinates resulting from the trajectory of the tracked particle for charge moving counter-clockwise

The CoC moves radially outward from the center of the mill with increases in mill speed. For all mill speeds, the more aggressive profile shifts the CoC further away from the mill center than the less aggressive profiles. The CoM moves radially inwards towards the center of the mill with increases in mill speed. For the 50% and 70% of critical speed, the more aggressive profile shifts the CoM further away from the mill center than the less aggressive profiles. At 90% of critical speed the opposite effect is observed. The angular position of the CoC and CoM displays a linear trend with mill speed. The angular and radial position of the CoM varies linearly with that of the CoC.

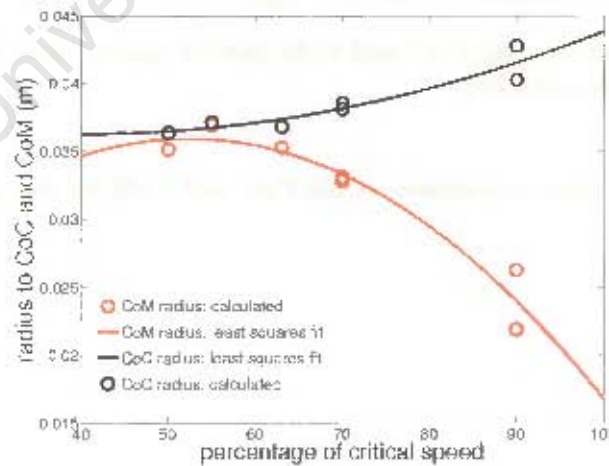


Figure 9.38: Variation of radial distance to CoM and CoC with mill speed

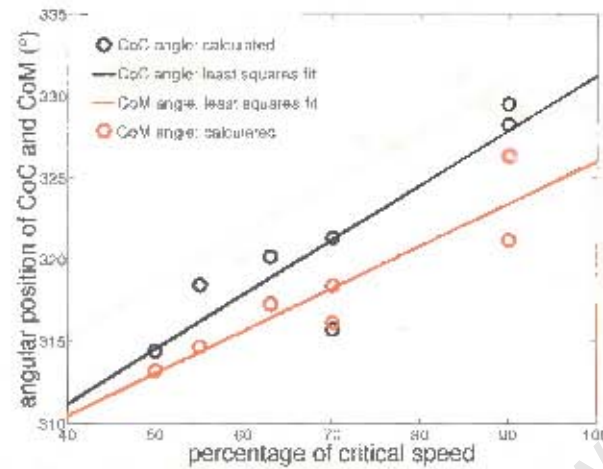


Figure 9.39: Variation of angular position of CoM and CoC with mill speed

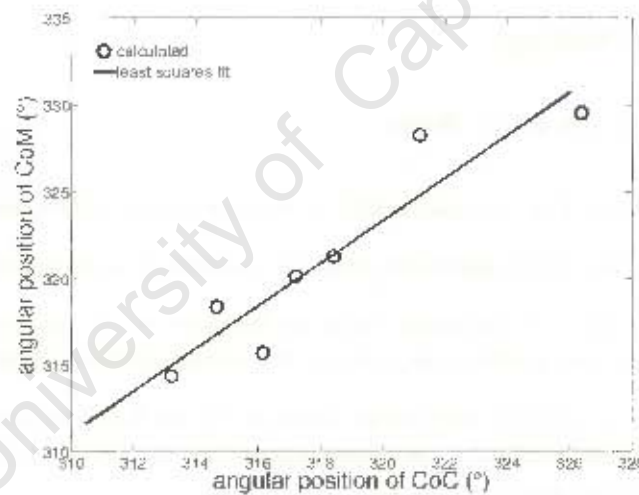


Figure 9.40: Relating the angular position of the CoM with that of the CoC

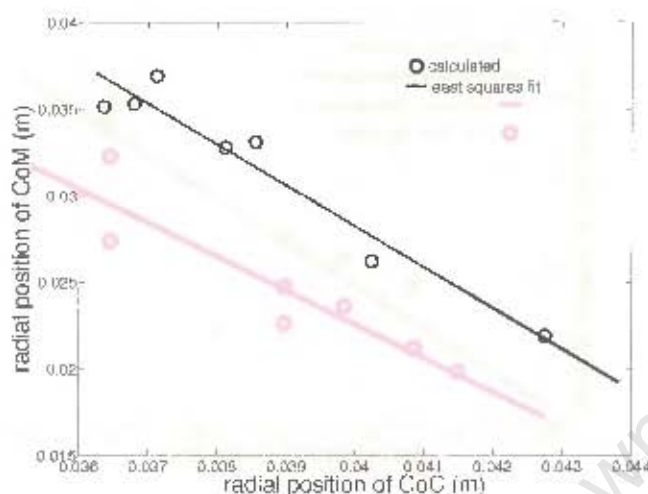


Figure 9.41: Relating the radial position of the CoM with that of the CoC

9.7.1 Summary of findings

For a mono-sized charge with a 40% filling:

1. Radial position of the CoC increase with in an increase in mill speed.
2. Radial position of the CoM decreases with an increase in mill speed.
3. Radial position of the CoC increases with an increase in the aggressiveness of lifter profile i.e. the more aggressive the profile, the greater the radial position of the CoC at a given speed.
4. For speeds < 90% of critical, the radial position of the CoM increases with an increase in the aggressiveness of lifter profile i.e. the more aggressive the profile, the greater the radial position of the CoM at a given speed.
5. At 90% of critical the radial position of the CoM drops off sharply, tending towards centrifuging. The more aggressive lifter profile pulls the CoM closer to the mill center than the less aggressive one.
6. Angular position of the CoC and CoM varies linearly with mill speed.
7. Angular position of the CoM increases linearly with that of the CoC.
8. Radius of the CoM decreases linearly with that of the CoC.

Based on these findings, an expression for the power drawn by the charge, as a function of mill speed, was derived.

The procedure used to obtain the power expression is as follows:

1. Using the trends observed in figures (9.40 and 9.41), fit straight lines, in the least squares sense, to the variation of the radial and angular position of the CoM with that of the CoC

$$\theta_{com}(s) = m_1\theta_{coc}(s) + c_1 \quad (9.33)$$

$$R_{com}(s) = m_2R_{coc}(s) + c_2 \quad (9.34)$$

where (m_1, m_2) and (c_1, c_2) are the slope and y-intercept of the straight lines respectively.

2. Fit a second order polynomial, in the least squares sense, to the variation of the radius of the CoC with speed, figure (9.38):

$$R_{coc}(s) = A_{coc}s^2 + B_{coc}s + C_{coc} \quad (9.35)$$

where s is the percentage of critical speed, A_{coc} , B_{coc} , C_{coc} are polynomial coefficients and $R(s)$ is the radial distance to the CoC. See Appendix (A.6) for a description of robust polynomial fitting.

3. Similarly, fit a straight line, in the least squares sense, to the variation of the angular position of the CoC with speed, figure (9.39).

$$\theta_{coc}(s) = D_{coc}s + E_{coc} \quad (9.36)$$

where D_{coc} , E_{coc} are constants of a straight line equation.

4. After solving for $(A_{coc}, B_{coc}, C_{coc})$ and (D_{coc}, E_{coc}) , use the speed range $s = [40; 100]$ to generate a set of $\theta_{coc}(s)$ and $R_{coc}(s)$.
5. Substitute $\theta_{coc}(s)$ and $R_{coc}(s)$ into equations (9.33 and 9.34) respectively to solve for $\theta_{com}(s)$ and $R_{com}(s)$.
6. The simple torque-arm method then gives:

$$\Gamma(s) = MgR_{com}(s) \cos[\theta_{com}(s)] \quad (9.37)$$

where $M = 4596 \times \left(\frac{0.0947}{1000}\right)$ is the mass of the 4596 balls constituting the total charge in kg, $g = 9.81 \text{ m.s}^{-1}$ and $\Gamma(s)$ is the torque delivered to the charge.

7. The power drawn by the charge is then computed as:

$$P(s) = \Gamma(s)\omega \quad (9.38)$$

where ω is the angular velocity of the mill in rad.s^{-1} .

The equation, in expanded form, is:

$$P = Mg\omega [m_2 \{A_{coc}s^2 + B_{coc}s + C_{coc}\} + c_2] \cos [360 - (m_1 \{D_{coc}s + E_{coc}\} + c_1)] \quad (9.39)$$

The variation of power draw with speed is illustrated in figure (9.42).

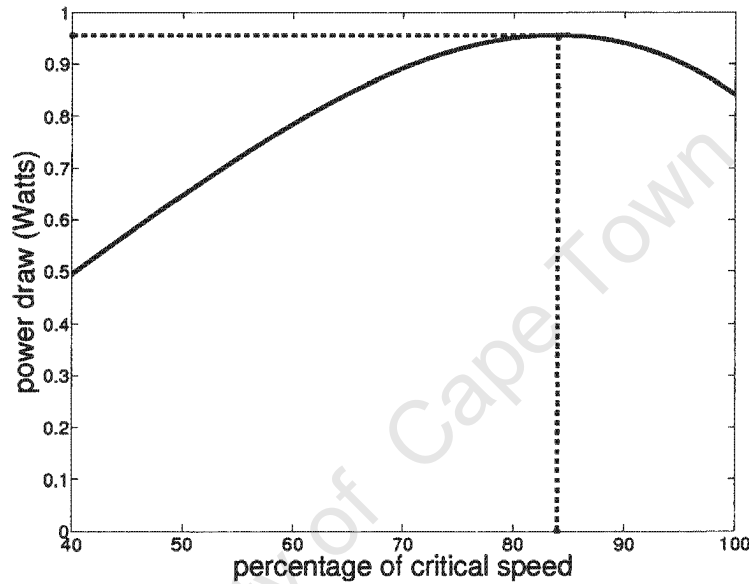


Figure 9.42: Illustration of the heuristic power model, equation (9.39), for a range of mill speed (% critical)

9.7.2 Error analysis of power formulation

The correct approach to assessing statistical validity of the power equation would be to compare the predicted power to measured data. Unfortunately, attempts to measure the dynamic power draw were unsuccessful, largely because of the minute torque drawn by the charge, which is 'lost' amidst the overall noise of the system. The errors associated with the model may be divided into two categories:

1. The error of fit associated with the heuristic trends
2. The error associated with the physical parameters

The error propagated into the the power predictions incorporate a combination of both errors, while the error of the calculated power, for each of the seven configurations, is determined, and dominated, by the power surging. The latter may be classified as an inherent systematic error of the system

that cannot be corrected⁴. The quantification of the power surge was addressed in section (9.6.3), and therefore not repeated here. The surge amplitudes given in table (9.5) represent the inherent systematic error of calculated power.

Error propagated into the power predictions

Assessing the statistical validity of the predicted power requires a base case. In the absence of measured data, the best estimate of the power was achieved by formulating the power equation directly from the CoM data and using it to benchmark the predictions of equation (9.39). The power equation derived directly from the CoM is given by:

$$P = Mg\omega [A_{com}s^2 + B_{com}s + C_{com}] \cos[360 - \{D_{com}s + E_{com}\}] \quad (9.40)$$

where $R_{com} = A_{com}s^2 + B_{com}s + C_{com}$ is modelled on the trend in figure (9.38) of the radius of the CoM with mill speed and $\theta_{com} = D_{com}s + E_{com}$ is the least squares fit to the variation of the angular location of the CoM with mill speed, figure (9.39).

Statistical agreement between equations (9.39 and 9.40) is said to occur if the error bars of the respective curves overlap. Additionally, if the error bars of equation (9.39) overlaps with the best fit curve of equation (9.40), a more rigorous statistical agreement is enforced. The latter case is illustrated in the ensuing error analysis.

A natural consequence of the polynomial technique, Appendix (A.6), employed in fitting the parameter trends of the power model is an estimation of the error for each prediction. This approach determines the error based on the location of the prediction relative to the input data, assigning relatively smaller errors to predictions made closer to the input data and relatively larger errors for extrapolatory predictions or those further away from the measured data. The technique assumes independent normal data for the input - essentially the measurement of the input data are not dependent on each other and the measurement error is normally distributed. The error bounds on individual data is a useful quantity to have when the input data to the fitting process is limited - the error estimates can be plotted as error bars about the polynomial prediction, giving a good visual assessment of the error distribution over the region of interest.

The error propagation into the power equation is given by:

$$\frac{\Delta P}{P} = \sqrt{\left(\frac{\Delta M}{M}\right)^2 + \left(\frac{\Delta \omega}{\omega}\right)^2 + \left(\frac{2\Delta \theta_{com}}{\theta_{com}^c}\right)^2 + \left(\frac{\Delta R_{com}}{R_{com}^c}\right)^2} \quad (9.41)$$

where $\Delta \theta_{com}$ and ΔR_{com} are calculated for each data point according to equation (A.28), θ_{com}^c and R_{com}^c refer to the calculated CoM coordinates and the factor of 2 comes from the Taylor series expansion of $\cos(\theta)$ to second order.

⁴In general, systematic errors should be identified and corrected, however for tumbling mills, the surging phenomenon is inherent

The error in the angular velocity is simply the mean of the standard deviations for each experimental configuration and is computed as:

$$\Delta\omega = \frac{\sum_{i=1}^P \delta\omega_i}{P} = 0.02 \text{ rad.s}^{-1} \quad (9.42)$$

$$\delta\omega = \sqrt{\frac{\sum_{j=1}^Q (\omega_j - \bar{\omega})^2}{Q - 1}} \quad (9.43)$$

where $\delta\omega_i$ is the standard deviation of the measured mill speed (tacho readings after converting to rad.s^{-1}) for configuration i , Q is the number of speed readings taken for each configuration and P is the number of experimental configurations investigated. The speed measurements were taken before and after each experimental run, at 10 second intervals for a duration of 1 minute.

The uncertainty in the mass of the charge is propagated as:

$$M = 4596\bar{m} \quad (9.44)$$

$$\delta m = \sqrt{\frac{\sum_{j=1}^L (m_j - \bar{m})^2}{L - 1}} \quad (9.45)$$

$$\Delta M = \sqrt{4596 (\delta m)^2} = 0.0067 \text{ kg} \quad (9.46)$$

where δm is the standard deviation of L particles of mean mass $\bar{m}$, and ΔM the propagated error for the 4596 particles. A sample of 100 particles ($L = 100$) were used to compute δm .

Error analysis findings

The error of each predicted CoM polar coordinate described by equation (A.28) is plotted in figures (9.43 and 9.44) as error bars about the mean predictions. The error bars are smaller when the predicted curve is closer to the measured data, becoming larger at speeds $< 45\%$ and $> 90\%$ of critical. The fit to the measured CoM polar coordinates, the red lines, are also included in these plots for statistical comparison. The error bars show clear overlap with the red curve, verifying that the predictions are in statistical agreement with the CoM-derived curve.

The propagation of these errors into the power model is shown in figure (9.45). The error bars of the predictions overlap with red curve for all speeds between 48 and 93% of critical, illustrating

statistical agreement for the entire range of speeds investigated. The overlap of the calculated error bars (blue bars) with that of the predicted error bars (black bars) also confirms that the modelled power (black curve) agrees with the calculated values (red circles) within the uncertainties, for the range of configurations investigated. To summarise:

1. The CoC based trends agree with the CoM trends within the uncertainties
2. The CoC derived, predicted power agrees statistically with that of the CoM predicted power
3. The CoC derived, power curve agrees with the calculated power within the uncertainties, for all configurations investigated.
4. The calculated power for the two different lifter configurations at 70% of critical speed agree with each other within the uncertainties
5. The calculated power for the two different lifter configurations at 90% of critical speed do not agree with each other within the uncertainties.
6. The lifter effect has an appreciable effect on the power draw at the very high speeds

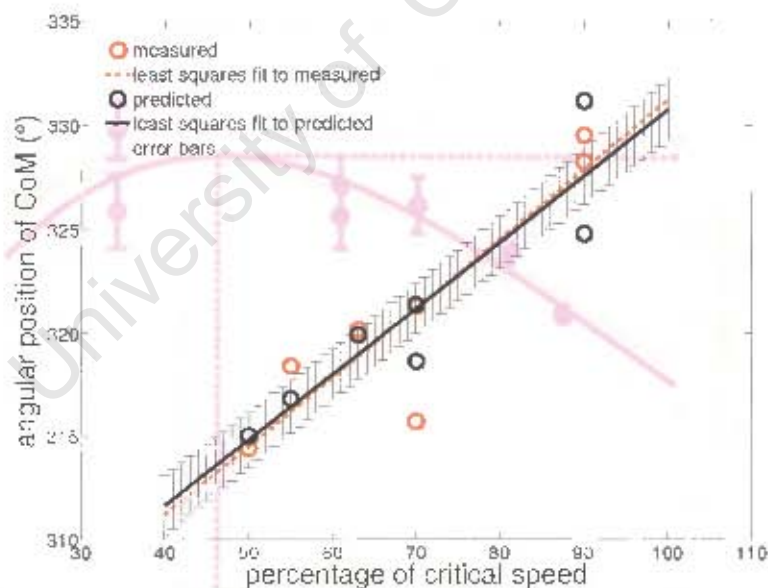


Figure 9.43: Error estimates of predicted angular position of the CoM plotted as error bars.

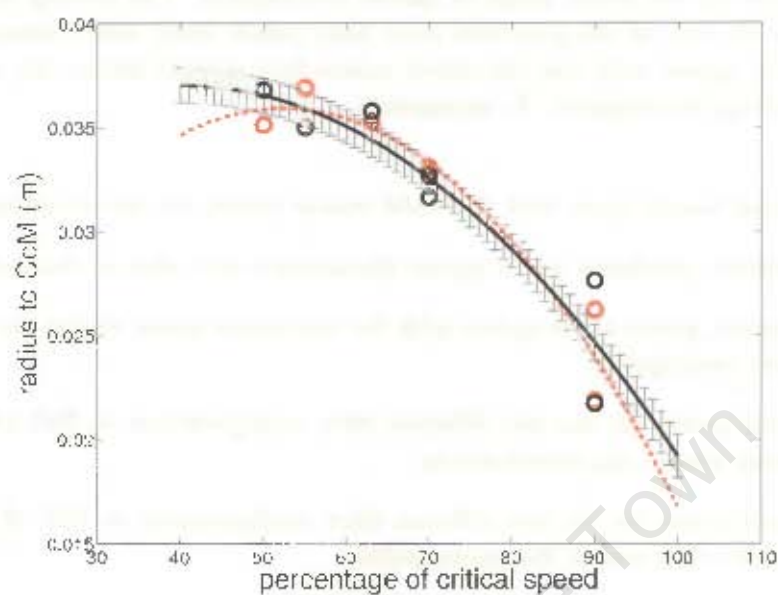


Figure 9.44: Error estimates of predicted radial position of the CoM plotted as error bars.

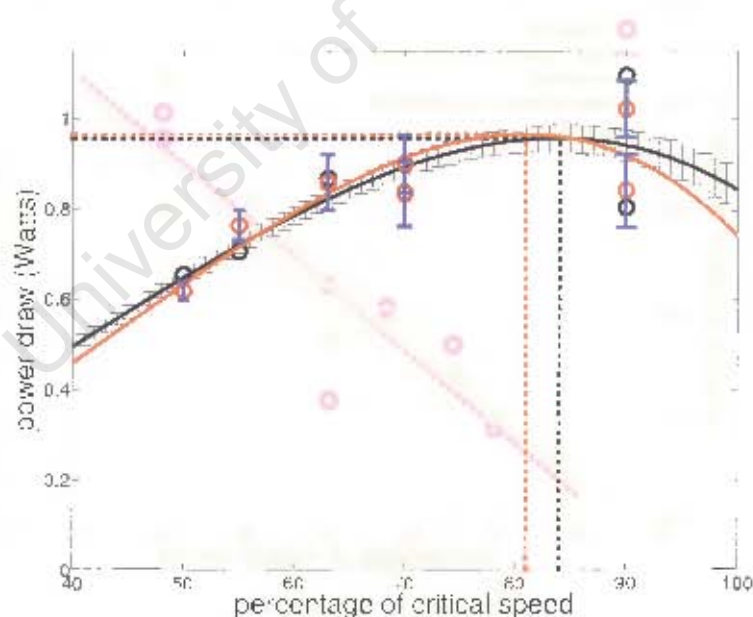


Figure 9.45: Power curves based on the CoM (red curve) and CoC (black curve). The error bars about the black curve is the prediction error based on the fitting of normal, independent data. The blue error bars about the actual power values (red circles) are the magnitudes of the power surge.

9.7.3 Discussion on Power draw formulation

The power draw increases with mill speed up to a maximum and then drops off with further increases in speed. The maximum power drawn by the charge is 0.95 Watts at 85% of critical speed. These observations are in accordance with practical experience by Morrell [42].

The error bars about the actual power values (blue bars) overlap with that of the heuristic curve (black bars) for all configurations investigated. The model is therefore said to agree with the calculated values within the uncertainties.

The overlap of the two error bars at 70% of critical speed indicate statistical agreement between the two configurations, while the two configurations at 90% of critical do not overlap. This implies that the latter speed produces statistically different power values for lifters that are sufficiently different⁵. It may thus be deduced, though not conclusively, that lifter geometry has a statistically significant influence on the power draw at the very high speeds, while the moderate speeds are less susceptible to them. Moderate mill speeds are typical of the Australian and North American mills, which is probably why power models from these parts of the world do not explicitly include lifter geometry into their models, and unfortunately why the same models are equally inappropriate to the high speed, South African mills. The North American models compound this problem by typically overestimating the power draw at high speeds, as observed by Liddell & Moys [37], while the Morrell [42] model, with empirical trends for the toe and shoulder angles that are well suited to Australian milling configurations, is equally inappropriate to the South African context.

9.7.4 Comparison with published models

The power curve given by equation (9.39) was compared to published power models that employ the torque-arm principle. In each case, the published model is implemented exactly, and where possible, the parameters are adjusted to achieve a more sensible prediction. The angle of repose, being a common thread in all power models investigated, was calculated from the line joining the shoulder and toe, section (2.2), with the toe and shoulder positions determined according to the numerical scheme of Powell & McBride [46]; see section (9.4) for details.

Classic torque-arm definition

The classic torque arm definition, as described in section (2.2), is the basis for most power models and is therefore included as part of the comparison. The power curve resulting from this definition is shown in figure (9.46).

⁵The question of sufficiency has not been addressed in this thesis

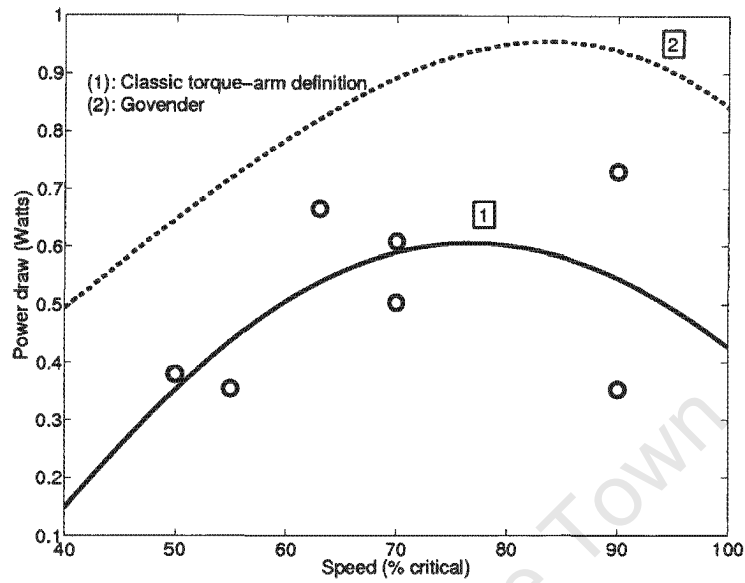


Figure 9.46: Comparison of the classic torque-arm definition with equation (9.39)

Comments

The idealised charge shape and distribution characterised by the classic torque-arm principle leads to an under-prediction of the power draw. It is however reassuring to note that the curve is realistic in shape, producing a power maximum.

Bond (1962)

Bond's suggested form of his power model is

$$P = 12.262\rho J(1 - 0.937J)LD^{2.3}\phi \left[1 - \frac{0.1}{2^{9-10\phi}} \right], \quad (9.47)$$

where

$$K = 12.262$$

$$\beta = 0.937$$

$$\gamma = 9$$

$$\zeta = 10$$

A plot of equation (9.47) as a function of mill speed, curve (1) in figure (9.47), indicates a gross over-estimation of the power when compared to the prediction of equation (9.39). The validity of this observation is reinforced by its similarity to Liddell & Moys [37] comparison of the Bond model with measured power draw on a 0.545×0.305 m mill. A more appropriate set of parameters was thus fitted by minimising, in the least squares sense, the difference between the predictions of equation (9.39) and Bond's general power equation:

$$P = K\rho \sin(\alpha) J(1 - \beta J)LD^{2.3}\phi \left[1 - \frac{0.1}{2^{\gamma-\zeta\phi}} \right] W \quad (9.48)$$

where

$$K = 1.4$$

α = angle of repose (method of calculation described above)

$$J = 0.4$$

$$\gamma = 8.8$$

$$\zeta = 10$$

$$\beta = 1.3$$

$$M = 0.435 \text{ kg}$$

$$g = 9.8 \text{ m.s}^{-1}$$

Figure (9.47) includes the power curves described by Bond's original equation, equation (9.47), the adjusted form, equation (9.48) and the heuristic power model, equation (9.39).

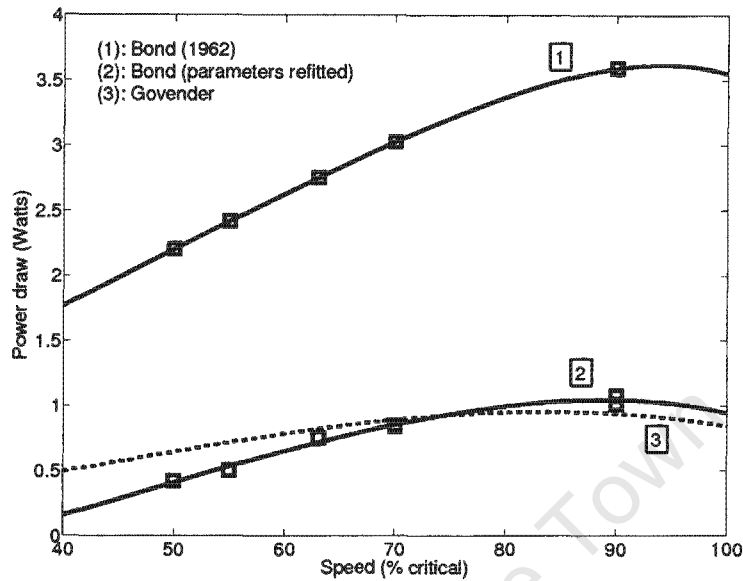


Figure 9.47: Comparison of Bond's power model (original and adjusted) with equation (9.39)

Comments

Aside from the obvious difference between curves (1) and (3), Bond's suggested parameters seem to smooth out the effect of the lifter geometry, presumably due to the lumping of $\sin(\alpha)$ into K . The use of equation (9.48) in conjunction with the adjusted parameters improves the fit dramatically, and also discerns the influence of the lifter geometry on power draw.

Arbiter and Harris (1982)

The power model of Arbiter & Harris, before the empiricism and lumping of parameters were introduced, is given by

$$P = \frac{1}{6} \pi \rho N L D^3 \sin^3(\theta) \sin(\alpha) \quad (9.49)$$

$$\sin^3(\theta) = 4L_f(1 - L_f) \quad (9.50)$$

Their model is very similar to that of Hogg & Feurstenau [30] who employed an energy approach. The usual lumping of parameters and omission of the speed function puts equation (9.49) in the more familiar form, which was also used by Liddell & Moys [37] in their assessment of power models, is given by

$$P = 1.88mgfL^{0.5} (1 - F) \sin(\alpha) \quad (9.51)$$

while the general structure, which was used to assess essentially equipment manufacturers' data, takes the form

$$P = KD^n LfY\lambda F(1 - aF) \quad (9.52)$$

where the symbols are described in section (2.2). Liddell & Moys [37] used the general structure of equation (9.52) with their own description of the speed function (Y). The comparison of equation (9.52) with the heuristic curve is therefore discussed under Liddell's model.

Figure (9.48) shows the comparison between equations (9.49 and 9.39).

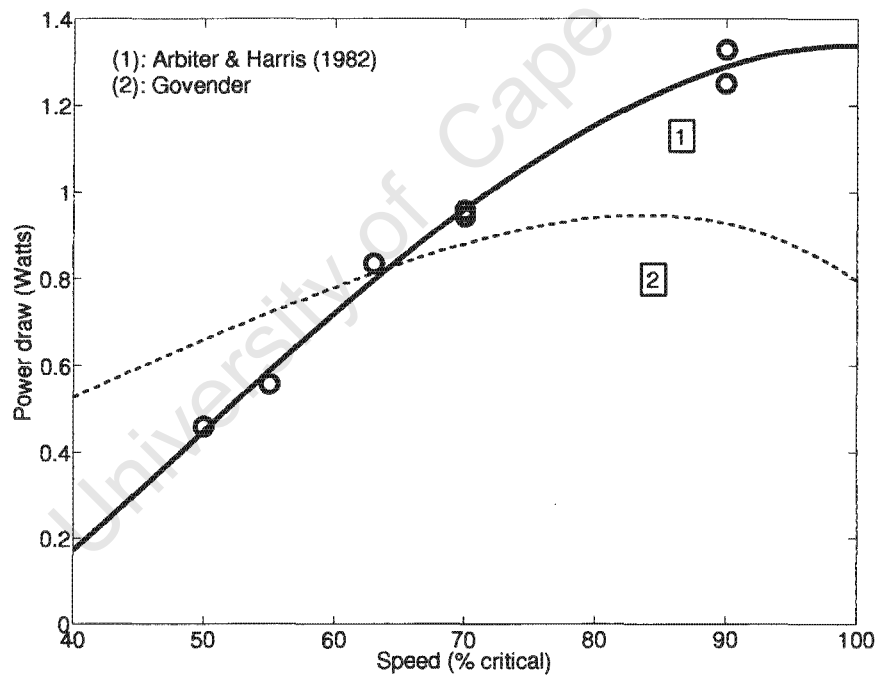


Figure 9.48: Comparison of the Arbiter & Harris power model with equation (9.39)

Comments

The model is able to distinguish the effect of the lifters. The distinct difference in shape is due to the simplistic charge shape that results from the angle of repose definition.

Liddell and Moys (1988)

Their model is given by

$$P = 9.69\rho\phi LD^{2.5}J(1 - 1.06J)Y \quad (9.53)$$

The speed function (Y), which was defined as the normalised, measured torque, was obtained as follows:

1. For each of the seven experimental configurations, calculate the torque using the torque-arm method and the true CoM
2. Heuristically fit a curve through the seven data points
3. Normalise the curve

Figure (9.49) shows a plot of the speed function used for Liddell & Moys's [37] model, while figure (9.50) illustrates the comparison of their power prediction with equation (9.39).

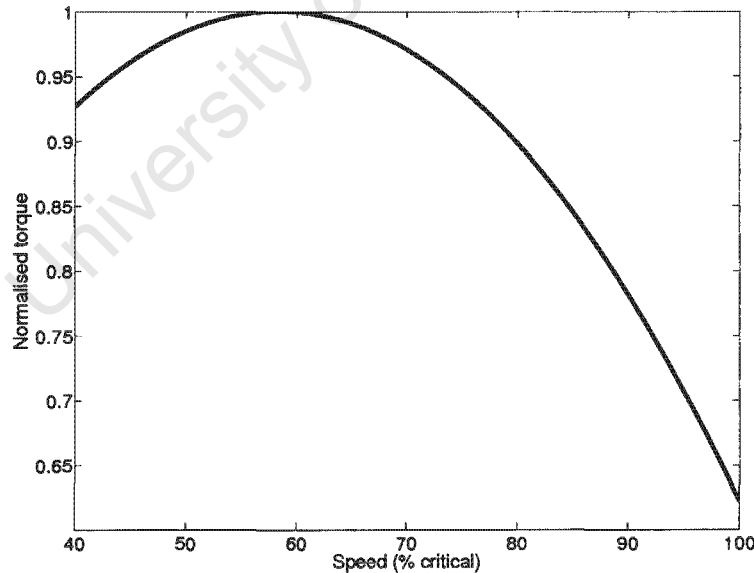


Figure 9.49: Speed function for the Liddell & Moys [37] model based on calculated torque data of tracked particle, using the torque-arm method and the true CoM.

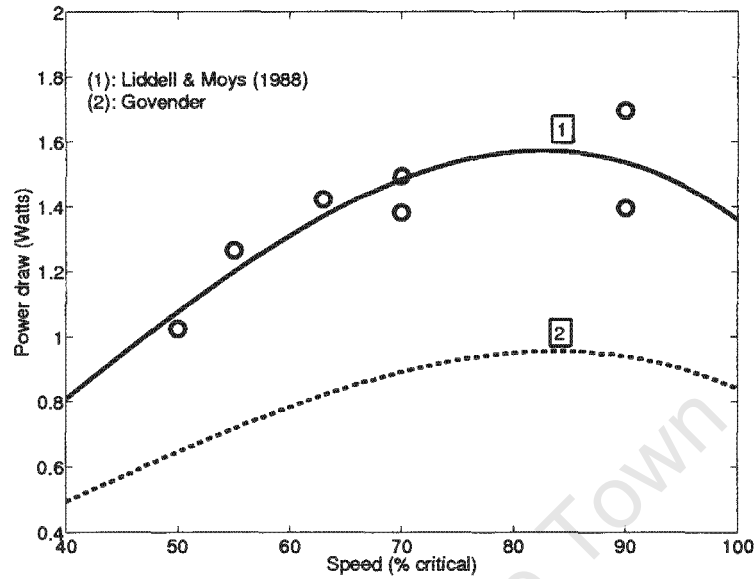


Figure 9.50: Comparison of the Liddell & Moys [37] power model with equation (9.39)

Comments

Even with the heuristic fit to the torque data, which is essentially the 'true' power per unit speed ($\frac{P}{S}$), Liddell's model overestimates the power. This implies that the empiricism, carried through from the Harris et al [29] equation, in particular their conclusion that after observing 39 operational mills, the average angle of repose was 42° , is inappropriate for the current data.

Morrell (1992)

Morrell's published equation is of the form

$$P_{net} = 0.5g^{1.5}L\rho\phi r_m^{0.5} \left(\frac{(2r_m^3 - 3r_m^2 r_i + r_i^3)}{3(r_m - r_i)} \right) [\sin \theta]_{\theta_T}^{\theta_S} \quad (9.54)$$

with

$$r_i = r_m \left(1 - \frac{(2\pi J_t)}{(2\pi + \theta_S - \theta_T)} \right)^{0.5} \quad (9.55)$$

and θ_S and θ_T are empirical relations for the shoulder and toe angle, given by

$$\theta_S = A_1 + A_2 J_t \quad (9.56)$$

where

$$\begin{aligned} A_1 &= 0.499\phi - 0.746 \\ A_2 &= 5.490\phi - 0.969 \end{aligned}$$

and

$$\theta_T = A_3 [1 - \exp \{-A_4 (A_5 - \phi)\}] + 0.5\pi \quad (9.57)$$

where

$$\begin{aligned} A_3 &= 2.321 (1.406 - J_t) \\ A_4 &= 23.2 \\ A_5 &= 0.75 (1.670 - J_t) & \phi \leq A_5 \\ A_5 &= \phi & \phi > A_5 \end{aligned}$$

Comments

Figure (9.51) illustrates the Morrell power curve. The adjusted curve is based on a heuristic fit to the toe and shoulder angles, figures (9.52 and 9.53). It is evident from the data that the trends observed by Morrell are not applicable to a charge comprised of mono-sized plastic beads. Morrell's empirical relations for the toe and shoulder angles tend to smooth out the lifter effect which can be differentiated with the heuristic approach. The adjusted curve agrees well with predictions of equation (9.39).

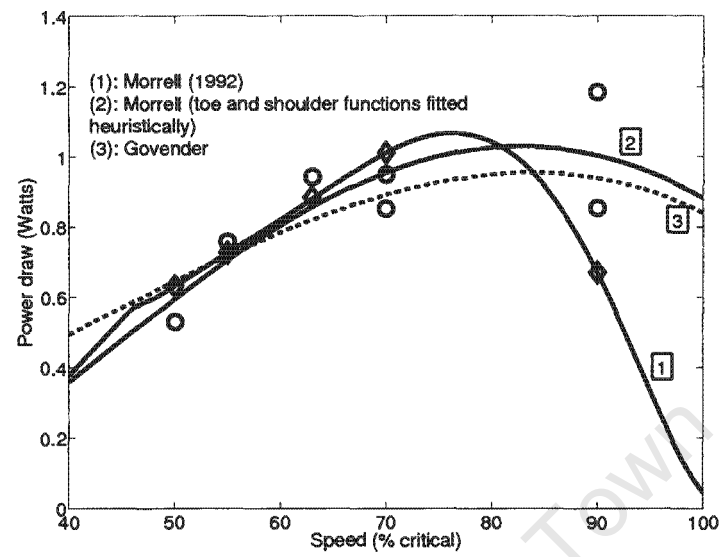


Figure 9.51: Comparison of the Morrell power model (original and adjusted) with equation (9.39)

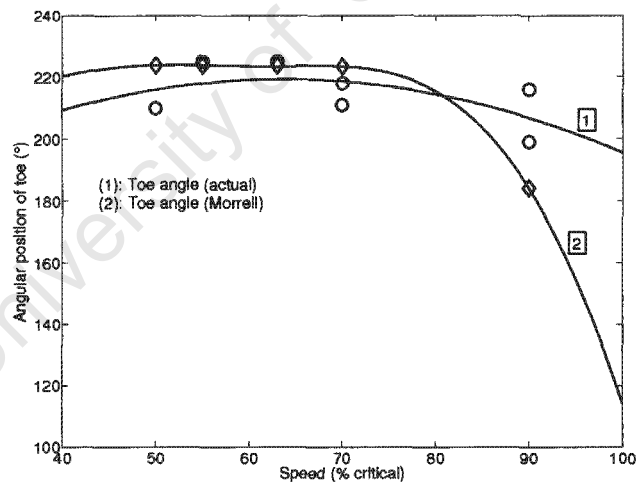


Figure 9.52: Empirical fit to toe angle

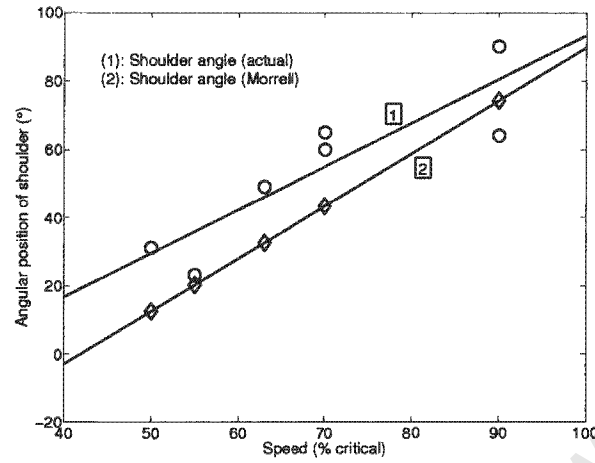


Figure 9.53: Empirical fit to Shoulder angle

Moys (1993)

Moys, like Feurstenau et al [39], partitioned the charge into two fractions: a centrifuging layer that drew no power and a non-centrifuging portion whose power draw was given by the torque-arm approach. The non-centrifuging layer impacted on the mill speed, diameter and load fraction by effectively reducing them according to empirical relations. The power draw was given by

$$P = K D_{eff}^{2.3} \sin(\alpha) \rho J_{eff} (1 - \beta J_{eff}) N_{eff} \quad (9.58)$$

where K and β are parameters, similar to that used by Bond, that were fitted using Powell's non-derivative regression technique [35] and measured power values at non-standard conditions. In the absence of a clear method to measuring the thickness of the centrifuging layer (δ_c), an estimate of one ball diameter was used for those milling configurations that depicted cataracting. The effective mill diameter (D_{eff}), critical speed (N_{eff}) and load fraction (J_{eff}) were then calculated by

$$\begin{aligned} D_{eff} &= (1 - 2\delta_c) D \\ J_{eff} &= J - 2\delta_c \\ N_{eff} &= \frac{\text{rpm} \sqrt{D_{eff}} 100}{42.3} \end{aligned}$$

where

rpm = revolutions per minute

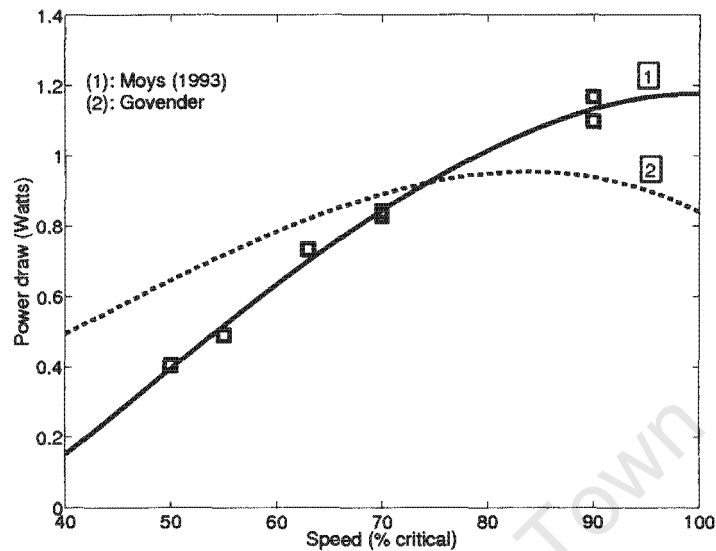


Figure 9.54: Comparison of the Moys power model with equation (9.39)

J = load fraction

D = mill diameter

and the optimisation gave

$K = 0.8$

$\beta = 0.97$

Figure (9.54) shows the resulting power curve.

Comments

The predicted curve is similar to that of Arbiter and Harris [36].

Powell and McBride (2004): DEM

The aim of this exercise was to compare the experimentally derived power curve with published predictions from DEM. The papers by McBride et al [15] and Powell & McBride [46] were selected, primarily because their simulation work was based on the experimental Perspex mill. The numerical code used by them was the particle flow code PFC. The power draw, which is a standard output from the PFC software, is calculated as the energy dissipation per unit time at steady state. Unfortunately,

Configuration	H9-W9-A60-S55	H9-W9-A60-S60
Position of CoC (mm)	(29, -22)	(29, -23)
Position of CoM (mm)	(26.4, -21.7)	(27.1, -19.9)

Table 9.7: CoC and CoM data from 3D DEM simulations of the experimental Perspex mill (after McBride et al [15])

the actual power predictions were not published. The power draw was thus calculated using the CoM coordinates and the appropriate heuristic trends in conjunction with equation (9.40).

The first paper was by McBride et al [15] who simulated two experimental conditions of the Perspex mill for the purpose of DEM validation. The data pertaining to the CoC coordinates were included in their publication, and is presented in table (9.7). The unpublished CoM data was obtained from the authors, and also included in table (9.7).

The second is attributed to Powell & McBride [46] who conducted 3D DEM simulations of the Perspex mill for the conditions shown in table (9.8). Included in their paper was the CoM and CoC coordinates. The CoM coordinates from tables (9.7 and 9.8) were then heuristically fitted against mill speed and then incorporated into equation (9.40), yielding the power curve shown in figure (9.55).

Configuration	H5-W9-A45-S50	H5-W9-A45-S70	H5-W9-A45-S90	H7-W9-A90-S90
Position of CoC (mm)	(21, -25)	(24, -23)	(28, -22)	(28, -21)
Position of CoM (mm)	(22, -24)	(24, -20)	(23, -14)	(18, -11)
Shoulder angle (°)	42	52	68	100
Upper toe angle (°)	214	214	204	140
Lower toe angle (°)	230	232	215	182

Table 9.8: Charge features derived from DEM simulations of an experimental Perspex mill (after Powell & McBride [46])

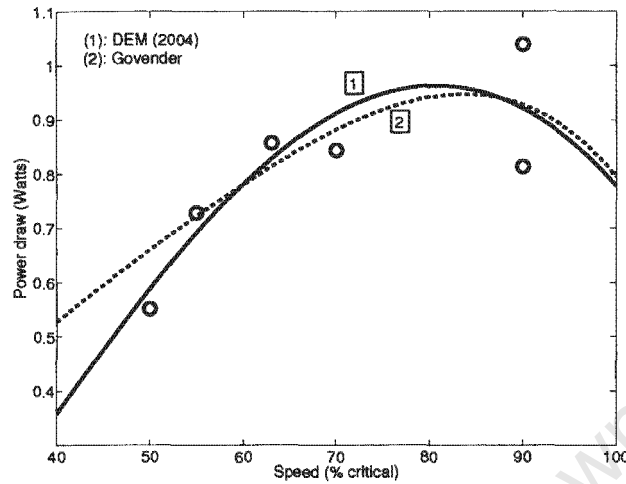


Figure 9.55: Comparison of DEM power with equation (9.39). The heuristic equation based on the CoM coordinates is used to obtain the DEM power curve.

Comments

The predicted power draw based on the DEM data compares well with that of the heuristic model. It should be noted that two of the DEM configurations (H5-W9-A45-S70 and H5-W9-A45-S90) are different to the experimental configurations (H7-W9-A90-S70 and H7-W9-A45-S90). The DEM seems to under-predict the power at speeds less than 55% of critical, while achieving the best comparison for speeds above 63% of critical.

Discussion on power models

The overall error of the various models, including the heuristic model given by equation (9.39), were assessed by comparing them to the calculated power based on the torque-arm principle and the true CoM. The residuals formed between each model prediction and the corresponding torque-arm power value then forms the basis for a RMS residual error analysis. The error thus formed, quantifies the overall error of each model, and is given by

$$\delta_i = \sqrt{\frac{\sum_{j=1}^7 (P_{ij}^{(m)} - P_j^{(ta)})^2}{7}} \quad (9.59)$$

where

δ_i = overall error of the i^{th} model

$P_{ij}^{(m)}$ = is the calculated power of the j^{th} experimental configuration as determined by the i^{th} model

$P_j^{(ta)}$ = is the calculated power of the j^{th} experimental configuration as determined by the torque-arm principle based on the true CoM coordinates

The overall error for the DEM calculation is based on the four configurations that match the experimental conditions. A histogram plot of the overall error, for each of the models investigated, is shown in figure (9.56) with the height of the bar quantifying the magnitude of the error. Figure (9.57) illustrates the same error plot, but includes the adjusted models (the shaded bars) where applicable.

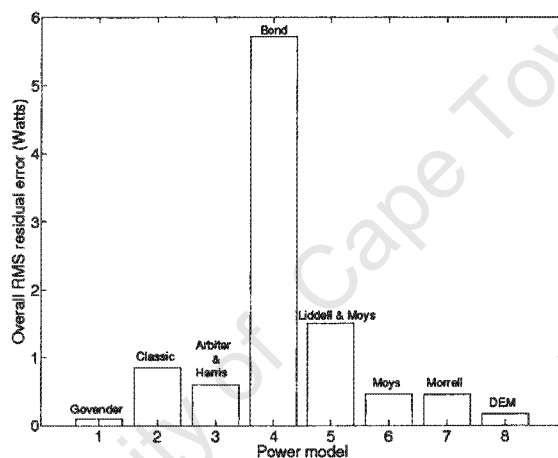


Figure 9.56: Overall error for the various models investigated in their original form

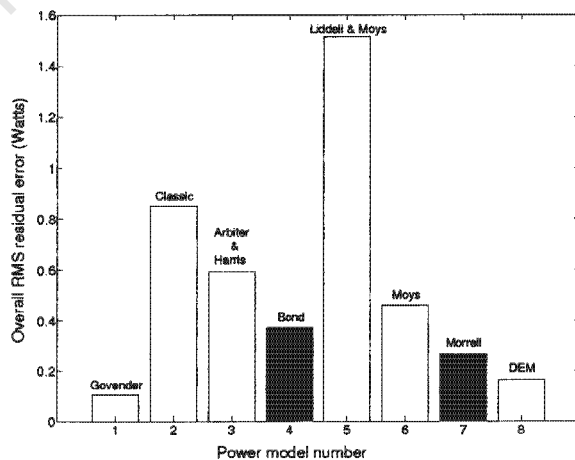


Figure 9.57: Overall error for the various models investigated (shaded bars = adjusted model)

Generally, the models that lump parameters, in particular the angle of repose, into a single scaling factor, suffer from a lack of sensitivity to lifter geometry. The repose angle, though crude in its characterisation of charge shape and position, is still more realistic than a single parameter. An obvious example of this effect is the popular form of the Bond model, equation (9.47). In its original form, Bond's model produces the highest overall error. The adjustment of the Bond parameters together with the angle of repose inclusion, equation (9.48), improves the error quite dramatically, while also depicting a slight discernment for the lifter influence. A major drawback of such adjustments is the lack of a clear minimisation strategy, which in the context of industrial operations, can only be achieved if the power is measured independently for a range of speeds. Consequently, the predictive capabilities become confined to the measured conditions. Ergo, Bond's model is not universally applicable, or easily reconfigurable, a fact which will eventually see its popularity dwindle.

The models of Arbiter & Harris, which Morrell notes to be similar to that of Hogg & Feurstenau, and consequently Liddell & Moys, are variants of the Bond model, with their predictions achieving less success than the adjusted implementation of Bond. Liddell's version of the speed function requires *a priori* knowledge of the torque, including the peak torque, in order to achieve a realistic profile of the power curve, a fact that greatly inhibits its implementation and applicability, and probably why it has achieved less popularity than Bond. The model of Arbiter & Harris, which eventually took the form of equation (9.51), has achieved similar success due to its linear form across the speed range.

The DEM method has an error approximately equal to that of the heuristic model, with an overall second best comparison - a glimpse at the future of comminution modelling. However, there is still a huge hurdle to be overcome in the way of incorporating slurries and a realistic size distribution into the DEM framework - a situation that is at least 5 to 10 years away from being efficiently simulated. Until then, models based on good fundamentals and physically quantifiable empiricism, will still be the best route to obtaining quick and accurate predictions of power. Unfortunately, none exist that can predict power draw for operations outside their window of design. Notwithstanding, Morrell's model is a good example of what a sound basis, founded on clear fundamentals, does for the predictive capability of an essentially empirical model. The empiricism used by Morrell for the toe and shoulder angular positions can be easily adjusted for operations outside the Australian context, however, these have yet to be addressed. The heuristic trends fitted to the angular positions of the toe and shoulder in this thesis clearly illustrate how Morrell's model can be adjusted to achieve predictions outside of its window of operation.

On the issue of sound fundamentals, Moys's model, with an error that is marginally higher than Morrell's, is difficult to implement, save for the unique data resulting from the current experimental program. His model relies on the mathematical construct of a 'centrifuging layer', even when this is clearly not occurring, with measurement of the layer's thickness impacting heavily on the model prediction. The model could have been useful had the empirical correlations between the centrifuging layer's thickness and mill speed been taken to fruition. Failing this, its usefulness in an industry where measurement of non-standard quantities are limited, must therefore be considered to be limited. The model of Feurstenau et al [39], which is probably what motivated a partitioning of the charge by Moys, suffers from the same difficulties.

The heuristic model produces the lowest overall error for the range of conditions investigated. The outcome is an affirmation of the torque-arm implementation with the proviso that the true CoM

and CoC coordinates can be determined. In general, the empirical trends pertaining to the polar coordinates of the CoC and their subsequent correlation with the CoM are not easily quantified in an industrial context. The use of the X-ray experimental program in this regard is limited due to the simplified configuration. The methodology, however, is fundamentally sound, and has been adequately demonstrated in this thesis. The way forward would require measurement of the 'true' CoM coordinates in the presence of a realistic charge, comprising a slurry, a size distribution of particles and possibly balls for SAG type conditions. Experiments similar to the X-ray system, but with a realistic charge and mill, are becoming more plausible, with the advent of positron emission particle tracking (PEPT). The proponents of PEPT have recently claimed to track a solid particle in a fluid medium for extended periods of time that far surpass the X-ray system. However, the accuracy of the particles location amidst strong attenuation of the emitted gamma rays is still not clearly known for the relatively high speeds encountered in tumbling mills.

9.8 Conclusions

1. The velocity and acceleration field of the tracked particle is best estimated by the LIP method.
2. Due to the relatively coarse sampling rate, the time-averaged, cumulative power was used to identify the energy ranges corresponding to useful comminution. These ranges were found to be statistically equivalent for all milling configurations investigated. The simplified experimental configuration of plastic beads tumbling within a Perspex mill do not produce a statistically varied range of interactions, making useful comminution energies difficult to discriminate for the various configurations.
3. The toe and shoulder regions are useful parameters for characterising the shape and position of the charge, but unnecessary for determining average power draw. The numerical technique of Powell & McBride [46] are the most user-independent schemes for calculating these quantities.
4. Slip within the charge was observed to follow a linear trend across the charge layers; an observation that is consistent with Morrell [42]. Slip was also found to increase linearly with mill speed, with the more aggressive lifters leading to greater slip than their less aggressive counterparts at a given speed. Lifter profile seems to have a greater influence on slip at the moderate speeds.
5. The fluctuation in the position of the charge, and the subsequent influence on power draw, was quantified across the speed range. A mechanism based on the dynamic angle of repose is proposed for the surging phenomenon.
6. A heuristic power model based on the simple torque-arm principle was formulated. The model uses the polar coordinates of the CoC and the mill speed to calculate power draw. A comparison with other torque-armed based models indicates that the heuristic model is the best power predictor for the configurations investigated.

Chapter 10

Conclusions

10.1 Development of an X-ray tracking system

The primary objective of this thesis was to develop an X-ray vision system that tracks the 3D motion of particles deep within the tumbling ball charge of an experimental Perspex mill. This entailed:

1. Designing a tumbling mill and drivetrain appropriate for the image acquisition
2. Developing image processing and target centering tools that ultimately determine the centroid of the tracked particle as accurately as possible, and with minimal user intervention
3. Mapping the centroidal locations from image space to a Lagrangian object space with the same accuracy and user dependency as the image processing

10.1.1 Experimental design

The design of the tumbling mill was governed by the X-ray beam characteristics and, to a lesser extent, the aspect ratio constraints associated with typical industrial mills. A parametric, optimisation scheme was developed in order to determine the maximum size mill that could be immersed completely within the conical, intersecting beams. The final size of the mill was a compromise between the predictions of the optimisation scheme, availability of Perspex tubes and to an extent price, which culminated in the mill being 0.142 mm long with a diameter of 0.142 mm. The lifter bars were designed such that their dimensions conformed to the Skega formula [81], commonly used in the comminution industry.

10.1.2 Image processing

A range of thresholding and edge detection algorithms, classified as state of the art in the machine vision world, were investigated to establish the most suitable feature extraction procedure for the X-ray data. The result was an image processing scheme that modifies the Canny edge detector, thought to be the standard method, with a more symmetric implementation of the Gaussian convolution, thereby nullifying the effects of subpixel shift. The two-fold variation is applied to all stages of edge discernment.

10.1.3 Target centering

An interrogation of the various centering routines commonly used in centroid location led to the ellipse specific conic fitting routine (ESCF) by Fitzgibbons [65]. This was shown to out-perform other conic fitting routine for all noise and occlusion conditions while doubling the accuracy usually achieved with area based centering of distinct targets. The ESCF algorithm did require additional metrics that leads to iterative improvement for the final implementation. The metrics are based on the global properties of the object's projection as discerned in image space.

10.1.4 3D reconstruction

The formalism adopted with regard to reconstructing the 3D trajectories of the tracked particle is based on the direct linear transformation (DLT). The choice was guided by the geometric similarities to the actual X-ray imaging process. The full DLT description is employed. A physically representative variance model of the overall distortion was developed through interpolation schemes based on the Delaunay, and its dual structure the Voronoi, tessellation. The final reconstruction error based on 18 control points that were treated as unknown, and 81 control points for the space parametrisation, was 0.15 mm. This was shown to be on average twice as accurate as the standard 11 parameter DLT solution (overall reconstruction error = 0.32 mm) and the full DLT equations solved via the Levenberg-Marquardt [63, 96] and singular valued decomposition (SVD) [96] numerical schemes. The robust numerical solvers do well to temper the inherent instability of the transformation matrix, a peculiarity of the DLT formulation, but fall short against the variance model implementation. The best accuracies achieved for the Levenberg-Marquardt [63, 96] and SVD [96] was 0.29 mm and 0.31 mm respectively. The validity of the variance model was demonstrated by comparing it to a physical quantification of the overall distortion errors known to afflict the X-ray imaging process. The average error in this regard was shown to be 0.13 mm. Subtracting this from the error associated with the 11 parameter formulation yields an answer of 0.19 mm. This is the error that should be expected of the full DLT implementation. The variance model implementation is clearly the only scheme that achieves better than 0.19 mm. In fact, the additional gains in accuracy may be attributed to partial correction of the unflatness associated with the image intensifier, an issue that has not been addressed in this thesis.

10.2 Validation tools for benchmarking DEM

The second objective dealt with the development of verification tools for the purpose of DEM validation. These tools were custom designed to address the particle trajectory data from the experimental mill, thereby providing an accurate benchmark against which to validate DEM simulations. The validation techniques presented characterise the mass behaviour of the charge based on observations of a limited number of particles. They are robust and noise free due to the inherent insensitivity to minor fluctuations in charge motion and the noise suppressing nature of the integration of experimental data. The verification tools form part of a larger project to develop a validation toolbox tailored to rigorously test the outputs of DEM, and include numerical schemes for computing the Center of Circulation (CoC), the equilibrium surface, angles of departure at the shoulder regions, and the position of the toe.

10.3 Analysis of trajectory data

The final objective surpasses the initial purpose of comparing with DEM and focusses on the analysis of the trajectory data with a view to discerning meaningful trends. A precursor to the analysis was to establish the most accurate numerical scheme for calculating the particle's velocity and acceleration from its position. The Lagrange interpolation polynomial proved to be best suited to the data, producing the most realistic behaviour in regions of known behaviour. The resulting kinematic information was then used to interrogate the data.

10.3.1 Circulation trends

The average axial displacement per circulation did not produce distinct trends. The findings were:

1. Longitudinal motion is clearly evident, even in the absence of the flow gradient along the mill length
2. Axial displacement decreases with speed up to 70% of critical speed
3. The aggressive lifters produce distinctly smaller axial displacements per circulation than their less aggressive counterparts.

The rate at which a particle circulates relative to the mill rotation was shown to decrease linearly with mill speed. The observations are contrary to current notions regarding energy transfer per mill circulation. It may be argued that the circulation rates on industrial mills are different from that of the Perspex mill, but the general motion of the media is the same (confirmed by looking through end-windows of pilot mills and down the trunnion of an industrial mill) so the range should be similar.

10.3.2 Toe and Shoulder estimates

The toe and shoulder positions were investigated using the numerical schemes of Powell & McBride [46]. The shoulder and impact toe can be numerically obtained according to the methodology suggested by the authors. The bulk toe location is not easily implemented in a numerical sense due to the noisy path followed by the horizontal equilibrium surface. It does however follow the trend of a third order polynomial for all charge shapes investigated in this thesis. It is therefore suggested that a third order polynomial be fitted to the data in the least squares sense and the lower inflection point used to determine the vertical position of the horizontal line that locates the bulk toe. The following summarise the findings of the analysis:

1. A conservative lifter profile has a greater influence on charge shape than high mill speeds, figure (9.23).
2. Aggressive lifter profiles seem to induce cataracting effects even if the speed is maintained at a moderate level, figures (9.24a and 9.25b).
3. Aggressive lifters combined with moderate or high mill speeds cause distinct cataracting effects. The cataracting stream is maintained even when the lifter begins to wear¹, figure (9.25b).

10.3.3 Slip within the charge

Slip between charge layers is known to be responsible for appreciable energy dissipation by the abrasion and attrition modes of particle interaction. A relative measure of slip was established by calculating the tangential velocity of the particle as it moved through the en-masse zone. The velocities decreased linearly with decreases in radial position, similar to that reported by Morrell [42]. All layers of the charge, irrespective of the speed and lifter configuration, slip relative to the mill - There is no perfectly keyed-in motion, even for the outermost layer of the charge. The degree of slip, represented by the slope of the linear trend, increased linearly with mill speed, figure (9.31). Aggressive lifters produced greater slip than their less aggressive counterparts. Mill speed was found to be the dominant influence on the degree of slip.

10.3.4 Charge surging

The fluctuation of the charge was characterised by looking at the movement of the center of mass per circulation of the tracked particle. The fluctuation impinges on the dynamic power draw, a phenomenon that is commonplace in industrial mills. The following summarise the findings on charge surging.

1. The surge amplitude increases rapidly with mill speed up to 60% of critical, with further increases in speed having less of an influence on the amplitude.

¹ a worn lifter profile is simulated by using a curved surface to the leading lifter face

2. The more aggressive the lifter profile, the greater the surge amplitude.
3. The influence of the lifter on the surge amplitude increases with increase in mill speed.

A mechanism of the surging phenomenon is proposed. The mechanism is analogous to a block placed on an inclined plane that oscillates according to the rotation of the mill. The block represents the charge body and the slope characterises the angle of repose of the charge. The slip-stick behaviour of the block, the source of the surging, is shown to vary periodically with the static and kinetic frictional force, like a block on an oscillating slope.

10.3.5 Heuristic power model

The final outcome of the data analysis was a heuristic power model based on the simple torque-arm principle. The heuristics relate to the best fit trend lines used in correlating the polar coordinates of the CoM with mill speed (% critical), which in turn were related to the heuristic curves governing the CoC as a function of speed. The true CoM coordinates take into account all the charge, cascading and cataracting, and its location is the culmination of the influences of mill speed, lifter profile, size distribution and charge type. The predictions of the heuristic model was compared to a range of published models based on the simple torque-arm principle and was shown to be the best power predictor for the configurations investigated.

10.4 Future work

One of the sources of error in the reconstruction of X-ray images to real space is the distortions due to image unflatness. A proper model of image unflatness that addresses the parabolic shape of the image intensifier is the only way to definitively resolve this issue. The work by Theodorou et al [121] indicate that the maximum deviation introduced by the parabolic shape of the intensifying screen is 0.1 mm which is relatively consistent with the error of 0.15 mm obtained for the final reconstruction achieved in this thesis. Toennies et al [122] proposed the use of a rectangular wire mesh to determine parabolic distortion quantitatively. Their work employed a first order photogrammetric solution based on the outputs of the angiographic system. Their final accuracy on distance measurements was quoted as 0.05 mm for the average error and 0.5 mm for the maximum. An analytic solution was promised in their paper, but is yet to be addressed. The pincushion, S-type error typified by the parabolic lens can be generated through standard algorithms employed by most imaging software. It is therefore a question of parametrisation that will make the general representation appropriate to the intensifying screen. This will be investigated at a later stage, via the rectangular mesh suggestion of Toennies et al [122].

The current experimental program can be extended to include different shape particles, a size distribution and a fluid medium to simulate the slurry found in typical mills. Except for the shape which might require modification of the feature extraction stage, all other extensions can be adequately

dealt with by the current imaging routines. A newer model of the BAE will be sought in favour of the higher image resolution of 1024×1024 . It is anticipated that sub 0.1 mm accuracies could be achieved with the current processing routines and the higher resolution. It is suggested that high density polyurethane (HDPE) be used instead of Perspex for the mill construction as the latter proved difficult to machine.

Experiments similar to the X-ray system, but with a realistic charge and mill, are becoming more plausible, with the advent of positron emission particle tracking (PEPT). The proponents of PEPT have recently claimed to track a solid particle in a fluid medium for extended periods of time that far surpass the X-ray system. However, the accuracy of the particle's location amidst strong attenuation of the emitted gamma rays is still not clearly known for the relatively high speeds encountered in tumbling mills. Such experiments warrant further investigation because they afford access to the true CoM and CoC, but for realistic charge. It is also anticipated that the extended filming duration of approximately 2 hours will on average counter-balance the location accuracy issue. An additional outcome from such experiments will be accurate quantifications of residence time with a view to fundamentally modelling discharge functions.

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Appendix A

Mathematical Details

A.1 Convolution and Correlation

The two Fourier transform relationships that constitute a basic link between the spatial and frequency domain are convolution and correlation. The definitions are taken from Digital Image Processing [87].

A.1.1 Convolution

Definition 1 *The convolution of two functions $f(x)$ and $g(x)$, denoted by $f(x) * g(x)$, is defined by the integral*

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(\alpha)g(x - \alpha)d\alpha \quad (\text{A.1})$$

where α is a dummy variable of integration. The importance of convolution in frequency domain analysis (that is the Fourier domain) lies in the fact that $f(x) * g(x)$ and $F(u)G(u)$ constitute a Fourier transform pair. In other words, if $f(x)$ has the Fourier transform $F(u)$ and $g(x)$ has the Fourier transform $G(u)$, then $f(x) * g(x)$ has the Fourier transform $F(u)G(u)$. Formally stated,

$$f(x) * g(x) \iff F(u)G(u) \quad (\text{A.2})$$

This indicates that convolution in the x domain can also be obtained by taking the inverse Fourier transform of the product $F(u)G(u)$. An analogous result is that convolution in the frequency domain reduces to multiplication in the x domain; that is,

$$f(x)g(x) \Longleftrightarrow F(u) * G(u) \quad (\text{A.3})$$

These two results are commonly referred to as the *convolution theorem*.

The convolution of 2D and 3D data is not always easy to visualise. However, it is completely analogous to one dimensional convolution, which has a clear geometric picture. In addition the results of convolution, in image processing, can be displayed pictorially for an appreciation of the process. In the one dimensional case, we consider the simple step functions $f(x)$ and $g(x)$ as illustrated in figures (A.1a) and (A.1b) respectively. Before performing the integration, the function $g(x - \alpha)$ must be formed. The two steps in figures (A.1c) and (A.1d) illustrate this process. It basically involves a reflection of $g(\alpha)$ about the vertical axis to give $g(-\alpha)$, followed by a displacement of x . Then, for any value of x , we multiply $f(\alpha)$ by the corresponding $g(x - \alpha)$ and integrate the product from $-\infty$ to ∞ . The product $f(\alpha)$ and $g(x - \alpha)$ is the shaded portion in figure (A.1e). This figure is valid for $0 \leq x \leq 1$. The product is zero for values of α outside the interval $[0, x]$, so $f(x) * g(x) = x/2$, which is simply the area of the shaded region in figure (A.1e). For x in the interval $[1, 2]$, figure (A.1f) applies, and $f(x) * g(x) = 1 - x/2$. Thus because $f(\alpha)g(x - \alpha)$ is zero for values of x outside the interval $[0, 2]$, we have finally

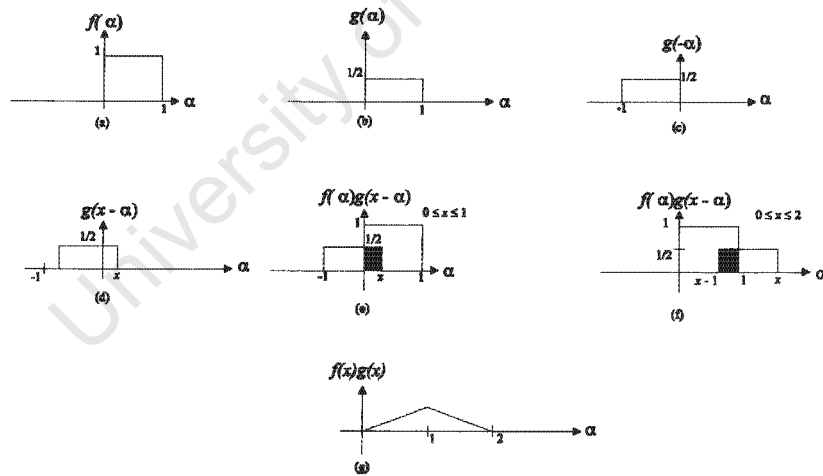


Figure A.1: Illustration of 1-D convolution

$$f(x)g(x) = \begin{cases} x/2 & 0 \leq x \leq 1 \\ 1 - x/2 & 1 \leq x \leq 2 \\ 0 & \text{elsewhere} \end{cases} \quad (\text{A.4})$$

A.1.2 Correlation

Definition 2 The correlation of two functions $f(x)$ and $g(x)$, denoted by $f(x) \circ g(x)$, is defined by the integral

$$f(x) \circ g(x) = \int_{-\infty}^{\infty} f^*(\alpha)g(x + \alpha)d\alpha \quad (\text{A.5})$$

where $*$ is the complex conjugate. The forms of equations (A.5) and (A.1) are similar, the only difference being that the function $g(x)$ is not folded about the origin. In terms of image processing, 2D correlation is related to 2D convolution by a 180 degree rotation of the filter matrix.

Correlation forms the basis for template matching in image processing. Given a digital image $f(x, y)$ of size $M \times N$, and some region $w(x, y)$ of size $J \times K$, where $J < M$, and $K < N$, then computing if $f(x, y)$ contains some region similar to $w(x, y)$ implies performing a correlation between $w(x, y)$ and $f(x, y)$. In its simplest form, the correlation between these two real functions is given by

$$R(m, n) = \sum_x \sum_y f(x, y)w(x - m, y - n) \quad (\text{A.6})$$

where $m = 0, 1, 2, \dots, M - 1$, $n = 0, 1, 2, \dots, N - 1$, and the summation is taken over the image region where $w(x, y)$ is defined. As m and n are varied, $w(x, y)$ moves around the image area, resulting in the function $R(m, n)$. The maximum value of $R(m, n)$ then indicates the position where $w(x, y)$ best matched $f(x, y)$. Note that accuracy will be lost for values of m and n near the edges of $f(x, y)$, with the amount of error being proportional to the size of $w(x, y)$.

The correlation function given in equation (A.6), although simple in nature, has the drawback that it is sensitive to scale changes in the amplitude $f(x, y)$ and $w(x, y)$. A method frequently used to overcome this difficulty is to perform matching via the *correlation coefficient*, defined as

$$r(m, n) = \frac{\sum_x \sum_y [f(x, y) - \bar{f}(x, y)] [w(x - m, y - n) - \bar{w}]}{\left[\sum_x \sum_y [f(x, y) - \bar{f}(x, y)]^2 \sum_x \sum_y [w(x - m, y - n) - \bar{w}]^2 \right]^{1/2}} \quad (\text{A.7})$$

where $m = 0, 1, 2, \dots, M - 1$, $n = 0, 1, 2, \dots, N - 1$, $\bar{w}$ is the average intensity of the mask (this value is computed only once), $\bar{f}(x, y)$ is the average value of $f(x, y)$ in the region coincident with $w(x, y)$, and the summations are taken over the coordinates common to both f and w . It is not difficult to show that $r(m, n)$ is scaled to the range from -1 to 1 , independent of scale changes in the amplitude of $f(x, y)$ and $w(x, y)$.

A.2 Neighbourhood search algorithms

Neighbourhood search algorithms are generally used to group pixels into objects. The process is normally implemented on binary images, i.e. after edge detection or thresholding. The grouping is based on how these pixels connect with each other and is usually used to group those pixels that are turned on (pixels with binary values of 1). There are two basic techniques used to achieve connectedness; the four-connected and eight-connected neighbourhood search algorithms.

1. *Four-connectedness*

Pixels are considered connected if their edges touch. This means that if the central pixel in figure (A.2) is on, then pixels 2, 4, 5 and 7 are possible candidates for being connected to the central pixel provided that they are also turned on.

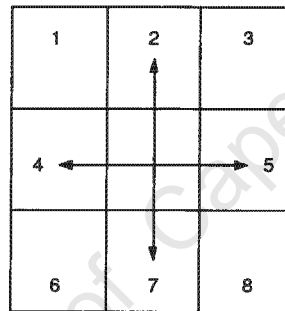


Figure A.2: Illustration of four-connectedness

2. *Eight-connectedness*

Pixels are considered connected if their edges or corners touch. This means that if the central pixel in figure (A.3) is on, then pixels 1, 2, 3, 4, 5, 6, 7 and 8 are possible candidates for being connected to the central pixel provided that they are also turned on.

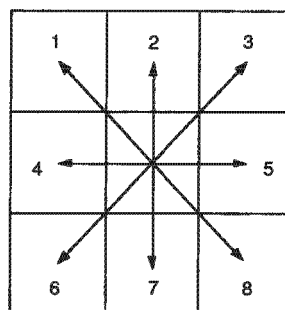


Figure A.3: Illustration of eight-connectedness

To illustrate the use of the connectedness criteria we consider the following simple case, figure (A.4). The figure illustrates an edge (pixels numbered 3) and two distinct regions (pixels numbered 1 and

2 define two different regions with the same intensity values). The eight-connected neighbourhood search algorithm is used to group the edge pixels into a single object by labelling them with some unique intensity, like 3. Now assume we wanted to perform some operation on region 2 only. The four-connected neighbourhood search algorithm can be used to identify regions 1 and 2 as two distinct regions. The region of interest is then processed according to its unique intensity label.

2	2	2	2	2	2	2
2	2	2		2	2	2
2	2		1		2	2
2		1	1	1		2
2	2		1	1		2
2	2	2			2	2
2	2	2	2	2	2	2

Figure A.4: Application of the four- and eight-connected algorithms

A.3 Geometric features of regions in image space.

A.3.1 Introduction

In all feature extraction processes, each region of interest is labelled and then processed in terms of its geometric properties. There are numerous geometric properties than can be computed, for example Area, Centroid, Second Moment of Area, Ellipse properties (including Major axis, Minor axis and Eccentricity), Pixel List and Orientation. The properties used to facilitate the target centering routines as applied to the tracked objects in the X-ray images will now be described.

A.3.2 Area

In the simplest sense the area of a labelled region is the sum of all pixels comprising the region.

A.3.3 Eccentricity

The eccentricity of a region is defined as the eccentricity of the ellipse that has the same second moments of area. Eccentricity is a scalar value with $0 \leq e \leq 1$. Accordingly we may define the eccentricity of a region (equivalent ellipse) as:

Definition 3 *The eccentricity is the ratio of the distance between the foci of the ellipse and its major axis length. (An ellipse whose eccentricity is 0 is actually a circle, while an ellipse whose eccentricity is 1 is a line segment.)*

A.4 Optimal Thresholding

Assume an image contains only two principal brightness regions. The histogram of such a picture may be considered as an estimate of the intensity probability density function, $p(x)$. This overall density function would be the sum or mixture of two unimodal densities, one for the light and one for the dark regions in the image. Furthermore, the mixing parameters would be proportional to the areas of the picture of each brightness. If the form of the densities is known or assumed, then it is possible to determine an optimal threshold (in terms of minimum error) for segmenting the image into the two brightness regions. A purely bimodal image with additive Gaussian noise has a mixture probability density function given by

$$p(x) = P_1 p_1(x) + P_2 p_2(x), \quad (\text{A.8})$$

which, for the Gaussian case, is

$$p(x) = \frac{P_1}{\sigma_1 \sqrt{2\pi}} \exp \left[-\frac{(x - \mu_1)^2}{2\sigma_1^2} \right] + \frac{P_2}{\sigma_2 \sqrt{2\pi}} \exp \left[-\frac{(x - \mu_2)^2}{2\sigma_2^2} \right], \quad (\text{A.9})$$

where μ_1 and μ_2 are the mean values of the two brightness levels, σ_1 and σ_2 are the standard deviations about the means, and P_1 and P_2 are the *a priori* probabilities of the two levels. Since the constraint

$$P_1 + P_2 = 1 \quad (\text{A.10})$$

must be satisfied, the mixture density has five unknown parameters. If all the parameters are known, the optimal threshold is easily determined.

Suppose that the dark regions correspond to the objects and the bright regions correspond to the background. In this case $\mu_1 < \mu_2$ and we may define a threshold T so that all pixels with a grey level above T are considered background points and all pixels with a level below T are considered object points. The probability of (erroneously) classifying an object point as a background point is

$$E_1(T) = \int_{-\infty}^T p_2(x) dx. \quad (\text{A.11})$$

Similarly, the probability of classifying a background point is

$$E_2(T) = \int_T^{\infty} p_1(x) dx. \quad (\text{A.12})$$

Therefore the overall probability of error is given by

$$E(T) = P_2 E_1(T) + P_1 E_2(T). \quad (\text{A.13})$$

To find the threshold value for which this error is minimal, we may differentiate $E(T)$ with respect to T (using Liebnitz's rule) and equate the result to zero. The result is

$$P_1 p_1(T) = P_2 p_2(T) \quad (\text{A.14})$$

Substituting the unimodal probability for p_1 and p_2 , and taking logarithms of both sides, we obtain a quadratic equation

$$AT^2 + BT + C = 0 \quad (\text{A.15})$$

where

$$A = \sigma_1^2 - \sigma_2^2 \quad (\text{A.16})$$

$$B = 2(\mu_1 \sigma_2^2 - \mu_2 \sigma_1^2) \quad (\text{A.17})$$

$$C = \sigma_1^2 \mu_2^2 - \sigma_2^2 \mu_1^2 + 2\sigma_1^2 \sigma_2^2 \ln \left(\frac{\sigma_2 P_1}{\sigma_1 P_2} \right) \quad (\text{A.18})$$

In the special case $\sigma = \sigma_1 = \sigma_2$, a single solution may be sufficient:

$$T = \frac{\mu_1 + \mu_2}{2} + \frac{\sigma^2}{\mu_1 - \mu_2} \ln \left(\frac{P_1}{P_2} \right) \quad (\text{A.19})$$

Further, if $P_1 = P_2$, the optimal threshold is simply the average of the means.

$$T = \frac{\mu_1 + \mu_2}{2} \quad (\text{A.20})$$

The threshold value for this special case is said to be symmetric about the mixed probability density function.

A.5 Conic section geometry

The general conic is described by

$$ax^2 + bxy + cy^2 + dx + ey + f = 0 \quad (\text{A.21})$$

From this we can easily extract the rotation of the ellipse

$$\theta = \frac{1}{2} \arctan\left(\frac{a-c}{b}\right) \quad (\text{A.22})$$

The rotation angle is measured counterclockwise from the positive x -axis to the first ellipse axis.

If we rotate the general conic so that the major and minor axes of the ellipse lie along the x - and y - axis respectively, then equation (A.21) becomes

$$A_{uu}u^2 + A_{vv}v^2 + A_uu + A_vv + A_0 = 0 \quad (\text{A.23})$$

with

$$\begin{aligned} A_0 &= f \\ A_u &= d \cos(\theta) + e \sin(\theta) \\ A_v &= -d \sin(\theta) + e \cos(\theta) \\ A_{uu} &= a \cos^2(\theta) + c \sin^2(\theta) + b \cos(\theta) \sin(\theta) \\ A_{vv} &= a \sin^2(\theta) + c \cos^2(\theta) - b \cos(\theta) \sin(\theta) \end{aligned}$$

Then the centre or principle point of the ellipse in the rotated coordinates is

$$\begin{aligned} u_0 &= -\frac{A_u}{2A_{uu}} \\ v_0 &= -\frac{A_v}{2A_{vv}} \end{aligned}$$

which is transformed back to

$$\begin{aligned} c_x &= u_0 \cos(\theta) - v_0 \sin(\theta) \\ c_y &= u_0 \sin(\theta) + v_0 \cos(\theta) \end{aligned}$$

and the radii of the major and minor axes of the ellipse are given by

$$k = A_0 + A_{uu}u_0^2 - A_{vv}v_0^2$$

$$\begin{aligned} R_x &= -\sqrt{\frac{k}{A_{uu}}} \\ R_y &= -\sqrt{\frac{k}{A_{vv}}} \end{aligned}$$

where negation is passed through the square root. The original conic is now of the form

$$\frac{(x - c_x)^2}{R_x |R_x|} + \frac{(y - c_y)^2}{R_y |R_y|} = 1$$

and the principle points of the ellipse are at $(c_x \pm R_x; c_y \pm R_y)$.

A.6 Polynomial fitting

Given a set of data $[x, y]$ such that the errors in y are independent normal with constant variance. Required to fit a polynomial $p(x)$ of degree n to the data.

$$p(x) = p_1x^n + p_2x^{n-1} + \dots + p_nx + p_{n+1} \quad (\text{A.24})$$

In matrix form

$$AX = B \quad (\text{A.25})$$

where A is the design matrix containing the unknown p_i terms, X contains the polynomial form of the x_i and B is the vector of the y_i . The numerical properties of the x_i can be improved by applying the appropriate centering and scaling. For independent normal data, this is usually achieved by replacing the x terms with $\hat{x}$:

$$x \rightarrow \hat{x} = \frac{x - \mu}{\delta} \quad (\text{A.26})$$

where μ is the mean of the x_i terms and δ the standard deviation of the x_i .

The following steps outline the procedure for implementing the least squares solution.

1. Now form the Vandermonde matrix V , whose elements are the powers of x . Therefore $v_{i,j} = x_i^{n-j}$, with each V_i of the form $V_i = [x_i^{n-1}, x_i^{n-2}, \dots, x_i, 1]$, and $n = \text{length}(x)$.
2. Perform QR decomposition on V such that $QR = V$ with R an upper diagonal and $Q^T Q = 1$.
3. Solve $A(QR) = B$ in the least squares sense

$$A = \left[(QR)^T (QR) \right]^{-1} \left[(QR)^T B \right] \quad (\text{A.27})$$

The statistical error of the predictions for input data that are independent normal with constant variance is as follows:

Define

$$E = VR^{-1}$$

$$e = \sqrt{1 + E^2}$$

$$r = B - VA$$

$$N = \max[\text{svd}(r)]$$

where $\text{svd}(r) = UWF^T$, with U a column orthogonal matrix (unitary matrix), W is a non-negative diagonal matrix, and F an orthogonal matrix. The error in the polynomial predictions are then

$$\Delta B = \left(\frac{N}{\sqrt{d}} \right) e \quad (\text{A.28})$$

where $d = \text{length}(B) - (n + 1)$.

The theory claims that if the errors in B are independent normal with constant variance, the error bounds contain at least 50% of the predictions. This is less than the usual statistically acceptable agreement of 68%, however the solution is far more stable. The theory and numerical implementation of polynomial fitting is described in Press et al [96]. Included there is a complete description of singular valued decomposition (*svd*) and QR decomposition.

A.7 Tangential velocity

The procedure for determining the magnitude of the tangential velocity is based on the positional and velocity data. The components of velocity are determined according to section (9.2). With the kinematic information as the input, the tangential velocity is calculated as follows:

$$\begin{aligned}
 R &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\
 p &= \begin{bmatrix} x & y \end{bmatrix} R \\
 \hat{p} &= \begin{bmatrix} \frac{p_x}{|p_x|} & \frac{p_y}{|p_y|} \end{bmatrix} \\
 v &= \begin{bmatrix} v_x & v_y \end{bmatrix} \\
 v_t &= v \cdot \hat{p}
 \end{aligned}$$

where R is the 90° counter-clockwise rotation matrix, v is the velocity vector of the particle at position $\begin{bmatrix} x & y \end{bmatrix}$, $\hat{p}$ is the unit vector orthogonal to particle position and v_t is the magnitude of the tangential velocity.

Appendix B

Control Frames and Calibration

B.1 Introduction

A control frame may be described as a rigid body with carefully chosen positions that have been measured using a surveying apparatus. The basic function of the control frame is to parametrise a given volume of space that has been described by some photogrammetric parametric model. The control points are used to solve for the unknown parameters of the model. The measured points used in the present work are denoted by ball bearings carefully positioned to span the control volume. Spheres have the property of becoming ellipses under a projective transformation and thus circular features (circles and ellipses) can be identified by their elliptical shape on an image. Highly attenuating targets, like ball bearings, provide a high contrast with the background, which eases the detection task and improves the centering precision.

The origin of the measured control points depend on the surveying apparatus and are said to be in a Lagrangian coordinate system. The Lagrangian origin is convenient for mathematical manipulation but is not necessarily a physically meaningful origin like the centre of the mill for example. The coordinates must therefore be transformed from the Lagrangian framework into an Eulerian system, where the Eulerian origin is some physically usable point of the system being modelled (like the centre of the mill).

This chapter describes the design of the control frame and the various surveying apparatus used to measure the control points.

B.2 Control Frame Design

The control frame design is governed by the mathematical requirements of the solution scheme used for the chosen photogrammetric model. Depending on the model used to describe the mapping of image space to real space, the following guide lines apply:

1. A minimum of six control points must be used.
2. The control points must not all lie on the same axial plane.
3. The control points must span the entire control volume evenly and sufficiently.

Six control points guarantees that solution matrices are not under-determined but is generally insufficient for a dynamic system like a tumbling mill. To ensure a well conditioned solution matrix it is generally acceptable to use an over-determined system. This implies having more control points than the number of unknown model parameters. Of equal importance is the positioning of the control points in the control volume. The governing principle is simply that the equations will produce meaningful convergence for points near the control points.

B.3 The Reflex Metrograph

The reflex metrograph belongs to a class of photogrammetric measuring instruments that use the principal of mirror reflection, illustrated in figure (B.1).

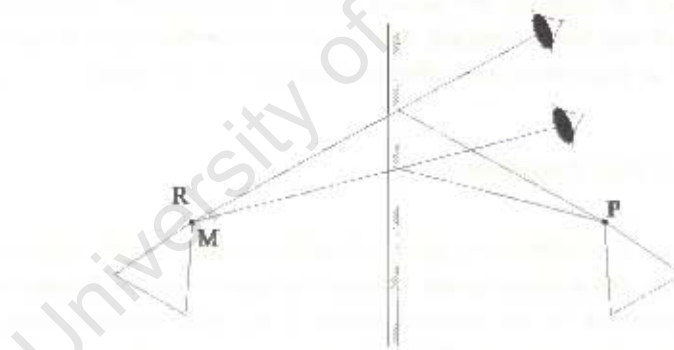


Figure B.1: The reflex principle

The object on the right is reflected in a semi-silvered mirror. An object point P creates an image of itself at R , where PR is normal to the mirror and P and R are equidistant from it. The virtual image is thus seen to contain all the three dimensional properties of the original object. An observer looking at R will be able also to see through the mirror to a measuring mark M and, when M is in coincidence with R , there will be no visual parallax. The observer can thus move M about the image, taking measurements as required, and the object is not there to obstruct the movement of M . The object itself must be stationary during the measurement and the size of the object is restricted by the practical consideration of the maximum mirror size possible. Also, with a large object, the distance from the mirror to the furthest points R may be so great as to make the eyebase:distance

ratio too small for accurate elimination of visual parallax. This may be overcome by extending the cycbase through the use of a device similar to a mirror stereoscope.

The reflex principle is incorporated into two measurement systems, the reflex metrograph and reflex microscope, at the Department of Biomedical Engineering, University of Cape Town. The latter has a least count of 0.01 mm, however it was too small to incorporate the control frame. The metrograph is a scaled up version of the microscope with a typical standard deviation of 1 mm. Both apparatus are connected to a personal computer that records the three dimensional coordinates once the user clicks them through, using a hand held switch.

In addition to the abovementioned problems, the coordinates measured correspond to the surface of the ball bearings. Depending on the operator the surface point will vary from ball to ball, resulting in low precision data. The measuring apparatus are however easy to use, and provide quick control data for analyses requiring accuracies of the order of 1 mm.

B.4 The Digitiser

The Digitiser is an instrument used to measure (X, Y, Z) of a point in space. A movable arm with a metal 'pencil' at the wrist end is attached to a (X, Y, Z) digitiser that has been calibrated. The pencil has five, equally spaced positions to rotate through. The arm rotates freely. The 3D co-ordinates of a point in space is obtained by placing the pencil tip on the required point and clicking on a control switch. The pencil tip need not be in contact with any solid surface for the point to be recorded. The orientations of the pencil or arm does not influence the (X, Y, Z) data.

B.4.1 Limitations of the Digitiser

The major disadvantage of the digitiser is that it is only accurate (and hence useful) if the pencil tip can be accurately placed on the desired point. However, most control frames are not adapted for the digitiser since the main function of the control frame is to have markers that are easily and clearly recognised on the image (in most cases an X-ray image). Accordingly most frames have ball bearings for markers. Although these markers show up as dark circular spots on the image, they are clearly not suited for the digitiser since the ball bearings define a spherical surface with an infinite number of points to place the pencil tip on. The following features make this apparatus unsuitable for obtaining measurements of the required accuracy.

1. The digitiser measures in meters with it's least count being 1 mm. This implies an inherent uncertainty of approximately 1 mm.
2. The pencil tip is 'blunt' - no sharp point to consistently place on ball bearings.
3. The five discrete slot positions of the digitiser wrist further complicates the procedure in point 2.
4. Measuring surface points of spheres, as opposed to their centres, introduces an uncertainty of 1.5 mm, for 3 mm ball bearings.

B.4.2 Solution

Although the digitiser proved unsuitable for the measurement accuracy that was required, its principle function provided the solution method required. If five or more surface points on the steel ball bearings could be measured, to within the required accuracy, it is a simple task to establish the centre of the sphere. This is desirable when establishing the transformation parameters as the centroids of objects in image space map to the centre of the sphere in real space. To these ends, a more accurate measuring instrument, similar in principle to the digitiser, was sought.

B.5 CMM: Coordinate Measuring Machine

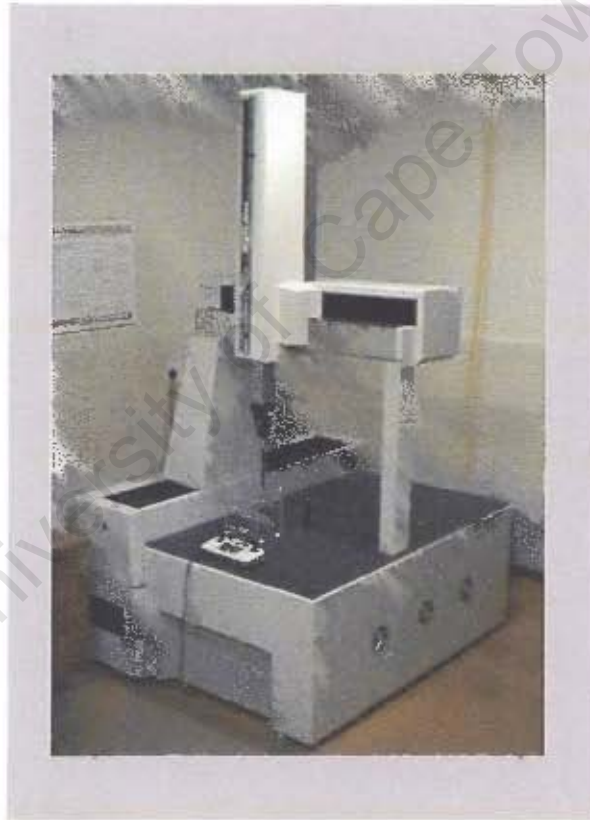


Figure B.2: Coordinate Measuring Machine (CMM)

The CMM, figure (B.2), is a computerised, industrial measuring tool. Its primary use is in the motor vehicle industry where large amounts of accurate profiles are required. Measurements are made by an electronically controlled probe that is mapped to a coordinate axes. Much like the digitiser, the probe is brought into contact with the object to record a measurement. However, the measurement is only made when a force deflection of 0.0687 N is registered by the probe. This ensures a consistently

accurate measurement. The particular value of the CMM is the ability to calculate the centre of a sphere by measuring surface points. The centre is obtained from a least squares solution, using at least 5 points measured on the surface of the ball bearing. The solution is included here for completeness, but is not necessary for obtaining the centre.

B.5.1 Determining the centre of spheres from surface points: An iterative least squares approach

The equation of a sphere of radius r centered on a point (h, k, l) is given by:

$$(x_i - h)^2 + (y_i - k)^2 + (z_i - l)^2 = r^2 \quad (\text{B.1})$$

where (x_i, y_i, z_i) are points on the surface of the sphere.

Expanding and rearranging equation (B.1)

$$F = 0 = x_i^2 + y_i^2 + z_i^2 - 2x_i h - 2y_i k - 2z_i l + h^2 + k^2 + l^2 + r^2 \quad (\text{B.2})$$

Taylor expanding F to first order with respect to the four parameters and rearranging gives

$$2x_i \partial x_i + 2y_i \partial y_i + 2z_i \partial z_i = \partial h(2x_i - 2h) + \partial k(2y_i - 2k) + \partial l(2z_i - 2l) + \partial r(2r) - (F)_0 \quad (\text{B.3})$$

In matrix notation

$$V = AX - I \quad (\text{B.4})$$

where

A is the coefficient matrix of the unknown residuals (the parameters need to have provisional values),

X is the vector containing the unknown residuals $(\partial h, \partial k, \partial l, \partial r)$, and

I is the vector containing the "free terms" (the $(F)_0$ terms).

The least squares estimation of X is then obtained as follows

$$\hat{X} = (A^T P A)^{-1} (A^T P I) \quad (\text{B.5})$$

where P is the *a priori* weight matrix of the sampled data. However, in most cases there is insufficient information to establish P . For this reason, we assume P to be of uniform weight. Equation (B.5) thus reduces to

$$\hat{X} = (A^T A)^{-1} (A^T I) \quad (\text{B.6})$$

The control frame coordinates derived from the CMM measurements is given in table (B1). Each point was measured three times and the mean and standard deviation computed. Those points producing relatively high standard deviations ($> 0.05\text{ mm}$) are most likely attributed to the difficulty in extracting five points on the surface of the ball bearing - an essential step to ensuring that the least squares solution to the centroid of the ball is accurate. This was noted by the CMM operator and was unfortunately unavoidable.

Table B1: 110 Control points with measured standard deviation

Point No.	$\bar{X} \pm \delta X$ (mm)	$\bar{Y} \pm \delta Y$ (mm)	$\bar{Z} \pm \delta Z$ (mm)
1	-15.11 ± 0.042	40.06 ± 0.010	29.69 ± 0.009
2	-30.13 ± 0.012	40.23 ± 0.048	29.69 ± 0.018
3	-45.07 ± 0.079	40.58 ± 0.045	29.73 ± 0.042
4	-60.06 ± 0.066	40.26 ± 0.068	29.73 ± 0.025
5	-75.12 ± 0.036	40.43 ± 0.065	29.77 ± 0.049
6	-90.11 ± 0.024	40.07 ± 0.053	29.81 ± 0.023
7	-105.14 ± 0.107	39.93 ± 0.063	29.66 ± 0.003
8	-120.15 ± 0.029	39.87 ± 0.034	29.61 ± 0.006
9	-15.14 ± 0.010	50.03 ± 0.041	-30.02 ± 0.023
10	-30.16 ± 0.011	50.01 ± 0.034	-30.10 ± 0.062
11	-45.08 ± 0.004	50.40 ± 0.017	-30.07 ± 0.046
12	-60.12 ± 0.054	49.87 ± 0.041	-30.15 ± 0.051
13	-75.12 ± 0.011	50.00 ± 0.011	-30.10 ± 0.015
14	-90.10 ± 0.047	50.04 ± 0.058	-30.20 ± 0.001
15	-105.11 ± 0.041	49.99 ± 0.050	-30.18 ± 0.034
16	-120.15 ± 0.000	49.79 ± 0.031	-30.15 ± 0.003
17	-15.15 ± 0.005	59.86 ± 0.063	-70.19 ± 0.061
18	-30.15 ± 0.016	59.87 ± 0.055	-70.28 ± 0.002
19	-45.08 ± 0.026	60.10 ± 0.023	-70.20 ± 0.056
20	-60.13 ± 0.001	59.81 ± 0.004	-70.23 ± 0.072
21	-75.13 ± 0.006	59.80 ± 0.041	-70.28 ± 0.030
22	-90.12 ± 0.036	59.78 ± 0.003	-70.26 ± 0.035
23	-120.12 ± 0.052	59.88 ± 0.016	-70.24 ± 0.021
24	-15.14 ± 0.041	69.99 ± 0.030	-60.31 ± 0.047
25	-30.17 ± 0.022	69.93 ± 0.021	-60.39 ± 0.090
26	-45.12 ± 0.042	70.11 ± 0.057	-60.35 ± 0.059
continue ...			

Table B1: 110 Control points with measured standard deviation

27	-60.12 ± 0.058	70.00 ± 0.024	-60.30 ± 0.010
28	-75.17 ± 0.020	69.74 ± 0.081	-60.36 ± 0.062
29	-90.11 ± 0.067	70.01 ± 0.034	-60.29 ± 0.045
30	-105.17 ± 0.046	69.79 ± 0.025	-60.31 ± 0.019
31	120.18 ± 0.074	69.76 ± 0.044	-60.22 ± 0.020
32	-15.20 ± 0.040	80.09 ± 0.021	9.56 ± 0.035
33	-30.25 ± 0.016	80.06 ± 0.040	9.51 ± 0.010
34	-45.15 ± 0.028	80.22 ± 0.038	9.48 ± 0.073
35	-60.16 ± 0.024	80.41 ± 0.050	9.62 ± 0.011
36	-75.22 ± 0.065	80.17 ± 0.085	9.52 ± 0.011
37	-90.14 ± 0.061	79.98 ± 0.023	9.54 ± 0.046
38	-105.22 ± 0.017	80.25 ± 0.023	9.54 ± 0.050
39	-120.21 ± 0.046	79.89 ± 0.025	9.71 ± 0.016
40	-15.19 ± 0.020	100.05 ± 0.017	-20.36 ± 0.062
41	-30.18 ± 0.007	100.06 ± 0.005	-20.45 ± 0.064
42	-45.18 ± 0.027	100.05 ± 0.053	-20.29 ± 0.029
43	60.13 ± 0.023	100.19 ± 0.005	-20.34 ± 0.023
44	-75.15 ± 0.071	99.99 ± 0.055	-20.27 ± 0.035
45	-90.16 ± 0.060	100.38 ± 0.011	-20.21 ± 0.052
46	-105.12 ± 0.100	100.09 ± 0.013	-20.29 ± 0.031
47	-120.15 ± 0.007	100.03 ± 0.028	-20.22 ± 0.038
48	-15.15 ± 0.128	90.14 ± 0.060	39.70 ± 0.027
49	-30.15 ± 0.001	90.26 ± 0.038	39.66 ± 0.035
50	-45.14 ± 0.018	90.38 ± 0.003	39.71 ± 0.006
51	-60.13 ± 0.051	90.48 ± 0.051	39.66 ± 0.030
52	-75.17 ± 0.002	90.29 ± 0.021	39.63 ± 0.061
53	-90.19 ± 0.013	90.21 ± 0.025	39.60 ± 0.021
54	-105.19 ± 0.074	90.02 ± 0.035	39.71 ± 0.038
55	-120.24 ± 0.029	89.88 ± 0.023	39.65 ± 0.055
56	-15.26 ± 0.046	110.05 ± 0.060	19.64 ± 0.087
57	-30.23 ± 0.013	110.16 ± 0.021	19.58 ± 0.042
58	-45.28 ± 0.043	110.35 ± 0.017	19.56 ± 0.023
59	-60.23 ± 0.048	110.44 ± 0.029	19.52 ± 0.020
60	-75.19 ± 0.027	110.08 ± 0.064	19.55 ± 0.001
61	-90.19 ± 0.034	110.25 ± 0.075	19.57 ± 0.028
62	-105.25 ± 0.056	110.09 ± 0.036	19.68 ± 0.025
63	-120.27 ± 0.037	109.94 ± 0.019	19.73 ± 0.015
64	-135.29 ± 0.026	109.91 ± 0.036	19.68 ± 0.041
65	15.18 ± 0.008	120.28 ± 0.002	-40.40 ± 0.021
66	-30.20 ± 0.045	119.92 ± 0.122	-40.46 ± 0.002
continue ...			

Table B1: 110 Control points with measured standard deviation

67	-45.15 ± 0.002	120.06 ± 0.048	-40.49 ± 0.039
68	-60.18 ± 0.003	119.96 ± 0.028	-40.41 ± 0.046
69	-75.09 ± 0.038	120.25 ± 0.046	-40.43 ± 0.017
70	-90.12 ± 0.077	120.41 ± 0.041	-40.58 ± 0.012
71	-105.17 ± 0.031	119.94 ± 0.044	-40.50 ± 0.051
72	-120.19 ± 0.010	120.15 ± 0.067	-40.42 ± 0.089
73	-15.20 ± 0.067	130.12 ± 0.079	-10.55 ± 0.074
74	30.17 ± 0.066	130.22 ± 0.062	-10.59 ± 0.051
75	-45.13 ± 0.017	130.45 ± 0.060	-10.62 ± 0.025
76	-60.16 ± 0.082	130.09 ± 0.040	-10.53 ± 0.013
77	-75.18 ± 0.015	130.04 ± 0.027	-10.53 ± 0.057
78	-90.20 ± 0.077	130.10 ± 0.053	-10.51 ± 0.009
79	-105.26 ± 0.012	129.94 ± 0.047	-10.44 ± 0.009
80	-120.23 ± 0.017	129.89 ± 0.012	-10.41 ± 0.020
81	0.00 ± 0.009	0.00 ± 0.031	0.00 ± 0.035
82	-15.03 ± 0.018	-0.03 ± 0.005	0.06 ± 0.020
83	-30.01 ± 0.049	0.01 ± 0.026	0.05 ± 0.035
84	-45.05 ± 0.018	-0.12 ± 0.064	-0.09 ± 0.043
85	-60.08 ± 0.044	-0.07 ± 0.030	-0.02 ± 0.025
86	-75.08 ± 0.033	-0.02 ± 0.036	-0.05 ± 0.022
87	-90.05 ± 0.016	-0.08 ± 0.035	-0.20 ± 0.079
88	-105.07 ± 0.023	-0.17 ± 0.009	-0.22 ± 0.037
89	-0.00 ± 0.034	19.95 ± 0.013	-49.95 ± 0.038
90	-15.09 ± 0.032	19.69 ± 0.030	-50.04 ± 0.026
91	-30.05 ± 0.009	19.76 ± 0.019	-50.00 ± 0.004
92	-45.04 ± 0.026	20.03 ± 0.022	-50.03 ± 0.017
93	-60.09 ± 0.030	19.51 ± 0.103	-50.16 ± 0.066
94	-75.06 ± 0.046	19.86 ± 0.092	-50.08 ± 0.048
95	-90.06 ± 0.015	19.70 ± 0.025	-50.11 ± 0.084
96	-105.07 ± 0.132	19.81 ± 0.023	-50.09 ± 0.006
97	-0.06 ± 0.028	30.15 ± 0.040	49.39 ± 0.035
98	-15.09 ± 0.048	30.10 ± 0.016	49.89 ± 0.060
99	-30.08 ± 0.027	30.15 ± 0.050	49.96 ± 0.001
100	45.03 ± 0.032	30.09 ± 0.018	49.97 ± 0.009
101	-60.12 ± 0.013	30.30 ± 0.048	49.81 ± 0.032
102	-75.13 ± 0.021	29.93 ± 0.053	49.81 ± 0.029
103	-90.14 ± 0.017	30.18 ± 0.069	49.82 ± 0.014
104	-0.08 ± 0.011	40.08 ± 0.007	29.80 ± 0.013
105	0.10 ± 0.029	50.01 ± 0.069	30.05 ± 0.053
106	-0.14 ± 0.063	80.09 ± 0.020	9.61 ± 0.007
continue ...			

Table B1: 110 Control points with measured standard deviation

107	-0.17 ± 0.044	100.05 ± 0.054	-20.33 ± 0.010
108	-0.12 ± 0.108	90.14 ± 0.003	39.62 ± 0.003
109	-0.20 ± 0.023	110.11 ± 0.013	19.64 ± 0.003
110	-0.11 ± 0.047	120.07 ± 0.067	-40.34 ± 0.061

Appendix C

X-rays and Angiography

A beam of X-ray radiation is in fact a ray of electromagnetic energy of wavelength from 1 to 10 Angstroms. As X-rays pass through matter they can penetrate completely through unaffected, they may be absorbed by the material and consequently pass their energy onto the absorbing material, or they may be scattered or redirected and likely lose energy in the process. As these photons (packets of electromagnetic energy) pass through matter they carry with them important information about the matter which is recorded on the X-ray image. The contrast recorded on the final image is directly related to the energy of the X-ray beams and the density of the material it passes through. Materials of higher density block some X-ray photons from reaching the film and therefore these areas of film appears lighter in colour since X-rays produce darkened images on film.

X-rays are widely used for non-invasive examinations because of their ability to penetrate objects easily and create an image - a function of their high energy levels. Diagnostic X-rays are produced with electrons having energy in the 150-250 keV range and X-rays resulting from them are readily stopped in matter. X-ray generators which are more commonly called X-ray machines, convert normal electrical current to high voltage direct current within a X-ray tube. This creates a controllable X-ray beam, which can be used to image all sorts of matter - either living or non-living.

X-rays are produced in vacuum-sealed tubes containing a cathode and an anode. These tubes are sealed because the presence of any gas, such as air, inhibits efficient X-ray production. Modern X-ray tubes allow for precise control of the number of electrons striking the anode therefore effectively controlling the strength of X-rays produced. Tubes are surrounded by oil and lead to protect the X-ray tube and prevent stray radiation from leaking out of the tube housing.

C.1 The cathode

An X-ray cathode is commonly made of a tungsten filament and a metallic focussing cup for focussing the electrons towards the anode target. The filament is a helical coil of tungsten wire of around 0.2 mm in diameter and about 10 mm long. Modern X-ray tubes usually have two or more filaments, which

effectively allows for a choice of focal spot sizes to allow the unit to have varied resolution levels and prolong the life of the tube. Focal spot size is an important factor determining the maximum spatial resolution recorded on the X-ray generator. Some X-ray generators use special X-ray tubes with very fine focal spots of approximately 0.1 mm for high resolution X-ray imaging where spatial resolution of the final X-ray image is crucial - such as for magnification mammography.

As electrical current is applied to the filament wire it becomes extremely hot - nearing 3370°C for tungsten. When this critical temperature is reached, electrons in their excited state overcome the surface barrier forces keeping them within the wire. These electrons leave the wire and form a cloud around the filament. This negatively charged electron cloud remains close to the filament because of its attraction to the positively charged filament. To free these electrons and give them sufficient kinetic energy the anode is given a strong positive charge by the high-tension circuit. When the exposure button is pressed, the circuit is closed and the electrons are accelerated towards the anode.

C.2 The anode

Tungsten and Rhenium are choice material of their incredible strength, high atomic number, high melting point, and it's high resistance to vaporising at these extreme temperatures. The target electrode, which is also called the anode, may be stationary or rotate at high speeds. An anode is normally made of two materials: the tungsten anode and a copper material holding it in place. Modern anodes used are composed of metal alloys; typically tungsten, rhenium, molybdenum and graphite. The additional metals serve to disperse heat making the anode more efficient.

Rotating anodes often have speeds of 3000 to 10 000 rpm and are better at dissipating heat than stationary anodes. In addition to their ability to radiate heat effectively, rotating anodes function to increase the surface area of the anode target - an advantage in producing an X-ray beam. Fans in the tube assembly help cool the X-ray tube.

By controlling the amount of electrical current and exposure time in the tube filament the intensity of the X-ray production is controlled. Changing either the exposure time or the loading current in the X-ray tube will change the output of X-ray photons proportionally.

Finally, X-rays are directed through a filter and collimator that serves to adjust the dimensions of the X-ray beam; that is to 'harden' and focus the beam. At this point radiography uses film to record the image, or as most digital systems currently use, a phosphor imaging plate is used to acquire the X-ray for digital storage and display.

Intensifying screens are built into the film cassettes that are used to hold sheets of unexposed X-ray film because their luminescent properties aid the formation of an X-ray image on the film. The phosphor layers emit light with a peak output of $4250 \text{ \AA}$. This wavelength is ideal for exposing X-ray film. In a typical film/screen system, 95% of the resulting image on the film is due to the light emitted from the intensifying screen, while the remaining 5% of the image is the effect of the direct exposure of the X-ray photons on the film.

C.3 Composition of intensifying screens

A typical intensifying screen is composed of four layers. A base layer acts as a support medium and is usually made of plastic or carbon fibre. It is invisible on the final image and serves as a protective covering for the screen. Next, a layer of reflective material is used to direct light emissions from the phosphor towards the film. The third layer is the most important component, the phosphor itself. It can be of various thicknesses depending on the actual application the screen is used for. Finally, a thin protective layer covers the phosphor to protect it from damage and reduces static electricity that could produce noise artifacts on the film.

Most X-ray films have emulsions on both sides, which is why an intensifying screen cassette actually has screen systems inside each cassette - one for imaging each side of the duplex film. Between these two layers is the slot into which the X-ray film is placed. The assembly as a whole is called an intensifying screen (or sometimes a cassette) and traditional X-ray systems using these are commonly referred to as film/screen systems.

The BAE is a diagnostic tool used to assess the condition of heart valves after a contrast medium (a semi-opaque dye) has been injected into the patient's heart valves. Consequently the image produced has the heart valves clearly marked as dark tubes. In normal radiography a person would be lying on a bed where the Perspex mill is positioned, figure (C.1).



Figure C.1: Angioscope in fully interlinked position

X-rays are produced and passed through the person resulting in scintillations of varying contrast on

the input screen of the image intensifier. This eventually gets relayed to an on-line television screen, providing a quick diagnosis of the heart valves. The biplanar mode is used to obtain perpendicular images of the heart. These images are rotated in space and time thus precluding a natural frame by frame comparison.

C.4 Absorption Coefficients and Decay of the Incident Beam Intensity through various media in a mill

The intensity of the incoming and outgoing radiation passing through a body of thickness χ is related by a simple exponential decay law, sometimes referred to as Lambert's Law:

$$I_{\chi} = I_0 \exp(-\mu\chi) \quad (C.1)$$

Where:

1. I_{χ} is the final intensity
2. I_0 is the initial intensity
3. μ is the linear absorption coefficient of the body
4. χ is the thickness (the longest line passing through the body)

For a single-element homogenous medium, the linear absorption coefficient is simply given by :

$$\mu = 12.3\rho \left(\frac{Z}{E}\right)^{2.83} \quad (C.2)$$

Where:

1. ρ is the density of the material
2. Z is the atomic number
3. E is the energy of the emission

If however the body is a chemical compound or a mixture of compounds, as is more often the case, the mass absorption coefficient (μ/ρ) becomes more useful. Unlike the linear absorption coefficient, it does not depend on the state of aggregation and chemical combination of the body.

The mass absorption coefficient for a compound M is calculated as:

$$\left(\frac{\mu}{\rho}\right)_M = \sum \left(\frac{\mu}{\rho}\right)_n W_n \quad (C.3)$$

Where:

1. W_n is the weight fraction of the n^{th} element present in the mixture
2. $(\mu/\rho)_n$ is the mass absorption coefficient of the n^{th} element

Lambert's Law can then be restated as:

$$I_x = I_0 \exp \left[- \left(\frac{\mu}{\rho}\right)_M \rho_M x \right] \quad (C.4)$$

Where ρ_M is the loose bulk density of the absorber; and all other symbols taking their usual meaning.

The above equations were used to calculate a suitable media for the mill shell and tumbling media. The results of these calculations constrain the material used as follows.

1. A low density ($0.1 - 1.5 \text{ g/cm}^3$)
2. Realistic particle behaviour (not like ping-pong balls)
3. Sufficiently strong to withstand typical particle interactions within a mill run at speeds of (60 - 120% of critical)
4. A low cumulative mass absorption coefficient such that an Aluminium-filtered beam will pass through 20 cm of charge particles and still be sufficiently strong to expose the X-ray film.

Of the various materials investigated, wood was found to best satisfy the above criteria. It is also known that plastic beads are sufficiently transparent to X-rays, Powell [1]. In fact, any carbon based material is virtually transparent to X-rays.

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Question 1: Discuss the role of the teacher in the classroom.

Answer: The role of the teacher in the classroom is to facilitate learning.

The teacher should create a supportive learning environment.

1.1.1. The teacher should assess the needs of the learners.

1.1.2. The teacher should provide feedback to the learners.

1.1.3. The teacher should encourage the learners to participate in the learning process.

1.1.4. The teacher should monitor the progress of the learners.

1.1.5. The teacher should evaluate the learning outcomes.

1.1.6. The teacher should reflect on their own practice.

1.1.7. The teacher should collaborate with other teachers.

1.1.8. The teacher should communicate with parents.

1.1.9. The teacher should be a role model for the learners.

1.1.10. The teacher should be a lifelong learner.

1.1.11. The teacher should be a professional.

1.1.12. The teacher should be a leader.

1.1.13. The teacher should be a change agent.