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Geometrical and Nonperturbative Aspects of Low Dimensional Field Theories

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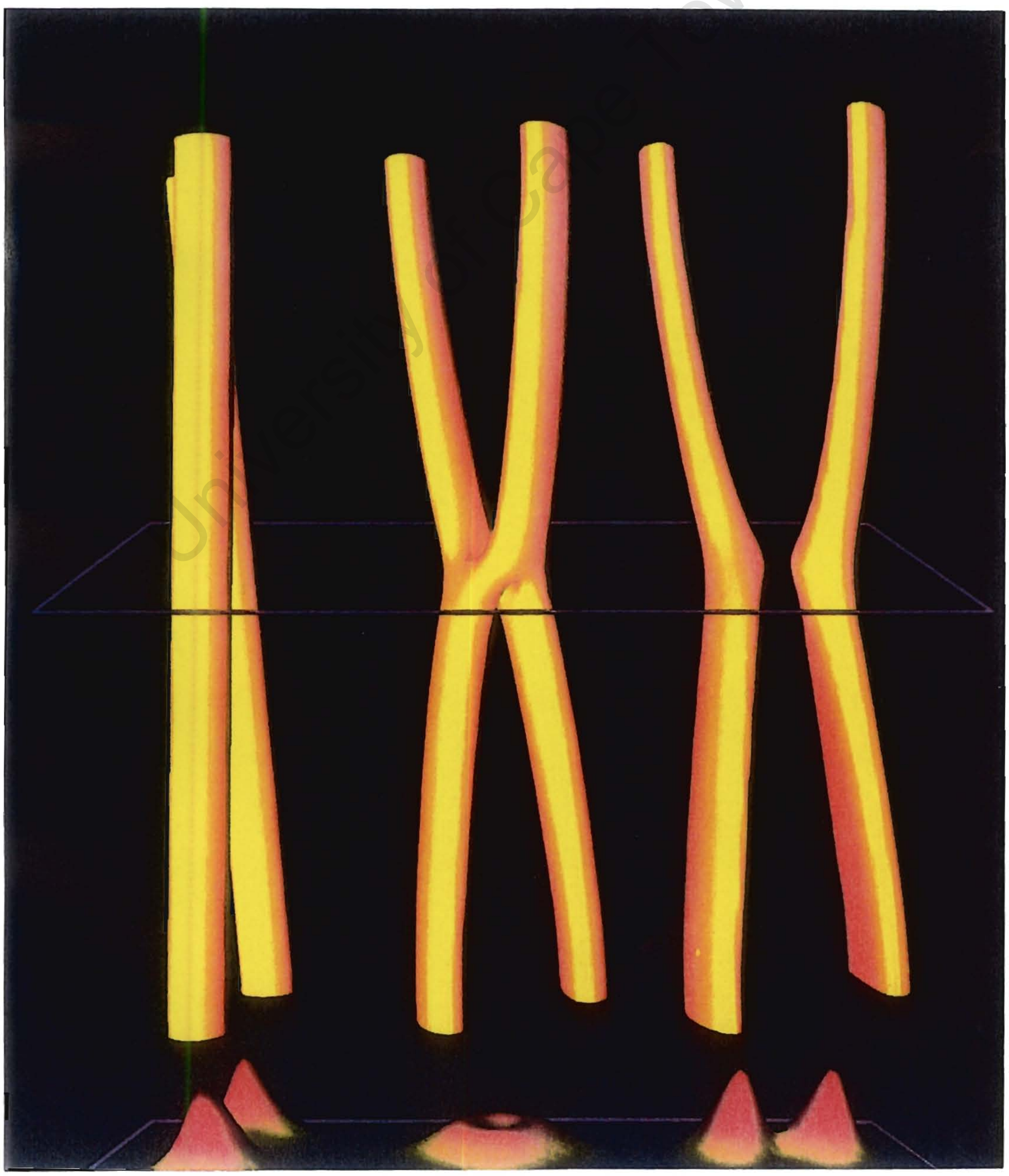
A thesis submitted in fulfillment of the requirements of the degree of Master of Science at the University of Cape Town.



Abstract

We present a collection of results on solitons in low-dimensional classical field theory. We begin by reviewing the geometrical setting of the nonlinear σ-model and demonstrate the integrability of the theory in two-dimensions on a symmetric target manifold. After reviewing the construction of soliton solutions in the $O(3)$ σ-model we consider a class of gauged nonlinear σ-models on two-dimensional axially-symmetric target spaces. We show that, for a certain choice of self-interaction, these models are all self-dual and analyze the resulting Bogomol'nyi equations in the BPS limit using techniques from dynamical systems theory. Our analysis is then extended to topologically massive gauge fields. We conclude with a deviation into exploring links between four-dimensional self-dual Yang-Mills equations and various lower-dimensional field theories. In particular, we show that at the level of equations of motion, the Euclidean Yang-Mills equations in light-cone coordinates reduce to the two-dimensional nonlinear σ-model.

Acknowledgements

This thesis is a brief account of some of the work I have been involved in over the past few years at the University of Cape Town (UCT). During these years I have had the pleasure of meeting and interacting with some unique people. Each has impacted on my life in some way. Regrettably I must constrain myself to those whose impact most directly relate to this period of my life. However, I do remember everyone and thank you all.

Firstly, I thank my family without whom none of this would have been possible...literally! I thank my supervisor, Prof. I.V. Barashenkov, whose encouragement, support and patience with my sometimes erratic progress is appreciated. To the faculty at the Department of Mathematics and Applied Mathematics at UCT, especially Profs. Ellis and Brink, your confidence in me is deeply appreciated. Many thanks too to the secretaries and staff in the department and to Marian Green at the Postgraduate Scholarships Office with out whom life would certainly have been less comfortable. I gratefully acknowledge financial support from the American-South African Scholarship Association and the National Reasearch Foundation (NRF) for their Prestigious M.Sc. Scholarship and the University of Cape Town.

Much of the initial work presented in Chapter 4 was carried out during a research visit to the University of Crete under the kind invitation of Prof. George Tsironis. I also thank Prof. Barashenkov again for arranging this visit. I would like to thank Prof. N.S. Manton for clarifying some points on the dimensional reduction procedure employed in Chapter 4 and Dr Georgi Kyrchev for valuable discussions on many of the group theoretical aspects of this work and for reading over parts of the manuscript. The idea of investigating wormhole solutions to a gauged nonlinear σ-model was sown by Prof Gérard Clément during a brief visit to the Université Pierre et Marie Curie and I thank him for his hospitality. I am grateful to Dr Peter Dunsby for his guidance during my years at UCT and (more immediately) for bringing to my attention the CONTENT dynamical systems package which, although just in its infancy proved invaluable for our work.

To Duncan Elliott, many thanks for my initiation into the 'Strong Programme' and for your unfailing encouragement! To my many friends in the Department, thank you for putting up with my many oddities and for the ever-stimulating conversations. To my many climbing partners, past and present...thank you for sharing the experience with me. I once read somewhere that a mathematician is God's invention for turning coffee into theorems...many thanks to everyone at the 'Coffee Bean' for feeding my habit. Thanks also to the Casanuevas: Juan, Ana and Clara - your friendship is truly valued.

Finally, to the one person who has most affected my life in the short time I have known her, Sara Norton. Thank you for being all that you are to me.

Contents

Introduction

...I was observing...a large solitary elevation, a rounded, smooth and well defined heap of water, which continued its course along the channel apparently without change of form or dimunition of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation, a name which it now very generally bears.

- John Scott Russel.

It's been said that no work involving solitons would be complete without a mention of John Scott Russel's remarkable observation of 1834 [1]: his so-called 'wave of translation'. Like many great scientific ideas, Russel's was met with much scepticism by the scientific community of the time, but science is a self-correcting organism and, as such, the idea of a localized 'bell-shaped' wave propagating with a velocity proportional to its amplitude attained its due resurrection.

What Russel observed was a wave propagating in a shallow channel. The resurrection of the solitary-wave idea was brought about by the mathematical formulation of the problem by Korteweg and de Vries who wrote down the nonlinear wave equation that now bears their names [2], and Boussiniesq [3] who found the hyperbolic secant squared solution for the free surface. However, the real impetus for the idea of the soliton only came in the 1960s when, with the developments in computing technology, Kruskal and Zabusky[4] were able to show that the solitary wave solutions of the Korteweg-de Vries (KdV) equation exhibited the property of reappearing almost unchanged after collisions. They coined the term 'soliton' to describe nonlinear waves that possess this remarkable property.

It was not long thereafter that the concept of a soliton found its way into the elementary particle physics and field theory literature. Most of the development in this field occurring in the mid-70s. Primarily because of its technical feasibility and the fact that many problems may be constructed as perturbations of simpler ones, diagrammatic perturbation theory is without a doubt the standard methodology of relativistic quantum field theory (QFT). However, it is becoming increasingly evident that embedded in gauge theory is a rich *non-perturbative* structure that may hold the key to several problems beyond the reach of perturbative methods. An example of this is the problem of confinement in quantum chromodynamics (QCD). The best that perturbative QFT offers in this regard are some qualitative, albeit plausible, arguments suggesting some confinement property. It is now clear that a complete understanding of this property may only be expected in the *non-perturbative* sector of QCD. That non-perturbative methods are crucial to a complete understanding of many nonlinear field theories may be further illustrated by the observation that not even the number of fundamental parameters in a theory may be certain when only the perturbative sector of the gauge theory is taken into account. A case in point is the uncovering of an additional (CP-violating) parameter θ in QCD which appears in an additional term in the effective QCD Lagrangian [5, 6, 7],

$$\mathcal{L}_{\text{QCD}}^{eff} = \sum_f \overline{\psi}_f (i \not{D} - m_f)\psi_f - \frac{1}{2} Tr(F_{\mu\nu}\tilde{F}^{\mu\nu}) + \frac{\theta}{16\pi^2} Tr(F_{\mu\nu}\tilde{F}^{\mu\nu}) \quad , \tag{1}$$

and plays a role analogous to a coupling constant. Different values of θ correspond to different Hilbert spaces and therefore different theories. In non-Abelian gauge theories, this term cannot be expressed as a surface term in the action (and hence discarded) due to the appearance of topologically non-trivial field configurations (instantons) that do not vanish suffiently rapidly at infinity. At present, the two main non-perturbative techniques used by field theorists are *lattice gauge theory* and *soliton methods*. The former [8, 9] pursues solutions of the field equations via impressive numerical methods like the Monte Carlo procedure and is probably the most powerful method of understanding the non-perturbative aspects of gauge theories. However, the calculations involved are numerically and computationally complex and very resource intensive. An alternative to this direct numerical assault on the non-perturbative sector of nonlinear field theories, and an approach that we will concentrate on in this thesis, is the use of soliton methods to construct exact solutions of the relevant nonlinear field equations.

Before proceeding any further we should define what solitary waves and solitons are, within the context of field theory as well as make some clear distinction between these concepts. Heuristicly, solitons and solitary waves are special solutions to nonlinear field equations that exhibit the following two properties:

- They are localized and propagate with velocity and shape retention, and
- the asymptotic shape velocity of several such solutions are preserved even after collision.

At first glance these properties do not seem to merit much attention. However, we should recall that we refer here to solutions of *nonlinear* equations. In this light, it becomes clear what is so remarkable about these objects: they are nonlinear objects that exhibit distinctly *linear* properties! Let us now formalize these two properties and state them in mathematical terms. In an area with as deep a literature as QFT, it should come as no surprise that there is no universally accepted definition of solitons (or even of solitary waves, for that matter). We will follow the treatment in [10] and choose a definition in terms of energy densities rather than wave fields. This, of course, restricts us to considering only those field equations that *have* an associated energy, $E[\phi_i]$, but this will be sufficient for our purposes.

Let $\mathcal{E}(\mathbf{x}, t)$ be the energy density associated with the set of fields $\phi_i(\mathbf{x}, t)$ so that $E[\phi_i] = \int d^n x \, \mathcal{E}(\mathbf{x}, t)$. We will call a solution *localized* if $\mathcal{E}(\phi_i) \to 0$ as $|\mathbf{x}| \to \pm\infty$ sufficiently rapidly at any instant of time. We now define a *solitary wave* as that localized non-singular solution of any nonlinear field equation whose energy density has a space-time dependence of the form

$$\mathcal{E}(\mathbf{x}, t) = \mathcal{E}(\mathbf{x} - \mathbf{u}t) \quad , \tag{2}$$

where $\mathbf{u}$ is some velocity vector. The energy density of a solitary wave thus propagates undistorted and with constant velocity. Let $\mathcal{E}_0(\mathbf{x} - \mathbf{u}t)$ be the (localized) energy density of a solitary wave solution of some nonlinear evolution equation and $\mathcal{E}(\mathbf{x}, t)$ the energy density of any other solution of the equation which, at some time in the past, consists of N such solitary wave solutions. $\mathcal{E}(\mathbf{x}, t)$ has the past asymptotic form

$$\lim_{t \to -\infty} \mathcal{E}(\mathbf{x}, t) = \sum_{i=1}^{N} \mathcal{E}_0(\mathbf{x} - \mathbf{a}_i - \mathbf{u}_i t) \quad , \tag{3}$$

where $\mathbf{a}_i$ and $\mathbf{u}_i$ are the arbitrary initial position and velocity of the ith solitary wave. The temporal evolution of this initial configuration is governed by the nonlinear equation under consideration. We will call this field configuration a *soliton* if its evolution is such that its energy density profile is asymptotically restored to its initial shape and profile i.e.,

$$\lim_{t \to \infty} \mathcal{E}(\mathbf{x}, t) = \sum_{i=1}^{N} \mathcal{E}_0(\mathbf{x} - \mathbf{a}_i - \mathbf{u}_i t + \mathbf{d}_i) \quad , \tag{4}$$

where the $\mathbf{d}_i$ are constant vectors allowing for the possibility that the solitons may undergo some bodily displacement relative to their pre-collision trajectories. This should be the *only* residual effect if the solutions are to be considered as solitons. While all solitons are clearly solitary waves, the stringency of the above additional constraint means that the converse is only very rarely true.

One of the most rapidly growing areas of modern mathematical physics involves the development of techniques for solving soliton-bearing equations and studying their many properties. While these often very powerful techniques, like the inverse scattering method and Bäcklund transformations offer elegant methods of studying such soliton-bearing systems they do not, as yet, allow the identification of new soliton-bearing equations. Furthermore, much of the mathematical theory of solitons was developed in the study of one dimensional propagation of waves. In that sense, the physical systems of interest to the mathematical physicist are essentially $(1+1)$ dimensional. Such low dimensional systems play a very important role to the field theorist as a test bed for new ideas. However, at the end of the day it is the physically realistic $(3+1)$-dimensional systems that these ideas (or some suitable generalizations thereof) must be applied to. We will

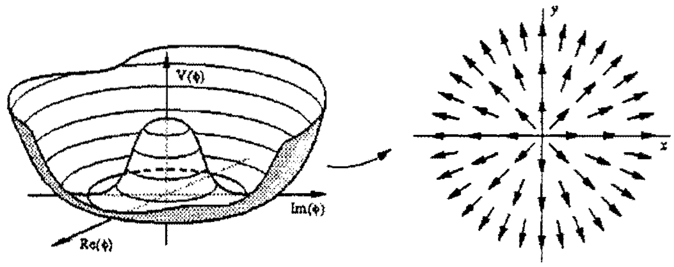


Figure 1: Cosmic strings are associated with models in which the set of minima are not simply-connected. The minimum energy states on the left form a circle and the string corresponds to a non-trivial winding around this. (Courtesy Cambridge Relativity and Cosmology group).

restrict our attention for the bulk of this work to considering systems in $(2+1)$ space-time dimensions and take consolation in the fact that such systems are not completely without phenomenology. Perhaps one of the simplest field theoretic models that exhibit solitonic solutions is the Higgs model. A theory of scalar electrodynamics, it is described by the Lagrangian,

$$\mathcal{L}_{\mathrm{AH}} = |D_A\phi|^2 - \frac{1}{4}F_A^2 - V(|\phi|^2) \quad , \tag{5}$$

where $V(|\phi|^2)$ is some polynomial function of ϕ, and was first shown to possess vortex-like solutions by Nielsen and Olesen in 1973 [11]. Solitonic solutions to the Higgs model have, in recent years, made a significant impact in the physics of the early universe as various types of topological defects [12]. Depending on the number of spatial dimensions and the particular gauge symmetry broken these are variously called domain walls, cosmic strings (see Fig. 1), monopoles and textures. Of these, we will be primarily interested in the Nielsen-Olesen cosmic strings whose spatial cross-sections are $(2+1)$-dimensional vortices.

Due to the extremely high energy scales involved (the characteristic energy density of a cosmic string, for example, is, in natural units, $(ct)T_{\mathrm{GUT}}^2/(ct)^3)$ where the horizon size is $ct \sim T_{\mathrm{P}}/T_{\mathrm{GUT}}^2$, T_{P} is the Planck temperature and T_{GUT}, the temperature at the GUT transition [13]) detection of such defects are rendered all but impossible by even the most powerful terrestrial accelerators. Topological defects, however, are not sole propriety of the realm of cosmological physics. Indeed, perhaps the most important realization, from the point of view of experimental verification, of low-dimensional solitons occurs in the physics of condensed matter systems. The interaction of superconducting matter and electromagnetic fields, for instance, is described by an effective field theory proposed by Ginzburg and Landau [14]. This phenomenological model is a nonrelativistic analogue of the Abelian-Higgs model. That the Ginzburg-Landau model supports localized field configurations was first pointed out by Abrikosov in 1957 [15] (sixteen years before their introduction into particle physics!). Since then solitonic configurations have proliferated condensed matter systems, from domains in ferromagnets; regions of magnetic dipole alignment separated by domain walls to string networks in liquid crystals (see Fig. 2) to vortex configurations in superfluid ^3He (see [16] and references therein).

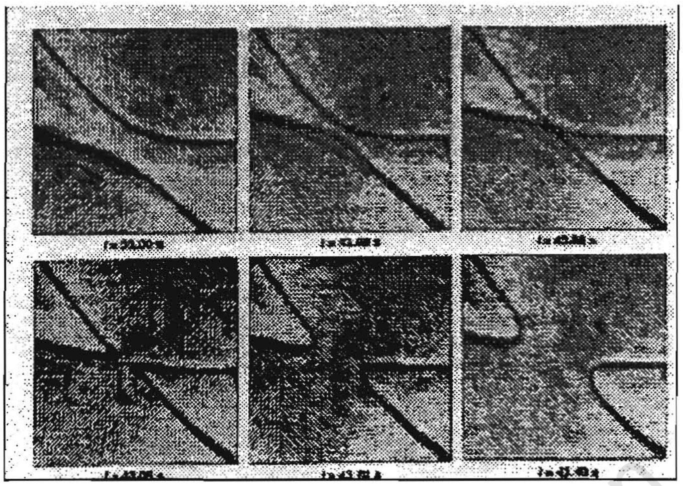
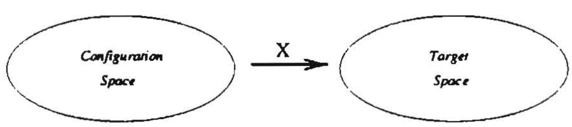


Figure 2: Optical microscope photographs of intercommuting stings in a nematic liquid crystal. From Chuang *et.al.* [17].

To say that this work is a collection of results in various aspects of soliton-bearing, low-dimensional field theoretic models is perhaps misleading because any such collection would easily merit several volumes - a daunting task indeed! We will therefore restrict ourselves to a small subset of such field theories. Much of this work is concerned with so-called non-ultralocal field theories, the generic example of which is the so-called non-linear σ model. As a rudimentary first 'definition' we will take a nonlinear σ-model to be a renormalisable scalar field theory in which the kinetic term has a field-dependent coefficient. The σ-model depends only on gauge-invariant objects built out of the metric and other fields and has a natural interpretation as a bosonic string in curved spacetime [18]

$$S_{\mathrm{P}} = \frac{1}{4\pi\alpha'}\int_M \sqrt{g}g^{ab}G_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu \, d^2\sigma \quad . \tag{6}$$

The massless σ-model fields, X^μ here, define an embedding of a configuration space into a target manifold or internal space:



and as such may be interpreted as coordinates on the target manifold. In string theory, the configuration space is the two-dimensional world sheet swept out by the string and the target space is spacetime itself. Our interests are, however, field theoretic and in this context the roles are reversed and it is the configuration space that is spacetime. Non-linear σ-models are an important exemplification of particle physics [19] and continuous spin systems and their excitations [20]. For example, with respect to the former; to a good approximation, the neutral and charged pion fields may be regarded as coordinates on the group manifold $SU(2)$.

Chapter 1 is devoted to a study of the nonlinear σ-model on a symmetric space and some of its geometrical properties. In particular, we review the integrability of the model in two-dimensions. In doing so we will find it necessary to exploit some internal degree of freedom in the model to couple the scalar field to a nondynamical gauge field. As such, the gauge field is treated as a fixed background and the σ-model fields themselves have nontrivial dynamics which corresponds to the mathematical theory of harmonic sections. Of the many remarkable properties of the nonlinear σ-model, the one in which we will be most interested in throughout this work is the existence of soliton solutions as first discovered by Belavin and Polyakov [21] for the case of the $O(3)$ σ-model. The rest of chapter 1 is used to review the construction of the self-dual solitons of the $O(3)$-model [10]. Of late, even these classical σ-model lumps have resurfaced in the context of 'the theory formerly known as string theory' in which it was shown [22, 23, 24] that these solitonic solutions can be understood as M-2-brane configurations.

One of the properties of the $O(3)$ sigma model solitons is their scale invariance. This property, although useful for finding solutions in the classical system, is preserved through quantization and consequently makes a particle interpretation problematic. One possible way to overcome this obstacle involves *gauging* the model, and recently much work has been done on the gauged nonlinear sigma model and its soliton solutions [25, 26, 27, 28]. In all of these works the global symmetry of the model is $O(3)$ and it is usually this or some subgroup, usually $U(1)$, that is gauged i.e. local $U(1)$ or $SU(2)$ invariance demands the introduction of a vector field that couples to the matter field. The difference between this and the former gauge mechanism is that here the gauge field is a dynamical one. The gauge dynamics, in fact, play a very important role in determining the nature of the soliton solutions. In fact, it can be shown [29] that in the case of a dynamical gauge field the nonlinear σ-model itself has no dynamics *i.e.* it can be 'gauged away' and its Lagrangian is essentially a gauge-invariant mass term for the gauge field.

In chapter 2 we investigate a class of gauged nonlinear σ-models formulated on two-dimensional target manifolds embedded in $\mathbb{E}^3$ with the aim of establishing a more or less unified framework from which soliton solutions of the model may be derived. In particular, we will focus on the relation between the topology of the target manifold and the existence and structure of self-dual solitons. We then introduce and study the gauged complex sine-Gordon model in chapter 3. Defined through the action

$$S_{\text{CSG}} = \int \frac{\partial_\mu \phi \overline{\partial^\mu \phi}}{1 - p^2|\phi|^2} + (v^2 - |\phi|^2)\, d^{n+1}x \quad , \tag{7}$$

the complex sine-Gordon model was first derived by Pohlmyer as a reduction of the $O(4)$ nonlinear σ-model [30] and later as a theory of dual strings interacting through a scalar field by Lund and Regge [31] and independently as a reduction of a theory of N classical two-dimensional massless Fermi fields interacting through symmetric couplings by Neveu and Papanicolaou [32]. It was recently shown to exhibit exact vortex solutions in closed analytic form [33]. These topological solitons were constructed by two independent methods, namely, (i) by auto-Bäcklund transformations resulting from a spinor representation of the complex sine-Gordon equation and (ii) by the Schlesinger transformation of the fifth Painlevé equation arising from a self-similar reduction of the energy functional associated with (7). These vortices are, however, energetically divergent. We investigate a mechanism to bring the divergences in the vortex energy under control by coupling the sine-Gordon matter field to a $U(1)$-gauge field. Apart from a brief review of the Barashenkov-Pelinovski contruction of the exact vortex solutions of the Euclidean complex sine-Gordon equation in chapter 3, the results of chapters 2 and 3, including the reductions to the Abelian Higgs and gauged $O(3)$ σ-models, are original work based for the most part on a forthcoming publication. The results reported in chapters 2 and 3 are thus new.

Most of the work in this thesis is concerned with the study of solitonic objects in low-dimensional relativistic field theories. In chapter 4 we deviate slightly and focus our attention on the relationship between the nonlinear σ-model and the self-dual Yang-Mills equation. After a short prelude in which we review the Witten-Forgács-Manton-Taubes reduction [35, 36, 37] of the Yang-Mills equations on $\mathbb{R}^2 \times S^2$ to a two-dimensional Higgs model on a space of constant negative curvature, we consider the Euclidean self-dual Yang-Mills equations. We show that, at least at the level of the equations of motion, the Euclidean Yang-Mills theory reduces to the nonlinear σ-model. While both the Abelian Higgs model and the principle chiral model have been assimilated into the Euclidean Yang-Mills hierachy we are not aware of any such work on the class of nonlinear σ-models that we have considered and we therefore emphasize that the work in section 4.4 is also new. We conclude with a discussion of other recent literature on the subject.

Chapter 1

The nonlinear σ-model

To a theoretical physicist, there is no greater joy than to see this curious activity we call calculation - the depositing of ink on paper, followed by throwing away the paper and depositing new ink on more paper - can actually tell us something about reality!

- Alan Guth

In this chapter we discuss a simple field theoretic model with highly non-trivial solutions called the nonlinear sigma-model (NSM). We show that the NSM on a symmetric target manifold is a completely integrable theory in 2-dimensions. Thereafter we review, in some detail, the 3-dimensional nonlinear sigma-model with target manifold S^2. We conclude with a derivation of the soliton solutions of this model.

1.1 Preliminaries

The NSM, originally introduced by Gell-Mann and Lévy [38] as a model for the generation of baryon mass by spontaneous symmetry breaking, is the simplest generic kinematically nonlinear field theory. It is essentially a scalar field theory in which the field interactions are (unlike, for instance, the Abelian-Higgs model (AHM)) not introduced via the addition of interaction terms to the action for the free field but rather completely in terms of intrinsic degrees of freedom. In other words, the field interaction is determined purely geometrically by the metric of the *target manifold* (see below). Structurally, the NSM is not unlike many non-Abelian gauge theories (such as the self-dual Yang-Mills theory) in the sense that it, too, is asymptotically free, has a non-trivial topological structure and, most importantly for us, exhibits soliton solutions. To demonstrate the integrability of the theory, it therefore makes sense to proceed in analogy with a non-Abelian gauge theory [39].

The field configurations of the NSM are maps from D-dimensional, Minkowskian space-time (configuration space) $\mathcal{C}_D$ into some n-dimensional Riemannian manifold Σ

(target space). We choose Σ to be a compact homogeneous space G/H with (field dependent) metric[1] $h_{ab}(\varphi)$. The action for the NSM is then given by

$$S_{NSM} = \frac{1}{2}\int_{\mathcal{C}^D} h_{ab}(\varphi)\partial_\mu\varphi^a(x)\partial^\mu\varphi^b(x)\, d^Dx \quad . \tag{1.1}$$

The σ-model space of field histories therefore coincide with the set of all smooth mappings

$$\varphi : \mathcal{C}^D \to \Sigma \quad . \tag{1.2}$$

The fields, $\{\varphi^a(x)\}$, are then local coordinates on Σ depending on points in $\mathcal{C}^D$. Note that the σ-model action eq.(1.1) is invariant under Poincaré transformations on the configuration space and the general coordinate transformations on Σ:

$$\begin{aligned} \varphi^a &\to \varphi'^a = f^a(\varphi) \\ h_{ab}(\varphi) \to h_{ab}(\varphi') &= \frac{\partial\varphi^c}{\partial\varphi^a}\frac{\partial\varphi^d}{\partial\varphi^b}h_{cd}(\varphi) \quad . \end{aligned} \tag{1.3}$$

1.2 Dynamics of the σ-model

The dynamics of the system is determined by the principle of least action. The equation of motion of the system is obtained, as usual, from requiring that the variation of the action vanish to first order, i.e., $\delta S_{NSM} = 0$. From eq.(1.1) we find

$$\begin{aligned} \delta S_{NSM}[\varphi] &= \frac{1}{2}\int_{\mathcal{C}^D}\Big\{\delta(h_{ab}(\varphi))\partial_\mu\varphi^a(x)\partial^\mu\varphi^b(x) + h_{ab}(\varphi)\delta(\partial_\mu\varphi^a(x))\partial^\mu\varphi^b(x) \\ &+ h_{ab}(\varphi)\partial_\mu\varphi^a(x)\delta(\partial^\mu\varphi^b(x))\Big\}\, d^Dx \\ &= \frac{1}{2}\int_{\mathcal{C}^D}\Big\{\frac{\partial h_{ab}}{\partial\varphi^c}\delta\varphi^c\partial_\mu\varphi^a(x)\partial^\mu\varphi^b(x) + 2h_{ab}\partial^\mu\varphi^b\delta(\partial_\mu\varphi^a)\Big\}\, d^Dx \\ &= -\int_{\mathcal{C}^D}\Big\{h_{cb}\partial_\mu\partial^\mu\varphi^b + \frac{\partial h_{cb}}{\partial\varphi^a}\partial_\mu\varphi^a\partial^\mu\varphi^b - \frac{1}{2}\frac{\partial h_{ab}}{\partial\varphi^c}\partial_\mu\varphi^a\partial^\mu\varphi^b\Big\}\delta\varphi^c\, d^Dx \\ &+ \int_{\partial\mathcal{C}^D} h_{ab}\partial_\mu\varphi^b\delta\varphi^a\, d^{D-1}x \quad ; \end{aligned} \tag{1.4}$$

where $\partial\mathcal{C}^D$ denotes the boundary of the D-dimensional Riemannian manifold $\mathcal{C}^D$. Restricting ourselves to those systems for which the field variations, $\delta\varphi^a$, vanish on the boundary ensures that the second integral above does not contribute to the variation of the action i.e.,

[1] Unless otherwise stated, the notation we adopt here is that Greek indices, $\mu\nu = 0, ..., D$; late Latin indices, $i, j = 1, ..., D$ and early Latin indices, $a, b = 1, ..., n$. The former two sets of indices are thus Lorentz indices on $\mathcal{C}^D$ while the latter are internal indices on Σ

$$\delta S_{NSM}[\varphi] = \int_{C^D} \left\{ h_{cb}\partial_\mu \partial^\mu \varphi^b + \frac{1}{2}\left(\frac{\partial h_{cb}}{\partial \varphi^a} + \frac{\partial h_{ca}}{\partial \varphi^b} - \frac{\partial h_{ab}}{\partial \varphi^c}\right)\partial_\mu \varphi^a \partial^\mu \varphi^b \right\} \delta\varphi^c \, d^D x \quad . \tag{1.5}$$

Setting the *functional derivative* $\delta S_{NSM}/\delta\varphi = 0$ yields the equation of motion

$$h_{cb}\partial_\mu \partial^\mu \varphi^b + \frac{1}{2}\left(\frac{\partial h_{cb}}{\partial \varphi^a} + \frac{\partial h_{ca}}{\partial \varphi^b} - \frac{\partial h_{ab}}{\partial \varphi^c}\right)\partial_\mu \varphi^a \partial^\mu \varphi^b = 0 \quad . \tag{1.6}$$

This can be rewritten in a form that better facilitates comparison with such nonlinear gauge theories as *Yang-Mills theories* (see e.g., [40]) and *General Relativity* (see [41] for a particularly lucid account) by contracting eq.(1.6) with $h^{bc} = (h_{bc})^{-1}$ and defining

$$\Gamma^b_{ad} := \frac{1}{2} h^{bc}\left(\frac{\partial h_{cd}}{\partial \varphi^a} + \frac{\partial h_{ca}}{\partial \varphi^d} - \frac{\partial h_{ad}}{\partial \varphi^c}\right) \quad . \tag{1.7}$$

The equation of motion eq.(1.6) thus becomes

$$\partial_\mu \partial^\mu \varphi^a + \Gamma^a_{bc}\partial_\mu \varphi^b \partial^\mu \varphi^c = 0 \quad . \tag{1.8}$$

This equation is reminiscent of the *geodesic equation* of general relativity which, in a coordinate basis, can be written as [42]

$$\frac{d^2 x^\mu}{dt^2} + \Gamma^\mu_{\sigma\nu}\frac{dx^\sigma}{dt}\frac{dx^\nu}{dt} = 0 \quad , \tag{1.9}$$

where $x^\mu(t)$ is a curve in some n-dimensional Riemannian manifold M^n. Eq.(1.8) is, in fact, a generalization of eq.(1.9) to the σ-model target manifold. This analogy allows us to identify Γ^a_{bc} as the Riemannian connection on Σ.

Our interest in the NSM lies primarily in the existence of static *solitons*[2] and as such, we will restrict our attention to the subset of solutions of eq.(1.8) for which the action (1.1) is finite. A necessary condition for the finiteness of the action is that the field values approach some constant values at spatial infinity, i.e.,

$$\lim_{|x|\to\infty} \varphi(x) = \varphi_0 \quad , \tag{1.10}$$

where $x \in \mathcal{C}_D$ is a generic point in the configuration space. This allows us to identify all points at spatial infinity and compactify the $(D-1)$-dimensional spatial-sections of Σ into the $(D-1)$ Riemann sphere $S^{D-1} = \mathbb{R}^{D-1} \cup \{\infty\}$. In this light, the static fields φ define a mapping, $\varphi : S^D \to \Sigma$. As is well known from *homotopy theory*[3]; all non-singular mappings from S^D into Σ may be classified into equivalence classes of the homotopy group $\Pi_D(\Sigma)$. We will be particularly interested in *non-trivial* such groups for which $\Pi_D(\Sigma) \neq 0$. The homotopy classification is valid for any finite action field configuration and does *not* require the fields, $\varphi(x)$ to be solutions of the field equations. However, obviously any finite action solution is also a finite action field configuration.

[2]We'll have a *lot* more to say about this a little later!

[3]See Appendix A for a summary of relevant results on homotopy theory.

1.3 Integrability

Having determined the dynamical properties of the σ-model and its effective configuration space structure (as required by the boundary conditions we impose on the fields) we turn now to the question of its integrability. Before addressing this issue within the context of the NSM, we beg the reader's patience to indulge in a small but important digression.

1.3.1 What is integrability?

What exactly does it mean to say that a dynamical system is 'completely integrable'? We will adopt the definition given in [43], namely that a *finite dimensional* dynamical system with, say $2N$-dimensions corresponding to the generalized coordinates q_i and momenta p_i $i = 1...N$, is called *completely integrable* if there exist N independent constants of motion $F_j(p_i, q_i)$, $i, j = 1...N$. Consequently, we can define N action-like variables and N angle-like variables which have linear temporal evolution. The above definition is easily generalized to a field theory, an infinite dimensional dynamical system. A field theory is thus called 'completely integrable' if we can find an infinite number of new variables, corresponding to the action-like variables of the finite dimensional system which represent the constants of motion of the system. How we find these new variables or even prove their existence is another matter altogether.

No discussion on the integrability of nonlinear equations would be complete without a study of the prototype nonlinear equation: the Kroteweg-de Vries (KdV) equation (see e.g., [44]). We see no reason to break with tradition here, so we consider the KdV equation in $(1+1)$-dimensions for the real field $u = u(t,x)$[4]:

$$\partial_t u + \alpha u \partial_x u + \partial_x^3 u = 0 \quad . \tag{1.11}$$

In 1967 Gardner *et al* [45] showed that eq.(1.11) possesses an infinite number of conservation laws and found all solutions $u(t,x)$ such that $\lim_{|x|\to\infty} u = u_0$. This work was closely followed by Miura's 1968 paper [46] in which he investigated a generalization of the KdV equation, the so-called *modified* KdV equation:

$$\partial_t v - \beta v^2 \partial_x v + \partial_x^3 v = 0 \quad , \tag{1.12}$$

derived the associated (infinite number of) conserved quantities and found the remarkable transformation that now carries his name and relates the solutions of the two equations:

$$u(t,x) = \frac{1}{\alpha}\Big(-\beta v^2 + \varepsilon\sqrt{6\beta}\,\partial_x v\Big), \quad \varepsilon = \pm 1 \quad . \tag{1.13}$$

This equation may be solved for $\partial_x v$ and substituted into eq.(1.12) to solve for $\partial_t v$. This procedure yields two evolution equations for $v(t,x)$ in space and time respectively:

[4] We *do* break from tradition in one small aspect: in order to maintain continuity of notation we do not resort to using subscripts to denote derivatives as is common practice in this field. Other than this we follow closely the treatment given in [43].

$$\partial_x v = \frac{\varepsilon}{\sqrt{6\beta}}(\alpha u + \beta v^2) \quad ,$$

$$\partial_t v = -\frac{\alpha\varepsilon}{\sqrt{6\beta}}\Big(\frac{1}{3}\beta v^2 u + \partial_x^2 u + \sqrt{\frac{2\beta}{3}}\varepsilon v \partial_x u + \frac{\alpha}{3}u^2\Big) \quad . \tag{1.14}$$

These equations may be linearized by exploiting the *Galilean invariance* of the KdV equation and by writing $v = -\varepsilon\sqrt{\frac{6}{\beta}}\partial_x \ln\psi$. In terms of this new field variables the evolution equations become

$$\partial_x^2\psi + \frac{\alpha}{6}u\psi - \frac{\alpha\lambda}{6}\psi = 0 \quad ,$$

$$\partial_t\psi + \frac{\alpha}{3}(u + 2\lambda)\partial_x\psi - \frac{\alpha}{6}\psi = 0 \quad , \tag{1.15}$$

where λ is an arbitrary real parameter. In 1968, Lax [47] showed that the integrability of the KdV system, and indeed a much larger class nonlinear models, is intrinsically tied to the existence of a set of *linear* operators, $\mathcal{L}$ and $\mathcal{B}$ which operate on elements of a Hilbert space[5] $\mathbb{H}$. For the KdV equation we choose $\alpha = -6$ and

$$\mathcal{L} := -\frac{\partial^2}{\partial x^2} + u \quad ,$$

$$\mathcal{B} := -4\frac{\partial^3}{\partial x^3} + 6u\frac{\partial}{\partial x} + 3\frac{\partial u}{\partial x} \quad . \tag{1.16}$$

It is evident that eqs.(1.15) can then be rewritten as

$$\mathcal{L}\psi = \lambda\psi \quad ,$$

$$\partial_t\psi = \mathcal{B}\psi \quad . \tag{1.17}$$

Differentiating the first of these with respect to time gives $\partial_t\mathcal{L}\,\psi + \mathcal{L}\partial_t\psi = \partial_t\lambda + \lambda\partial_t\psi$. Substituting from the second equation and rearranging terms produces

$$\partial_t\lambda\,\psi = (\partial_t\mathcal{L} + [\mathcal{L},\mathcal{B}])\psi \quad . \tag{1.18}$$

Invariance of the parameter λ under time translations then implies the compatibility condition

$$\partial_t\mathcal{L} + [\mathcal{L},\mathcal{B}] = 0 \quad . \tag{1.19}$$

[5] Note that the operators $\mathcal{L}$ and $\mathcal{B}$ are not unique as the addition of any constant to $\mathcal{B}$ would still ensure that eq.(1.17) is satisfied.

From eq.(1.16) it is obvious that $\partial_t \mathcal{L} = u$. Continuing further we have:

$$
\begin{aligned}
\mathcal{L}\mathcal{B}\psi &= \left(-\partial_x^2 + u\right)\left(-4\partial_x^3 + 6u\partial_x + 3\partial_x u\right)\psi \\
&= 4\partial_x^5\psi - 6\partial_x^2 u\,\partial_x\psi - 12\partial_x u\,\partial_x^2\psi - 6u\,p_x^3\psi - 3\partial_x^2\psi\,\partial_x u \\
&- \; 6\partial_x\psi\,\partial_x^2 u - 3\psi\,\partial_x^3 - 4u\,\partial_x^3 u + 6u^2\partial_x\psi + 3u\psi\,\partial_x u \quad ,
\end{aligned} \tag{1.20}
$$

and

$$
\begin{aligned}
\mathcal{B}\mathcal{L}\psi &= \left(-4\partial_x^3 + 6u\partial_x + 3\partial_x u\right)\left(-\partial_x^2 + u\right)\psi \\
&= 4\partial_x^5\psi - 4\partial_x^3 u\,\psi - 12\partial_x^2\,\partial_x\psi - 12\partial_x u\,\partial_x^2\psi - 4u\,\partial_x^3\psi \\
&- \; 6u\,\partial_x^3\psi + 6u\psi\,\partial_x u + 6u^2\,\partial_x\psi - 3\partial_x u\,\partial_x^2\psi + 3u\psi\partial_x u \quad ,
\end{aligned} \tag{1.21}
$$

so that that the commutator $[\mathcal{L}, \mathcal{B}] = -\frac{\partial^3 u}{\partial x^3} + 6u\frac{\partial u}{\partial x}$. The above compatibility condition is then equivalent to the Korteweg-de Vries equation eq.(1.11). These ideas were later extended to include field theoretic models in 1978 by Zakharov and Mikhailov [48]. In the context of a field theory one has to consider a generalization of the Lax-pair problem in which the equation for the auxiliary field ψ involving the $\mathcal{L}$ operator is not an eigenvalue equation like the first equation of eq.(1.17) but rather resembles the second equation. Having said all of this, let us now look at the problem of the integrability of the nonlinear σ-model.

1.3.2 Integrability of the σ-model

The discussion in this section closely follows the treatment given in [39]. The main goal of this section will be to establish a Lax pair for the NSM. Keeping this in mind we assume that, in addition to its homogeneity property, Σ is a *symmetric* space. To emphasize its coset structure we will write it as $\Sigma = G/H$ where G is a connected Lie group that is the global symmetry group of the NSM with Lie algebra $\mathbf{g}$ and $H \subset G$, a closed subgroup of G that is its *gauge group*. On $Adj(G)$[6] there exists an automorphism σ of G [49] satisfying $\sigma^2 = 1$ with $\sigma \neq 1$ that permits a splitting of the Lie algebra $\mathbf{g}$ into $\mathbf{g} = g_- \oplus g_+$ such that $[\sigma, g_-] = 0$ and $\{\sigma, g_+\} = 0$.

Let $g(x) : X \to G$ where $X = x^\mu, \mu = 1, 2$ denotes the base manifold of $\varphi(x)$. The fields $\varphi(x) \in G/H$ are related to the G-valued fields $g(x)$ by a (local) lifting, i.e.

$$
\varphi(x) = g(x)H \quad . \tag{1.22}
$$

Using the operator σ on G we can define the G/H-valued field $\Phi(x)$ by

[6] We will denote by $Adj(\cdot)$ the adjoint representation space; [A,B]:=AB - BA and {A,B} := AB + BA are the usual commutator and anticommutator respectively.

$$\Phi(x) = g(x)\sigma g^{-1}(x) \quad . \tag{1.23}$$

By construction, it is clear that $\Phi^2(x) = (g(x)\sigma g^{-1}(x))^2 = 1$. We can now rewrite the action for the NSM in terms of the (constrained) dynamical field, $\Phi(x)$ as

$$S_{NSM} = \frac{1}{8}\int_{C_D} Tr(\partial_\mu \Phi(x)\partial^\mu \Phi(x))\, d^D x \quad . \tag{1.24}$$

The derivation of the equations of motion proceed in exactly the same manner as previously, only this time subject also to the constraint equation $\Phi^2(x) = 1$ which we implement via a Lagrange multiplier. The Lagrange multiplier, λ is an internal space *scalar* and so commutes with the G/H-valued fields, $\Phi(x)$. Varying the action (1.24) we find

$$\begin{aligned} \delta S_{NSM} &= \frac{1}{8}\int_{C_D} \delta Tr(\partial_\mu \Phi(x)\partial^\mu \Phi(x) + \lambda(\Phi^2(x) - 1))\, d^D x \\ &= \frac{1}{4}\int_{C_D} Tr\Big\{\partial^\mu \Phi \delta \partial_\mu \Phi + \lambda \Phi \delta \Phi\Big\}\, d^D x \quad , \end{aligned} \tag{1.25}$$

where we have used the cyclic property of the trace. Integrating the first term by parts and discarding the boundary term we find

$$\delta S_{NSM} = -\frac{1}{4}\int_{C_D} Tr\Big\{(\partial_\mu \partial^\mu \Phi - \lambda \Phi)\delta \Phi\Big\}\, d^D x \quad , \tag{1.26}$$

from whence we read off the equation of motion as

$$\partial_\mu \partial^\mu \Phi(x) - \lambda \Phi(x) = 0 \quad . \tag{1.27}$$

The Lagrange multiplier may be eliminated by noting that $\lambda = \lambda\Phi^2 = \Phi\lambda\Phi = \Phi\partial_\mu\partial^\mu\Phi$. Substituting into eq.(1.27) allows us to write the equation of motion for the nonlinear σ-model as

$$[\partial_\mu \partial^\mu \Phi(x), \Phi(x)] = 0 \quad . \tag{1.28}$$

Differentiating the constraint equation, $\Phi^2 = 1$ twice yields the identity

$$\partial_\mu \Phi \partial^\mu \Phi = -\frac{1}{2}\Big(\Phi \partial_\mu \partial^\mu \Phi + (\partial_\mu \partial^\mu \Phi)\Phi\Big) \quad , \tag{1.29}$$

which allows us to rewrite the field equation, eq.(1.28) as a *conservation law*

$$\partial_\mu K^\mu = 0 \quad , \tag{1.30}$$

for the Noether current

$$K_\mu(x) := -\frac{1}{2}\Phi(x)\partial_\mu\Phi(x) \quad . \tag{1.31}$$

The conservation law, eq.(1.30), follows essentially as a consequence of the invariance of the NSM action, eq.(1.24), under (local) left G translations

$$\begin{aligned} g(x) &\rightarrow g'(x) = g(x)h(x) \quad h(x) \in H \\ \Phi(x) &\rightarrow \Phi'(x) = g'(x)\sigma(g'(x))^{-1} \\ &= g(x)h(x)\sigma h^{-1}(x)g^{-1}(x) \\ &= g(x)\sigma g^{-1}(x) = \Phi(x) \quad , \end{aligned} \tag{1.32}$$

where we have used the fact that σ is an involutive automorphism [49] on the embedding space G of G/H and hence, $\sigma h(x)\sigma = h(x)$ for $h(x) \in H$ from which it easily follows that $h(x)\sigma h^{-1}(x) = \sigma$. There exists a globally G-invariant, g-valued Maurer-Cartan 1-form $A_\mu(x) = g^{-1}(x)\partial_\mu g(x)$[7] which can be Cartan decomposed, using the automorphism operator as

$$\begin{aligned} A_\mu &= \frac{1}{2}A_\mu + \frac{1}{2}A_\mu = \frac{1}{2}A_\mu + \frac{1}{2}A_\mu + \frac{1}{2}\sigma A_\mu\sigma - \frac{1}{2}\sigma A_\mu\sigma \\ &= \frac{1}{2}A_\mu\sigma\sigma + \frac{1}{2}\sigma A_\mu\sigma + \frac{1}{2}A_\mu\sigma\sigma - \frac{1}{2}\sigma A_\mu\sigma \\ &:= h_\mu(x) + k_\mu(x) \quad , \end{aligned} \tag{1.33}$$

where we define

$$\begin{aligned} h_\mu(x) &:= \frac{1}{2}\{A_\mu(x),\sigma\}\sigma \quad , \\ k_\mu(x) &:= \frac{1}{2}[A_\mu(x),\sigma]\sigma \quad . \end{aligned} \tag{1.34}$$

It now follows simply that h_μ and k_μ take values in the subalgebras g_- and g_+ respectively, since

$$\begin{aligned} [\sigma, h_\mu] &= \sigma h_\mu - h_\mu\sigma = \frac{1}{2}\Big(\sigma\{A_\mu(x),\sigma\}\sigma - \{A_\mu(x),\sigma\}\Big) \\ &= \frac{1}{2}\Big(\{A_\mu(x),\sigma\} - \{A_\mu(x),\sigma\}\Big) = 0 \quad \Rightarrow h_\mu \in g_- \end{aligned} \tag{1.35}$$

[7]Note that this simplified product notation really represents the action of G on its tangent bundle, $T(G)$.

and similarly

$$\begin{aligned}
\{\sigma, k_\mu\} &= \sigma k_\mu + k_\mu \sigma = \frac{1}{2}\Big(\sigma[A_\mu,\sigma]\sigma + [A_\mu,\sigma]\Big) \\
&= \frac{1}{2}\Big(-[A_\mu,\sigma] + [A_\mu,\sigma]\Big) = 0 \quad \Rightarrow k_\mu \in g_+ \quad . \qquad (1.36)
\end{aligned}$$

The g_+-valued 1-form k_μ is related to the principal chiral field $\Phi(x)$ through the Noether current K_μ since

$$\begin{aligned}
k_\mu &= \frac{1}{2}[A_\mu,\sigma]\sigma = -\frac{1}{2}\Big(\sigma A_\mu \sigma - A_\mu \sigma\sigma\Big) \\
&= -\frac{1}{2}\Big(\sigma g^{-1}\partial_\mu g \sigma - g^{-1}\partial_\mu g \sigma\sigma\Big) \\
&= -\frac{1}{2}\Big(\sigma g^{-1}\partial_\mu g \sigma + \partial_\mu g^{-1} g \sigma\sigma\Big) \\
&= -\frac{1}{2}\Big(g^{-1}(g\sigma g^{-1})\partial_\mu(g\sigma g^{-1})g\Big) = g^{-1}K_\mu g \quad . \qquad (1.37)
\end{aligned}$$

So far what we have described is nothing more than the standard prescription for introducing the concept of a *gauge* into the NSM and the 1-form A_μ is identified as a pure gauge connection on the internal manifold [8]. We are now in a position to rewrite the equation of motion to make this more clear. Starting from eq.(1.30) we find that

$$\begin{aligned}
\partial_\mu K^\mu &= \partial_\mu\Big(g(x)k^\mu g^{-1}(x)\Big) \\
&= (\partial_\mu g(x))k^\mu g^{-1}(x) + g(x)(\partial_\mu k^\mu)g^{-1}(x) + g(x)k^\mu(\partial_\mu g^{-1}(x)) = 0 \quad , \\
&\Rightarrow (\partial_\mu k^\mu)g^{-1}(x) + g^{-1}(x)(\partial_\mu g(x))k^\mu g^{-1}(x) + k^\mu(\partial_\mu g^{-1}(x)) = 0 \quad , \\
&\Rightarrow \partial_\mu k^\mu + g^{-1}(x)(\partial_\mu g(x))k^\mu + k^\mu(\partial_\mu g^{-1}(x))g(x) = 0 \quad , \\
&\Rightarrow \partial_\mu k^\mu + A_\mu k^\mu - k^\mu A_\mu = 0 \quad . \qquad (1.38)
\end{aligned}$$

Thus defining a *covariant derivative* $D_\mu := \partial_\mu + [A_\mu, \cdot]$ we write the equation of motion as

$$D_\mu k^\mu = 0 \quad . \qquad (1.39)$$

In other words the equation of motion for the NSM implies that the g_+-valued form, k^μ is covariantly conserved. Using the decomposition of the pure gauge A_μ, eq.(1.33), we can write

[8]Note that this is to be distinguished from the use of the term in the next chapter in the sense that the A_μ is *not* a dynamical field.

$$\begin{aligned} D_\mu k^\mu &= \partial_\mu k^\mu + [A_\mu, k^\mu] = \partial_\mu k^\mu + [h_\mu + k_\mu, k^\mu] \\ &= \mathcal{D}_\mu k^\mu := \partial_\mu k^\mu + [h_\mu, k^\mu] = 0 \quad , \end{aligned} \tag{1.40}$$

The NSM equation of motion eq.(1.39), or equivalently eq.(1.40), may also be derived along similar lines as previously, by variation of the action

$$S_{NSM} = -\frac{1}{2}\int_{\mathcal{C}_D} (k_\mu(x), k^\mu(x))\, d^D x \quad , \tag{1.41}$$

where we denote the G-invariant metric on the coset space G/H by $(\cdot,\cdot)$.

Now consider the g-valued Maurer-Cartan form $A = g^{-1}dg$

$$\begin{aligned} dA + A\wedge A &= d(g^{-1}dg) + g^{-1}dg \wedge g^{-1}dg \\ &= dg^{-1}\wedge dg + g^{-1}\wedge d^2 g + g^{-1}dgg^{-1}\wedge dg \\ &= dg^{-1}\wedge dg - g^{-1}gdg^{-1}\wedge dg = dg^{-1}\wedge dg - dg^{-1}\wedge dg \\ \Rightarrow dA + A\wedge A &= 0 \quad . \end{aligned} \tag{1.42}$$

This is the Maurer-Cartan structure equation satisfied by the Maurer-Cartan form. In keeping with our analogy with non-Abelian gauge theory we recognize $F = dA + A\wedge A$ as the *curvature* form corresponding to A. The Maurer-Cartan equation then, is just a statement of *vanishing curvature*, $F = 0$. In terms of the gauge 1-form A_μ this becomes

$$\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = 0 \quad . \tag{1.43}$$

Using the Cartan decomposition of A_μ we find

$$\begin{aligned} &\partial_\mu (h_\nu + k_\nu) - \partial_\nu (h_\mu + k_\mu) + [(h_\mu + k_\mu), (h_\nu + k_\nu)] \\ &= \Big(\partial_\mu h_\nu - \partial_\nu h_\mu + [h_\mu, h_\nu] + [k_\mu, k_\nu]\Big) + \Big(\partial_\mu k_\nu + [h_\mu, k_\nu]\Big) \\ &- \Big(\partial_\nu k_\mu + [h_\nu, k_\mu]\Big) \quad . \end{aligned} \tag{1.44}$$

The Maurer-Cartan equation can then be written as one first-order equation, the Codazzi equation, and one first-order constraint

$$\left\{ \begin{array}{c} \partial_\mu h_\nu - \partial_\nu h_\mu + [h_\mu, h_\nu] + [k_\mu, k_\nu] = 0 \\ \mathcal{D}_\mu k_\nu - \mathcal{D}_\nu k_\mu = 0 \end{array} \right. \tag{1.45}$$

The second of these equations may be written in a more compact form if we define the *dual form* to k_μ as ${}^*k_\mu := \epsilon_{\mu\nu}k^\nu$, where $\epsilon_{\mu\nu}$ is the two-dimensional Levi-Civita tensor[9]. In terms of this dual form, the constraint equation may be written simply as

$$\mathcal{D}_\mu {}^*k^\mu = 0 \quad . \tag{1.46}$$

Furthermore, since $[k_\mu, k_\nu] = -[{}^*k_\mu, {}^*k_\nu]$, the Codazzi equation becomes

$$\partial_\mu h_\nu - \partial_\nu h_\mu + [h_\mu, h_\nu] - [{}^*k_\mu, {}^*k_\nu] = 0 \quad . \tag{1.47}$$

Clearly, eqs.(1.46) and (1.47) are dual to the equation of motion, eq.(1.40) and the Codazzi equation, (the first of) eq.(1.45) under the duality transformation, $k_\mu \leftrightarrow i\,{}^*k_\mu$. If, in addition the k_μ 1-form is *self dual* i.e., $k_\mu = \pm i\,{}^*k_\mu$ then the equations of motion and Codazzi equations are equivalent to their dual counterparts so that k_μ and $i\,{}^*k_\mu$ must satisfy the same linear equations. Let

$$\tilde{k}_\mu(x,\lambda) := \cosh\vartheta k_\mu(x) + \sinh\vartheta {}^*k_\mu(x) \quad , \tag{1.48}$$

where

$$\begin{cases} \cosh\vartheta = (\lambda + \lambda^{-1})/2 \\ \sinh\vartheta = (\lambda - \lambda^{-1})/2 \end{cases} \tag{1.49}$$

and λ is a real parameter. This defines a *continuous* dual transformation on g_+ such that $\tilde{k}_\mu(x,1) = k_\mu(x)$ and $\tilde{k}_\mu(x,-1) = {}^* k_\mu(x)$. Using the equation of motion (1.39) and Codazzi equation (1.45) it is easily seen that

$$\mathcal{D}_\mu \tilde{k}^\mu(x,\lambda) = \cosh\vartheta\, \mathcal{D}_\mu k^\mu(x) + \sinh\vartheta\, \mathcal{D}_\mu {}^*k^\mu(x) = 0 \quad , \tag{1.50}$$

and

$$\begin{aligned}
[\tilde{k}_\mu(x,\lambda), \tilde{k}_\nu(x,\lambda)] &= [\cosh\vartheta\, k_\mu(x) + \sinh\vartheta\, {}^*k_\mu, \cosh\vartheta\, k_\nu(x) + \sinh\vartheta\, {}^*k_\nu] \\
&= \cosh^2\vartheta\, [k_\mu(x), k_\nu(x)] + \sinh^2\vartheta\, [{}^*k_\mu(x), {}^* k_\nu(x)] \\
&+ \sinh\vartheta\, \cosh\vartheta \Big([k_\mu(x), {}^* k_\nu(x)] + [{}^*k_\mu(x), k_\nu] \Big) \\
&= \Big(\cosh^2\vartheta\, - \sinh^2\vartheta \Big) [k_\mu(x), k_\nu(x)] \\
&+ \sinh\vartheta\, \cosh\vartheta \Big([k_\mu(x), {}^* k_\nu(x)] - [k_\mu(x), {}^* k_\nu] \Big) \\
&= [k_\mu(x), k_\nu(x)] \quad .
\end{aligned} \tag{1.51}$$

[9]We use the normalization, $\epsilon_{01} = -\epsilon_{10} = 1$.

Clearly the set of 1-forms $(h_\mu(x), \tilde{k}_\mu(x,\lambda))$ satisfy the same set of linear equations as $(h_\mu(x), k_\mu(x))$

$$\left\{ \begin{array}{c} \partial_\mu h_\nu(x) - \partial_\nu h_\mu(x) + [h_\mu(x), h_\nu(x)] + [\tilde{k}_\mu(x,\lambda), \tilde{k}_\nu(x,\lambda)] = 0 \quad , \\ \mathcal{D}_\mu \tilde{k}^\mu(x,\lambda) = 0 \quad , \end{array} \right. \tag{1.52}$$

and $\{\sigma, \tilde{k}_\mu(x,\lambda)\} = \cosh\vartheta\,\{\sigma, k_\mu\} + \epsilon_{\mu\nu}\sinh\vartheta\,\{\sigma, k^\nu\} = 0 \Rightarrow \tilde{k}_\mu(x,t) \in g_+$. We can therefore write $h_\mu(x) + \tilde{k}_\mu(x,\lambda)$ as the Cartan decomposition of a pure-gauge (Maurer-Cartan form) $B = \psi d\psi^{-1}$ i.e.,

$$h_\mu(x) + \tilde{k}_\mu(x,\lambda) = \psi(x,\lambda)\partial_\mu\psi^{-1}(x,\lambda) \quad . \tag{1.53}$$

From this it follows that

$$\begin{aligned} (\partial_\mu\psi)\psi^{-1} &= -\Big(h_\mu(x) + \tilde{k}_\mu(x,\lambda)\Big) \\ \Rightarrow \mathcal{D}_\mu\psi(x,\lambda) &= -\tilde{k}_\mu(x,\lambda)\psi \quad , \end{aligned} \tag{1.54}$$

which defines the action of the covariant derivative $\mathcal{D}_\mu := \partial_\mu + h_\mu$ on $\psi(x,\lambda)$.

We now look at gauge transformations of the Maurer-Cartan 1-form A_μ and its composition forms h_μ and k_μ. Let $v(x) \in H$. We define the local gauge transformation of $g(x) \in G$ through right multiplication of $g(x)$ with $v(x)$ i.e., under an H-gauge transformation $g(x)$ transforms as

$$g(x) \to g'(x) = g(x)v(x) \quad . \tag{1.55}$$

The resulting transformations on A_μ and its projections are easily calculated

$$\begin{aligned} A_\mu \to A'_\mu(x) &= h'_\mu + k'_\mu \\ &= g'^{-1}(x)\partial_\mu g'(x) = v^{-1}(x)A_\mu(x)v(x) + v^{-1}(x)\partial_\mu v(x) \\ &= v^{-1}(x)h_\mu(x)v(x) + v^{-1}(x)\partial_\mu v(x) + v^{-1}(x)k_\mu(x)v(x) \quad . \end{aligned} \tag{1.56}$$

The correct transformation properties of the projection 1-forms can be deduced by noting that the involution operator σ on G commutes and anticommutes with the first and second terms respectively so

$$\left\{ \begin{array}{l} A_\mu(x) \to A'_\mu = v^{-1}(x)A_\mu(x)v(x) + v^{-1}(x)\partial_\mu v(x) \\ h_\mu(x) \to h'_\mu = v^{-1}(x)h_\mu(x)v(x) + v^{-1}(x)\partial_\mu v(x) \\ k_\mu(x) \to k'_\mu = v^{-1}(x)k_\mu(x)v(x) \end{array} \right. \tag{1.57}$$

Pursuing our analogy with non-Abelian gauge theories to the fullest, $A_\mu(x)$ and $h_\mu(x)$ evidently transform like gauge connection 1-forms while k_μ is gauge covariant.

In the above, we considered $v(x) \in H$ and as such we were able to interpret the transformation, $g(x) \to g(x)v(x)$ as an H-gauge transformation. We can generalize this to consider the case when $v(x)$ is some element of the embedding group G. In this case the above transformation corresponds to a change of frame. Without loss of generality, we can choose $v(x) = g^{-1}(x) \in G$. Under this transformation then, the Maurer-Cartan 1-form and its projections transform as

$$\begin{cases} A_\mu(x) \to A'_\mu = g(x)A_\mu(x)g^{-1}(x) + g(x)\partial_\mu g^{-1}(x) \\ h_\mu(x) \to h'_\mu = H_\mu(x) \\ k_\mu(x) \to k'_\mu = K_\mu(x) \end{cases} \tag{1.58}$$

where $H_\mu := g(x)h_\mu g^{-1}(x) + g(x)\partial_\mu g^{-1}(x)$ and K_μ is the Noether current defined in eq.(1.31). These two 1-forms are not, in fact, unrelated since

$$\begin{aligned} H_\mu(x) &= g(x)h_\mu g^{-1}(x) + g(x)\partial_\mu g^{-1}(x)g(x)g^{-1}(x) \\ &= g(x)\Big(h_\mu(x) - g^{-1}(x)\partial_\mu g(x)\Big)g^{-1}(x) = -g(x)k_\mu g^{-1}(x) \\ &= -K_\mu(x) \quad . \end{aligned} \tag{1.59}$$

Define $U(x,\lambda) := g(x)\psi(x,\lambda)$. We can use eq.(1.53) to calculate the evolution equation for $U(x,\lambda)$ as follows

$$\begin{aligned} \partial_\mu U(x,\lambda) &= \partial_\mu g(x)\,\psi(x,\lambda) + g(x)\partial_\mu\psi(x,\lambda) \\ &= \partial_\mu g(x)\,\psi(x,\lambda) - g(x)\Big(h_\mu(x) + \tilde{k}_\mu(x,\lambda)\Big)\psi(x,\lambda) \\ &= \Big\{-g(x)h_\mu g^{-1}(x) - g(x)\partial_\mu g^{-1}(x) - g(x)\tilde{k}_\mu(x,\lambda)g^{-1}(x)\Big\}g(x)\psi(x,\lambda) \\ &= -\Big(H_\mu(x) + \tilde{K}_\mu(x,\lambda)\Big)U(x,\lambda) \quad , \end{aligned} \tag{1.60}$$

where $\tilde{K}_\mu(x,\lambda) := g(x)\tilde{k}_\mu(x,\lambda)g^{-1}(x)$. Writing this in terms of K_μ we find

$$\tilde{K}_\mu(x,\lambda) = \cosh\vartheta\, K_\mu + \epsilon_{\mu\nu}\sinh\vartheta\, K^\nu(x) \quad . \tag{1.61}$$

We can write eq.(1.60) in terms of the constrained dynamical field $\Phi(x)$ by substituting $-\frac{1}{2}\Phi(x)\partial_\mu\Phi(x)$ for $K_\mu(x)$. We finally arrive at

$$\partial_\mu U(x,\lambda) = -\frac{1}{2}\Big\{(\cosh\vartheta - 1)\Phi(x)\partial_\mu\Phi(x) + \epsilon_{\mu\nu}\sinh\vartheta\,\Phi(x)\partial^\nu\Phi(x)\Big\}U(x,\lambda) \quad . \tag{1.62}$$

This, in turn can be brought into canonical form by defining the operator

$$L_\mu(x,\lambda) := -\frac{1}{2}\Big\{(\cosh\vartheta - 1)\Phi(x)\partial_\mu\Phi(x) + \epsilon_{\mu\nu}\sinh\vartheta\,\Phi(x)\partial^\nu\Phi(x)\Big\} \quad , \tag{1.63}$$

in terms of which the linear evolution equation for $U(x,\lambda)$ becomes

$$\partial_\mu U(x,\lambda) = L_\mu(x,\lambda)U(x,\lambda) \quad . \tag{1.64}$$

This is the *Lax pair representation* for the NSM; a classical indicator of its integrability. It is a set of *linear* equations (also called the auxiliary linear equations of the NSM) containing an arbitrary real parameter λ, the spectral parameter. As far removed as the above linear auxiliary equation seems from the original *nonlinear* equations of motion for the principal chiral field $\Phi(x)$; this distinction is merely illusory as the field equations eq.(1.28) are embedded in the Lax pair. Differentiating eq.(1.64) we find

$$\partial_\nu\partial_\mu U(x,\lambda) = \partial_\nu L_\mu(x,\lambda)\,U(x,\lambda) + L_\mu(x,\lambda)\partial_\nu U(x,\lambda) \quad . \tag{1.65}$$

Subtracting this from the corresponding equation with $\mu \to \nu$ and using the commutativity of the differentiation operators we find that the compatibility condition on eq.(1.64) is

$$\partial_\mu L_\nu(x,\lambda) - \partial_\nu L_\mu(x,\lambda) - [L_\mu(x,\lambda), L_\nu(x,\lambda)] = 0 \quad . \tag{1.66}$$

We note again the similarity that eq.(1.66) bears to the vanishing curvature condition (for the two-form $\mathcal{M} := dL - L\wedge L$) of non-Abelian gauge theory. In two dimensions, $\mu = 0,1$ and the evolution equation for $U(x,\lambda)$ becomes the set of two equations

$$\begin{cases} \partial_0 U(x,\lambda) = L_0 U(x,\lambda) \\ \partial_1 U(x,\lambda) = L_1 U(x,\lambda) \quad , \end{cases} \tag{1.67}$$

and the compatibility condition eq.(1.66) becomes

$$\partial_0 L_1(x,\lambda) - \partial_1 L_0(x,\lambda) - [L_0(x,\lambda), L_1(x,\lambda)] = 0 \quad . \tag{1.68}$$

Using the definition of L_μ, eq.(1.63), we find

$$\begin{aligned} \partial_0 L_1(x,\lambda) &= -\frac{1}{2}\Big\{(\cosh\vartheta - 1)(\partial_1\Phi(x)\partial_0\Phi(x) + \Phi(x)\partial_1\partial_0\Phi(x)) \\ &+ \sinh\vartheta\,(\partial_1\Phi(x)\partial^1\Phi(x) + \Phi(x)\partial_1\partial^1\Phi(x))\Big\} \quad , \\ \partial_1 L_0(x,\lambda) &= -\frac{1}{2}\Big\{(\cosh\vartheta - 1)(\partial_0\Phi(x)\partial_1\Phi(x) + \Phi(x)\partial_0\partial_1\Phi(x)) \\ &+ \sinh\vartheta\,(\partial_0\Phi(x)\partial^0\Phi(x) - \Phi(x)\partial_0\partial^0\Phi(x))\Big\} \quad , \end{aligned} \tag{1.69}$$

and

$$
\begin{aligned}
[L_0(x,\lambda), L_1(x,\lambda)] &= \frac{1}{4}\Big\{(\cosh\vartheta - 1)^2[\Phi(x)\partial_1\Phi(x), \Phi(x)\partial_1\Phi(x)] \\
&- \sinh^2\vartheta\,[\Phi(x)\partial^0\Phi(x), \Phi(x)\partial^1\Phi(x)] \\
&+ (\cosh\vartheta - 1)\sinh\vartheta\,[\Phi(x)\partial_1\Phi(x), \Phi(x)\partial^1\Phi(x)] \\
&- (\cosh\vartheta - 1)\sinh\vartheta\,[\Phi(x)\partial^0\Phi(x), \Phi(x)\partial_0\Phi(x)]\Big\} \quad . \qquad (1.70)
\end{aligned}
$$

Now using the constraint condition $\Phi^2 = 1$ and its derivatives it is easily seen that the commutator, $[L_0, L_1]$ and the first terms in the derivatives of L_0 and L_1 vanish. The compatibility condition then becomes

$$
\begin{aligned}
\partial_1 L_0 - \partial_0 L_1 &= \frac{1}{2}\sinh\vartheta\,\Big\{(\partial_0\partial^0\Phi(x) + \partial_1\partial^1\Phi(x))\Phi(x) \\
&- \Phi(x)(\partial_0\partial^0\Phi(x) + \partial_1\partial^1\Phi(x))\Phi(x))\Big\} = 0, \\
\Rightarrow [\partial_\mu\partial^\mu\Phi(x), \Phi(x)] &= 0 \quad . \qquad (1.71)
\end{aligned}
$$

This is, as claimed, nothing but the equation of motion for the principal chiral field $\Phi(x)$. We have thus shown that the equations of motion for the NSM correspond to the integrability problem for the Lax-pair and in principle, at least, reduced the problem to one in algebra. This however does not imply that the equations of motion are *simple* to solve. We will now turn our attention to a nonlinear σ-model which does exhibit an interesting class of exact solutions.

1.4 The O(3) σ-model

Famed for its ubiquity in physics, where it appears in a vast range of fields from high energy physics to condensed matter theory, the $O(3)$ σ-model has a target manifold $\Sigma = S^2$ which we can describe by 3 real, scalar fields $\varphi^a, a = 1...3$. In $(d+1)$-dimensions these fields define maps from a $(d+1)$-dimensional Minkowskian configuration space into the target space, $\varphi : \mathcal{C}_{d+1} \to \Sigma$. The global symmetry is implemented by the nonlinear constraint that the fields must lie on the unit sphere S^2 i.e., $\varphi^a\varphi^a = 1$. The dynamics of the model is determined by the action

$$
S_{O(3)} = \frac{1}{2\lambda^2}\int_{\mathcal{C}_{d+1}} \partial_\mu\varphi^a\partial^\mu\varphi^a\, d^{d+1}x \quad , \qquad (1.72)
$$

where λ is a coupling constant (which we set to unity in the rest of this chapter without any loss of generality). That this action is equivalent to our definition eq.(1.1) may be seen [43] by noting that we may 'solve' the constraint equation for one of the field components, say φ^3 in terms of the others as $\varphi^3_\pm = \pm\sqrt{1-\varphi^\alpha\varphi^\alpha}$[10]. We can then expand the above action as

$$\begin{aligned} S_{O(3)} &= \frac{1}{2}\int_{C_{d+1}} \left\{\partial_\mu\varphi^\alpha\partial^\mu\varphi^\alpha + \partial_\mu\varphi^3_\pm\partial^\mu\varphi^3_\pm\right\} d^{d+1}x \\ &= \frac{1}{2}\int_{C_{d+1}} \left\{\delta_{\alpha\beta}\partial_\mu\varphi^\alpha\partial^\mu\varphi^\beta + \frac{1}{1-\varphi^\eta\varphi^\eta}\varphi_\alpha\varphi_\beta\partial_\mu\varphi^\alpha\partial^\mu\varphi^\beta\right\} d^{d+1}x \\ &= \frac{1}{2}\int_{C_{d+1}} h_{\alpha\beta}(\varphi)\partial_\mu\varphi^\alpha\partial^\mu\varphi^\beta \, d^{d+1}x \quad , \end{aligned} \tag{1.73}$$

where we identify the metric on the internal manifold as $h_{\alpha\beta} = \delta_{\alpha\beta} + \varphi_\alpha\varphi_\beta/(1-\varphi^\eta\varphi^\eta)$ with a representation

$$h_{\alpha\beta} = \begin{pmatrix} 1+\frac{\varphi_1^2}{1-\varphi^\eta\varphi^\eta} & \frac{\varphi_1\varphi_2}{1-\varphi^\eta\varphi^\eta} \\ \frac{\varphi_2\varphi_1}{1-\varphi^\eta\varphi^\eta} & 1+\frac{\varphi_2^2}{1-\varphi^\eta\varphi^\eta} \end{pmatrix} . \tag{1.74}$$

From this we can calculate $h := det(h_{\alpha\beta}) = 1/(1-\varphi^\eta\varphi^\eta)$ and $h^{\alpha\beta} := (h_{\alpha\beta})^{-1} = \delta^{\alpha\beta} - \varphi^\alpha\varphi^\beta$. Using the definition, eq.(1.7) and the metric we can also calculate the connection coefficients $\Gamma^\alpha_{\beta\eta}$ on the target space as

$$\Gamma^\alpha_{\beta\eta} := \frac{1}{2}h^{\alpha\varsigma}\left(\frac{\partial h_{\varsigma\eta}}{\partial\varphi^\beta} + \frac{\partial h_{\varsigma\beta}}{\partial\varphi^\eta} - \frac{\partial h_{\beta\eta}}{\partial\varphi^\varsigma}\right) = \varphi^\alpha h_{\beta\eta}(\varphi) \quad . \tag{1.75}$$

Again, we are interested in describing solitons; extended objects that are spatially localized with finite dynamical characteristics so we will consider solutions that satisfy the boundary condition eq.(1.10). At any fixed time t the field φ defines a map from an effective configuration space S^d into the target space S^2. The fixed value φ^a_0 is assigned to the image of spatial infinity. This fixes the configuration space for the $O(3)$ NSM as the set of based maps $C_d = Map_0[S^d, S^2]$. As in the case of the principal chiral field $\Phi(x)$ in the previous section, we impose the field constraint via a Lagrange multiplier λ, in terms of which we write the $O(3)$ NSM action as

$$S_{O(3)}[\varphi] = \frac{1}{2}\int\left\{\partial_\mu\varphi^a\partial^\mu\varphi^a + \lambda(\varphi^a\varphi^a - 1)\right\} \quad . \tag{1.76}$$

As usual the dynamical equations for the fields are obtained by requiring that the action be stationary with respect to variations in φ. The equations of motion are thus

$$(\partial_\mu\partial^\mu - \varphi^b\partial_\mu\partial^\mu\varphi^b)\varphi^a = 0 \quad , \tag{1.77}$$

where we have used the constraint equation $\varphi^a\varphi^a = 1$ to eliminate the Lagrange multiplier.

[10]Note that early Greek indices, $\alpha, \beta = 1, 2$ unless otherwise stated.

In what follows, we shall restrict ourselves to a $(2+1)$-dimensional configuration space i.e., $d=2$. Since eq.(1.77) is manifestly covariant and Lorentz invariant, it suffices to consider static solutions ($\partial_0\varphi^a(x) = 0$) from which time-dependent solutions may be obtained by Lorentz boosting the static solutions. Under this condition, the d'Alembertian $\partial_\mu\partial^\mu \to -\nabla^2$ and the field equations become

$$(\nabla^2 - \varphi^b\nabla^2\varphi^b)\varphi^a = 0 \quad . \tag{1.78}$$

Eq.(1.78) is a set of three (one for each component of $\vec{\varphi} = \{\varphi^a\}$) nonlinear coupled partial differential equations which we now have the epic task of solving. We choose not to! Rather than attack this problem by a brute force attempt at solving the full nonlinear equations we will adopt the following strategy:

- Find the classical vacuum solutions,
- Identify the set of boundary conditions to be satisfied by any finite energy field configuration,
- Make a homotopy classification of these boundary conditions, and
- In each "homotopy sector" find the finite energy solutions.

1.4.1 Classical Vacuum Solutions

The Hamiltonian for the $O(3)$-NSM, $\mathcal{H}$ is obtained from the classical Lagrangian, $\mathcal{L} = \frac{1}{2}\partial_\mu\varphi^a\partial^\mu\varphi^a$ by the usual Legendre transformation, $\mathcal{H} = \pi^0\partial_0\varphi^a - \mathcal{L}$, where $\pi^0 := \partial\mathcal{L}/\partial_0\varphi^a$ is the canonically conjugate momentum field. Substituting the above Lagrangian and integrating over $\mathbb{R}^2$ gives the energy of static configurations as

$$E_{O(3)} = \frac{1}{2}\int_{\mathbb{R}^2} \nabla\vec{\varphi}\cdot\nabla\vec{\varphi}\, d^2x \quad . \tag{1.79}$$

Let us first look at the classical vacuum. This corresponds to the zero-energy point of the theory. From eq.(1.79) we see that this requires that $\partial_i\vec{\varphi} = 0$ i.e., that the field be spatially constant, $\vec{\varphi}(x) = \vec{\varphi}_0\, \forall x \in \mathbb{R}^2$. Note that the only requirement on $\vec{\varphi}_0$ is that it be a unit vector satisfying $\vec{\varphi}_0^2 = 1$. There are no restrictions on its direction in internal space and thus there exists a family of continuous degenerate zero energy solutions corresponding to the continuous set of directions in the internal space in which $\vec{\varphi}_0$ may point and related to each other by $O(3)$ rotations in S^2. In order to pick a vacuum, we need to *arbitrarily choose one* of these directions. Having done this, we break the global $O(3)$ symmetry of the theory. This is clearly an example of spontaneous symmetry breaking (SSB) at the classical level i.e., the classical vacuum does not exhibit the same symmetry as the full theory.

1.4.2 Soliton Solutions

We now look at *soliton* solutions of the $O(3)$-NSM. In keeping with our definition thereof, we look for finite (non-zero) energy, spatially localized solutions of the field equations. Reparametrizing $\mathbb{R}^2$ by plane polar coordinates (r, θ) so that the $\mathbb{R}^2$ volume element $d^2x \to r\,dr\,d\theta$, we can rewrite the energy of the system as

$$\begin{aligned} E_{O(3)} &= \frac{1}{2}\int_0^{2\pi}\int_0^{\infty}\left[(\partial_r\vec{\varphi})^2 + \frac{1}{r^2}(\partial_\theta\vec{\varphi})^2\right] r\,dr\,d\theta \\ &= \frac{1}{2}\int_0^{2\pi}\int_0^{\infty} r|\nabla\vec{\varphi}|^2\,dr\,d\theta \quad . \end{aligned} \tag{1.80}$$

The energy is thus finite if $\lim_{r\to\infty}|\nabla\vec{\varphi}|^2 = 0$ i.e., if $\lim_{r\to\infty}\vec{\varphi}(x) = \vec{\varphi}_0\,\forall\theta$. Thus, since the limiting field value is the same along all directions of approach to spatial infinity, we identify all points at spatial infinity and the configuration space, $\mathbb{R}^2$ is essentially compactified into a 2-sphere with all points at infinity identified as the north pole. This, then justifies our earlier claim that all finite energy solutions of the $O(3)$-NSM in 2 spatial dimensions are maps from an effective configuration space, S^2, into a target space S^2. As is well known from homotopy theory, all such non-singular mappings may be classified according to their *homotopy* or *Chern-Pontryagin* classes (see Appendix A) which form a group $\Pi_2(S^2) = \mathbb{Z}$, where the equality here denotes group isomorphism. Fields in each of the different homotopy classes are distinguished by their topological index (or charge) Q, a homotopic *dynamical variable* that is conserved *independent* of the dynamical symmetries of the classical theory. For the $O(3)$-NSM we define the topological current

$$\mathcal{K}^\mu := \frac{1}{8\pi}\epsilon^{\mu\nu\sigma}\vec{\varphi}\cdot\partial_\nu\vec{\varphi}\times\partial_\sigma\vec{\varphi} \quad . \tag{1.81}$$

That this current is conserved *geometrically* and is not a Noether current may be seen by differentiating the constraint equation $\vec{\varphi}\cdot\vec{\varphi} = 1$ to give $\vec{\varphi}\cdot\partial_\mu\vec{\varphi} = 0$. All field derivatives therefore live in a plane orthogonal to the field vectors in internal space. Clearly then, the internal space vector product between any two field derivatives is parallel to the field vector itself. Thus

$$\partial_\mu\mathcal{K}^\mu = \frac{1}{8\pi}\epsilon^{\mu\nu\sigma}\partial_\mu\vec{\varphi}\cdot\partial_\nu\vec{\varphi}\times\partial_\sigma\vec{\varphi} = 0 \quad . \tag{1.82}$$

The topological charge for the $O(3)$-NSM is constructed by integrating the time component of the current over $\mathbb{R}^2$ i.e.,

$$Q := \int_{\mathbb{R}^2} K^0\,d^2x = \frac{1}{8\pi}\int_{\mathbb{R}^2}\epsilon^{ij}\vec{\varphi}\cdot\partial_i\vec{\varphi}\times\partial_j\vec{\varphi}\,d^2x \quad . \tag{1.83}$$

A short calculation shows that this expression is in fact integer-valued i.e., $Q = n \in \mathbb{Z}$, and is a measure of the number of times the target space S^2 is traversed as the (effective) configuration space S^2 is spanned. At this stage the above construction might seem rather *ad hoc.* In fact, it is quite difficult, in general, to obtain an explicit expression for the topological conserved quantities in a nonlinear field theory in terms of the classical field variables. In 1977 Isham [50] proposed one such construction algorithm based on the relationships between the homotopy and *cohomology* groups of the maps from configuration into internal manifolds. Using this construction it can be shown that the above expression for the topological current arises quite naturally. The topological charge is essential in establishing a *finite* lower bound on the energy as follows: We begin with the trivial identity

$$\int_{\mathbb{R}^2} \left\{ (\partial_i \vec{\varphi} \mp \epsilon_{ij} \vec{\varphi} \times \partial^j \vec{\varphi}) \cdot (\partial^i \vec{\varphi} \mp \epsilon^{ik} \vec{\varphi} \times \partial_j \vec{\varphi}) \right\} d^2x \geq 0 \quad , \tag{1.84}$$

where $i, j = 1, 2$ and the above identity is just a statement of the positivity of the inner product on $\mathbb{R}^2$. We then expand this as

$$\int \left\{ \partial_i \vec{\varphi} \cdot \partial^i \vec{\varphi} + \epsilon_{ij} \epsilon^{ik} (\vec{\varphi} \times \partial^j \vec{\varphi}) \cdot (\vec{\varphi} \times \partial_k \vec{\varphi}) \mp 2\epsilon_{ij} \partial^i \vec{\varphi} \cdot (\vec{\varphi} \times \partial^j \vec{\varphi}) \right\} d^2x \geq 0 \quad . \tag{1.85}$$

Contracting the skew-symmetric tensors in the second term and using some trivial vector identities as well as the constraint condition on the field, we can write the second term as

$$\begin{aligned} \epsilon_{ij} \epsilon^{ik} (\vec{\varphi} \times \partial^j \vec{\varphi}) \cdot (\vec{\varphi} \times \partial_k \vec{\varphi}) &= \delta_j^k \vec{\varphi} \cdot [\partial_k \vec{\varphi} \times (\vec{\varphi} \times \partial^j \vec{\varphi})] \\ &= \delta_j^k \vec{\varphi} \cdot [\vec{\varphi}(\partial_k \vec{\varphi} \cdot \partial^j \vec{\varphi}) - \partial^j \vec{\varphi}(\partial_k \vec{\varphi} \cdot \vec{\varphi})] \\ &= \partial_i \vec{\varphi} \cdot \partial^i \vec{\varphi} \quad , \end{aligned} \tag{1.86}$$

so that the above identity eq.(1.84) becomes

$$\int_{\mathbb{R}^2} (\partial_i \vec{\varphi} \cdot \partial^i \vec{\varphi}) \, d^2x \geq \mp \int_{\mathbb{R}^2} \epsilon_{ij} \vec{\varphi} \cdot \partial^i \vec{\varphi} \times \partial^j \vec{\varphi} \quad . \tag{1.87}$$

We recognize immediately the energy (up to a constant) on the left-hand side of the inequality and a term proportional to the topological charge on the right. In fact

$$E \geq 4\pi |Q| \quad . \tag{1.88}$$

So finally, the importance of the topological charge becomes clear: it establishes a lower bound on the energy of the solutions. In a given homotopy sector (characterized by Q) the energy is minimized when the inequality (1.88) is saturated. This occurs if the field configurations satisfy the so-called *self-duality* equations

$$\partial_i\vec{\varphi} = \pm\epsilon_{ij}\,\vec{\varphi}\times\partial^j\vec{\varphi} \quad . \tag{1.89}$$

Note that this is a *first order* set of equations which are potentially easier to solve than the full *second order* set of Euler-Lagrange equations. That the solutions of the above self-duality equations also satisfy the Euler-Lagrange equations is easily seen since

$$\begin{aligned} \nabla^2\vec{\varphi} &= \partial_i\partial^i\vec{\varphi} = \partial^i(\pm\epsilon_{ij}\vec{\varphi}\times\partial^j\vec{\varphi}) = \pm\epsilon_{ij}(\pm\epsilon^{ik}\vec{\varphi}\times\partial^k\vec{\varphi})\times\partial^j\vec{\varphi} \\ &= \delta_j^k(\vec{\varphi}\times\partial_k\vec{\varphi})\times\partial^j\vec{\varphi} = -\vec{\varphi}(\partial_i\vec{\varphi}\cdot\partial^i\vec{\varphi}) = \vec{\varphi}(\vec{\varphi}\cdot\partial^i\partial_i\vec{\varphi}) \quad , \end{aligned} \tag{1.90}$$

where we have used the identity $\partial^i\vec{\varphi}\cdot\partial_i\vec{\varphi}+\vec{\varphi}\cdot\partial^i\partial_i\vec{\varphi}=0$ in the last step. The converse, clearly, need not be true i.e., all solutions of the second order field equations need not satisfy the self-duality equations. However, the problem of solving a set of nonlinear coupled (albeit first order) equations still remains. We may glean more insight into the self-duality equations by making a change of variables. We define

$$\omega_1 := \frac{2\varphi_1}{1-\varphi_3}, \qquad \omega_2 := \frac{2\varphi_2}{1-\varphi_3} \quad , \tag{1.91}$$

which are just the coordinates of S^2 stereographically projected onto $\mathbb{R}^2$. Defining also the complex valued fields, $\phi := \varphi_1 + i\varphi_2$ and $\omega := \omega_1 + i\omega_2$, we find that

$$\partial_i\omega = \frac{2}{(1-\varphi_3)^2}[\partial_i\phi + (\phi\partial_i\varphi_3 - \varphi_3\partial_i\phi)], \quad i = 1,2 \quad . \tag{1.92}$$

From the self-dual equation eq.(1.89), we find that

$$\partial_i\phi = \pm i\epsilon_{ij}(\phi\partial^j\varphi^3 - \varphi^3\partial^j\phi) \quad . \tag{1.93}$$

Substituting this into eq.(1.92) reduces the self-duality equations to

$$\partial_1\omega = \mp i\partial_2\omega \quad . \tag{1.94}$$

Expanding this gives $\partial_1\omega_1 + i\partial_2\omega_2 = \mp i\partial_2\omega_1 \pm \partial_2\omega_2$ which we recognize as the Cauchy-Riemann equations for the analytic function ω being a function of $\overline{z}$ or $z := x^1 + ix^2$ corresponding to the upper or lower signs respectively:

$$\begin{cases} \partial_1\omega_1 = \pm\partial_2\omega_2 \\ \partial_1\omega_2 = \mp\partial_2\omega_1 \quad . \end{cases} \tag{1.95}$$

Therefore, any analytic function $\omega(z)$ is a solution of the self-dual equations and thus also the full field equations[11]. Choosing $\omega = \omega(z)$ analytic, we find that $d\omega/dz =$

[11] Note that while ω must be analytic, it need not be an entire function: isolated poles in ω are permitted. Divergences in ω correspond to the north pole on S^2 or $\varphi = 1$.

$(\partial_1 - i\partial_2)\omega = \partial_1\omega$, so $|d\omega/dz|^2 = (\partial_1\omega_1)^2 + (\partial_1\omega_2)^2$. Substituting for these expressions from eq.(1.92) we can rewrite the expression for the energy of the solutions in terms of ω as

$$E = \int \frac{|d\omega/dz|^2}{(1+|\omega|^2/4)^2}\, d^2x \quad . \tag{1.96}$$

Since we assumed that ω was an analytic function it is clear that, when written in terms of $\vec{\varphi}$, it satisfies the self-dual equations, eq.(1.89) and is hence a self-dual solution of the field equations. Furthermore, the topological index associated with such solutions is related to their energy by

$$|Q| = \frac{1}{4\pi}E \quad . \tag{1.97}$$

What do such solutions look like? As a simple prototype of a possible (static) solution for arbitrary positive topological number n we take

$$\omega(z) = \frac{1}{\lambda^n}(z - z_0)^n \quad , \tag{1.98}$$

where $n \in \mathbb{Z}^+$, $\lambda \in \mathbb{R}$ is a scale parameter and $z_0 \in \mathbb{C}$ represents the 'location' of the solution in $\mathbb{R}^2$. It is not difficult to see that the above solution corresponds to the $Q = n$ vacuum sector of the theory since

$$\begin{aligned} Q &= \frac{1}{4\pi}\int \frac{|d\omega/dz|^2}{(1+|\omega|^2/4)^2}\, d^2x \\ &= \frac{1}{4\pi}\int \frac{n^2|z-z_0|^{2n-2}\lambda^{-2n}}{(1+\frac{1}{4}|z-z_0|^{2n}\lambda^{-2n})^2}\, d^2x \\ &= \frac{n^2}{4\pi}\int_0^{2\pi} d\theta \int_0^{\infty} \frac{\lambda^{2n}\varrho^{2n-1}}{(\lambda^{2n}+\frac{1}{4}\varrho^{2n})^2}\, d\varrho \quad , \end{aligned} \tag{1.99}$$

where, in the last step, we have used the polar representation $z - z_0 = \varrho\, e^{i\theta}$. This integration is easily carried out, yielding $Q = n$, as claimed. The energy $E = 4\pi n$ of these soliton solutions are thus finite as our analysis would lead us to expect. We conclude then that $\omega(z)$ represents explicit static soliton solutions of the $O(3)$ NSM in $(2+1)$ dimensions. Dynamical solitons may be generated by Lorentz boosting the above static solitons as a result of the Lorentz invariance of the parent theory.

One remarkable point about the above solutions is that they are *scale invariant*. In other words the self-dual solutions remain solutions under an arbitrary rescaling. This seemingly benign property of the classical solutions clearly becomes problematic after quantization (through which it is preserved) when we would like to have some particle-like interpretation of these solitons. However this problem is not intractable, as we shall see in the next chapter.

1.5 Summary

We have reviewed some basic properties of the nonlinear σ-model, emphasizing the rich geometrical structure that makes it almost uniquely ubiquitous in physics. After introducing the model in terms of an embedding of a configuration space (usually identified with spacetime) into a topologically nontrivial target space, we show that the coordinates on the target manifold, Σ, interpreted as free dynamical fields satisfy a geodesic-like equation of motion.

Section 1.3 was devoted to a study of integrability. A brief discussion of the general idea of integrability of nonlinear partial differential equations and the method of Lax is followed by a detailed construction of a Lax pair (1.64) for the nonlinear σ-model, thus demonstrating its integrability in two dimensions.

Some time was then devoted to the study of the σ-model under the gauge group $O(3)$. Following [10], we have shown that not only does the $O(3)$ model possess localized solitonic solutions but that these solutions saturate a finite, nonzero lower bound on their energy. Such solitons solve a subset of the σ-model equations - the self-dual equations - which will remain a focal point of this work. In terms of stereographic coordinates on the Riemann sphere that is the $O(3)$ model target space, these self-duality equations are just the Cauchy-Riemann equations (1.95) whose solutions, the σ-model solitons, are holomorphic functions.

Chapter 2

The gauged σ-model

I find physics is a wonderful subject. We know so very much and then subsume it into so very few equations that we can say we know very little.

- Richard Feynman

In this chapter we consider a class of 2-dimensional gauged nonlinear σ-models on axisymmetric target manifolds embedded in a 3-dimensional Euclidean space. We show that, subject to not too stringent constraints on the metric of the target space of the model, any σ-model in this class possess a self-dual point in its parameter space at which the corresponding energy of the field configuration saturates a Bogomol'nyi-Prasad-Sommerfeld (BPS) bound [51]. By regarding the resulting first order self-dual field equations as a dynamical system we are able to find several classes of spatially localized static solutions which we identify as rings, lumps and vortices. Specifying the target space metric allows us to correlate our results with several well known models. We show also that this analysis breaks down when the gauge dynamics is of the mixed Maxwell-Chern-Simons type.

2.1 Introduction.

The Abelian-Higgs model is without doubt, the prototype gauge field theory. It is a theory of scalar electrodynamics described by an action (in $(2+1)$-dimensions):

$$S_{\rm AH} = \int \left[D_\mu\phi\overline{D^\mu\phi} - \frac{\mu}{4}F_{\mu\nu}F^{\mu\nu} - V(|\phi|^2)\right] d^3x \quad , \tag{2.1}$$

where the charged scalar field, $\phi = \phi_1 + i\phi_2$, and massless vector field, A_μ transform as

$$\begin{aligned} \phi(x) &\to \phi'(x) = e^{i\lambda(x)}\phi(x) \quad , \\ A_\mu(x) &\to A'_\mu(x) = A_\mu - \partial_\mu\lambda(x) \quad , \end{aligned} \tag{2.2}$$

making (2.1) locally $U(1)$ invariant. Not only does the model exhibit spontaneous symmetry breaking, it also (and more importantly, for our purposes) possesses vortex solutions

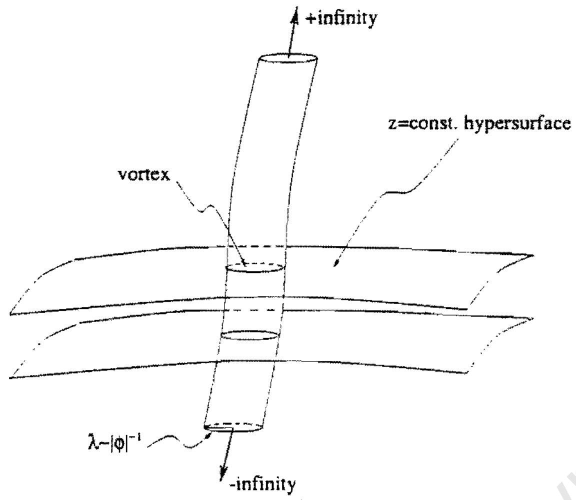


Figure 2.1: Vortices as spatial cross sections on an infinite length cosmic string with radius equal to the characteristic length of the vortex. The string endpoints are taken to be located at spatial infinity.

[11] in the BPS limit of the theory *i.e.*, there exists a point in the parameter space of the model at which a critical relation among the coupling constants in the model permits a reduction of the second order Euler-Lagrange equations to a set of first order equations - the Bogomol'nyi equations [51] which may be solved (numerically or asymptotically) to yield topologically stable solitons. These vortices are usually interpreted as spatial cross-sections of (infinite length) cosmic strings [12] (see fig.2.1) of radii equal to the characteristic length, $\lambda = (e|\phi|)^{-1}$, of the vortex solution *i.e.*, the region of space over which the magnetic field B is appreciably different from zero.
In this chapter, we propose to study a generalization of the Abelian-Higgs model [34] which differs from it by a factor of a scalar function of the field amplitude, $h(|\varphi|^2)$, in the kinetic term for the matter field, φ. Although a seemingly trivial extension, this modifies the model quite significantly; in fact, as we will see, the resulting field theory actually belongs to a class of gauged nonlinear σ-models and, as such, carries with it all the geometrical splendor of the latter. We will focus our attention on the possible extraction of BPS vortex-like solutions and the constraints this places on the theory. We show that although the model *does* support a self-dual limit (under appropriate restrictions of the target space metric), the first order Bogomol'nyi equations are not, in general analytically solvable. Nevertheless, it is possible to extract valuable qualitative information about the self-dual system by treating it as a dynamical system and analyzing the *spatial* field configuration as the *temporal* evolution of the dynamical system. Thus we are able to construct a general picture of the phase space of the associated dynamical system and deduce from it the possible existence of several classes of localized solutions.

A key feature of our analysis is its generality; the scalar field self-interaction, for example, is completely determined by imposing self-duality on the model thereby eliminating the almost *ad hoc* usual treatment. The model reduces, under specific metric choices, to several well known low dimensional field theories; the Abelian-Higgs model and gauged

$O(3)$ σ-model [25, 26, 27, 28] among them.

Not content to quit while ahead, we then attempt to extend our analysis to include Chern-Simons gauge interactions. Peculiar to odd-dimensional spacetimes, the Chern-Simons gauge field, A_μ, takes values in a finite representation of the Lie algebra, $\mathbf{g}$, corresponding to the gauge group G. The Chern-Simons equations of motion are derived from a term proportional to the wedge product of the Chern-Simons gauge connection and its associated curvature 2-form *i.e.*, $\mathcal{L}_{\text{CS}} \sim \epsilon^{\mu\nu\lambda} A_\mu F_{\nu\lambda}$. As such, the equations of motion are *gauge-covariant* under the gauge transformation $A_\mu \to g^{-1}A_\mu g + g^{-1}\partial_\mu g$. For an Abelian gauge theory ($G = U(1)$, for example) the *action* $S_{\text{CS}} = \int \mathcal{L}_{\text{CS}}\, d^3x$ is gauge invariant so even though the Chern-Simons Lagrangian itself is not gauge-invariant it *does* define a sensible gauge theory. Because the Chern-Simons term may be written directly as a 3-form, $Tr(A \wedge dA + A \wedge A \wedge A)$, it does not depend explicitly on the metric. Consequently, the Chern-Simons term describes a topologically massive gauge field.

Coupling the Chern-Simons field to a charged scalar field taking values on a two-dimensional σ-model target manifold embedded in $\mathbb{E}^3$, we ask whether the gauged model allows for the existence of BPS solitons? Our findings are then tested against known results for the Abelian-Higgs and the $O(3)$ σ-models.

2.2 The Model

In this section we introduce the class of nonlinear σ-models that will occupy our attention for at least the next two chapters. We assume that spacetime is a differentiable $(n+1)$ manifold $\mathcal{M} = \Pi \times \mathbb{R}$ with metric $g_{\mu\nu}(x)$ with signature (+,-,...,-). Although the case $n = 2$ will be our primary concern we can - and will - retain as much generality as possible for as long as possible. For the target manifold, Σ, of the theory we shall consider a set of two-dimensional axisymmetric smooth surfaces embedded in a three-dimensional Euclidean (or pseudo-Euclidean) space. A vanishing Weyl tensor on Σ [52] implies that any such surface must be conformally flat and we can always choose a system of isothermal coordinates on Σ such that its metric, $h_{ab}(\phi)$ is conformal *i.e.*,

$$h_{ab}(\phi) = h(\phi)\delta_{ab} \qquad \{a, b = 1, 2\} \quad , \tag{2.3}$$

where $h(\phi)$ is a smooth positive definite function. Let $f_t : \Pi \to \Sigma$ be a smooth mapping between spatial sections of $\mathcal{M}$ at fixed t and the target manifold such that each point in the coordinate space, labelled by the coordinate vector $\mathbf{x}$ with respect to some coordinate chart on $\mathcal{M}$ is mapped onto some point on Σ with coordinates $\phi(\mathbf{x}, t)$ with respect to some local chart on Σ. We define the gauged nonlinear σ-model on Σ by its action as:

$$S_{\text{NSM}} := \int_{\mathcal{M}} \sqrt{|g|}\left\{h_{ab}(\phi) g^{\mu\nu} D_\mu\phi^a(x) D_\nu\phi^b(x) - \frac{\mu}{4} g^{\mu\sigma} g^{\nu\rho} F_{\mu\nu} F_{\sigma\rho} - V(\phi)\right\} d^{n+1}x, \tag{2.4}$$

where $g := \det(g_{\mu\nu})$. The gauge field $A_\mu(x)$ takes values in some finite representation of the Lie algebra, $\mathbf{g}$, of the gauge group G which, for definiteness, we will assume to be Abelian and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the associated field strength tensor. We shall consider a minimally coupled model so that, as usual, the gauge-covariant derivative is $D_\mu = \partial_\mu + ieA_\mu$ where e is the gauge-matter coupling constant. Using (2.3) we can write the σ-model action (2.4) in terms of the complex scalar field $\varphi = \phi_1 + i\phi_2$ as

$$S_{\rm NSM} = \int_{\mathcal{M}} \sqrt{|g|}\left\{h(|\varphi|^2)g^{\mu\nu}D_\mu\varphi\overline{D_\nu\varphi} - \frac{\mu}{4}g^{\mu\sigma}g^{\nu\rho}F_{\mu\nu}F_{\sigma\rho} - V(|\phi|^2)\right\} d^{n+1}x \quad . \tag{2.5}$$

Although entirely equivalent to the formulation in terms of the real constrained fields, this latter form will prove more convenient for our purposes and henceforth we will take (2.5) as the defining action for the gauged nonlinear σ-model. Note that the scalar field self-interaction, $V(|\varphi|^2)$, is an as yet undetermined function. We will show that the form of V is completely determined by imposing self-duality on the model. An important point to consider in any realistic gauge theory is that of renormalisability; for instance, when $n = 2$ and 3 it is well known that renormalisability constrains the scalar potential to be a polynomial in $|\varphi|$ not exceeding degree 6 and 4 respectively. Our treatment will, however, be unabashedly classical and, as such, we will be unbridled by the requirement of renormalisability (see, for example, the complex sine-Gordon model in chapter 3 where V must be chosen as a logarithmic function of $|\varphi|$).

2.2.1 Equations of Motion

The equations of motion for the charged matter field and gauge field result from varying the action (2.5) with respect to $\overline{\varphi}$ and A_μ respectively:

$$\begin{aligned}(\delta S)_{\overline{\varphi}} &= \int_{\mathcal{M}} \left\{\sqrt{|g|}g^{\mu\nu}D_\mu\varphi\overline{D_\nu\varphi}\frac{\partial h}{\partial\overline{\varphi}} - \sqrt{|g|}\frac{\partial V}{\partial\overline{\varphi}}\right. \\ &\quad \left. - \; D_\nu[\sqrt{|g|}g^{\mu\nu}hD_\mu\varphi]\right\}\delta\overline{\varphi}\, d^{n+1}x \quad ,\end{aligned} \tag{2.6}$$

after integrating by parts. Defining the 'force density' by $F := g^{\mu\nu}D_\mu\varphi\overline{D_\nu\varphi}\frac{\partial h}{\partial\overline{\varphi}} - \frac{\partial V}{\partial\overline{\varphi}}$ and setting $\delta S/\delta\overline{\varphi} = 0$ yields

$$\frac{1}{\sqrt{|g|}}D_\nu\left[\sqrt{|g|}g^{\mu\nu}h(|\varphi|^2)D_\mu\varphi = F\right] \quad . \tag{2.7}$$

Similarly; varying $S_{\rm NSM}$ with respect to A_ρ and integrating by parts gives:

$$\begin{aligned}(\delta S)_{A_\rho} &= \int_{\mathcal{M}} \left\{\sqrt{|g|}g^{\mu\nu}h(|\varphi|^2)[-i\overline{\varphi}D_\nu\varphi + i\varphi\overline{D_\nu\varphi}] - \partial_\nu[\sqrt{|g|}g^{\mu\sigma}g^{\nu\rho}F_{\sigma\rho}]\right. \\ &\quad \left. - \; \sqrt{|g|}\frac{\partial V}{\partial\overline{\varphi}}\right\}\delta^\rho_\mu\delta A_\nu\, d^{n+1}x \quad ,\end{aligned} \tag{2.8}$$

from which the corresponding Euler-Lagrange equations for the gauge field are

$$\frac{1}{\sqrt{|g|}}\partial_\nu\left[\sqrt{|g|}g^{\mu\sigma}g^{\nu\rho}F_{\sigma\rho}\right] = h(|\varphi|^2)g^{\mu\nu}J_\nu \quad , \tag{2.9}$$

where the gauge-covariant current density, J_ν, is defined by $J_\nu := -i[\overline{\varphi}D_\nu\varphi - \varphi\overline{D_\nu\varphi}]$. Treating the metric, $g_{\mu\nu}$, on $\mathcal{M}$ as a nondynamical background field means that equations (2.9) and (2.9) completely determine the dynamics of the gauged nonlinear σ-model on Σ. All that remains is to solve them! However, eqs.(2.9) and (2.9) form a set of coupled, nonlinear second order equations which, even in the simplest case of $h(|\varphi|^2) = 1$ (the Abelian-Higgs model) is analytically intractable in its most general form. The aim of this chapter is to show that under certain conditions it becomes possible to reduce the above equations to an equivalent set of first order equations in much the same way as was achieved for the Abelian-Higgs model [51]. This reduction occurs at a point in the space of parameters of the model - the self-dual point - and the reduced set of equations is called the self-dual or Bogomol'nyi equations. The existence of a self-dual limit in a theory is more than just a coincidental tool that facilitates solution of the equations of motion (or at least some subset thereof); it was shown that self-duality is in fact intimately related to the embedding properties of the gauge group of the model into a supersymmetric gauge group [53].

2.3 A self-dual limit

Proceeding in analogy with the Abelian-Higgs model we will consider static field configurations and use the gauge degree of freedom permitted by the gauge-invariance of (2.5) under $U(1)$ transformations to fix the gauge as the temporal gauge, $A_0 = 0$. Clearly the electric field $E^i = F^{0i}$ vanishes and the static energy as constructed from the energy-momentum tensor

$$\begin{aligned} T_{\mu\nu} &= h(|\varphi|^2)D_\mu\varphi\overline{\partial_\nu\varphi} + h(|\varphi|^2)\overline{D_\mu\varphi}\partial_\nu\varphi \\ &- \mu F_{\mu\alpha}\partial_\nu A^\alpha - g_{\mu\nu}\mathcal{L}_{\mathrm{NSM}} \quad , \end{aligned} \tag{2.10}$$

is

$$E[\varphi, \mathbf{A}] = \int_\Pi \left\{h(|\varphi|^2)|\mathbf{D}\varphi|^2 + V(|\varphi|^2) + \frac{1}{2}F_{12}^2\right\} d^2x \quad , \tag{2.11}$$

where we have fixed the configuration space, $\mathcal{M}$, to be a (2+1)-dimensional Minkowskian manifold *i.e.*, $g_{\mu\nu} = \eta_{\mu\nu}$ and the magnetic field $B = F_{21} = -F_{12}$.

The construction of a self-dual limit for the Abelian-Higgs theory hinges on the so-called Bogomol'nyi identity [51]

$$|\mathbf{D}\varphi|^2 = |(D_1 \pm iD_2)\varphi|^2 \pm \frac{1}{2}\nabla \times \mathrm{J} \pm e|\varphi|^2 B \quad , \tag{2.12}$$

which, it can be shown [34], generalizes to

$$h(|\varphi|^2)|\mathbf{D}\varphi|^2 = h(|\varphi|^2)|(D_1 \pm iD_2)\varphi|^2 \pm e|\varphi|^2 B \pm \nabla \times \frac{\alpha \mathbf{J}}{2} \quad , \tag{2.13}$$

when the target manifold of the theory is nonflat. Denoting by ϱ the square of the field amplitude, we define $\alpha(\varrho)$ in (2.13) by

$$\alpha(\varrho) := \frac{1}{\varrho}\int_0^{\varrho} h(\varrho')\,d\varrho' \quad > 0 \quad , \tag{2.14}$$

and the spatial component of the covariant matter current density as $\mathbf{J} := -i(\varphi\overline{\mathbf{D}\varphi} - \overline{\varphi}\mathbf{D}\varphi)$. The lower limit of integration in (2.14) ensures that $\alpha\mathbf{J}$ exists and is differentiable at zeros of ϱ thereby satisfying the requirements of Green's theorem. Applying this to the last term in (2.13) yields:

$$\int_{\Pi} \nabla \times \frac{\alpha\mathbf{J}}{2}\, d^2x = \int_{\partial\Pi} \alpha\mathbf{J}\cdot d\mathbf{S} \quad \to 0 \quad , \tag{2.15}$$

since $\mathbf{J} \to 0$ at spatial infinity. Using (2.13) we rewrite the energy (2.11) after a little algebra, as

$$\begin{aligned} E &= \int_{\Pi} \Big\{ h(|\varphi|^2)|(D_1 \pm iD_2)\varphi|^2 + \frac{\mu}{2}\Big[B \pm \Big(\frac{e\alpha\varrho}{\mu} - K\Big)\Big]^2 \\ &- \frac{\mu}{2}\Big(\frac{e\alpha\varrho}{\mu} - K\Big) + V\Big\}\, d^2x \pm \mu K \Phi \quad , \end{aligned} \tag{2.16}$$

where $\Phi := \int_{\Pi} B\, d^2x$ is the magnetic flux through the plane and K is an, as yet, undetermined constant. Note that the fact that we have not as yet specified the form of the self interaction $V(|\varphi|^2)$ permits us a degree of freedom which we now fix by requiring that the theory possess a self-dual limit. Choosing the self-interaction to be

$$V(|\varphi|^2) = \frac{\mu}{2}\Big(\frac{e\alpha\varrho}{\mu} - K\Big)^2 \tag{2.17}$$

reduces (2.16) to

$$E = \int_{\mathbb{R}^2} \Big\{ h|(D_1 \pm iD_2)\varphi|^2 + \frac{\mu}{2}\Big[B \pm (\frac{e\alpha\varrho}{\mu} - K)\Big]^2 \Big\}\, d^2x \pm \mu K \Phi \quad . \tag{2.18}$$

Assuming that the field amplitude, ϱ, approaches some constant value, v, as $|\mathbf{x}| \to \infty$ then the finiteness of the energy of the solutions requires that the self-dual potential tend to one of its zeros. This fixes $K = \frac{e}{\mu}\alpha(v)v$ and hence (2.17) becomes

$$V(\varrho) = \frac{e^2}{2\mu} f^2(\varrho) \quad , \tag{2.19}$$

where $f(\rho) = \alpha(v)v - \alpha(\varrho)\varrho$. Since μ and, by construction, $h(|\varphi|^2)$ are positive definite the first term in (2.18) is positive definite. Consequently, the energy of the model is bounded from below by the magnetic flux through the plane. This gives the Bogomol'nyi-Prasad-Sommerfeld (BPS) bound on the energy: $E \geq \mu K \Phi$ that is saturated for field configurations satisfying the self-duality equations:

$$(D_1 \pm iD_2)\varphi = 0 \quad , \tag{2.20}$$

$$B \pm \frac{e}{\mu}\Big(\alpha\varrho - \alpha(v)v\Big) = 0 \quad . \tag{2.21}$$

Solving (2.20) for B and substituting into (2.21) gives a Liouville-like equation

$$-\nabla^2 \ln \varrho = \frac{2e}{\mu} f(\varrho) \quad , \tag{2.22}$$

or, equivalently,

$$-\nabla^2 \varrho + \frac{(\nabla\varrho)^2}{\varrho} = \varrho f(\varrho) \quad , \tag{2.23}$$

where the independent variable $\mathbf{x}$ has been rescaled to absorb the coefficient $2e/\mu$.

Equations (2.22),(2.22) admit several classes of localized solutions which can be easily classified if the system is treated as a (nonautonomous) dynamical system. We will restrict ourselves to radially symmetric solutions, $\varrho = \varrho(r)$. Renaming ϱ through x and r through t, eq.(2.23) is rewritten as

$$\dot{x} = y \quad , \tag{2.24}$$

$$\dot{y} = \frac{y^2}{x} - \frac{y}{t} - xf(x) \quad . \tag{2.25}$$

Certain facts about solutions of (2.24), (2.25) can be established without specifying the form of $f(\varrho)$. First of all, since $\varrho\alpha(\varrho)$ is a monotonically growing function, $f(\varrho)$ has only one zero: $f(v) = 0$. Hence the system (2.24), (2.25) will always have only two fixed points: $(x, y) = (0, 0)$ and $(v, 0)$. Localized solutions are described by homoclinic and heteroclinic trajectories of (2.24),(2.25) interpolating between these two fixed points.

The point $(v, 0)$ is always a saddle point. Indeed, small perturbations about $(v, 0)$ satisfy $\nabla^2 \delta\varrho = \nu^2 \delta\varrho$, where

$$\nu^2 = v\frac{d}{d\varrho}(\alpha\varrho)\Big|_{\varrho=v} > 0 \tag{2.26}$$

It is not too difficult to see that the origin is also a saddle point. Let, first, $x < 0$. Writing $x = -e^q$, eq.(2.23) becomes an equation for a damped particle in a potential U:

$$\ddot{q} + \frac{\dot{q}}{t} = -f(-e^q) := -\frac{\partial U}{\partial q} \quad . \tag{2.27}$$

Since $f(\varrho)$ is positive for negative ϱ, the function $U(q)$ is monotonically growing for all q. Therefore as $t \to +\infty$, the particle will roll downhill in the direction of negative q. For large negative q the slope becomes constant:

$$f(-e^q) \to f(0) = v\alpha(v) \quad \text{as} \quad q \to -\infty \quad , \tag{2.28}$$

and eq.(2.27) gives

$$q(t) \to C_1 \ln t + C_2 - \frac{v\alpha(v)}{4} t^2 \quad \text{as} \quad t \to +\infty \quad , \tag{2.29}$$

where C_1, C_2 are constants determined by the initial conditions. Consequently, all trajectories starting at $x(t_0) < 0$ at some $t_0 > 0$, flow to the origin:

$$x(t) \to -e^{C_1 \ln t + C_2 - t^2/4} \to 0 \quad \text{as} \quad t \to +\infty \quad . \tag{2.30}$$

Since (2.27) is time-reversible, the backwards-in-time evolution of any initial condition with $x(t_0) < 0$ for some $t_0 < 0$, will also end up at the origin: $x(t) \to 0$ as $t \to -\infty$.

Let now $x > 0$. Here we write $x = e^q$ and eq.(2.23) becomes

$$\ddot{q} + \frac{\dot{q}}{t} = -f(e^q) := -\frac{\partial U}{\partial q} \quad . \tag{2.31}$$

For $q < \ln v$ the potential $U(q)$ has positive slope, and for $q > \ln v$ the slope is negative. Hence a trajectory with initial conditions $x(t_0) > 0$, $y(t_0) \leq 0$ for some $t_0 \geq 0$, will end up at the origin as $t \to +\infty$. The same is true for the backwards evolution of the initial conditions $x(t_0) > 0$, $y(t_{0)} \geq 0$ for some $t_0 \leq 0$. As $t \to -\infty$, we have $x, y \to (0,0)$ for such a trajectory. In essence, then, we can isolate three possible sets of trajectories that correspond to possible physical field configurations:
(i) the homoclinic orbits emanating from the origin, passing through $y = 0$ at $x < v$ and flowing to $(0,0)$ as $t \to \infty$; *(ii)* a set of trajectories with initial condition $(x_0, 0)$, $0 < x_0 < v$ which flow to the origin as $t \to \infty$ and *(iii)* a set of heteroclinic trajectories that emanate from the origin at $t = -\infty$ and flow to $(v, 0)$ as $t \to \infty$. It is not difficult to see that these families can be identified with 'ring', 'lump' and 'vortex'-like solutions respectively.

The above analysis is admittedly rather qualitative but in order to get a more clear picture of the phase portrait we need to specify the metric of the target space of the theory. Furthermore, without explicit knowledge of $f(x)$ we are unable to comment on the stability of each type of solution. Nevertheless, we are at least able to classify possible localized solutions. In the next chapter we apply this prescription to find finite energy localized solutions of the gauged complex sine-Gordon model in $(2+1)$-dimensions but before we do so, it will be useful to check the compatibility of our results with known results for the Abelian-Higgs model and gauged $O(3)$ σ-model.

2.3.1 Reductions of the model

As a consistency check of our model we treat two models with well known solutions.

1. $h(|\varphi|^2) = 1$: In this case the action (2.5) (with $n = 2$ and $g_{\mu\nu} = \eta_{\mu\nu}$) reduces trivially to (2.1) and the σ-model self-duality equations (2.20),(2.21) become

$$(D_1 \pm iD_2)\varphi = 0 \quad , \tag{2.32}$$

$$B \mp (1 - |\varphi|^2) = 0 \quad , \tag{2.33}$$

   which are just the Bogomol'nyi equations for the Abelian-Higgs model [51] corresponding to the symmetry breaking self-interaction $V(|\varphi|^2) = \frac{1}{2}(1 - |\varphi|^2)^2$. The solutions of the above system are well known and we refer the reader to the extensive literature on the Abelian-Higgs model for further details.

2. $h(|\varphi|^2) = 2/(1 + |\varphi|^2)^2$: This metric choice corresponds to $\Sigma = S^2$ and reduces (2.5) to

$$S_{O(3)} = \int_{\mathcal{M}} \sqrt{|g|}\Big\{\frac{2g^{\mu\nu}}{(1+|\varphi|^2)^2}D_\mu\varphi\overline{D_\nu\varphi} - \frac{\mu}{4}g^{\mu\sigma}g^{\nu\rho}F_{\mu\nu}F_{\sigma\rho} - V(|\varphi|^2)\Big\}\, d^3x \quad , \tag{2.34}$$

   which is nothing but the action for the $O(3)$ σ-model covariantly coupled to the metric, $g_{\mu\nu}$. For a flat Minkowski manifold, $\mathcal{M}$, eqs.(2.20) and (2.21) reduce to

$$(D_1 \pm iD_2)\varphi = 0 \quad , \tag{2.35}$$

$$B = \pm\Big\{\frac{1 - |\varphi|^2}{1 + |\varphi|^2}\Big\} \quad , \tag{2.36}$$

   which, comparing with eqs.(13) and (14) of [26] (with $\eta = 0$) are evidently the Bogomol'nyi equations for a $U(1)$ gauged $O(3)$ nonlinear σ-model. The above system of self-dual equations was shown to support coexisting vortices and anti-vortices with a quantized magnetic flux, Φ, that is proportional to the difference between the number of vortices and anti-vortices. With the coupling of gravity these solutions represent cosmic strings and anti-strings with opposite magnetic charge. An important point to note is that our prescription naturally picks out the maximally symmetry breaking potential

$$V = \frac{1}{2}\Big(\frac{1-|\varphi|^2}{1+|\varphi|^2}\Big)^2. \tag{2.37}$$

This is not the only form of the potential that has been studied [28]. One may also *choose* a potential $V = |\varphi|^2/(2+2|\varphi|^2)$ in which the ground state is equivalent to the symmetric vacuum $\varphi = 0$, $A_\mu = 0$. The resulting gauged theory is slightly problematic though, in that the energy of the model fails to account for vortices with opposing magnetic alignments *i.e.*, only vortices with negative local magnetic charge contribute to the total energy of the vortex-anti-vortex system.

2.4 Chern-Simons gauge field interactions

It is a well known fact that odd-dimensional spacetimes allow for the possibility of topologically massive gauge dynamics [54] with the introduction of a Chern-Simons term proportional to $\epsilon^{\mu\nu\lambda}A_\mu F_{\nu\lambda}$ in the action. Moreover, several of the low-dimensional field theoretic models known to exhibit vortex solutions when the gauge dynamics is of the Maxwell type also admit so-called Chern-Simons or Maxwell-Chern-Simons vortices [55, 56] which, unlike Nielsen-Olesen vortices of the Abelian-Higgs model, carry both electric charge and magnetic flux. We now gauge the above $(2+1)$-dimensional general model with a pure Chern-Simons field and find that most of the preceding analysis on the Maxwell-gauged theory remain valid. The fact that we retain a dynamical gauge theory even in the absence of a Maxwell term is justified by the presence of a Higgs mechanism in the classical theory [57] as suggested by the symmetry-breaking self-interactions induced by self-duality. We conclude by using our procedure to reconstruct the vortex solutions of the Chern-Simons gauged $O(3)$ σ-model as well as to construct new vortex solutions of the gauged complex sine-Gordon equation.

The $(2+1)$-dimensional model of a charged scalar field on a target manifold with metric $h(|\varphi|^2)$ coupled to a Chern-Simons gauge field is defined by the action

$$S = \int_{\mathcal{M}^3}\Big\{h(|\varphi|^2)D_\mu\varphi\overline{D^\mu\varphi} + \frac{\kappa}{4}\epsilon^{\alpha\beta\gamma}A_\alpha F_{\beta\gamma} - V(|\varphi|^2)\Big\}\,d^3x \quad . \tag{2.38}$$

Variation of (2.38) with respect to φ and A_ν produce the scalar and Chern-Simons field equations respectively:

$$D_\mu\Big(hD^\mu\varphi\Big) = -\frac{\partial V}{\partial\overline{\varphi}} + \frac{\partial h}{\partial\overline{\varphi}}D_\mu\varphi\overline{D^\mu\varphi} \quad , \tag{2.39}$$

$$\frac{\kappa}{2}\epsilon^{\nu\alpha\beta}F_{\alpha\beta} + eh(|\varphi|^2)J^\nu = 0 \quad . \tag{2.40}$$

Suppressing all time derivatives in the action (2.38), the Chern-Simons static energy functional is

$$E[\varphi, B] = \int_{\mathbb{R}^2} \left(h(|\varphi|^2)|\mathbf{D}\varphi|^2 + \frac{\kappa^2}{4e^2}\frac{B^2}{|\varphi|^2 h} + V(|\varphi|^2) \right) d^x \quad , \tag{2.41}$$

where we have used the Chern-Simons Gauss law,

$$A^0 = \left(\frac{\kappa}{2e^2}\right)\frac{B}{h|\varphi|^2} \tag{2.42}$$

to eliminate A^0. Using the generalized Bogomol'nyi identity (2.13) and choosing the scalar field self-interaction as

$$V(|\varphi|^2) = \frac{e^4}{\kappa^2}|\varphi|^2 h(|\varphi|^2) f^2(|\varphi|^2) \quad , \tag{2.43}$$

where, as before, $f(|\varphi|^2) := (\alpha(v)v - \alpha(|\varphi|^2)|\varphi|^2)$, allows us to further rewrite the static energy (2.41) in a self-dual form,

$$\begin{aligned} E &= \int_{\mathbb{R}^2} \left\{ h(|\varphi|^2)|(D_1 \pm iD_2)\varphi|^2 + \left(\frac{\kappa}{2e}\right)^2 \left(\frac{1}{|\varphi|^2 h}\right) \left[B \mp \frac{2e^3}{\kappa^2}|\varphi|^2 h(|\varphi|^2) f(|\varphi|^2) \right]^2 \right\} d^2x \\ &\pm \; K\Phi \quad , \end{aligned} \tag{2.44}$$

where $\Phi = \int B \, d^2x$ is, as usual, the magnetic flux through the plane, $K = e\alpha(v)v$ is a constant fixed by the boundary conditions on φ. With $h(|\varphi|^2) > 0$ the first term in the static energy functional (2.44) is manifestly positive giving a lower bound on the energy of $E \geq K|\Phi|$ for a fixed value of the flux. The bound is achieved by field configurations satisfying the self-duality equations

$$(D_1 \pm iD_2)\varphi = 0 \quad , \tag{2.45}$$

$$B \mp \frac{2e^3}{\kappa^2}|\varphi|^2 h(|\varphi|^2) f(|\varphi|^2) = 0 \quad , \tag{2.46}$$

where the upper (lower) sign corresponds to positive (negative) flux values. Restricting ourselves to axisymmetric field configurations, $\varphi(r,\theta) = \sqrt{\varrho(r)}e^{i\omega(\theta)}$, reduces eq.(2.46) to

$$-\nabla^2 \varrho + \frac{\nabla \varrho}{\varrho} = \varrho^2 h(\varrho) f(\varrho) \quad , \tag{2.47}$$

after rescaling $\mathbf{x}$ to $\sqrt{\frac{\kappa^2}{2e^4}}\mathbf{x}$.

As in the Maxwell case the self-duality equations (2.45) and (2.46) are equivalent to a nonautonomous dynamical system that we get by renaming ϱ through x and r through t:

$$\dot{x} = y \quad , \tag{2.48}$$

$$\dot{y} = \frac{y^2}{x} - \frac{y}{t} - x^2 h(x) f(x) \quad . \tag{2.49}$$

In fixing $K = e\alpha(v)v$, we have implicitly restricted ourselves to field configurations that approach a constant value v at spatial infinity. Since $h(\rho)$ is strictly positive $f(\rho)$, as in the Maxwell case, has only one zero[1] at $\rho = v$. Writing $\tilde{f}(x) := x\,h(x)\,f(x)$ casts (2.49) into a similar form as (2.25) with $\tilde{f}$ satisfying similar properties to $f(x)$ in the latter system. Hence the system (2.48),(2.49) for the Chern-Simons case will also have only two fixed points (for nonzero Dirichlet boundary conditions on φ): $(x, y) = (0, 0)$ and $(v, 0)$ with the homoclinic and heteroclinic orbits that interpolate between them describing localized solutions of (2.45),(2.46). In fact an analysis similar to the preceding one for Maxwell gauge dynamics shows that the structure of the phase space of the Chern-Simons system is at least qualitatively the same as the former and we will not repeat the calculation here. Suffice it to say that we expect similar localized solutions for the Chern-Simons gauged σ-model.

2.5 Self-duality in Maxwell-Chern-Simons σ-models

In this section we investigate the possibility of extending our previous results to a class of $(2+1)$-dimensional gauged nonlinear σ-models coupled to a spin-1 field in a region of its parameter space in which the Maxwell and Chern-Simons terms in its action are of the same order.

In a recent paper, Lee *et.al.* [58] consider two self-dual Higgs models: $U(1)$ and $U(1) \times U(1)$ gauge theories with one and two Maxwell-Chern-Simons gauge fields respectively. A common ingredient in each model was a neutral scalar field N which coupled to the matter fields of each model only via the potential terms. Although, as in our prescription, the only constraint that is imposed on the models is that they possess a self-dual limit, it is sufficient to completely determine the auxiliary field in terms of the charged matter fields of each theory. Evidently, the neutral field is crucial for each model to possess self-dual limits. These (gauge invariant) results were also shown to be consistent with previously known results for self-dual models [51, 11, 56].

In what follows, we continue to consider a class of gauged nonlinear σ-models on two-dimensional target manifolds embedded in $\mathbb{E}^3$, only now retaining both Maxwell *and* Chern-Simons terms in the action and ask whether the phenomenon reported by Lee *et.al.* (namely, that their models are not manifestly self-dual when the gauge field dynamics is of the mixed Maxwell-Chern-Simons type)is specific to the Higgs model (corresponding to the flat target-space case of our models) and its variations (like the aforementioned

[1]Note K was fixed so that the scalar potential, V, vanishes as $|\mathbf{x}| \to \infty$. In the Chern-Simons case, this is not the only way that V might do so, since requiring that $|\varphi| \to 0$ as $|\mathbf{x}| \to \infty$ will also ensure that V go to zero at spatial infinity for *arbitrary* K.

$U(1) \times U(1)$ model) or does it apply to a wider class of theories? Stated another way; do the Maxwell-Chern-Simons gauged σ-models on embedded two-dimensional target s-paces possess self-dual limits?

To answer this question we consider a nonlinear σ-model on a two-dimensional smooth surface, Σ, embedded in $\mathbb{E}^3$. A vanishing Weyl tensor for such a surface [52] implies that it is conformally flat, which in turn allows us to write the metric on Σ as $h_{ab} = h(|\varphi|^2)\eta_{ab}$ where $h(|\varphi|^2)$ is a C^∞, positive definite function on Σ. As such the action for the Maxwell-Chern-Simons gauged Σ-model on Σ reads

$$S = \int_{\mathcal{M}} d^3x \left\{ h(|\varphi|^2) D_\mu\varphi\overline{D^\mu\varphi} - \frac{\mu}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\kappa}{4}\epsilon^{\alpha\beta\gamma}A_\alpha F_{\beta\gamma} - V(|\varphi|^2)\right\} \quad . \tag{2.50}$$

The matter field equations remain the same as in the pure Maxwell (or pure Chern-Simons) cases:

$$D_\mu\Big(h(|\varphi|^2)D^\mu\varphi\Big) = -\frac{\partial V}{\partial\overline{\varphi}} + \frac{\partial h}{\partial\overline{\varphi}}D_\mu\varphi\overline{D^\mu\varphi} \quad . \tag{2.51}$$

Variation of the action with respect to the gauge field A_ν gives the Maxwell-Chern-Simons gauge field equations

$$h(|\varphi|^2)J^\nu + \mu\partial_\lambda F^{\lambda\nu} + \frac{\kappa}{2}\epsilon^{\nu\alpha\beta}F_{\alpha\beta} = 0 \quad , \tag{2.52}$$

where $J_\mu = -ie[\overline{\varphi}D_\mu\varphi - \varphi\overline{D_\mu\varphi}]$ is the gauge-covariant current density. The corresponding Gauss law can be read from the zeroth component of (2.52) as

$$\kappa B = 2e|\varphi|^2 h\,\partial_0 Arg\varphi + 2e^2|\varphi|^2 h\,A_0 - \mu\nabla\cdot(\nabla A_0 + \partial_0\mathbf{A}) \quad , \tag{2.53}$$

where $B = F_{21}$ is the magnetic field strength and $Arg\varphi = -i(\varphi - \overline{\varphi})/(\varphi + \overline{\varphi})$. Our interest in the model lies in whether or not it supportsa self-dual limit; in other words, whether the corresponding energy functional, $E[\varphi, A_0, \mathbf{A}]$ is bounded from below by some topological quantity. The energy functional derives from the Lagrangian via a Legendre transformation as

$$E[\varphi, A_0, \mathbf{A}] = \int_{\mathcal{M}^3}\left\{h\,|D_0\varphi|^2 + h\,|\mathbf{D}\varphi|^2 + \frac{\mu}{2}[(\partial_0\mathbf{A} + \nabla A_0)^2 + B^2] + V(|\varphi|^2)\right\} d^3x \tag{2.54}$$

Restricting ourselves to static field configurations $\varphi = \varphi(\mathbf{x}), A_0 = A_0(\mathbf{x})$ and $\mathbf{A} = \mathbf{A}(\mathbf{x})$ and suppressing all time derivatives reduces the energy (2.54) to

$$E = \int_{\mathcal{M}^2}\left\{h\,|\mathbf{D}\varphi|^2 + e^2A_0^2|\varphi|^2h + \frac{\mu}{2}(\nabla A_0)^2 + \frac{\mu}{2}B^2 + V(|\varphi|^2)\right\} d^3x \quad , \tag{2.55}$$

and the Maxwell-Chern-Simons-Gauss law to

$$\kappa B = 2e^2|\varphi|^2 A_0 h - \mu\nabla\cdot(\nabla A_0) \quad . \tag{2.56}$$

Applying the Bogomol'nyi identity

$$h(|\varphi|^2)|\mathbf{D}\varphi|^2 = h(|\varphi|^2)|(D_1 \pm iD_2)\varphi|^2 \pm e\alpha|\varphi|^2 B \pm \nabla\times\frac{\alpha\mathbf{J}}{2} \tag{2.57}$$

where $\alpha|\varphi|^2 = \int_0^{|\varphi|^2} h(x)\,dx > 0$, to the static energy (2.55) and recalling that, for sufficiently smooth α, the last term in the above identity integrates to zero over $\mathbb{R}^2$, we find that

$$E = \int_{\mathbb{R}^2} \left\{ h|D_1 \pm iD_2)\varphi|^2 \pm e\alpha|\varphi|^2 B + \frac{\kappa}{2}A_0 B + \frac{\mu}{2}B^2 + V(|\varphi|^2) \right\} d^2x \quad . \tag{2.58}$$

The temporal component of the gauge field, A_0, may be completely elliminated from the expression for the energy (2.58) by appealing to the Gauss law (2.56) and imposing the gauge condition $\nabla A_0 = 0$ so that, after a little algebra, we may write (2.58) in the self-dual form

$$E = \int_{\mathbb{R}^2} \left\{ h(|\varphi|^2)|D_1 \pm iD_2)\varphi|^2 + \frac{1}{2G}(B \mp fG)^2 \right\} d^2x \pm K\Phi \quad , \tag{2.59}$$

where $G := (2e^2|\varphi|^2 h)/(\kappa^2 + 2\mu e^2|\varphi|^2 h) > 0$; $f := K - e\alpha|\varphi|^2$, K is a constant and Φ is the magnetic flux through $\mathbb{R}^2$. Note that, by virtue of (2.59), the form of the scalar field self-interaction is determined as

$$V = \frac{1}{2}f^2 G = \frac{e^2|\varphi|^2 h}{\kappa^2 + 2\mu e^2|\varphi|^2 h}(K - e\alpha|\varphi|^2)^2 \quad . \tag{2.60}$$

We can now read off the self-duality equations that saturate the BPS energy bound, $E \geq K|\Phi|$, from (2.59) as:

$$(D_1 \pm iD_2)\varphi = 0 \quad , \tag{2.61}$$

$$B \mp \left(\frac{2e^2|\varphi|^2 h}{\kappa^2 + 2\mu e^2|\varphi|^2 h}\right) f = 0 \quad . \tag{2.62}$$

Returning now to the gauge condition $\nabla A_0 = 0$, we find after integration that $A_0 = k$ where k is some non-zero constant. From the Gauss law we then can read off an expression for the magnetic field:

$$B = \frac{2e^2}{\kappa} k|\varphi|^2 h \quad . \tag{2.63}$$

So, comparing with (2.62) we find that

$$\frac{2e^2}{\kappa}k|\varphi|^2h = \pm\frac{2e^2|\varphi|^2h}{\kappa^2+2\mu e^2|\varphi|^2h}f \quad . \tag{2.64}$$

Writing $\varphi = \sqrt{\varrho}e^{i\omega(\theta)}$, this may be written as

$$\frac{k}{\kappa} = \pm\frac{f}{\kappa^2+2\mu e^2\varrho h} \quad , \tag{2.65}$$

or, equivalently, recalling that $h = -f'$

$$f' \pm \frac{A}{\varrho}f = \frac{B}{\varrho} \tag{2.66}$$

where $A := \kappa/2\mu e^2 k$ and $B = \kappa^2/2\mu e^2 > 0$. Eq.(2.66) is easily integrated, giving

$$\left[\varrho^{\pm A}f\right]' = B\varrho^{\pm A-1} \quad , \tag{2.67}$$

or $f = \pm(B/A)+C\varrho^{\mp A}$. Finiteness of the energy of the solutions require that φ evolve to one of the zeros of the potential at spatial infinity, thus fixing K as $K = e\alpha(\varrho_0)\varrho_0$ where $\varrho_0 > 0$ is the field amplitude at spatial infinity. Clearly then, $f(\varrho_0) = K - e\alpha(\varrho_0)\varrho_0 = 0$ fixing $C = \mp(B/A)\varrho_0^{\pm A}$. Finally, then we express f as

$$f = \pm\frac{B}{A}\left[1 - \left(\frac{\varrho}{\varrho_0}\right)\right] \quad , \tag{2.68}$$

and find $h = -f' = -B(\varrho/\varrho_0)^{\mp A-1} < 0$. The inconsistency here is clear! We began by assuming that h was positive definite and applied the gauging prescription advocated for the Maxwell and Chern-Simons gauged σ-models to the Maxwell-Chern-Simons sector of the same class of nonlinear σ-models only to find that consistency of the prescription requires that the metric h on the target manifold of the σ-model be negative definite. The above argument is admittedly gauge dependent but is consistent with the results of Lee *et.al.* [58] for the $h = 1$ case. As in their $U(1)$ and $U(1)\times U(1)$ Higgs models, this problem may be circumvented by the addition of an auxiliary neutral scalar field that couples to the charged matter content of the model only through the self-dual potential. As we have seen, the form of the self-dual potential is completely determined by imposing self-duality on the model so the coupling of such an auxiliary field should also be determined at the self-duality point in the parameter space of the model. Hence, we would also expect the target manifold of the auxiliary field to be topologically nontrivial. Details of this analysis, as well as a fully gauge invariant version of the preceding argument are currently being completed and we hope to report our findings in a forthcoming publication.

2.6 Summary and conclusions

This chapter was devoted to a study of a class of $U(1)$-gauged nonlinear σ-models on 2-dimensional axisymmetric target manifolds. Our results have been two-fold: firstly; we have shown that, given certain restrictions on the target manifold (manifest as constraints on the metric) such σ-models possess self-dual limits in which the second order equations of motion for the scalar field reduce to a set of first order equations that saturate a lower bound on the energy. Secondly; we have shown that by treating the self-duality equations as a dynamical system we are able to extract valuable qualitative information about localized field configurations.

The latter parts of the chapter have focused on extending the formalism to include topologically massive gauge dynamics. We have shown that while a nonlinear σ-model coupled to a pure Chern-Simons gauge field also exhibits a self-dual limit; the same cannot be said for mixed Maxwell-Chern-Simons dynamics in which the existence of a self-dual limit cannot be established without the introduction of a neutral auxiliary scalar field. As such this is an extension of the work of Lee *et.al.* [58] to theories with nontrivial target manifolds.

Chapter 3

The complex sine-Gordon model

Must you be so linear Jean-Luc?

- Q to Picard in 'All good things...'

We investigate some properties of the planar complex sine-Gordon model. The recent Barashenkov-Pelinovsky construction of exact vortices for the Euclidean formulation of the theory is reviewed. We then investigate the existence of vortices in the BPS limit of a $U(1)$-gauged complex sine-Gordon model in $(2+1)$-dimensions.

3.1 Introduction.

The complex sine-Gordon model (CSG), like the nonlinear σ-model is a non-ultralocal field theory with a nontrivial target manifold. In fact, it was originally derived from a reduction of the $O(4)$ nonlinear σ-model [30]. It differs, however, from the nonlinear models we have encountered thus far in that it describes a *massive* scalar field. As an application of the general formalism developed in the previous chapter we will investigate some properties of the $(2+1)$ dimensional complex sine-Gordon model. The $(2+0)$ Euclidean formulation has received considerable attention of late and was recently shown to exhibit exact vortex solutions [33]. The simplicity manifest in this solution merits further attention so we begin by reviewing the construction of the exact vortices of the Euclidean complex sine-Gordon theory.

3.2 Exact Vortex Solutions of the $O(2)$ Complex sine-Gordon Theory

The complex sine-Gordon theory in two Euclidean dimensions is defined by its action as

$$S_{\mathrm{CSG}} := \int_{\mathbb{E}^2} \left\{ \frac{|\nabla\varphi|^2}{1-|\varphi|^2} + (1-|\varphi|^2) \right\} d^2x \quad . \tag{3.1}$$

Although not as extensively studied as its $(1+1)$ dimensional Minkowskian formulation the Euclidean CSG nevertheless exhibits rich geometric and nonperturbative structure and merits further study. Recent interest in the Euclidean formulation has also been sparked by the realization that it resurfaces in Euclidean conformal field theories. In particular, it may be derived as a reduction of the integrably perturbed $SU(2)_N$ gauged Wess-Zumino-Witten model [59, 60, 61].

The equations of motion (the CSG equation) follows, as usual, from the least action principle[1]

$$\frac{\delta S_{\mathrm{SG}}}{\delta \varphi} = 0 \quad \Rightarrow \quad \nabla^2 \varphi + \frac{(\nabla\varphi)^2 \overline{\varphi}}{1-|\varphi|^2} + \varphi(1-|\varphi|^2) = 0 \quad . \tag{3.2}$$

Note that the action, eq.(3.1) and hence the CSG equation, eq.(3.2) is clearly invariant under global $O(2)$ transformations.

3.2.1 The Barashenkov-Pelinovsky construction

It is well known [62] that there exists a one-to-one correspondence between the $O(2)$ CSG model and the massive Thirring model on $\mathbb{E}^2$ as defined by its action

$$S_{\mathrm{MTM}} = \int_{\mathbb{E}^2} \left\{ i\Psi^\dagger \partial\!\!\!/ \Psi + \Psi^\dagger \Psi - \frac{1}{4}(\Psi^\dagger \gamma_i \Psi)^2 - 1 + c.c \right\} d^2x \tag{3.3}$$

where $\partial\!\!\!/ := \gamma^i \partial_i$ and $\gamma_i := \sigma_i$ are the Dirac gamma matrices on $\mathbb{E}^2$. $\Psi = (u, v)^T$ is a Dirac 2-spinor. A simple calculation shows that, in terms of the spinor components, u and v, the action for the massive Thirring model, eq.(3.3) becomes

$$S_{\mathrm{MTM}} = \int_{\mathbb{E}^2} \left\{ i\overline{u}\partial v + i\overline{v}\overline{\partial} u + |u|^2 + |v|^2 - |uv|^2 - 1 + c.c \right\} d^2x \tag{3.4}$$

where we have defined the complex derivative operators as $\partial := \partial_1 - i\partial_2$ and $\overline{\partial} := \partial_1 + i\partial_2$. Carrying out the necessary functional differentiation yields the massive Thirring equations as a system of first order coupled partial differential equations for the spinor components

$$\begin{aligned} \frac{\delta S_{\mathrm{MTM}}}{\delta \overline{u}} = 0 \quad &\Rightarrow \quad i\partial v + u - u|v|^2 = 0 \quad , \\ \frac{\delta S_{\mathrm{MTM}}}{\delta \overline{v}} = 0 \quad &\Rightarrow \quad i\overline{\partial} u + v - v|u|^2 = 0 \quad . \end{aligned} \tag{3.5}$$

Solving the first of eqs.(3.5) for v in terms of u and substituting into the second yields the second order equation satisfied by u

$$\partial\overline{\partial} u + \frac{\partial u \overline{\partial} u}{1-|u|^2}\overline{u} + u(1-|u|^2) = 0 \quad . \tag{3.6}$$

[1] We omit details of the calculation and refer the reader to previous, similar calculations in chapter 1.

Similarly, the second order equation satisfied by v is

$$\partial\overline{\partial}v + \frac{\partial v \overline{\partial} v}{1-|v|^2}\overline{v} + v(1-|v|^2) = 0 \quad . \tag{3.7}$$

It is clear from a comparison of these equations with the planar CSG equation, eq.(3.2) that the spinor components satisfy the CSG equation on $\mathbb{E}^2$. The crux of the BP construction is the fact that the massive Thirring equations of motion for the spinor components of Ψ, eqs.(3.5) may be interpreted as an auto-Bäcklund transformation between solution surfaces of the CSG theory. Given any solution, φ_0, of the CSG equation we may construct an entire hierarchy of solutions via the Bäcklund transformation in a *purely algebraic* manner. We now proceed to construct a particular solution of eq.(3.2).

In trying to construct vortex solution to the planar CSG model, we restrict ourselves to axisymmetric multivortex solutions of the form $\varphi(r) = \Phi_n(r)e^{in\theta}$ where r and θ are a polar parameterization of $\mathbb{E}^2$ and n labels the vorticity of the solutions [33]. The field amplitudes $\Phi_n(r) \to 1$ at spatial infinity i.e., as $r \to \infty$. Substituting this form into eq.(3.2) and simplifying yields an equivalent form of the CSG equation for the vortex amplitude

$$\frac{d^2\Phi_n(r)}{dr^2} + \frac{1}{r}\frac{d\Phi_n(r)}{dr} + \frac{\Phi_n(r)}{1-\Phi_n^2(r)}\left\{\left(\frac{d\Phi_n(r)}{dr}\right)^2 - \frac{n^2}{r^2}\right\} + \Phi_n(r)(1-\Phi_n^2(r)) = 0 \quad . \tag{3.8}$$

Note that while we have converted the CSG equation to an ordinary differential equation; eq.(3.8) is not much simpler than the original CSG equation. However, recalling that we have two equivalent first order equations at our disposal in the form of the Bäcklund transformation equations (3.5). We assume that each of the spinor components have the appropriate vortex form and choose $u = -i\Phi_{n-1}(r)e^{i(n-1)\theta}$ and $v = \Phi_n(r)e^{in\theta}$. In polar coordinates the derivative operators are $\partial = (\partial_r - \frac{i}{r}\partial_\theta)e^{-i\theta}$ and $\overline{\partial} = (\partial_r + \frac{i}{r}\partial_\theta)e^{i\theta}$ so that the spinor equations, eqs.(3.5), yield a system of equations relating the vortex with vorticity n to that with vorticity $n-1$

$$\begin{aligned} \frac{d\Phi_n(r)}{dr} + \frac{n}{r}\Phi_n(r) &= (1-\Phi_n^2(r))\Phi_{n-1}(r) \quad , \\ -\frac{d\Phi_{n-1}(r)}{dr} + \frac{n-1}{r}\Phi_{n-1}(r) &= (1-\Phi_{n-1}^2(r))\Phi_n(r) \quad . \end{aligned} \tag{3.9}$$

For the case $n = 1$ the second of eqs.(3.9) reduces to

$$-\frac{d\Phi_0}{dr} = (1-\Phi_0^2)\Phi_1 \quad , \tag{3.10}$$

which is clearly solved by the constant solution $\Phi_0 = 1$. This is the vacuum solution which may now be used in conjunction with the Bäcklund transformation to generate less trivial solutions with nonzero vorticities. To derive the 1-vortex solution we substitute

$\Phi_0 = 1$ into the second of eqs.(3.9) with $n = 1$. This reduces to a Ricatti equation of the form

$$\frac{d\Phi_1}{dr} + \frac{1}{r}\Phi_1 = 1 - \Phi_1^2 \quad . \tag{3.11}$$

Defining $\Gamma'(r)/\Gamma(r) := \Phi_1$ where the prime denotes differentiation with respect to r brings eq.(3.11) into a linear equation in $\Gamma(r)$

$$\Gamma''(r) + \frac{1}{r}\Gamma'(r) - \Gamma(r) = 0 \quad . \tag{3.12}$$

We recognize eq.(3.12) as the zero order modified Bessel's equation whose solutions are modified Bessel's functions of zero order $\Gamma(r) = AI_0(r) + BK_0(r)$. Boundary conditions on the field amplitude $\Phi_1(r)$ require that $\Gamma(r)$ be bounded at the origin. This fixes $B = 0$. Without loss of generality we can choose $A = 1$ thereby normalizing Γ at the origin so that $\Gamma(r) = I_0(r)$. The $n = 1$ vortex solution of the planar CSG model is then

$$\Phi_1(r) = \frac{I_0'(r)}{I_0(r)} = \frac{I_1(r)}{I_0(r)} \quad . \tag{3.13}$$

Rearranging the spinor equations in the form eq.(3.9) provides a recursion relation

$$\Phi_{n+1} = \Phi_{n-1} - \frac{2}{1 - \Phi_n^2}\frac{d\Phi_n}{dr} \quad , \tag{3.14}$$

that may be used to generate a hierarchy of higher order exact vortex solutions to the Euclidean CSG model. For instance, when $n = 1$, eq.(3.14) gives an expression for the 2-vortex as

$$\Phi_2 = \Phi_0 - \frac{2}{1 - \Phi_1^2}\frac{d\Phi_1}{dr} = 1 - \frac{2I_0^2}{I_0^2 - I_1^2}\left(\frac{I_1'}{I_0} - \frac{I_1 I_0'}{I_0^2}\right) = \frac{I_1^2 - I_0 I_2}{I_0^2 - I_1^2} \quad . \tag{3.15}$$

Similar calculations yield expressions for $\Phi_n(r), n \geq 3$. As advertised, this construction is completely algebraic. We plot the vortex solutions for the case $n = 1, 2$ and $n = 3$ in figs. 3.1 and 3.2 respectively.

3.2.2 Properties of the vortex solutions

Before proceeding to analyze the $U(1)$-gauged CSG model, we note several interesting properties of the exact CSG vortices. Firstly, the CSG model is, like the nonlinear σ-model, a non-ultralocal field theory. While the two theories differ quite significantly in the topology of the respective target manifolds they are structurally very similar. As such, we might anticipate the existence of internal gauge degrees of freedom in the CSG model also. It turns out that not only is this the case, but the Euclidean CSG theory

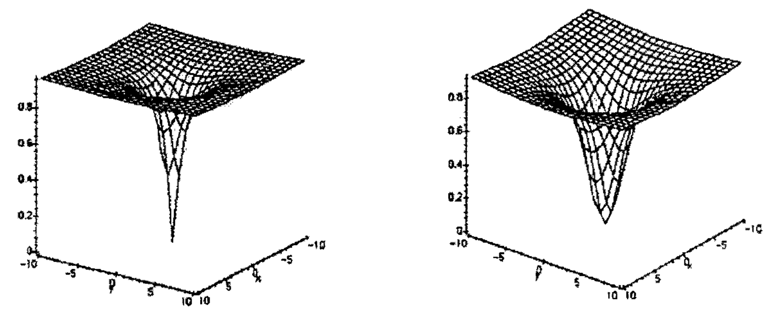


Figure 3.1: Vortex solutions of the planar complex sine-Gordon equation for $n = 1, 2$.

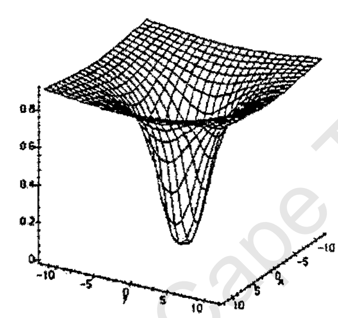


Figure 3.2: Complex sine-Gordon vortex for $n = 3$.

also posses a self-dual limit not unlike that exhibited by the nonlinear sigma model. Analogous to the holomorphic functions which saturate the Bogomol'nyi bound on the energy in the $O(3)$ σ-model, the $n = 1$ vortex of the Euclidean CSG model minimizes the action, eq.(3.1) and may be interpreted as a self-dual soliton of the model. To show this we note that $|\nabla\psi|^2 = |\partial\psi|^2 - 2(\partial_1\psi_1\partial_2\psi_2 - \partial_1\psi_2\partial_2\psi_1)$ where $\psi = \psi_1 + i\psi_2$ so that after a little algebra we can rearrange the action, eq.(3.1) as

$$S_{\text{ESG}} = \int_{\mathbb{E}^2} \frac{|\partial\psi - (1-|\psi|^2)|^2}{1-|\psi|^2}\, d^2x + \int_{\mathbb{E}^2} \partial_i\Big\{ \ln(1-|\psi|^2)\epsilon_{ij}\partial_j(\text{Arg}\psi) + 2\psi_i \Big\}\, d^2x \quad (3.16)$$

Defining the "gauge potential" $A_i = \ln(1-|\psi|^2)\epsilon_{ij}\partial_j(\text{Arg}\psi) + 2\psi_i$ allows us to write this in a more familiar form as

$$S_{\text{ESG}} = \int_{\mathbb{E}^2} \frac{|\partial\psi - (1-|\psi|^2)|^2}{1-|\psi|^2}\, d^2x + \int_{\mathbb{E}^2} \nabla \cdot \mathbf{A}\, d^2x \quad . \quad (3.17)$$

Implicit in this construction of the above form of the action is the constraint that $|\psi|^2 < 1$. Thus, unlike the nonlinear σ-model, self-dual solutions of the Euclidean CSG model do not exist on the entire target space, Σ, but are restricted to that submanifold, $\Sigma_{\text{SD}} \subset \Sigma$ on which the coordinates satisfy the above inequality. As we will see, this constraint is intimately tied to the reformulation of the CSG model as a nonlinear sigma model on a noncompact target space. We point out also that the vector field A_i is completely determined by the the matter field ψ and has no dynamical properties of its own. It corresponds to an internal degree of freedom that is manifest in the action, eq.(3.17). As such it differs significantly from the gauge field to be introduced in the gauged CSG model in which we couple the complex sine-Gordon matter field to a dynamical $U(1)$-vector field whose dynamics play an important role in determining the nature of any solitonic solutions. In Euclidean space, the action and energy are identical so we see that on Σ_{SD} the energy, eq.(3.17) is bounded from below by a flux-like term representing the divergence of the "gauge field" A_i through $\mathbb{E}^2$ i.e., $E_{\text{CSG}} \geq F$ where $F := \int \nabla \cdot \mathbf{A}\, d^2x$. This is the Bogomol'nyi bound for the CSG model. This bound is saturated in the self-dual limit i.e. when

$$\partial\psi - (1 - |\psi|^2) = 0 \quad . \tag{3.18}$$

It is not difficult to see that the $n = 1$ vortex, eq.(3.13), is a solution of eq.(3.18) and is consequently a self-dual soliton of the Euclidean CSG model.

The second point of interest is that, unlike the solitons of the nonlinear sigma model; the energy of the Euclidean CSG vortices is divergent. This is easily seen from the Bogomol'nyi inequality. The flux term, F, may be written as

$$F = \int_{\mathbb{E}^2} \nabla \cdot \mathbf{A}\, d^2x = \int_{\partial\mathbb{E}^2} \mathbf{A} \cdot d\mathbf{l} \quad , \tag{3.19}$$

where the boundary of $\mathbb{E}^2$, $\partial\mathbb{E}^2$ is a circle, C_R, of radius $R \to \infty$. Using the asymptotic form of the nth order vortex configuration $\Phi_n(r) \sim 1 - \frac{n}{2r} - \frac{n^2}{8r^2} + \mathrm{O}(r^{-3})$ we find

$$F = 2\pi(2R - \ln R - 1) + \mathrm{O}(r^{-1}) \quad , \tag{3.20}$$

which clearly diverges as $R \to \infty$ and the vortex energy also. This divergence of the vortex energy points to a natural cut-off radius in the system. Physically this could correspond to the radial distance between adjacent vortex lines in a multivortex system.

3.3 The gauged complex sine-Gordon model

We proceed now to the CSG model in $(2+1)$ Minkowski spacetime, $\mathcal{M}^3$, as defined by its action

$$\mathcal{S}_{\text{CSG}} := \int_{\mathcal{M}^3} \frac{\partial_\mu\varphi\partial^\mu\overline{\varphi}}{1 - |\varphi|^2} + (1 - |\varphi|^2). \tag{3.21}$$

The equations of motion for the system are, as usual, derived from the variation of the above action. Varying[2] eq.(3.21)with respect to $\overline{\varphi}(x)$ gives the relativistic complex sine-Gordon equation on $\mathcal{M}^3$

[2]Again, we omit calculational details and refer the reader to chapter 1 for similar calculations

$$\partial_\mu \partial^\mu \varphi + \frac{\overline{\varphi} \partial_\mu \varphi \partial^\mu \varphi}{1 - |\varphi|^2} + \varphi(1 - |\varphi|^2) = 0 \tag{3.22}$$

Static solutions of this model essentially solve the planar Euclidean CSG equations. The CSG model on $\mathcal{M}^3$ thus also exhibits localized, stable, static vortex solutions[33]. The action, eq.(3.21), and thus the equations of motion eq.(3.22) are clearly invariant under global $U(1)$ transformations $\varphi(x) \rightarrow \varphi'(x) = e^{i\alpha}\varphi$ (α = constant). Physical considerations, however, motivate that we consider a *local* theory for which α is a function of the spacetime coordinates x_μ. Invariance under local $U(1)$ transformations is implemented by requiring that the derivative operator be gauge covariant under local transformations $\varphi(x) \rightarrow e^{i\alpha(x)}\varphi(x)$. This couples the matter field $\varphi(x)$ to a spin-1 gauge field with the appropriate transformation properties. Physically, as in for instance the theory of superfluids, this corresponds to immersing the matter field (the superfluid) in an electromagnetic field.

As we will show, the dynamics of the gauge field play an important role in determining the structure of the of the solutions to the field equations. In odd dimensional spacetime the gauge form may also be a topologically massive Chern-Simons 1-form [54, 63]. In this case, the gauge dynamics is of the Chern-Simons type. With the recent upsurge in interest surrounding low dimensional systems in general and anyonic matter in particular, Chern-Simons theories are now receiving more attention than ever before (see, for instance [64] and references therein). In this chapter we investigate in detail the existence of vortex-like solutions to the Maxwell-gauged CSG model on $\mathcal{M}^3$. Although we omit treating the case when the gauge dynamics is of the Chern-Simons type, we know from our previous analysis that our prescription for constructing localized self-dual solutions extends without much difficulty to topologically massive gauge dynamics. This work is based primarily on a forthcoming paper [34] in which the geometry of the σ-model surface and Chern-Simons dynamics is treated more thoroughly.

3.4 The Maxwell-gauged complex sine-Gordon model

Beginning with the action, eq.(3.21), we impose local $O(2)$ invariance, *i.e.*, we require that under local $U(1) \sim O(2)$ transformations $\varphi(x) \rightarrow e^{i\alpha(x)}\varphi(x)$ the action transform as $\mathcal{S}[\varphi] \rightarrow \mathcal{S}[\varphi]$. Invariance of the action demands the matter field be coupled to a $U(1)$ gauge field $A_\mu(x)$ which transforms as $A_\mu \rightarrow A'_\mu = A_\mu - i\partial_\mu\alpha(x)$ with α a $U(1)$ generator. We then construct the minimally coupled gauge-covariant derivative as $D_\mu := \partial_\mu + ieA_\mu$ where e is a coupling constant. We consider the case where the gauge field dynamics is of the Maxwell type. Defining the gauge curvature as usual by $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$ we find that the action for the (locally $O(2)$ invariant) CSG model becomes

$$\mathcal{S}_{\mathrm{CSG}} = \int_{\mathcal{M}^3} \left\{ \frac{D_\mu\varphi\overline{(D^\mu\varphi)}}{1 - |\varphi|^2} + (1 - |\varphi|^2) - \frac{\mu}{4}F_{\mu\nu}F^{\mu\nu} - V(|\varphi|^2) \right\} d^3x \quad . \tag{3.23}$$

Since the theory is both Lorentz and gauge invariant we may restrict ourselves to static, finite energy solutions in the temporal gauge, so $\varphi = \varphi(\mathbf{x})$, $A_\mu = A_\mu(\mathbf{x})$, $A_0 = 0$. Under these restrictions it is clear that $\partial_0\varphi = \partial_0 A_\mu = 0$ and $F_{0\mu} = F_{\mu 0} = 0$ and the solutions are electrically neutral. The related energy momentum tensor $T_{\mu\nu}$ derives from

$$T^{\mu\nu} \;=\; \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\partial^\nu\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\overline{\varphi})}\partial^\nu\overline{\varphi} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\lambda)}\partial^\nu A_\lambda - g^{\mu\nu}\mathcal{L} \quad . \tag{3.24}$$

where $\mathcal{L}$ is the Lagrangian corresponding to the gauged CSG action eq.(3.23). Using the explicit form in eq.(3.23) in the definition of the energy-momentum tensor yields

$$T^{\mu\nu} = \frac{(D^\mu\varphi)}{1-|\varphi|^2}\partial^\nu\overline{\varphi} + \frac{\overline{(D^\mu\varphi)}}{1-|\varphi|^2}\partial^\nu\varphi - \mu F^{\mu\alpha}\partial^\nu A_\alpha - g^{\mu\nu}\mathcal{L} \tag{3.25}$$

The $(0,0)$th component of the energy-momentum tensor gives the energy of static solutions to the CSG system as

$$E = \int_{\mathbb{E}^2} T^{00}\, d^2x = \int_{\mathbb{E}^2}\left\{\frac{(D_i\varphi)\overline{(D_i\varphi)}}{1-|\varphi|^2} - (1-|\varphi|^2) + \frac{\mu}{4}F_{\mu\nu}F^{\mu\nu} + V(|\varphi|^2)\right\} d^2x \quad , \tag{3.26}$$

where $i = 1,2$. Any static solution $\varphi(\mathbf{x})$ of the CSG model in $\mathcal{M}^3$ must satisfy eq.(3.26). We now consider the subset of those solutions that satisfy a Bogomol'nyi type bound on the energy. Such solution must satisfy en equivalent first order set of equations (the self-dual equations). Obviously while all self-dual solutions are also solutions of the CSG equations, the reverse need not be true.

3.4.1 Self-dual limit

Let us now apply the formalism developed in the previous chapter to the complex sine-Gordon model in order to establish a self-dual limit. First of all, from the action (3.23) we identify (in our previously defined notation) $h(|\varphi|^2) = 1/(1-|\varphi|^2)$. Using this together with the generalized Bogomol'nyi identity

$$h(|\varphi|^2)|\mathbf{D}\varphi|^2 = h(|\varphi|^2)|(D_1 \pm iD_2)\varphi|^2 \pm e|\varphi|^2 B \pm \nabla \times \frac{\alpha\mathbf{J}}{2} \quad , \tag{3.27}$$

allows us to rewrite the complex sine-Gordon energy (3.26) as

$$\begin{aligned} E \;&=\; \frac{1}{2}\int_{\mathbb{R}^2}\left\{\frac{2|(D_1 \pm iD_2)\varphi|^2}{1-|\varphi|^2} + \mu\Big[F_{12} \pm \frac{e}{\mu}\Big(1+\ln(1-|\varphi|^2)\Big)\Big]^2\right. \\ &+\; \left. 2V(|\varphi|^2) - 2(1-|\varphi|^2) - \frac{e^2}{\mu}\Big[1+\ln(1-|\varphi|^2)\Big]^2\right\} d^2x \mp e\Phi \quad , \end{aligned} \tag{3.28}$$

where $\Phi := \int_{\mathbb{R}^2} F_{12}\, d^2x$ is the magnetic flux through $\mathbb{R}^2$. Note that, as in the general model, we have not, as yet specified the form of the self interaction $V(|\varphi|^2)$. This permits us a degree of freedom which we now fix by requiring that the theory posses a self-dual limit. Choosing the self-interaction to be

$$V(|\varphi|^2) = \frac{e^2}{2\mu}\Big[\ln(1-|\varphi|^2)+1\Big]^2 + (1-|\varphi|^2) \quad , \tag{3.29}$$

allows us to write the energy as

$$\begin{aligned} E &= \int \frac{1}{2}\Big\{\frac{2|(D_1 \pm iD_2)\varphi|^2}{1-|\varphi|^2} + \mu\Big[F_{12} \pm \frac{e}{\mu}\Big(\ln(1-|\varphi|^2)+1\Big)\Big]^2\Big\}\, d^2x \\ &\mp e\Phi \quad . \end{aligned} \tag{3.30}$$

Since $\mu > 0$, the first term in eqn.(3.30) is positive definite in on the self-dual subspace $\Sigma_{\rm SD} = \{\varphi \in \Sigma \mid |\varphi|^2 < 1\}$ of the CSG target space Σ. For $p = 1$, this is precisely the constraint on the (standard) CSG model that permits a σ-model reformulation [33]. On the self-dual submanifold, the CSG energy is bounded from below by the magnetic flux. This gives the Bogomol'nyi-Prasad-Sommerfeld (BPS) bound on the energy: $E \geq e|\Phi|$. From eq.(3.30) we see that the BPS bound is saturated for field configurations that satisfy the self-dual equations:

$$(D_1 \pm iD_2)\varphi = 0 \tag{3.31}$$

$$F_{12} \pm \frac{e}{\mu}\Big(1+\ln(1-|\varphi|^2)\Big) = 0 \quad . \tag{3.32}$$

We now attempt to solve the above system for classical, localized, axisymmetric field configurations and as such we can (without loss of generality) normalize all arbitrary constants and write $\varphi(r) = \sqrt{\varrho(r)}e^{i\omega(\theta)}$ where (r,θ) are polar coordinates on $\mathbb{R}^2$. The first of eqs.(3.32) gives a relation between the matter field amplitude and the gauge field components as $A_i = \mp(1/2)\epsilon_{ij}\partial_j \ln\varrho - \partial_i\omega(\theta)$. We then calculate F_{12} and substitute into the second of eqs.(3.32). This gives a second order differential equation for the field amplitude

$$-\frac{1}{2}\nabla^2 \ln\varrho(r) = 1 + \ln(1-\varrho(r)) \quad , \tag{3.33}$$

Unfortunately, unlike the ungauged case, eq.(3.33) is devoid of exact analytic solutions[3]. Nevertheless, we can still extract valuable phase and asymptotic behavior from it to supplement numerical integration of the equation.

3.4.2 Fictitious Particle Interpretation

We begin our qualitative analysis of the self-dual equation, eq.(3.33) by drawing an analogy with classical mechanics. Let $\chi := \ln\varrho(r)$. Then the self-dual equation for the

[3] As far as we are aware.

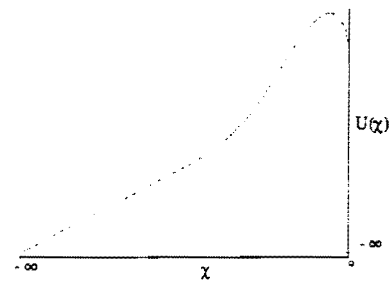


Figure 3.3: Plot of the fictitious particle potential. Note that we have scaled the potential so as to include spatial infinity in the fictitious particle space.

gauged CGS equation becomes $\chi''(r) + (1/r)\chi'(r) = -2(1 + \ln(1 - e^{\chi})) = -dU(\chi)/dr$, where $U(\chi) := \chi - \text{dilog}(1 - e^{\chi})$ and the dilogarithm function is defined as $\text{dilog(x)} := \int_1^x \ln t/(1-t)\, dt$. This may be interpreted as the equation of motion of a fictitious classical particle undergoing *damped* motion under the action of the potential $U(\chi)$. We conformally rescale $U(\chi)$ and plot it in fig.3.3. Note that in this classical mechanical analogy, the damping of the system is inversely proportional to the time evolution so we expect the particle 'energy' loss to be large early on in it's time evolution. Returning to the field theoretic system; this observation allows us to determine which of the possible field configurations have finite energy.

3.4.3 Phase Behavior

In order to determine the qualitative behavior of the solutions to eq.(3.33) we rewrite it (recalling that $\varrho = \varrho(r)$) as

$$r\varrho\frac{d^2\varrho}{dr^2} - r(\frac{d\varrho}{dr})^2 + \varrho\frac{d\varrho}{dr} = -2r\varrho^2[1 + \ln(1-\varrho)] \quad , \tag{3.34}$$

and define $t := r$, $x := \varrho$ and $y = \dot{x} := \frac{d\varrho}{dt}$. In terms of these new variables eq.(3.34) becomes

$$\begin{aligned} \dot{x} &= y \quad ; \\ \dot{y} &= -2x(1 + \ln(1-x)) + \frac{y^2}{x} - \frac{y}{t} \quad . \end{aligned} \tag{3.35}$$

This is a nonautonomous system of differential equations with fixed points at $(0,0)$ and $(1 - 1/e, 0)$. As discussed earlier, we restrict ourselves to field configurations that take values on the self-dual submanifold of the target space. This restriction effectively compactifies the phase space of the system to $\{(x,y) \in \mathbb{R}\times\mathbb{R} \mid 0 \le x < 1\}$. In the phase portrait for this system, given in fig.3.4, we include only the self-dual solution curves. From the phase portrait we are able to distinguish several interesting trajectories:

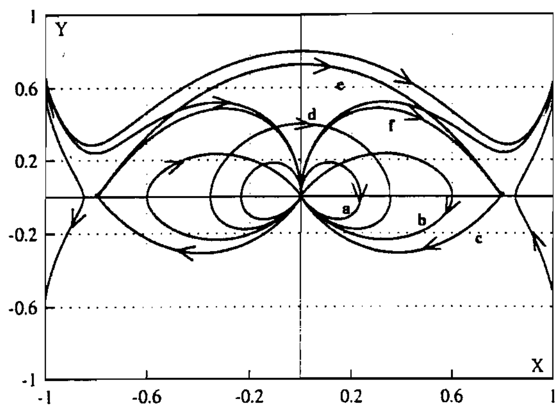


Figure 3.4: Trajectories of the dynamical system.**a**-**d**: homoclinic trajectories emanating from $(0,0)$ at $t = -\infty$ and flowing into $(0,0)$ as $t \to +\infty$. The parts of the curves with parameter values $(0 \leq t < \infty)$ describe the $n = 0$ (**d**) and $n = 2$ (**a**-**c**) lump solitons.

- Solution curves, **a-c**, that begin at $t = 0$ at points along the line $y = 0$ and end at the origin as $t \to \infty$. These curves form a continuous family for all initial x values between $x = 0$ and $x = 1 - 1/e$. Essentially, these solution curves are symmetric about $y = 0$ reflecting the fact that the dynamical system is manifestly invariant under a $\{t \to -t, y \to -y\}$ transformation. The reflected curves correspond to $r < 0$ and are inadmissible.

- Another continuous family of trajectories, **d**, that, at $t = 0$, begin at $x = 0$ with finite,nonzero y values, cross the $y = 0$ line at some finite t and terminate at (0.0) as $t \to \infty$ and

- Two special solution curves, **e** and **f**, beginning, at $t = 0$, at $x = 0$ and $y \simeq 0.702$ and $y \simeq 0.00013$ respectively and evolving to the saddle point at $(1 - 1/e, 0)$ as $t \to \infty$. The second of these trajectories is the hetroclinic trajectory joining the two fixed points of the system.

3.4.4 Asymptopia

The phase behavior of the self-dual system indicates that at least two of the three families of solutions, decay to zero amplitude at large radius while the third approaches some finite non zero value, ϱ_0 asymptotically. We now analyze each family of curves separately.

1. <u>*Rings*</u>: These solutions correspond to the family of trajectories labelled **a-c** in fig.3.4 The asymptotic forms for such ring-like solutions is easily found, from eq.(3.34) to be

$$\begin{aligned} \varrho(r) &\sim ar^n - \frac{a}{4}r^{n+2} + \ldots \quad \text{as} \quad r \to 0 \quad , \\ \varrho(r) &\sim e^{-r^2/4} \quad \text{as} \quad r \to \infty \quad . \end{aligned} \tag{3.36}$$

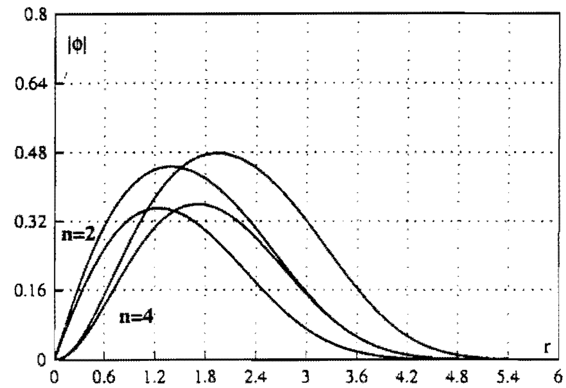


Figure 3.5: Plot of the field amplitude, $\varrho(r)$, for ring-like self-dual solutions of the gauged CSG model for $n = 2$ and $n = 4$ and various values of a.

These solutions involve two parameters, n and a, vanish for large r and are non-singular at the origin for $n \geq 1$. Pursuing our analogy with the classical particle, these solutions correspond to a particle starting at $\chi = -\infty$ at $t = 0$ with a positive velocity. As a result of its large energy losses in the early stages of its evolution, if its velocity is less than a certain critical velocity it will stop short of the top of the potential at some finite 'time', t, and roll back down to $\chi = -\infty$ as $t \to \infty$. In fig.3.5 we plot the field amplitude, $\sqrt{\varrho(r)}$, for ring-like solutions for $n = 2$ and 4 and $a = 0.5$ and 0.6.

2. <u>*Lumps*</u>: Self-dual field configurations corresponding to the family of trajectories, **d**, in fig.3.4 all have nonzero field amplitude at the origin. These solutions, plotted in fig.3.6, have the same asymptotic form as the ring-like solutions above with $n = 0$, *i.e.*,

$$\begin{aligned} \varrho(r) &\sim a - \frac{1}{4}ar^2 + \ldots \quad \text{as} \quad r \to 0 \quad , \\ \varrho(r) &\sim e^{-r^2/4} \quad \text{as} \quad r \to \infty \quad . \end{aligned} \tag{3.37}$$

In terms of the classical particle analogy, such lump-like solutions correspond to the fictitious particle beginning at some finite position on the potential hill at $t = 0$. All initial positions, χ_0 to the left (see fig.3.3) of the saddle point at $\chi_* = \ln(1 - 1/e)$ will evolve to $\chi = -\infty$ as $t \to \infty$ while any initial particle position $\chi_0 > \chi_*$ will evolve off the self-dual manifold. The former behavior corresponds to the lump solutions of the self-dual equation, eq.(3.33).

3. <u>*Vortices*</u>: The vortex solution of the model (trajectories **e** and **f** in fig.3.4) corresponds to the particle in our classical mechanical analogy beginning at $\chi_0 = -\infty$ at $t = 0$ with some critical initial velocity such that it evolves to a final state at $\chi = \chi_*$. The large r asymptotic form of this solution is determined by rewriting the self-dual equation in terms of $\chi(r)$ as

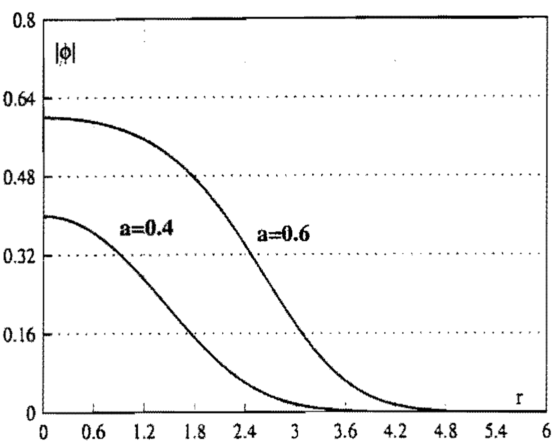


Figure 3.6: Plot of the field amplitude, $\varrho(r)$, for lump-like self-dual solutions of the gauged CSG model for $n = 0$ and various values of a.

$$\nabla^2 \chi = f(\chi) \quad , \tag{3.38}$$

where we define $f(\chi) := -2(1 + \ln(1 - e^{\chi}))$. Linearizing about the fixed point $\chi_* = 1 - 1/e$ and changing variables to $z := \chi - \chi_*$ we find that $z(r)$ obeys

$$\frac{d^2 z}{dr^2} + \frac{1}{r}\frac{dz}{dr} - \lambda^2 z = 0 \quad , \tag{3.39}$$

where we have exploited the axial symmetry of z and defined $\lambda^2 := (e - 1)$. This is easily recognized as the modified Bessel's equation of the zeroth order with solution $z(r) = AI_0(\lambda r) + BK_0(\lambda r)$, where A and B are integration constants. The boundary condition on $\varrho(r)$ imposes that $z \to 0^-$ as $r \to \infty$ which fixes $A = 0$ and $B = -k^2$ for some $k \in \mathbb{R}$. Thus , using the asymptotic form of the modified Bessel's function of the second kind, we find $\varrho(r) \sim \varrho_0 \exp\{e^{-\lambda r}/\sqrt{r}\}$ as $r \to \infty$. The behavior of the solution close to the origin is easily found to be the same as for the ring-like solutions, *i.e.* $\varrho(r) \sim ar^n - \frac{a}{4}r^{n+2} + \ldots$ as $r \to 0$. The vortex solution is plotted in fig.3.7 for $n = 2$ and 4 (see below).

3.4.5 Fluxes and energies of self-dual solutions

While the field amplitude, $\varrho(r)$, of self-dual field configurations remains nonsingular for all radial values and, the gauge field, A_i, develops a singularity at the origin for $n > 0$ as a result of the $\ln \varrho$ term:

$$A_i \to \mp \frac{n}{2r}\epsilon_{ij}\hat{r}_j - \partial_i \omega(\theta) \quad \text{as} \quad r \to 0 \quad . \tag{3.40}$$

A judicious choice of $\omega := \mp n\theta/2$, however, removes this divergence in the gauge field. The field profile for the self-dual solutions is $\varphi(r) = \sqrt{\varrho(r)}e^{\mp in\theta/2}$. It is clear also that

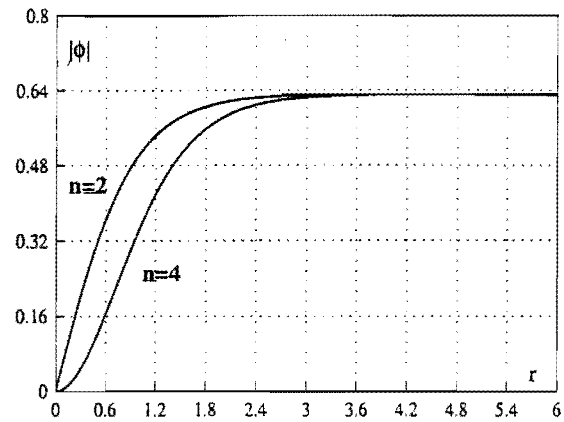


Figure 3.7: Plot of the field amplitude, $\sqrt{\varrho(r)}$, for the vortex solutions of the gauged CSG model for $n = 2$ and $n = 4$.

for φ to be single-valued when zeros of $\varrho(r)$ are encircled $n/2$ must be an integer and hence $n = 2N$, $N \in \mathbb{Z}$. These self-dual field configurations are BPS solitons. As such their energies are proportional to the flux of the magnetic field through $\mathbb{R}^2$. This is easily calculated as follows: Let C_R be a circle of sufficiently large radius, R, centered at the origin, then

$$
\begin{aligned}
\Phi &= \lim_{R\to\infty} \int_{C_R} \left(\pm\nabla^2 \ln\varrho + \nabla\times\nabla\omega\right) d^2x \\
&= \lim_{R\to\infty} \int_{\partial C_R} \left(\pm\nabla\times\ln\varrho + \nabla\omega\right)\cdot dl \quad , \qquad (3.41)
\end{aligned}
$$

where our use of Green's theorem in the last step is justified, having eliminated divergences in A_i in the integration domain. Since the field amplitude for both ring and lump solutions approach zero as Gaussian functions of the radius asymptotically far away from the origin; we need only distinguish between two cases:

1. *Rings and lumps*: In this case the dominant contribution to the flux integral, eq.(3.41), comes from the first term:

$$
\int_{\partial C_R} (\nabla\times\ln\varrho)|_{r=R}\cdot dl = -\int_{\partial C_R} \frac{d}{dr}\ln\varrho(r)\,\hat{e}_\theta|_{r=R}\cdot R\,d\theta\,\hat{e}_\theta = -\pi R^2 \quad , \qquad (3.42)
$$

   where we have used the asymptotic form, $\varrho(r) \sim e^{-r^2/4}$ for the field amplitude. The magnetic flux thus diverges quadratically in the limit as R becomes infinite. Consequently, both lumps and rings correspond to *infinite-energy* solutions.

2. *Vortices*: The vortex solution is characterized by the fact that $\varrho(r) \to \varrho_0$ far away from the origin. The first term in eq.(3.41) thus rapidly decays to zero for sufficiently large r and the contribution to the magnetic flux is dominated by the

second term. The flux for vortex solutions is calculated to be $\Phi = 2\pi n = \pi N$ for $\{N \in \mathbb{Z} \mid |N| \geq 1\}$. We interpret the integer, N, as the *vorticity* of the vortex solutions. Furthermore, as the vortex solutions saturate a BPS bound, their energy is proportional to Φ and therefore both finite and quantized.

3.4.6 Comments

A few comments are now in order. First, we note that the primary difference between the static vortices in the $(2+1)$-dimensional gauged model and those of the ungauged complex sine-Gordon model in two Euclidean dimensions is the vortex energy. As was shown in [33], the energy of the latter diverges. The above construction seems to indicate that divergences in the energy of complex sine-Gordon vortices can be controlled by coupling the nonlinear scalar field to a magnetic field. In fact, since the gauged complex sine-Gordon vortices saturate a BPS bound; the vortex energy is not only finite; it is also quantized (not unlike the Nielsen-Olesen strings in $(2+1)$-dimensional Abelian-Higgs theory). Similarly, the stability of the vortex solutions of the gauged complex sine-Gordon model is topological in nature and related to the fact that each vortex may be uniquely labelled by its 'vorticity'. That this is so is not surprising [65]. The action, eq.(3.23), is manifestly $U(1)$-invariant and from the form of the self-interaction of φ, eq.(3.29), we identify the coset space, G/H, of the gauged complex sine-Gordon as S^1. Every finite-energy non-singular solution of the field equations defines a map from S^1 (in physical space) into $G/H \simeq S^1$ (in field space). As is well known, such maps fall into homotopy classes characterized by an integer 'winding number' which effectively partitions the space of non-singular finite-energy solutions to the gauged complex sine-Gordon equations into an infinite number of disjoint sectors. The vorticity of the vortex solutions in the above construction is precisely the (topologically invariant) winding number of the field maps.

The next point of interest is that, unlike other low dimensional models that also exhibit vortices [11, 28], self-dual solutions in the complex sine-Gordon model only occur for $|\varphi| < 1$ effectively reducing the volume of physical space in which vortices arise. The reduced physical space is the image of the inverse map, $\varphi^{-1} : \Sigma_{\text{SD}} \to \mathcal{M}^3$. This restriction was also acknowledged in [33] where it was pointed out that it is a necessary condition for the exact vortex solution to render the Euclidean action a minimum. Furthermore; imposing $|\varphi| < 1$ allows one to reformulate the complex sine-Gordon theory as a σ-model on a two-dimensional surface embedded in a three-dimensional non-compact pseudo-Euclidean space. Evidently then only the self-dual submanifold, Σ_{SD}, permits interpretation as a nonlinear σ-model. Why this is so is not entirely clear to us at this stage but we postpone further analysis of the geometry of the complex sine-Gordon theory to a future publication [34].

3.5 Summary

We have reviewed the construction procedure of exact vortex solutions of the planar complex sine-Gordon equation. After rederiving the exact solution we point out, as in [33] that the vortex energy is linearly divergent.

We then proposed a mechanism to bring the vortex energies under control by coupling the sine-Gordon matter field to a $U(1)$ Maxwell gauge field. Bearing in mind the structural similarity of the complex sine-Gordon and nonlinear σ-models, we have shown that, by treating the complex sine-Gordon model as a σ-model on a 2-dimensional surface embedded in a pseudo-Euclidean 3-space (see [33] for details), we can apply the formalism developed in the previous chapter. A full phase space analysis was then carried out on the resulting Bogomol'nyi equations using the CONTENT dynamical systems package and three classes of localized solutions extracted. Two of these (the 'lump' and 'ring'-like solutions) are energetically divergent while the third corresponds to finite energy vortices. A simple homotopic argument shows that these are topologically stable solutions.

Chapter 4

The σ-model and self-dual Yang-Mills theory

Many (and perhaps all?) of the ordinary and partial differential equations that are regarded as being integrable or solvable may be obtained from the self-dual gauge field equations (or its generalisations) by reduction.

- R. Ward (1985)

We investigate the four-dimensional self-dual Yang-Mills equations. Beginning with a review of Witten's dimensional reduction of the four-dimensional Euclidean Yang-Mills action to a $(1+1)$-dimensional Abelian-Higgs model we proceed to show that the procedure may be generalized to include field theories , like the principal chiral model, with nonlinear target manifolds.

4.1 Introduction

The self-dual Yang-Mills system is arguably one of the most interesting and important multidimensional integrable systems. Initially manifest in field theory [66, 67, 68] and relativity [69, 70] it was subsequently realized that the self-dual Yang-Mills equations are quite ubiquitous. Indeed, in 1985 Ward conjectured that every completely integrable equation arises as a reduction of the self-dual Yang-Mills equations [71, 72] although at the time it was not known whether two of the most important completely integrable equations, the Korteweg-de Vries (KdV) and Kadomtsev-Petviashvili (KP) equations could be derived from the self-dual Yang-Mills equations. Later work by Mason and Sparling [73] has however added strength to the conjecture showing that for an $sl(2,\mathbb{C})$ valued gauge form in a particular gauge, the self-dual Yang-Mills equations (in lightcone coordinates) do indeed reduce to the KdV equation

$$4\partial_\tau u - \partial_\xi^3 u - 12u\partial_\xi u = 0 \quad , \tag{4.1}$$

and the nonlinear Schrödinger equation,

$$i\partial_\tau \phi = \partial_\xi^2 \phi \mp 2|\phi|^2 \phi. \tag{4.2}$$

Moreover, recent work by Strachan [74] has shown that methods developed to construct monopole solutions to the four-dimensional self-dual Yang-Mills equations [75] may be extended to extract the soliton solutions of eq.(4.2).

The last fifteen years have seen a proliferation of the number of equations that arise from the self-dual Yang-Mills system either by dimensional reduction (see for instance [36] for a development of the dimensional reduction procedure based on the gauge symmetries of the Yang-Mills system) or in the asymptotic limit. A full discussion of these reductions is beyond the scope of this thesis and we refer the interested reader to [76] and the references therein for a more complete discussion. In keeping with the spirit of the thesis we focus on the nonlinear σ-model and its "derivation" as a reduction from the self-dual Yang-Mills theory in four Euclidean dimensions. This chapter is structured as follows: In the next section we recapitulate some basic facts regarding the self-dual Yang-Mills model and, in particular, Pohlmeyer's reformulation of the Yang-Mills equations [30, 77]. Section 4.3 offers a small but significant diversion in which we look more closely at the dimensional reduction procedure pioneered by Witten [35], Forgács and Manton [36] and Taubes [37]. In doing so we will adopt the differential form notation of [37]. We then proceed to show that the nonlinear σ-model in $(1+1)$-dimensions may be derived from a dimensional reduction of the self-dual Yang-Mills system. We conclude with a proof of the equivalence of the nonlinear σ-model and the principal chiral model actions.

4.2 The self-dual Yang-Mills model

In this section we recall some basic facts about Yang-Mills gauge theory and its (anti) self-dual structure. Yang-Mills theory is essentially a study of connections on the principal bundle over some base manifold $\mathcal{M}$. The Yang-Mills gauge potential, A_μ, is identified with the connection 1-form, $A = A_\mu dx^\mu$ taking values in a Lie algebra $\mathbf{g}$ (with corresponding Lie group G)[1]. The field strength tensor corresponds to the associated curvature 2-form, $F := dA + A \wedge A$ and may be regarded as a $\mathbf{g}$-valued rank-2 skew symmetric contravariant tensor on $\mathcal{M}$. The dynamics of the Yang-Mills field is determined by the Yang-Mills action

$$S_{\mathrm{YM}} := \frac{1}{e^2} \int_{\mathcal{M}} Tr\,(F \wedge {}^*F) \quad , \tag{4.3}$$

where * is the Hodge star operator on $\mathcal{M}$ and e is a coupling constant. Variation of (4.3) with respect to $A(x)$ yields the (sourceless) Yang-Mills equations

$$D^*F = 0 \quad , \tag{4.4}$$

[1] Although this is quite general, it is also possible for A to take values in an algebra related to diffeomorphisms as in the gauge theoretic formulation of gravity.

where the gauge-covariant derivative operator is defined through its action on some 2-form Ω as $D\Omega := d\Omega + [A, \Omega]$. In terms of the curvature 2-form F the self-dual Yang-Mills equations are $F = \pm {}^*F$, where the $(-)+$ sign corresponds to the (anti self-) self-dual equations. In matrix form this becomes

$$F_{AB} = \pm\frac{1}{2}\epsilon_{ABCD}F^{CD} \quad , \tag{4.5}$$

where $F_{AB} = \partial_A A_B - \partial_B A_A - [A_A, A_B]$ and the coordinates on $\mathbb{E}^4$ are x_A $(A = 1, 2, 3, 4)$. Lightcone (null) coordinates on $\mathbb{E}^4$ are defined, as usual, by

$$x_\alpha := x_1 + ix_4, \quad x_{\bar{\alpha}} := x_1 - ix_4, \quad x_\beta := x_2 + ix_3, \quad x_{\bar{\beta}} := x_2 - ix_3 \quad , \tag{4.6}$$

so that the metric on $\mathbb{E}^4$ becomes $ds^2 = dx_\alpha dx_{\bar{\alpha}} + dx_\beta dx_{\bar{\beta}}$. Furthermore, in null coordinates,

$$\partial_\alpha := \partial_1 - i\partial_4, \quad \partial_{\bar{\alpha}} := \partial_1 + i\partial_4, \quad \partial_\beta := \partial_2 - i\partial_3, \quad \partial_{\bar{\beta}} := \partial_2 + i\partial_3 \quad , \tag{4.7}$$

and the self-dual equations (4.5) become

$$\begin{aligned} F_{\alpha\beta} &= 0 \quad , \\ F_{\bar{\alpha}\bar{\beta}} &= 0 \quad , \\ F_{\alpha\bar{\alpha}} + F_{\beta\bar{\beta}} &= 0 \quad , \end{aligned} \tag{4.8}$$

where $A_\alpha, A_{\bar{\alpha}}, A_\beta$ and $A_{\bar{\alpha}}$ are defined in a manner analogous to (4.6). As in the original Yang-Mills equations, the self-dual field equations, eqs.(4.5) and (4.9) are invariant under the gauge transformation $A \to gAg^{-1} - (dg)g^{-1}$.

For later convenience it will prove useful to recast the self-duality equations into a form due to Pohlmeyer [30]. To do this we choose $\mathbf{g} = gl(N, \mathbb{C})$ (the Lie algebra of $N \times N$ complex matrices) associated with the Lie group $Gl(N, \mathbb{C})$. We now choose the components of the connection 1-form in null coordinates as

$$A_\alpha = -\mathbf{H}^{-1}\partial_\alpha\mathbf{H}, \quad A_{\bar{\alpha}} = -\mathbf{K}^{-1}\partial_{\bar{\alpha}}\mathbf{K}, \quad A_\beta = -\mathbf{H}^{-1}\partial_\beta\mathbf{H}, \quad A_{\bar{\beta}} = -\mathbf{K}^{-1}\partial_{\bar{\beta}}\mathbf{K}, \tag{4.9}$$

where $\mathbf{H}, \mathbf{K} \in gl(N, \mathbb{C})$. In this parameterization we find after a little algebra that

$$\begin{aligned} \mathbf{K}F_{\alpha\bar{\alpha}}\mathbf{K}^{-1} &= \partial_{\bar{\alpha}}(\mathbf{K}\mathbf{H}^{-1})\partial_\alpha(\mathbf{H}\mathbf{K}^{-1}) + \mathbf{K}\mathbf{H}^{-1}\partial_{\bar{\alpha}}\partial_\alpha(\mathbf{H}\mathbf{K}^{-1}) \\ &= \partial_{\bar{\alpha}}\Big\{(\mathbf{H}\mathbf{K}^{-1})^{-1}\partial_\alpha(\mathbf{H}\mathbf{K}^{-1})\Big\} \quad . \end{aligned} \tag{4.10}$$

After a judicious choice of $\mathbf{J} := \mathbf{H}\mathbf{K}^{-1} \in gl(N, \mathbb{C})$ this yields $\mathbf{K}F_{\alpha\bar{\alpha}}\mathbf{K}^{-1} = \partial_{\bar{\alpha}}(\mathbf{J}^{-1}\partial_\alpha \mathbf{J})$. It is obvious also that a similar result holds with α replaced by β. In terms of the Pohlmeyer variables, then, the last of eq.(4.9) becomes

$$\partial_{\bar{\alpha}}(\mathbf{J}^{-1}\partial_\alpha \mathbf{J}) + \partial_{\bar{\beta}}(\mathbf{J}^{-1}\partial_\beta \mathbf{J}) = 0 \quad . \tag{4.11}$$

It is not difficult to see that the first and second of eqs.(4.9) are nothing but the compatibility conditions for the Pohlmeyer parameterization and so are identically satisfied. While this particular form of the self-dual equations proves very convenient in many of the reduction schemes (as it will in our reduction to the nonlinear σ-model) we note, following Ward [71] that having specified the gauge group (as $Gl(N, \mathbb{C})$) it is not as general as eq.(4.5) nor does it preserve the manifest $SO(4)$ invariance of the latter.

4.3 The Abelian-Higgs model and self-dual Yang-Mills theory

We now investigate the possibility of deriving the (Euclidean) Abelian-Higgs model by dimensional reduction of an $SU(2)$ Yang-Mills system on the four dimensional manifold $\mathbb{R}^2 \times S^2$ with Riemannian line element $ds^2 = dx_1^2 + dx_2^2 + \Omega^2(d\theta^2 + \sin^2\theta d\varphi^2)$. It was shown by Witten [35] and then by Forgács and Manton [36] that an $SU(2)$ Yang-Mills theory on $\mathbb{R}^4$ could be reduced to an Abelian Higgs theory on a 2-dimensional spacetime with constant negative curvature by imposing spherical symmetry on the gauge fields. This work was recently extended [78, 79] to a class of Higgs models in which the gauge field dynamics occur in the presence of some (field-dependent) dielectric medium. In the language of differential forms, the reduction procedure requires the connections for the Yang-Mills field to be invariant under a lifting of the group of rotations, $O(3)$, about a fixed line in $\mathbb{R}^4$ to an action on the bundle $\mathbb{R}^4 \times SU(2)$. It was proved by Taubes [37] that all solutions to the $SU(2)$ Yang-Mills equations in the set of C^∞ lifting-invariant connections are either self dual or anti-self dual. More importantly, this is also the case for $O(3)$ invariant solutions to the $SU(2)$ Yang-Mills equations on $\mathbb{R}^2 \times S^2$. It is instructive to first review the reduction procedure [37].

Consider the $SU(2)$ Yang-Mills equations on the 4 dimensional manifold $\mathbb{R}^2 \times S^2$ with a line element, $ds^2 = dx_1^2 + dx_2^2 + \Omega^2(d\theta^2 + \sin^2\theta d\varphi^2)$, with Ω possibly dependent on the connection components. Let $\mathcal{C}_{O(3)}$ be the set of C^∞ Yang-Mills connections invariant under a lifting of the rotation group, $O(3)$, acting on S^2 onto the principal bundle and $\{\sigma^j\}_{j=1}^3$ be the set of 2×2 Pauli matrices. Then

$$Q = i\{\cos\theta\,\sigma^3 + \sin\theta(\cos\varphi\,\sigma^1 + \sin\theta\,\sigma^2)\}, \tag{4.12}$$

is an element of the Lie algebra of $SU(2)$ with $Q^2 = -1$. Any Yang-Mills connection $A \in \mathcal{C}_{O(3)}$ has the form

$$A = \frac{1}{2}A_i dx^i Q + \frac{1}{2}(\phi_1 - 1)QdQ + \frac{1}{2}\phi_2 dQ\,, \tag{4.13}$$

where $\phi_1 = \phi_1(x_1, x_2)$, $\phi_2 = \phi_2(x_1, x_2)$ and $A(x_1, x_2)$ is a $U(1)$-valued gauge field. This gauge connection is clearly cylindrically symmetric and ϕ_i and A are independent of θ and ϕ. This will permit us to integrate out any polar degrees of freedom which are allocated to the scalar matter field. The associated curvature two-form is given by,[2]

$$\mathcal{F} = dA + \frac{1}{2}[A, A] = dA + A \wedge A\,. \tag{4.14}$$

Now, defining the curvature of the $U(1)$ sector as $F = \partial_i A_j dx^i \wedge dx^j$, we get

$$dA = \frac{1}{2}FQ + \frac{1}{2}dQ \wedge a + \frac{1}{2}Qd\phi_1 \wedge dQ + \frac{1}{2}(\phi_1 - 1)dQ \wedge dQ + \frac{1}{2}d\phi_2 \wedge dQ\,, \tag{4.15}$$

where the one-form $a = A_i dx^i$. Now

$$\begin{aligned} A \wedge A &= \frac{1}{4}\{Qa + (\phi_1 - 1)QdQ + \phi_2 dQ\} \wedge \{Qa + (\phi_1 - 1)QdQ + \phi_2 dQ\} \\ &= \frac{1}{4}\{aQ \wedge aQ + (\phi_1 - 1)^2 QdQ \wedge QdQ + (\phi_1 - 1)\phi_2 QdQ \wedge dQ + \phi_2 dQ \wedge aQ \\ &+ (\phi_1 - 1)\phi_2 dQ \wedge QdQ + \phi_2^2 dQ \wedge dQ\} \end{aligned} \tag{4.16}$$

Using the identities

$$\begin{aligned} dQ \wedge dQ &= -2Q \sin\theta d\theta \wedge d\phi & (4.17)\\ dQ \wedge QdQ &= dQ \wedge d(Q^2) - dQ \wedge (dQ)Q = -QdQ \wedge dQ & (4.18)\\ dQ \wedge aQ &= a \wedge QdQ\,, & (4.19) \end{aligned}$$

we can write this as

$$\begin{aligned} A \wedge A &= -\frac{1}{2}\phi_1 a \wedge dQ + \frac{1}{2}a \wedge dQ + \frac{1}{2}\phi_2 a \wedge QdQ + \frac{1}{4}(\phi_1^2 + \phi_2^2 + 1)dQ \wedge dQ \\ &- \frac{1}{2}\phi_1 dQ \wedge dQ \quad . \end{aligned} \tag{4.20}$$

The curvature two-form is

$$\begin{aligned} \mathcal{F} &= \frac{1}{2}FQ + \frac{1}{2}(d\phi_1 + a\phi_2) \wedge QdQ + \frac{1}{2}(d\phi_2 - a\phi_1) \wedge dQ \\ &+ \frac{1}{4}(\phi_1^2 + \phi_2^2 - 1)dQ \wedge dQ \end{aligned} \tag{4.21}$$

[2]In what follows, unless explicitly stated otherwise, late Latin indices (i,j,k)∈{1,2}; early Latin indices (a,b, etc)∈ {3,4} and upper case Latin indices (A ,B, etc) ∈ {1,2,3,4}.

To facilitate comparison with the standard formulation of the Higgs action for a charged scalar field we define the complex field $\phi = \phi_1 + i\phi_2$ and $D\phi = d\phi - ia\phi = (d\phi_1 + a\phi_2) + i(d\phi_2 - a\phi_1)$. We can now write the curvature more compactly as[3],

$$\mathcal{F} = \frac{1}{2}FQ + \frac{1}{2}\Re(D\phi) \wedge QdQ + \frac{1}{2}\Im(D\phi) \wedge dQ + \frac{1}{4}(\phi\bar{\phi} - 1)dQ \wedge dQ \tag{4.22}$$

Using,

$$^*(dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}) = \frac{|g|^{1/2}}{(n-p)!}\epsilon^{\mu_1 \ldots \mu_p}{}_{\mu_{p+1} \ldots \mu_n} dx^{\mu_{p+1}} \wedge \ldots \wedge dx^{\mu_n}, \tag{4.23}$$

where $g = det(g^{\mu\nu})$ and n is the dimensionality of the space, we get the duality relations on $\mathbb{R}^2 \times S^2$:

$$^*(dx^i \wedge dx^j) = \epsilon^{ij}\Omega^2 \sin\theta \, d\theta \wedge \varphi \tag{4.24}$$

$$^*(dx^i \wedge d\theta) = -\epsilon^{ij} \sin\theta \, dx^j \wedge d\varphi \tag{4.25}$$

$$^*(dx^i \wedge d\varphi) = \frac{1}{\sin\theta}\epsilon^{ij} \, dx^j \wedge d\theta \tag{4.26}$$

$$^*(d\theta \wedge d\varphi) = \Big(\frac{1}{2\Omega^2 \sin\theta}\Big)\epsilon_{ij} \, dx^i \wedge dx^j \tag{4.27}$$

Substituting these relations into

$$\begin{aligned} ^*\mathcal{F} &= \frac{1}{2}\partial_i A_j Q \, ^*(dx^i \wedge dx^j) + \frac{1}{2}\Big\{\Re(D_i\phi)Q\partial_\theta Q + \Im(D_i\phi)\partial_\theta Q\Big\} \, ^*(dx^i \wedge d\theta) \\ &+ \frac{1}{2}\Big\{\Re(D_i\phi)Q\partial_\varphi Q + \Im(D_i\phi)\partial_\varphi Q\Big\} \, ^*(dx^i \wedge d\varphi) \\ &+ \frac{1}{2}(1 - \phi\bar{\phi})Q \, ^*(\sin\theta \, d\theta \wedge d\varphi), \end{aligned} \tag{4.28}$$

and defining the matrix valued quantities

$$P_i = \frac{1}{2}\Big\{\Re(D_i\phi)Q\partial_\theta Q + \Im(D_i\phi)\partial_\theta Q\Big\} \tag{4.29}$$

$$R_i = \frac{1}{2}\Big\{\Re(D_i\phi)Q\partial_\varphi Q + \Im(D_i\phi)\partial_\varphi Q\Big\}, \tag{4.30}$$

we write the curvature and its dual as

[3] In this section we will denote complex conjugation by an overbar to avoid confusion with duality operation

$$
\begin{aligned}
\mathcal{F} &= \frac{1}{2}\partial_i A_j Q\, dx^i \wedge dx^j + P_i\, dx^i \wedge d\theta + R_i\, dx^i \wedge d\varphi \\
&+ \frac{1}{2}(1-\phi\bar{\phi})Q \sin\theta\, d\theta \wedge d\varphi
\end{aligned} \tag{4.31}
$$

$$
\begin{aligned}
{}^*\mathcal{F} &= \frac{1}{4\Omega^2}(1-\phi\bar{\phi})Q\epsilon^{ij}\, dx^i \wedge dx^j + \frac{1}{\sin\theta}\epsilon^{ij} R_i\, dx^j \wedge d\theta - \sin\theta\, P_i \epsilon^{ij}\, dx^j \wedge d\varphi \\
&+ \frac{1}{2}\partial_i A_j Q \Omega^2 \epsilon^{ij} \sin\theta\, d\theta \wedge d\varphi.
\end{aligned} \tag{4.32}
$$

It is then easy to see that

$$
\begin{aligned}
\mathcal{F} \wedge {}^* \mathcal{F} &= \Big(-\frac{1}{4}{F_{12}}^2 \Omega^2 \sin\theta + P_i P^i \sin\theta + \frac{1}{\sin\theta} R_i R^i \\
&- \frac{1}{4\Omega^2}(1-\phi\bar{\phi})^2 \sin\theta \Big) dx^1 \wedge dx^2 \wedge d\theta \wedge d\varphi\,.
\end{aligned} \tag{4.33}
$$

Using the definition of P_i and R_i we can evaluate the second and third terms in the above expression as follows:

$$
\begin{aligned}
P_i P^i &= -\frac{1}{4}\Big\{\Re(D_i\phi)(\sin\varphi\,\sigma^1 - \cos\varphi\,\sigma^2) + \Im(D_i\phi)(\cos\theta\,\sin\varphi\,\sigma^2 + \cos\theta\,\cos\varphi\,\sigma^1 - \sin\theta\,\sigma^3)\Big\} \\
&\Big\{\ \Re(D^i\phi)(\sin\varphi\,\sigma^1 - \cos\varphi\,\sigma^2) + \Im(D^i\phi)(\cos\theta\,\sin\varphi\,\sigma^2 + \cos\theta\,\cos\varphi\,\sigma^1 - \sin\theta\,\sigma^3)\Big\} \\
&= -\frac{1}{4}\Big\{\Re(D_i\phi)\Re(D^i\phi)(\sin\varphi\,\sigma^1 - \cos\varphi\sigma^2)^2 \\
&+ \Im(D_i\phi)\Im(D^i\phi)(\cos\theta\sin\varphi\,\sigma^2 + \cos\theta\cos\varphi\,\sigma^1 - \sin\theta\,\sigma^3)^2 \\
&+ \Re(D_i\phi)\Im(D^i\phi)(\sin\varphi\,\sigma^1 - \cos\varphi\,\sigma^2)(\cos\theta\sin\varphi\,\sigma^2 + \cos\theta\cos\varphi\,\sigma^1 - \sin\theta\,\sigma^3) \\
&+ \Re(D^i\phi)\Im(D_i\phi)(\cos\theta\sin\varphi\,\sigma^2 + \cos\theta\cos\varphi\,\sigma^1 - \sin\theta\,\sigma^3)(\sin\varphi\,\sigma^1 - \cos\varphi\,\sigma^2)\Big\} \\
&= -\frac{1}{4}\{\Re(D_i\phi)\Re(D^i\phi) + \Im(D_i\phi)\Im(D^i\phi)\} = -\frac{1}{4}(D_i\phi)\overline{(D^i\phi)}\,.
\end{aligned} \tag{4.34}
$$

A similar calculation yields $R_i R^i = -\frac{1}{4}\sin^2\theta(D_i\phi)\overline{(D^i\phi)}$.

$$
\begin{aligned}
\Rightarrow \quad \mathcal{F} \wedge {}^* \mathcal{F} &= -\Big\{\frac{1}{8}\Omega^2 F_{ij}F^{ij} + \frac{1}{2}(D_i\phi)\overline{(D^i\phi)} \\
&+ \frac{1}{4\Omega^2}(1-\phi\bar{\phi})^2\Big\} \sin\theta\, dx^1 \wedge dx^2 \wedge d\theta \wedge d\varphi\,.
\end{aligned} \tag{4.35}
$$

The Yang-Mills action for the connection A on $\mathbb{R}^2 \times S^2$ is

$$
\begin{aligned}
S[A] &\doteq -Tr \int\limits_{\mathbb{R}^2\times S^2} \mathcal{F}\wedge^* \mathcal{F} \\
&= 2\int\limits_{\mathbb{R}^2\times S^2} \{\frac{1}{8}\Omega^2 F_{ij}F^{ij} + \frac{1}{2}(D_i\phi)\overline{(D^i\phi)} \\
&+ \frac{1}{4\Omega^2}(1-\phi\bar{\phi})^2\}\sin\theta d\theta\wedge d\phi\wedge dx^1\wedge dx^2
\end{aligned}
\tag{4.36}
$$

The integration over the polar coordinates is easily carried out, giving

$$
S = 8\pi\int\limits_{\mathbb{R}^2}\{\frac{1}{8}\Omega^2 F_{ij}F^{ij} + \frac{1}{2}(D_i\phi)\overline{(D^i\phi)} + \frac{1}{4\Omega^2}(1-\phi\bar{\phi})^2\}\, dx^1\wedge dx^2\,. \tag{4.37}
$$

This is, as claimed, nothing but the action for the Abelian Higgs theory in two dimensional Euclidean space! The inherent elegance and geometric insight offered by this reduction mechanism poses the question as to whether the procedure may be extended to models other than the Abelian-Higgs model. In this regard, one of the strongest points of this Kaluza-Klein type reduction turns out to be its Achilles' heel; the constraints imposed on the Yang-Mills connection forms are *linear.* Consequently the resulting scalar fields in the reduced theory live in linear vector spaces. How then do we impose *nonlinear* constraints on the Yang-Mills fields so that the reduced low-dimensional scalar field theory takes values on a topologically nontrivial target manifold? Moreover, can this be done without breaking the gauge invariance of the Yang-Mills system while still retaining the geometrical interpretation (in terms of connections and curvature) of the above Witten-Forgács-Manton-Taubes formalism? It may be shown (for a review see [43]) that at least one class of 2-dimensional nonlinear σ-models, the principal chiral models, may be derived from a reduction of 4-dimensional Euclidean Yang-Mills theory. In the next section we review this reduction and rederive the σ-model field equations in 2-dimensional Euclidean and Minkowski space as a first step in arriving at a coherent, self-consistent unification of self-dual nonlinear σ-model type theories and self-dual Yang-Mills theory.

4.4 Reduction to the principal chiral model

Consider the self-dual Yang-Mills equations in the Pohlmeyer J-field form (4.11). Constraining $\mathrm{J}\in SU(N)$ to depend on only two of the four Euclidean coordinates, t and x say, simplifies the lightcone partial derivatives (4.7) to $\partial_{\bar{\alpha}}\to\partial_t$, $\partial_{\bar{\beta}}\to i\partial_x$, etc., where $x_\alpha = t$ and $x_\beta = ix$. As such, the Yang-Mills equation (4.11) reduces to

$$
\partial_t\Big\{\mathrm{J}^{-1}\partial_t\mathrm{J}\Big\} + \partial_y\Big\{\mathrm{J}^{-1}\partial_x\mathrm{J}\Big\} = \partial_A\Big\{\mathrm{J}^{-1}\partial_A\mathrm{J}\Big\} = 0, \quad A = t, x \quad . \tag{4.38}
$$

Equation (4.38) is, of course, nothing but the 2-dimensional chiral model field equations [43] whose finite action solutions correspond to the finite charge solutions of the nonrelativistic self-dual Chern-Simons equations [64]. Imposing $\mathrm{J}^2 = \mathbb{1}$, we expand (4.38) as

$$\partial_A \mathbf{J} \partial_A \mathbf{J} + \mathbf{J} \partial_A^2 \mathbf{J} = 0 \quad , \tag{4.39}$$

which upon using the identity $(\partial \mathbf{J})^2 = -(\mathbf{J}\partial^2 \mathbf{J} + \partial^2 \mathbf{J}\, \mathbf{J})/2$ (which derives trivially from differentiation of the constraint equation) and rescaling the independent coordinates t and x to absorb a superfluous factor of $1/2$ reduces to

$$[\Phi, \partial_A \partial^A \Phi] = 0 \quad , \tag{4.40}$$

where we have identified $\Phi = \mathbf{J}$ for convenience. In this form eq.(4.40) is readily identified as the equations of motion of the principal chiral model in Euclidean 2-space. Reduction to the principal chiral model in 2 Minkowski dimensions is carried out is exactly the same manner. Restricting $\Phi = \mathbf{J}$ to be independent of two of the Euclidean coordinates, imposing $\Phi^2 = 1$ and Wick rotating the 'time' coordinate $t \to it$, we rewrite the self-dual Yang-Mills equations (4.11) as

$$[\Phi, \partial_\mu \partial^\mu \Phi] = 0 \quad , \tag{4.41}$$

which is the Euler-Lagrange equations that yield the principal chiral model action

$$S_{\mathrm{PCM}}[\Phi] = \frac{1}{8} \int_{\mathcal{M}} g^{\mu\nu}\, Tr\left(\partial_\mu \Phi(x) \partial_\nu \Phi(x)\right) {}^*1 \quad , \tag{4.42}$$

stationary under variations of Φ. Here ^{*1} is the volume element in a 2-dimensional Minkowski spacetime, $\mathcal{M}$, with metric $g_{\mu\nu} = diag(1,-1)$. That the action (4.42) is equivalent to the defining action for the nonlinear σ-model

$$S_{\mathrm{NSM}} = \frac{1}{2} \int_{\mathcal{M}} g^{\mu\nu} \partial_\mu \varphi^a \partial_\nu \varphi^b h_{ab} {}^*1 \quad , \tag{4.43}$$

is a result often quoted in literature and an interesting exercise in its own right which we present here in the interests of making this section as self contained as possible.

To begin with, let G be a compact Lie group and G/H a compact homogeneous symmetric space with a G-invariant Riemannian structure so that there exists an involutive automorphism operator, σ, on G/H satisfying $\sigma^2 = 1$ and $\sigma \neq 1$. Using the involution operator we can represent the chiral field $\Phi(x)$ on G/H in terms of the map $g : x \to G$ as

$$\Phi(x) = g(x) \sigma g^{-1}(x) \; \in \; G/H \quad , \tag{4.44}$$

with $\Phi^2(x) = \mathbb{1}$. The chiral G/H-valued nonlinear σ model is then defined by its action (4.42). To demonstrate the equivalence of this action with (4.43) we note that, choosing canonical (normal) coordinates, any element $g \in G$ may be written as

$g = exp(\varphi \cdot c) = exp(\varphi^i c_i)$ where the $c_i = \text{Adj}(T_i)$ are the generators of G in the adjoint representation. We now need only to rewrite (4.42) in terms of the canonical coordinates $\{\varphi^i(x)\}$ to complete the proof.

Using the identity $dg^{-1} = -g^{-1}\, dg\, g^{-1}$ we find the action of the differential operator on the chiral field to be $d\Phi = g\,[g^{-1}\,dg, \sigma]\,g^{-1}$ where $[\cdot,\cdot]$ is the usual Lie bracket on G. Substituting this relation into (4.42) we evaluate the trace as:

$$\begin{aligned} Tr(d\Phi \cdot d\Phi) &= Tr(g\,[g^{-1}dg,\sigma]\,g^{-1}g\,[g^{-1}dg,\sigma]\,g^{-1}) \\ &= Tr(g\,[g^{-1}dg,\sigma]\,\mathbb{1}\,[g^{-1}dg,\sigma]\,g^{-1}) \\ &= Tr([g^{-1}dg,\sigma][g^{-1}dg,\sigma]) \quad , \end{aligned} \tag{4.45}$$

where, in the last step we have used the invariance of the trace under cyclic permutation of its arguments. To evaluate $g^{-1}dg$ we recall that [63] the inverse of the left generators of the global symmetry group G are given by

$$L^{-1}(\varphi)^i_j = \left[\frac{\mathbb{1} - e^{-\varphi\cdot c}}{\varphi \cdot c}\right] \tag{4.46}$$

which, upon Taylor expansion becomes

$$\begin{aligned} L^{-1}(\varphi)^i_j &= \left\{(\varphi\cdot c)^{-1}[\varphi\cdot c - \frac{1}{2!}(\varphi\cdot c)^2 + \frac{1}{3!}(\varphi\cdot c)^3 + \mathrm{O}(\varphi^4)]\right\}^i_j \\ &= \delta^i_j - \frac{1}{2!}\varphi^k c^i_{kj} + \frac{1}{3!}\varphi^k\varphi^l c^i_{kr} c^r_{lj} + \mathrm{O}(\varphi^3) \quad , \end{aligned} \tag{4.47}$$

where the c^i_{jk} are the structure constants of the Lie algebra $\mathbf{g}$ associated with G. Using the expression for the group elements of G in normal coordinates we can now calculate $g^{-1}dg$ in terms of the $\varphi(x)$ field as

$$\begin{aligned} g^{-1}dg &= e^{-\varphi\cdot c} de^{\varphi\cdot c} \\ &= \left\{\mathbb{1} - (\varphi\cdot c) + \frac{1}{2!}(\varphi\cdot c)^2 + \mathrm{O}(\varphi^3)\right\}\left\{d\varphi\cdot c + \frac{1}{2}(d\varphi\cdot c)(\varphi\cdot c)\right. \\ &+ \left.\frac{1}{2}(\varphi\cdot c)(d\varphi\cdot c) + \mathrm{O}(\varphi^3)\right\} \\ &= d\varphi\cdot c - \frac{1}{2!}[\varphi\cdot c, d\varphi\cdot c] + \frac{1}{3!}[\varphi\cdot c,[\varphi\cdot c, d\varphi\cdot c]] + \mathrm{O}(\varphi^4) \quad . \end{aligned} \tag{4.48}$$

The commutators in the above expression are easily seen to be $[\varphi\cdot c, d\varphi\cdot c] = d\varphi^k\varphi^j c^i_{jk} c_i$ which brings $g^{-1}dg$ into the form:

$$\begin{aligned} g^{-1}dg &= d\varphi^k\delta^i_k c_i - \frac{1}{2!}d\varphi^k\varphi^j c^i_{jk} c_i + \frac{1}{3!}d\varphi^k(\varphi\cdot c)^2{}^i_k c_i \\ &= d\varphi^k L^{-1}(\varphi)^i_k c_i \quad , \end{aligned} \tag{4.49}$$

where, in the last step, we have factored out $d\varphi^k$ and used the above Taylor-expanded form for the inverse left generators of G. Substituting (4.49) into (4.45) reduces the trace to

$$Tr(d\Phi \cdot d\Phi) = d\varphi^k d\varphi^l L^{-1}(\varphi)^i_k L^{-1}(\varphi)^j_l \, Tr([c_i, \sigma][c_j, \sigma]) \quad . \tag{4.50}$$

In the adjoint representation of G the corresponding Lie algebra $\mathbf{g}$ Cartan-decomposes as $\mathbf{g} = \mathbf{g}_- \oplus \mathbf{g}_+$ such that $[\mathbf{g}_-, \sigma] = 0 = \{\mathbf{g}_+, \sigma\}$. This, in turn, induces a Cartan splitting of the c_i into those generators that commute and those that anti-commute with the involutive automorphism operator; thus we write $c_i = c_i^- + c_i^+$. As such, we find that

$$\begin{aligned} Tr([c_i, \sigma][c_j, \sigma]) &= Tr([c_i^- + c_i^+, \sigma][c_j^- + c_j^+, \sigma]) \\ &= Tr([c_i^+, \sigma][c_j^+, \sigma]) \\ &= Tr(c_i^+ \sigma c_j^+ \sigma + \sigma c_i^+ \sigma c_j^+ - c_i^+ \sigma^2 c_j^+ - \sigma c_i^+ c_j^+ \sigma) \\ &= -4Tr(c_i^+ c_j^+) = -4\delta_{ij} \quad . \end{aligned} \tag{4.51}$$

Substituting this latter result into the expression for the Lagrangian of the chiral model (4.50) and using the identity $L^{-1}(\varphi)^i_k L^{-1}(\varphi)^j_l \delta_{ij} = h_{kl}(\varphi)$ yeilds

$$Tr(d\Phi \cdot d\Phi) = -4 d\varphi^k \, d\varphi^l \, h_{kl}(\varphi) \quad , \tag{4.52}$$

where $h_{kl}(\varphi)$ may be interpreted as the metric on the target manifold of the model. Thus, discarding an overall sign factor (which has no effect on the equations of motion anyway), the action for the chiral G/H-valued nonlinear σ-model may be written as (4.43) as claimed.

That the nonlinear σ-model and higher-dimensional theories are related should not be unexpected nor has it gone unnoticed in the recent literature [80, 81, 82, 83, 84]. In [80], for example, it was shown that starting from a five-dimensional Einstein-Hilbert action with three commuting Killing vectors, the ansatz

$$ds^2 = g_{ij}(x^k) \, dx^i dx^j + \left(2\phi_a(x^k)\phi_b(x^k) - \delta_{ab}\right) dx^a dx^b \quad , \tag{4.53}$$

where $(i, j, k = 1, 2; a, b = 3, 4, 5)$ and the $\{\phi_a\}$ take values on S^2, reduces the five-dimensional action to

$$S = -\frac{1}{2} \int d^3x \int d^2x \sqrt{|g|} \left[\frac{1}{\kappa} R + \frac{2}{\kappa} g^{ij} \partial_i \phi^a \partial_j \phi^a\right] \quad , \tag{4.54}$$

which up to a rescaling of the scalar field is recognized as the $O(3)$ σ-model repulsively coupled to gravity.

A recent result that *is* unexpected is that the G/H-valued nonlinear σ-model may be regarded as a limiting case of a Higgs model [28, 29]. Why is this unexpected? The nonlinear σ-model carries a *nonlinear* realization of G *i.e.* there exists a homomorphism $\rho : G \to Diff(G/H)$ while the Higgs model taking values in a linear vector space, V, carries a *linear* realization of G *i.e.* a homomorphism $\rho : G \to GL(V)$. Since, for some m, V is isomorphic to $\mathbb{R}^m$, $GL(V) \subset Diff(\mathbb{R}^m)$ is a subgroup. Hence a representation of G is a special case of a realization. Clearly then, a nonlinear σ-model should be a more general theory than a Higgs model. Or should it? It is known (see [29] and references therein) that (i) every nonlinear realization can be equivariantly embedded in a linear one and (ii) every manifold can be isometrically embedded into a linear space of sufficiently high dimension.

Denoting by $\Gamma(X, Y)$ the manifold of maps from X to Y such that $\Gamma \subset C^2$, the embedding $j_* : \Gamma(M, G/H) \to \Gamma(M, \mathbb{R}^m)$ defined by $j_*\varphi := j \circ \varphi$ is an equivariant and isometric map. The G-invariant Higgs model is defined by the action

$$S_{\rm H} = \frac{1}{2}\int_M \left[g^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^a + W\right]{}^*1 \quad , \tag{4.55}$$

where $W = P \circ \phi : M \to \mathbb{R}$ is the Higgs potential and P is a G-invariant polynomial of order p on $\mathbb{R}^m$. The Riemannian structure, h, on G/H is the pullback of the inner product on $\mathbb{R}^m$ by j so (see [29] for details)

$$\frac{1}{2}\int_M g^{\mu\nu}\partial_\mu(j_*(\varphi))^a\partial_\nu(j_*(\varphi))^a\,{}^*1 = \frac{1}{2}\int_M g^{\mu\nu}h_{ab}(\varphi)\partial_\mu\varphi^a\partial_\nu\varphi^b\,{}^*1 \quad , \tag{4.56}$$

which is exactly the nonlinear σ-model action if we restrict $P(y) = 0$ if $y \in G(y_0)$, $P(y) > 0$ elsewhere and assume that the image of the embedding is the orbit at which P attains its minimum. The action of the nonlinear σ-model is thus just the restriction of $S_{\rm H}$ to $im\, j_* = \Gamma(M, G(y_0)) \approx \Gamma(M, G/H)$. Physically, the nonlinear σ model can be regarded as an approximation to a linear Higgs model at energies much lower than the characteristic mass scale $\partial^2 W/\partial y^2|_{y_0}$.

4.5 Summary

In this chapter, we have studied the self-dual Yang-Mills equations in four Euclidean dimensions and its relation to the 2-dimensional nonlinear σ-model.

As a prelude, we considered the $SU(2)$ Yang-Mills equations on $\mathbb{R}^2 \times S^2$ and, following [37] imposed invariance of the Yang-Mills connection under a lifting of the rotation group, $O(3)$, about a fixed line in $\mathbb{R}^4$ to an action on the $\mathbb{R}^4 \times SU(2)$ bundle. This constraint effectively reduced the 4-dimensional Yang-Mills action to an Abelian-Higgs model on a 2-dimensional spacetime of constant negative curvature.

In order to reduce the 4-dimensional Yang-Mills equations to a σ-model we found it necessary to impose nonlinear constraints on the connection forms. Taking the self-dual Yang-Mills equations, we have recast them in null coordinates in terms of which they become eq.(4.9). Sacrificing the manifest gauge invariance of the self-dual equations, these were then rewritten in terms of Pohlmeyer's J variables. Constraints were then easily imposed on the $SU(N)$-valued J field to yield the principal chiral equations (4.41) which are derived from variation of the action (4.42). The final part of this section was devoted to a short proof of the equivalence of the principal chiral model action and the nonlinear σ-model action.

Conclusion

But I canna change the laws of physics, Captain!

- Scotty to Kirk (innumerable times).

Our aim in this thesis has been primarily exploratory. We have set out to introduce and study several low-dimensional soliton bearing field-theoretic models, most notably the nonlinear σ-model and in the course of doing so we have arrived at several, previously unnoticed results which, if nothing else, certainly merit further investigation. That we focus on the nonlinear σ-model is both obvious and justified: It is quite unique, even among nonlinear field theories in that it may be formulated directly in terms of intrinsic degrees of freedom.

Chapter 1 is devoted to the study of the integrability properties of a G/H-valued σ-model in two-dimensions. We demonstrate the existence of a Lax pair for the σ-model (for the case when the target space is homogeneous and symmetric) whose compatibility condition is exactly the σ-model field equations.This analysis hinges on the exploitation of a gauge degree of freedom in the theory. What distinguishes the gauged model in chapter 1 from that in chapter 2 is that the gauge field in the former is treated as a fixed background; consequently the nonlinear σ-model has nontrivial dynamics which corresponds to the mathematical theory of harmonic sections. The rest of chapter 1 used to give an elaboration of the standard derivation of the soliton solutions of the $O(3)$ σ-model. As was shown, these solutions (a) saturate a lower bound on the energy that is proportional to a topologically conserved quantity - the winding number - and (b) are solutions of a set of first order self-duality equations which, in this case are just the Cauchy-Riemann equations for holomorphic functions of a complex variable.

In chapter 2 we study the coupling between a G/H-valued nonlinear σ-model and a dynamical $U(1)$ gauge field. Gauging the nonlinear σ-model effectively means replacing the maps $\varphi : M \to \Sigma$ by sections of Σ-bundles over M. With this replacement comes possible new topological sectors (corresponding to nontrivialisable bundles) which must be accounted for; this amounts to an enlargement of the configuration space. We have shown that for certain two-dimensional target spaces, Σ, such gauged nonlinear σ-models exhibit self-duality in the sense that the second order Euler-Lagrange equations of motion may be reduced to an equivalent set of first order Bogomol'nyi equations. No more amenable to exact solutions in the general case, these equations do however permit a dynamical systems interpretation which allows us to extract valuable qualitative information about the field configurations. In particular, we were able to associate with the homoclinic and heteroclinic trajectories of the dynamical system, three families of localized field configurations - 'rings', 'lumps' and 'vortices'. We were able also to extend the analysis without much modification to the case where the gauge field is a topologically massive Chern-Simons field.

In all our analysis thus far, the metric on the configuration space was treated as a fixed background and hence the gravitational field was nondynamical. The simplest way to include a dynamical metric into the model in three spacetime dimensions is to conformally couple the matter field, φ, to a nonfixed gravitational field which, for the $O(3)$ σ-model, for instance, gives rise to the action

$$S = \frac{1}{2}\int d^3x\sqrt{|g|}\left[g^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^a + \lambda(\phi^a\phi^a - 1) + \frac{1}{8\pi G}R\right] \quad , \tag{4.57}$$

where R is the Ricci scalar constructed from $g_{\mu\nu}$. This model was shown [85, 86] to possess multi-soliton gravitating solitons as well as [87] multi-wormhole solutions given by the metric

$$ds^2 = dt^2 - l^2(\cosh X)^{2\kappa\nu^2}\frac{d\zeta d\bar{\zeta}}{|\zeta_1^2 - 1|} \quad . \tag{4.58}$$

This construction was shown also to extend almost trivially to a gravitating σ-model gauged with a Chern-Simons field

$$\begin{aligned} S &= \frac{1}{2}\int_M \sqrt{|g|}\left[g^{\mu\nu}D_\mu\phi^a D_\nu\phi^a + \lambda(\phi^a\phi^a - 1) + \frac{1}{8\pi G}R\right. \\ &\quad \left. - \mu\epsilon^{\mu\nu\rho}\Big(\partial_\mu\mathbf{A}_\nu\cdot\mathbf{A}_\rho + \frac{1}{3}\epsilon^{abc}A^a_\mu A^b_\nu A^c_\rho\Big)\right] d^3x \quad , \end{aligned} \tag{4.59}$$

since the Chern-Simons term is topological in nature and hence does not couple to the gravitational field. A natural question would be whether this construction may also be applied to the class of gauged σ-models introduced in chapter 2. In other words, does a gauged nonlinear σ-model of the form:

$$\begin{aligned} S &= \int_M \sqrt{|g|}\Big[h(|\varphi|^2)g^{\mu\nu}D_\mu\varphi\overline{D_\nu\varphi} - \frac{\mu}{4}g^{\mu\sigma}g^{\nu\rho}F_{\mu\nu}F_{\sigma\rho} \\ &\quad - V(|\varphi|^2) - \frac{1}{8\pi G}R\Big]\, d^{n+1}x \quad , \end{aligned} \tag{4.60}$$

admit black/worm-hole solutions for $n = 2$ or 3 and target spaces other than S^2?

In particular, we are interested in the case when the target manifold is an asymptotically conical smooth surface [33] with metric $h(|\varphi|^2) = 1/(1 - |\varphi|^2)$. This is just the gauged complex sine-Gordon model, to which chapter 3 was devoted, and which we showed exhibits finite energy BPS vortices in its self-dual limit. An essential point of our treatment of the gauged complex sine-Gordon equations is the interpretation of the self-duality equations as a dynamical system. This was the first real test of the ideas developed in chapter 2 and the extraction of finite energy vortices from this model essentially provides a mechanism for controlling the divergences in the energy of the vortices of the ungauged theory.

In chapter 4 we have pursued possible links between higher-dimensional theories and the nonlinear σ-model. We have looked in detail at two possibilities: The first begins with a Yang-Mills theory on $\mathbb{R}^2 \times S^2$ and by imposing lifting invariance on the connection 1-forms reduces the Yang-Mills action to a two-dimensional Higgs model on a space of constant negative curvature. Lifting invariance about a fixed line in $\mathbb{R}^4$ is, however, a linear constraint so the resulting low-dimensional theory has a target space that is expectedly flat. We then considered a system of self-dual Yang-Mills equations on $\mathbb{E}^4$ and showed that in null coordinates and with a Yang-Mills gauge group of $SU(N)$ these reduce naturally to the nonlinear σ-model equations so that, at least at the level of equations of motion, the nonlinear σ-model derives from the Yang-Mills system. As noted, this reduction by no means defines a unique map from Yang-Mills theory to the space of nonlinear σ-models. Indeed there has been much work devoted to exploring the connection between the Yang-Mills model and the nonlinear σ-model and some of these works have been mentioned in closing chapter 4.

Another idea that pervades this work is that of self-duality; so much so that it is the central constraint that we place on the gauged nonlinear σ-models that we have investigated in chapter 2. As shown in chapter 2, it turns out that this is a sufficient condition to determine (up to an additive constant) the scalar field self-interaction for both Maxwell and Chern-Simons gauged models. More than just a state of the system for which the second order equations of motion reduce to an equivalent first order subset; self-duality in a classical field theory points to an embedding of the theory into a supersymmetric form [53]. The BPS states that saturate the BPS bound are invariant under a nontrivial subalgebra of the full supersymmetric algebra [18] and the mass spectrum of such states is exactly determined by the supersymmetry algebra in terms of its conserved charges. This extension was recently carried out for the Maxwell-gauged $O(3)$ σ-model [88] and it was shown that the bosonic sector of that model was just the Maxwell-gauged $O(3)$ σ-model of [28]. However, as was argued in [26] and in chapter 2 of this work, the scalar field self-interaction of this model implies a ground state that is equivalent to the symmetric vacuum $\varphi = 0, A_\mu = 0$. To the best of our knowledge, then, a consistent supersymmetric extension of a $U(1)$-gauged $O(3)$ σ-model with maximal symmetry-breaking has yet to be constructed. We speculate, but have not yet proven, that such a supersymmetric extension exists for the class of self-dual σ-models considered in chapter 2. Should this indeed be the case, then not only would we have achieved the former result for the $O(3)$ model but we would also have done so for, among others, the gauged complex sine-Gordon model. These questions are currently under study and we hope to report our results in a future publication.

Appendix A

Homotopy theory

In this appendix we collect some useful results on homotopy theory. We make no claims (nor do we need) to be completely rigorous in our treatment; in fact the average algebraic topologist reading this would no doubt cringe at the lack of mathematical rigor.

We begin by introducing the concept of a path and a closed path on a manifold.

Definition A.0.1 *A path is a continuous mapping, f from a real line segment $I = [0,1]$ onto a manifold $\mathcal{M}$,*

$$f : I \to \mathcal{M} \tag{A.1}$$

The image $f(I)$ of f is a curve on $\mathcal{M}$. Two paths f_1 and f_2 on a manifold $\mathcal{M}$ are said to be *homotopic*, and denoted $f_1 \sim f_2$ if they can be continuously deformed into each other.

Definition A.0.2 *A path is called* **closed** *if the image of the corresponding map f on $\mathcal{M}$ is the same for both end points of I,*

$$f(0) = f(1) = x_0 \tag{A.2}$$

Any two closed paths[1] (or loops) are called *homotopically equivalent* if they may be continuously deformed into each other while keeping the base point x_0 unchanged. Homotopy is an equivalence relation with all closed paths in a given equivalence class being homotopic to each other. We denote by $[f]$ the equivalence class of all closed paths that are homotopic to f. As an example, consider two regions $D_1, D_2 \subset \mathbb{R}^2$ with the region D_2 containing a hole. Clearly, all closed paths in D_1 may be continuously contracted to the base point x_0 while the closed path f_1 in D_2 cannot be contracted to the base point x_0 without making a cut. The two paths f_1 and f_2 in D_2 do not fall into the same homotopy class. Since the starting points and end points of a closed path are the same point, one may define the multiplication of two loops with the same base point. Let f

[1] Note that where there is no chance of any confusion occurring, we will label paths on a manifold by the corresponding map

and g be two closed paths on a manifold $\mathcal{M}$ with the same base point x_0; the product $h := f \cdot g$ is defined as,

$$h(t) := \begin{cases} f(2t), & 0 \leq t \leq \frac{1}{2} \\ g(2t-1), & \frac{1}{2} \leq t \leq 1 \end{cases} \tag{A.3}$$

Since $h(0) = h(1) = x_0$ it is clear that the product of two loops with base point x_0 is also a loop with base point x_0. If $f \sim f'$ and $g \sim g'$ then $f \cdot g \sim f' \cdot g'$ so the product operation is homotopy preserving. For each closed path f we can also define the inverse path f^{-1} by,

$$f^{-1} := f(1-t), \quad 0 \leq t \leq 1, \tag{A.4}$$

such that the identity map $e : I \to \mathcal{M}$ is homotopic to the product of f and f^{-1}, i.e., $f \cdot f^{-1} \sim e$. The product of homotopy classes is defined as the homotopy class determined by the product of the representative elements of each of the homotopy classe: i.e.,

$$\prod_{i=1}^{N} [f_i] := \Big[\prod_{i=1}^{N} f_i\Big] \tag{A.5}$$

The identity element $[e]$ is taken as the homotopy class of identity maps e that map the interval I onto the base point x_0. Clearly then, the set of homotopy classes with base point x_0 form a *group* under group multiplication as defined above. This is called the (first)*homotopy group* on $\mathcal{M}$ with base point x_0, denoted $\Pi_1(\mathcal{M}, x_0)$. A useful result is the following theorem:

Theorem A.1 *Let $\mathcal{M}$ and $\mathcal{N}$ be two path connected manifolds. If they have the same homotopy type then their homotopy groups are isomorphic,*

$$\Pi_1(\mathcal{M}, x_0) = \Pi_1(\mathcal{N}, y_0), \quad x_0 \in \mathcal{M},\ y_0 \in \mathcal{N}. \tag{A.6}$$

A corollary to this is that the homotopy groups on a path connected manifold $\mathcal{M}$ corresponding to any two base points on $\mathcal{M}$ are isomorphic and hence depend only on the manifold. We will therefore makes sense to speak about the abstraction $\Pi_1(\mathcal{M})$ called the fundamental group of the manifold.

Definition A.0.3 *A manifold $\mathcal{M}$ is called* **simply connected** *if its fundamental group is trivial i.e.,* $\Pi_1(\mathcal{M}) = 0$

An example of a simply connected manifold is the Euclidean n-space, $\mathbb{E}^n$. The non-triviality of the fundamental group of a two dimensional path connected surface is an indication of the existence of holes in the surface. To study higher dimensional manifolds one introduces higher order homotopy groups in analogy with the fundamental group. These are denoted, $\Pi_k(\mathcal{M}, x_0)$ and are the set of equivalence classes of mappings of the k-loop $f : I_k \to \mathcal{M}$ with base point x_0, where I_k is a closed interval in $\mathbb{E}^k$. It is not difficult to show (although we omit the proof here) that

1. while the fundamental group on $\mathcal{M}$, $\Pi_1(\mathcal{M}, x_0)$ could be a non-Abelian group; all higher order homotopy groups $\Pi_k(\mathcal{M}, x_0)$, $k \geq 2$ must be Abelian.

2. for any path connected manifold $\mathcal{M}$ the kth homotopy groups, $\Pi_k(\mathcal{M}, x_0)$ and $\Pi_k(\mathcal{M}, y_0)$ with different base points are isomorphic so again we need concern ourselves with the abstract group $\Pi_k(\mathcal{M})$.

Sometimes, in order to calculate the homotopy groups on a manifold it becomes necessary to decompose the manifold in question into a product of other manifolds for which the homotopy groups are known. As such, the following theorem proves rather useful.

Theorem A.2 *The homotopy group of the product two manifolds $\mathcal{M}$ and $\mathcal{N}$ is isomorphic to the direct product of the corresponding homotopy groups of $\mathcal{M}$ and $\mathcal{N}$ i.e.,*

$$\Pi_k(\mathcal{M} \times \mathcal{N}, p \times q) = \Pi_k(\mathcal{M}, p) \otimes \Pi_k(\mathcal{N}, q) \tag{A.7}$$

where $p \in \mathcal{M}$ and $q \in \mathcal{N}$ and $\otimes$ denotes the direct product of the two groups. Although the definition of the fundamental and higher-order homotopy groups are straightforward enough, calculation of the homotopy groups of a manifold can be quite difficult as can be noted by the fact that the homotopy groups even for S^2 have not as yet been computed to all orders! What we need, therefore are some additional calculational tools with which to tackle the problem. These come in the form of the *relative homotopy group* and *exact homotopy sequence.* A complete discussion of these concepts are beyond the scope of this appendix and we refer the reader to the relevant literature [39],[40].

We now use the existence of exact sequences to derive some results that will prove useful in our search for solitons in nonlinear field theories. We know that $S^1 = \mathbb{E}^1/\mathbb{Z}$ and it can be shown that there exists an exact sequence for $k \geq 2$,

$$... \to \Pi_k(\mathbb{Z}) \to \Pi_k(\mathbb{E}^1) \to \Pi_k(S^1) \to \Pi_{k-1}(\mathbb{Z}) \to ... \tag{A.8}$$

but $\Pi_k(\mathbb{Z}) = 0 = \Pi_{k-1}(\mathbb{Z})$ so the above sequence becomes the short exact sequence,

$$0 \to \Pi_k(\mathbb{E}^1) \to \Pi_k(S^1) \to 0 \tag{A.9}$$

and we have that $\Pi_k(\mathbb{E}^1) = \Pi_k(S^1)$. However, since $\mathbb{E}^1$ is a contractible manifold, the homotopy group on $\mathbb{E}^1$ is zero to any order so,

$$\Pi_k(S^1) = 0, \quad \forall k \geq 2 \tag{A.10}$$

Similarly, for the coset space G/H of some group G there exists an exact sequence,

$$... \to \Pi_2(G) \to \Pi_2(G/H) \to \Pi_1(H) \to \Pi_1(G) \to ... \tag{A.11}$$

If G is any simple Lie group, the fact that its first and second homotopy groups are trivial ($\Pi_1(G) = \Pi_2(G) = 0$) reduces the above long exact sequence to the following short exact sequence,

$$0 \to \Pi_2(G/H) \to \Pi_1(H) \to 0 \tag{A.12}$$

which gives the group isomorphism $\Pi_2(G/H) = \Pi_1(H)$. Let G be a Lie group of transformations acting on a simply connected manifold $\mathcal{M}$ (for which $\Pi_1(\mathcal{M}) = 0$). If the action of G on $\mathcal{M}$ is *effective* then there exists a short exact sequence $0 \to G_x \to G$, where G_x is the isotropy subgroup of G corresponding to any point $x \in \mathcal{M}$. For the fundamental group $\Pi_1(\mathcal{M}/G, x)$ of the *orbit space* $\mathcal{M}/G$ there exists a homomorphism mapping from G to $\Pi_1(\mathcal{M}/G, x)$ whose kernel is G_x which gives the relation $\Pi_1(\mathcal{M}/G, x) = G/G_x$. In particular, if G acts *freely* on $\mathcal{M}$ then G_x contains only the identity element and

$$\Pi_1(\mathcal{M}/G, x) = G \tag{A.13}$$

We can now use this result to calculate the (very important) fundamental group of S^1 since $\Pi_1(S^1) = \Pi_1(\mathbb{E}^1/\mathbb{Z}) = \mathbb{Z}$. Using *this* result together with theorem (A.2) we can also, for instance, calculate the fundamental group of the torus as $\Pi_1(T^2) = \Pi_1(S^1 \times S^1) = \Pi_1(S^1) \otimes \Pi_1(S^1) = \mathbb{Z} \otimes \mathbb{Z}$.

The isomorphism between groups for any manifold $\mathcal{M}$ with a non-trivial fundamental group is realized by a characteristic number Q called the *winding number* or *topological index*. For instance, let $\theta(x)$ be the inverse stereographic projection $\mathbb{R}^1 \to S^1$ and consider the mapping,

$$f[\theta(x)] = \exp\{i\,\alpha(\theta(x))\} : S^1 \to S^1, \tag{A.14}$$

with $\alpha(2\pi) - \alpha(0) = 2\pi n$. All of these maps may be classified according to their homotopy classes $\Pi_1(S^1)$ which we have shown is isomorphic to $\mathbb{Z}$. It can be shown [89] that this isomorphism may be realized by the homotopically dynamic variable,

$$Q := \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial \alpha}{\partial \theta}\, d\theta. \tag{A.15}$$

Unfortunately, it is rather difficult, in general, to obtain an explicit expression for this topologically conserved quantity in terms of the fields of the classical field theory. One very powerful method of solving this problem was proposed by Isham in 1977 [50] based on the relationships between homotopy and cohomology groups. This algorithm is not restricted to the characteristic numbers of the fundamental group and was, in fact, successfully used to derive the known topological current (and hence charge also) in the

Skyrme model [90] which has an $SU(2)$-valued group manifold. We conclude this appendix by listing, for future reference, the lower order homotopy groups of the n-sphere S^n.

$\Pi_k \backslash S^n$	1	2	3	4	5	6	7	8	9	10	11
1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0
2	0	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0
3	0	$\mathbb{Z}$	$\mathbb{Z}$	0	0	0	0	0	0	0	0
4	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	0	0	0	0
5	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	0	0	0
6	0	$\mathbb{Z}_{12}$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	0	0
7	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}+\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	0
8	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2+\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
9	0	$\mathbb{Z}_3$	$\mathbb{Z}_3$	$\mathbb{Z}_2+\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
10	0	$\mathbb{Z}_{15}$	$\mathbb{Z}_{15}$	$\mathbb{Z}_3+\mathbb{Z}_{24}$	$\mathbb{Z}_2$	0	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
11	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

Figure A.1. Homotopy groups of S^n.

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