Complex Algebras, Varieties and Games

Hendrik Schalekamp

A dissertation prepared under the supervision of
Dr P. Jipsen to fulfill the requirements for the degree
of Master of Science in Mathematics.

Author address:

Department of Mathematics and Applied Mathematics, University of Cape Town, Rondebosch, 7700, South Africa

E-mail address: carnun@hivemind.net

Acknowledgements

I wish to express my thanks to my supervisor Dr. Peter Jipsen who has provided me with invaluable support and guidance. Without his insights, explanations and corrections this dissertation would never have come into being.

Furthermore I would like to express my appreciation to Dr. Ingrid Rewitzky for the time she spent reading through my drafts for me. I am also very greatful to her for the effort she put into the administrative requirements that allowed this work to become a reality.

To Prof. Valentin Goranko and Prof. Chris Brink my heartfelt thanks for introducing me into the wonderful field of modern mathematical logic.

I am very greatful to the South African National Research Foundation for the financial support they provided me with, and to the Mathematics Department of the University of Cape Town for employing me during the time I was working on this dissertation.

Lastly I wish to thank my friends for their moral and motivational support, and, most importantly, my family for their constant backing of this endeavor. A special word of thanks is also due to duck for the inspiration it has provided me throughout.

Contents

Introduction

Complex algebras have proven very useful in presenting the modern day logician with a tool to approach a wide variety of problems in the field of algebraic logic. This dissertation is intended as an exploration of various approaches to the study of complex algebras. In particular we will take a look at the logical and semantic views of complex algebras, as well as logical games involving these algebras.

This introduction aims to provide the reader with a cursory overview of the history of algebraic logic and in particular those parts of the history that relate to varieties, logical games and complex algebras. (For an insightful analysis of the early development of algebraic logic we refer the reader to [**AnH88**].) We round off the introduction with an overview of the work that is to be presented in this dissertation. The reader familiar with the historical development of these fields might do well to skip directly to this general overview.

1.1. History

Varieties and universal algebra. Before we launch ourselves into the history of universal algebra it might be a good idea to quickly discuss the concept universal algebra. At present this field of study, along with its sister field of algebraic logic, is one of the more active areas of research in modern mathematical logic with many implications to the fields of philosophy, natural language and computer science. The following quote, as stated in [**BuS81**], quite succinctly describes the field at hand.

> "One of the aims of universal algebra is to extract, whenever possible, the common elements of several seemingly different types of algebraic structures."

From this vantage point the researcher can come to discover general concepts that quite often lead to a concise way of communicating such ideas, but also the level of abstraction lends itself to, sometimes surprising, application of these results to disparate areas of interest.

By the beginning of the twentieth century mathematicians were confronted with a wide variety of algebraic systems, including Peano's arithmetic of natural numbers and Schröders algebra of binary relations, not to even mention the field of group theory, the study of rings, vector spaces, Hamilton's quaternians and Boolean algebras. The earliest attempt to unify the study of algebraic systems was undertaken by A. N. Whitehead in his work "A Treatise on Universal Algebra" [**Whi98**] and was focused on the study of systems of formal axiomatic reasoning about equations.

Another author to touch on the topic of putting algebraic systems into a generalized framework is B. L. van der Waerden, in his book "Moderne Algebra" [**vdW31**].

However the coherent study of generalised algebraic systems is generally accepted to start with the work of Birkhoff (c.f. [**Bir33**] and [**Bir35**]). In the first ([**Bir33**]) of these works Birkhoff explicitly introduces the modern notions of an algebra and a subalgebra. [**Bir35**] includes the ideas of congruences, free algebras and varieties. In this paper he also proves the fundamental result connecting the study of varieties with the field of equational logic, often referred to as Birkhoff's Theorem.

In their excellent book [**MMT87**] McKenzie, McNulty and Taylor divide the further development of universal algebra into four periods. The first of these last up until 1950 and includes the above mentioned papers by Birkhoff. This introductory phase covers the introduction of the basic concepts of free algebras, the isomorphism theorems, congruence lattices, and subalgebra lattices, along with the first use of this new level of abstraction to generalize results from group theory (c.f. [**Ore35**] and [**Ore36**]) and to analyse the lattice of clones on a two element set (c.f. [**Pos41**]). In this phase the first works of A. Tarski ([**Tar31**] and [**Tar35**]) studying the notion of a relational structure appears, which, as we will subsequently see, is a fertile field in the study of complex algebras. During this period lattice theory is expanded in the works of Birkhoff, Dilworth, Frink, von Neuman, Ore and Whitman to name a few. Another important result during this period comes from Stone on the representability of Boolean algebras in [**Sto36**].

During the second era, lasting until about 1963, the influence of mathematical logic, especially model theory, comes to the fore. The principal driving forces behind this period are A. Tarski and A. I. Maltsev, along with the wide array of young mathematicians attracted by the fame of their various schools. The main conceptualisation of the model theory of first-order languages, or simply model theory, is expressed in Tarski's address to the International Congress of Mathematicians in 1952. Importantly, especially from an algebraic logic perspective, this period includes the publication of two seminal works by B. Jónsson and A. Tarski (c.f. [**JoT51**] and [**JoT52**]) introducing the concept of a Boolean algebra with operators. Other highlights of this period include the ultraproduct construction of J. Łos (c.f. [**Łoś55**]); further developments on the ideas of free algebras in Poland and the ideas of A. L. Foster to extend the work of Stone on the representability of Boolean algebras to more general classes of structures. In [**Gau57**] N. D. Gautam also published an important work on the validity of equations in complex algebras that would lead to some fruitful research.

The third period sees the divergence of the fields of universal algebra and model theory and stretches from the early 1960's to the late 1970's. This era results such as the abstract characterisation of the congruence lattices of arbitrary algebras by Grätzer and Schmidt (c.f. [**GrS63**]), Jónsson's study of congruence distributive varieties (c.f. [**Jón67**]) as well as various ways of classifying varieties by the behaviors of congruences in their members, are presented. During this period several important books on the topics of the general theory of algebras were published. These include the books by Cohn [**Coh65**], Grätzer [**Grä68**], Pierce [**Pie68**] and Maltsev [**Mal73**]. Due to the earlier work of Foster the important ideas of primal and functionally complete algebras were introduced and studied during the early to late 60's. Out of these studies and the work of mathematicians like A. F. Pixley, H. Werner and R. McKenzie grew an interest in discriminator varieties. As we shall see in later chapters the existence of a discriminator in a class of algebras simplifies a lot of the results we wish to express (c.f. [**McK75**]). For an excellent survey of this field we refer the reader to the monograph [**Wer78**] of Werner.

The final era, not directly relevant to this thesis, starts with the introduction of commutator theory (c.f. [**Smi76**] and [**HaH79**]). Flowing from this theory were insights into varieties that satisfy congruence modularity as well as other strong Maltsev properties. The theory of tame congruences also belong to this period as does the hierarchy of congruence ditributive varieties (c.f. [**BlP82**] and [**BKP84**]).

Complex Algebras. Given any group **G** with universe G there is a natural way to lift the group multiplication '$\cdot$' on G to subsets A, B of G:

$$A \cdot B = \{a \cdot b : a \in A,\ b \in B\}.$$

According to G. Birkhoff (c.f. [**Bir48**]) an early application of this idea originates from Frobenius, with the multiplication of subgroups. At that time any subset of a group was referred to as a complex, and the above mentioned "lifting" of operations gave rise to a calculus of complexes.

The idea of lifting multiplication in group theory can be seen in such commonplace ideas as the coset construction. In lattice theory the set of ideals of any lattice L again forms a lattice, and if the original lattice Λ is distributive the meet and joins of our new structure turn out to be exactly the lifted meets and joins from L. Another general example derives from viewing the image of a function f as an operation on subsets.

However natural this idea of lifting operations to subsets seems, it is important to note that these lifted operations do not always satisfy the same properties as their original counterparts. For example, the lifted group operations do not in fact again give us group operations. In contrast lifting the semigroup operation will again give us semigroup operations (c.f. [**TaS67**] for more on lifted semigroups).

Of course the power set of any set itself has a natural Boolean structure, if we take set theoretic union and complementation as the Boolean join and complementation of our new algebraic structure, From an algebraic logicians perspective is it often natural to include these boolean operations in our standard signature from complex algebras, along with the other lifted operations. However, much work has been done in languages that do not contain the underlying Boolean symbols, the structures arrived at via this perspective are often referred to as power algebras, power structures or globals. For an excellent introduction to this field the reader is referred to [**Bri93**].

Given an axiomatisation of a particular class of algebras an important question has been whether an algebra in such a class can be presented as an algebra over a field of sets. Such an algebra is then referred to as a representable algebra. Since complex algebras have fields of sets as their universe this is equivalent to seeing whether an algebra is isomorphic to some complex of an algebra.

From Boolean algebras to fields of sets. As was later to be seen the study of complexes of algebras has a very close link to the study of Boolean algebras. Boolean algebras have their origin in the works by G. Boole (c.f. [**Boo54**]) and A. de Morgan (c.f. [**DeM47**]) concerned with a general analysis of thought. In these works both of the authors took a keen interest in the formalisation of the logic of propositions.

The further research of C. S. Peirce (c.f. [**Pei80**]) and E. Schröder (c.f. [**Sch95**]) led to the formation of the related field of lattice theory, which has had a great impact on the development of universal algebra.

From our current perspective the most important work in the development of the theory of Boolean algebras comes from the result of M. H. Stone demonstrating how any Boolean algebra is isomorphic to some field of sets (c.f. [**Sto36**]). This representation of a Boolean algebra by a field of sets is one of the earliest examples of representation algebras and hence a close link to the study of complex algebras. (For an in depth survey of Boolean algebras the reader is referred to [**Sik64**] and Chapter IV of [**BuS81**].)

Algebras of relations. Another field, adjacent to the study of Boolean algebra, that is related to our study here is that of relation algebra. This field evolved out of the study of the properties of binary relations on sets. The origin of this field can be traced back to the late 19th century and the works of de Morgan ([**DeM64**]), Schröder ([**Sch95**]) and Peirce ([**Pei83**]). However the modern definition of a relation algebra has its roots in Tarski's axiomatisation of the calculus of relations (c.f. [**Tar41**]). Not long hereafter the first formal definition of relation algebras, as the expansion of a *Boolean algebra*, appears (c.f. [**JoT48**]).

Since Tarski's calculus of relations could derive essentially all of the known results about sets of relations it was natural to ask whether this axiomatisation was complete. In [**Lyn50**] we start seeing the first connections between the representability of structures and their (finite) axiomatisability, and in this paper R. Lyndon showed that there exist non-representable relation algebras. In 1964 J. D. Monk (c.f. [**Mon64**]) used Lyndon's work (c.f. [**Lyn61**]) to show that there is in fact an infinite class of non-representable relation algebras with a representable ultraproduct. This then demonstrated that the class of representable relation algebras (RRA) was in fact not finitely axiomatisable. In [**JoT52**] Jónsson and Tarski demonstrated that all proper relation algebras can be represented over the class of generalised Brandt groupoids.

From Kripke semantics to modal algebras. The introduction of relational semantics, also known as Kripke frames, for certain modal logics by S. Kripke in [**Kri59**] was fundamental in the move toward a more coherent study of such logics. E. J. Lemmon and D. Scott, in their famed monograph [**LeS77**] (written in 1966), extended Kripke's work to cover a wide class of modal logics; to such an extent that they were led to conjecture that every modal logic was complete with respect to its relational semantics. This was however refuted a few years later by K. Fine (c.f. [**Fin74**]) and S. K. Thomason (c.f. [**Tho74**]).

Maybe it was the intuitiveness of Kripke's approach that lent it such popularity, in contrast to the earlier algebraic semantics introduced by J. C. C. McKinsey and A. Tarski (c.f [**McT41**] and [**McT48**]). It took the complex algebraic perspective to bring these approaches together. In particular the complex algebras of Kripke frames turn out to be modal algebras (the algebraic counterpart of modal logics). In fact, by a result of Jónsson and Tarski in [**JoT51**], every modal algebra is a subalgebra of a complex algebra of some Kripke frame.

For a survey of the algebraisation of logics we refer the reader to the excellent monograph [**BlP89**] by W. J. Blok and D. Pigozzi.

From complex algebras of relational structures to BAO*s.* The above mentioned result for modal algebras is in fact only a special case of the result by Jónsson and Tarski. Kripke frames turn out to be a special class of relational structures. As was clearly demonstrated by Jónsson and Tarski (c.f. [**JoT51**]) the complex algebras of relational structures are in fact nothing but Boolean algebras with operators. This is closely related to a result by J. F. A. K. van Benthem (c.f. [**vBe80**]), proven from a model- theoretic perspective, that if the class of Kripke frames for a modal logic is elementary then it is validated by its canonical frames.

Complex algebras of groups. As early as the works of McKinsey and Tarski, and Jónsson and Tarski the class of complex algebras of groups was discussed. In particular these complex algebras give rise to an important class of representable relation algebras called group relation algebras (GRA). As mentioned above [**Mon64**] addresses the question raised by Tarski as to whether RRA is finitely axiomatisable. In this paper Monk not only shows that RRA is non-finitely axiomatisable, but also that group relation algebras are non-finitely axiomatisable. The situation is in fact far worse than this. In [**McK70**] R. McKenzie showed that no finite set of axioms can be added to any axiomatisation for representable integral relation algebras to axiomatise GRA, i.e. GRA is not finitely axiomatisable over representable integral relation algebras. Later, in the work [**Com86**] of S. D. Comer, the connection between the complex algebras of polygroups and integral relation algebras as well as connections to multi-valued loops is evident. (We refer the reader to [**AGN97**] for even more recent results on GRA.)

Cylindric algebras. The field of cylindric algebras, closely related to the theory of first order languages, has also led to the development of many tools and insights into the study of the representability of algebras. Especially important to us since the relation algebras of Tarski and Jónsson turn out to be related to cylindric algebras. For a comprehensive survey of this question and many others relating to cylindric algebras the reader is referred to [**HMT71**] and [**HMT85**].

Logical Games. In the study of structures from a model theoretic perspective the use of formalised games have come to play a very important role. In particular the use of back-and-forth games have proven successful in studying and understanding questions relating to the representability, axiomatisation and decidability of specific classes of structures. (The reader is referred to Chapter 3 of W. Hodges' book [**Hod94**] for an excellent model theoretic survey of such games.)

A first important contribution in this line appears in the work of A. Ehrenfeucht [**Ehr61**] in which such games are used to show certain required conditions for the elementary equivalence of structures, extending work originally done by Fraïssé [**Fra52**]. Such games are thus often referred to as Ehrenfeucht-Fraïssé games.

A closely connected idea, for our purposes here, is that of the method of step-by-step construction used by R. Lyndon in [**Lyn50**] (with corrections in [**Lyn56**]). In these constructions the objects to be represented are constructed one by one in a possibly infinite sequence of steps. As is demonstrated clearly by R. Hirsch and I. Hodkinson, in their book [**HiH99**], and in the article [**HMV99**], by Hodkinson, Mikulas and Venema, this type of construction can easily be viewed as a two player game. [**HiH99**] also contains an excellent historical introduction and justification for the use of these, and similar, games in the setting of relation algebra.

1.2. General overview

We now turn to an overview of the specific theory to be covered in this dissertation. The reader will find that each chapter is presented as a conceptual unit with Chapters 2 and 3 providing us with respectively the general theory and underlying paradigm on which Chapters 4 and 5 are based. The theory discussed in Chapters 4 and 5 is then applied to a few case studies in Chapter 6. Each of the latter chapters are concluded with a short section describing open problems or avenues for further research relating to the theory covered in that chapter.

Preliminaries. We start off with an overview of the basic theory underlying complex algebras, varieties and games. Chapter 2 is a cursory introduction to fields such as *cardinal arithmetic*, *model theory*, *category theory* and the main *universal algebraic* constructions and results that will be used in later chapters. The theory in Chapter 2 should be well known to graduate students from the fields of logic, model theory and universal algebra. Most proofs are omitted, except where there are no standard references.

Chapter 2 also demonstrates the notation, mainly model theoretic and category theory based, which we will use for the rest of this work. Most of the notational conventions used are in the style of [**HiH99**] or taken from R. Goldblatt's approach (c.f. [**Gol89**] and [**Gol99**]).

Polymodal logics and their semantics. The material covered in Chapter 3 will be very familiar to modal logicians. In Chapter 3 we demonstrate the categorical connections between *modal logic*, *Kripke frames* and *algebraic semantics*. We introduce the standard notions of the *Lindenbaum-Tarski construction* and the related ideas of *canonical models* and *frames* in the general setting of n-ary modal logics, in most cases following [**Gol99**]. The interaction between a logic, its relational semantics and algebraic semantics can be viewed as a paradigm triangle. As this triangle seemingly has a plethora of notational conventions, we try to present this body of work in a uniform way by sticking close to a mixture of categorical and model theoretic notation.

In the last section of Chapter 3 we introduce the concepts of *canonical* and (strongly) *complete logics*. We then use the characterisation of these logics to motivate the definitions of canonical, complex and complete varieties.

Complex algebras and varieties. Chapter 4 focuses on the beautiful result by *Fine* [**Fin75**], *van Benthem* [**vBe80**] and *Goldblatt* [**Gol89**] connecting relational structures and canonical embedding algebras, and in this way showing us how to determine when a logic is canonical. In the lead up to this result we introduce the model theoretic concepts of *saturated models* and *good ultrafilters* which are crucial to prove this result. We present this result in the style of Jónsson (c.f. [**Jón89**]). (This exposition is intended as a complete and self contained initiation into the deep theory underpinning this result and yet still make it accessible to graduate students.)

Apart from the obvious applications of this result to algebraic logic we also use it to prove the famous *Kiesler-Shelah Theorem* and deduce several criteria for the semantic characterisation of universal and existential classes.

We conclude Chapter 4 by elaborating on the hierarchy formed by *canonical*, *complex* and *complete varieties* and prove a few new (c.f. Theorem 4.3.17 p. 85), and not so new, results about complex varieties using the tools at hand. We also introduce the reader to the work of N. D. Gautam on the *preservation of equations* between an algebra and its complex algebra (c.f. [**Gau57**]) and to some results on *discriminator varieties* that are useful in the study of boolean algebras with operators.

Complex algebras and logical games. As we have mentioned, the question of representability of certain classes of algebras is closely tied up with the construction of complex algebras. In Chapter 5 we explore this area using game theoretic techniques, i.e. *representation games.* We set up a generic game in the style of [**HMV99**] to check the representability of a structure over an equational class of algebras. The approach described takes on a less general form than that of [**HiH99**], but lends itself to more concrete examples.

We then proceed to demonstrate how these games can be used to axiomatise certain classes of structures. We conclude Chapter 5 with a discussion on two modifications of the original game presented by [**HMV99**] that extends their application to certain classes of partial algebras and to representations over products of algebras.

Case studies. In the final chapter of this dissertation we apply some of the results and techniques presented to certain classes of algebras. In particular these algebras are either subvarieties of groups or are well-known classes of expansions of groups (e.g. Boolean rings and algebras).

Semilattices and bands. In this case study we will look at an application of the theory of preserving equations, discussed in Chapter 4, to a subclass of semigroups called rectangular bands. We then present a result by P. Jipsen giving a finite axiomatisation of the variety generated by the complex algebras of rectangular bands, c.f. [**Jip01**].

Semigroups. In [**Rei96**] P. J. Reich studied the class of complex algebras of semigroups. In this short case study we present some further comments and results relating to this class of algebras.

Boolean algebras and rings. In this section we take a look at the varieties generated by complex algebras of boolean rings and boolean algebras. These varieties are respectively referred to as hyperboolean algebras (HBA) and hyperboolean rings (HBR).

The first important result we prove is that the varieties of hyperboolean algebras and hyperboolean rings are not term definitionally equivalent, although HBA is interpretable in HBR.

We also investigate whether HBA is finitely based. While this is still an open problem, we give a list of equations such that for any algebra $\mathbf{A}$ of the HBA similarity type and of size $\leq 2^4$, the equations hold if and only if $\mathbf{A} \in$ HBA. In conclusion we observe that HBR has an undecidable equational theory, whereas for HBA this is still an open problem. (Most of these results were originally published in [**Sch00**].)

Preliminaries

For the purpose of this dissertation we will assume the reader is familiar with the general theory of universal algebra as well as a basic understanding of Boolean Algebras (BA). For a more detailed account of these topics we refer the reader to [**BuS81**], [**MMT87**] and [**Sik64**]. (An updated version of [**BuS81**] is currently available on the Internet at http://www.thoralf.uwaterloo.ca/ualg.html.) Notationally we will follow [**HiH99**] as closely as possible.

2.1. Set Theory

A familiarity with the basic notions of set theory is assumed. We will use the standard notions of *membership* ($\in$), *subset* ($\subseteq$), *finite subset* ($\subseteq_\omega$), *union* ($\cup$), *intersection* ($\cap$), *difference* ($\setminus$), *products* of sets ($X \times Y, \prod_{\lambda\in\Lambda} X_\lambda$) and (*direct*) *powers* of sets (X^Λ). We define the *disjoint union*, $\biguplus_{\lambda\in\Lambda} X_\lambda$, of sets X_λ, to be the set $\bigcup_{\lambda\in\Lambda}(X_\lambda \times \{\lambda\})$. We denote the collection of all subsets of a set X by $\mathcal{P}(X)$, also called the *power set* of X, and the collection of all finite subsets by $\mathcal{P}_\omega(X)$. Ordered tuples will be presented between angle brackets. e.g. $\langle x_0, \ldots, x_{n-1}\rangle$. For a more complete introduction to set theory we refer the reader to [**Jec78**].

DEFINITION 2.1.1. A set M is said to have the *finite intersection property* if for all $x_0, x_1, \ldots, x_{n-1} \in M$ and all $n \in \omega$, $x_0 \cap x_1 \cap \ldots \cap x_{n-1} \neq \emptyset$.

2.1.1. Cardinal Numbers.

The reader is assumed to be familiar with the notions of ordinals and cardinals and their arithmetic. We will however recall a few definitions and results that are directly related to results presented in later chapters of this dissertation.

We will denote the *cardinality* of a set X by $|X|$ and define addition, multiplication and exponentiation of cardinal numbers in the standard way. If $|X| \leq \kappa$ for some cardinal κ we will say X *is of power* κ. We will use the Greek letter ω when referring to the smallest infinite cardinal.

For any cardinal κ its successor is written as κ^+. The *Generalised Continuum Hypothesis* (GCH) states that $\kappa^+ = 2^\kappa$.

We recall the following results of cardinal arithmetic. (We denote the greater of two cardinals λ and κ by $\max(\lambda, \kappa)$.)

LEMMA 2.1.2. *Let κ, λ, μ be cardinals and X be a set.*

(i) *If $\lambda \geq \omega$ and $\lambda \geq \kappa$ then $\kappa + \lambda = \kappa \cdot \lambda = \lambda$ and $\kappa^\lambda = 2^\lambda$.*
(ii) *$(\kappa^\lambda)^\mu = \kappa^{\lambda \cdot \mu}$.*
(iii) *If $|X| = \kappa$ then $|\mathcal{P}(X)| = 2^\kappa$.*
(iv) *If $\lambda, \kappa \geq \omega$ then $\lambda + \kappa = \lambda \cdot \kappa = \max(\lambda, \kappa)$.*
(v) *If $\lambda \geq \omega$ and $n < \omega$ then $\lambda^n = \lambda$.*
(vi) *If $|X| = \lambda$ and $\lambda \geq \omega$ then $|\mathcal{P}_\omega(X)| = \lambda$.*

PROOF. (i): C.f. [**Dev93**] Corollary 3.7.8 and Theorem 3.9.3.

(ii): Let X, Y and Z be sets such that $\kappa = |X|$, $\lambda = |Y|$ and $\mu = |Z|$. We need to find a function $\gamma : (X^Y)^Z \longrightarrow X^{Y \times Z}$ that is bijective. Take γ to be defined as follows, for $y \in Y$, $z \in Z$ and $f \in (X^Y)^Z$, let

$$\gamma(f)(y, z) = f(z)(y).$$

Thus $\gamma(f) \in X^{Y \times Z}$.

Let $f, g \in (X^Y)^Z$ be such that $f \neq g$. Then for some $y \in Y$ and $z \in Z$ it must be the case that $f(z)(y) \neq g(z)(y)$ so

$$\gamma(f)(y, z) = f(z)(y) \neq g(z)(y) = \gamma(g)(y, z).$$

Hence γ is injective. For any $h \in X^{Y \times Z}$ define $f_h(z) = h(\_, z)$ then clearly $f_h(z) \in X^Y$ and hence $f_h \in (X^Y)^Z$. Furthermore $\gamma(f_h) = h$ making γ surjective.

(iii): C.f. [**Dev93**] Lemma 3.9.1.

(iv): C.f. [**Jec78**] (6.1).

(v): We prove this by finite induction. Trivially $\lambda^1 = \lambda$. So let $\lambda^k = \lambda$. Since λ is infinite $\lambda^{k+1} = \lambda^k \cdot \lambda = \max(\lambda^k, \lambda) = \lambda^k = \lambda$, where the second equality follows from (iv).

(vi): Let X be a set such that $|X| = \lambda$. Then, for any $n \in \omega$, X has at most λ subsets of size n. Thus $\lambda \leq |\mathcal{P}_\omega(X)| \leq \omega \times \lambda \leq \lambda^2 = \lambda$, where the final equality follows from (iv). □

Let $\alpha > 0$ be a limit ordinal. A set $X \subseteq \alpha$ is said to be *bounded* in α if, and only if, there is a $\beta < \alpha$ such that $X \subseteq \beta$, we say X is *unbounded* in α otherwise.

Now let λ be a limit ordinal, and let $\beta_0, \beta_1, \ldots, \beta_\xi, \ldots$, with $\xi < \lambda$, be an increasing sequence of ordinals in α. We say this sequence is *cofinal* in α if, and only if, the set $\{\beta_\xi : \xi < \lambda\}$ is unbounded in α. The *cofinality* of α, denoted $\mathrm{cf}\,\alpha$, is the least limit ordinal λ such that there is an increasing λ-sequence that is cofinal in α.

LEMMA 2.1.3. *For any infinite cardinal κ, $\kappa^{\mathrm{cf}\,\kappa} > \kappa$.*

For a proof of this result we refer the reader to [**Jec78**] (Corollary 4).

THEOREM 2.1.4. *For any infinite cardinal κ, $\mathrm{cf}\,2^\kappa > \kappa$.*

PROOF. Suppose $\mathrm{cf}\,2^\kappa \leq \kappa$. Then by letting $\lambda = 2^\kappa$, we get

$$\lambda^{\mathrm{cf}\,\lambda} = \lambda^{\mathrm{cf}\,2^\kappa} \leq \lambda^\kappa = (2^\kappa)^\kappa = 2^{\kappa \cdot \kappa} = 2^\kappa = \lambda.$$

But this contradicts Lemma 2.1.3 and hence the result follows. □

This theorem and its proof is taken from [**Dev93**] (Theorem 3.9.7).

2.2. Structures

2.2.1. Syntax.

By a *signature* $L = R \uplus F$ we mean a fixed set of *relational symbols* R and *functional symbols* F (also called *operation symbols*), where each symbol $\sigma \in L$ has an associated arity $ar(\sigma) \in \omega$. We assume $ar(r) \geq 1$ for each $r \in R$. When $R = \emptyset$ we call L a *functional signature* and conversely when $F = \emptyset$ we call L a *relational signature*. A symbol $f \in F$, with $ar(f) = 0$, is called a *constant symbol*.

The *first order language* associated with a signature L is constructed by forming terms and formulas from the symbols of the signature in the usual way, using countably many variables (e.g. $\{x_0, x_1, \ldots, x_n\}$), the boolean connectives $\vee$, $\neg$ and quantifier $\exists$. The symbols $\top$, $\bot$, $\rightarrow$, $\leftrightarrow$, $\forall$ are defined by their usual abbreviations. We will generally write L for the language as well as the signature. We call a language functional if its signature is functional and relational if its signature is relational. We define (atomic) formulas and sentences in the usual way.

2.2.2. Semantics.

Each signature L has an associated class of L-*structures*, where an L-structure is an object of the form $\mathbf{M} = \langle M, \langle \sigma^{\mathbf{M}} : \sigma \in L \rangle \rangle$, for some non-empty set M (called the *domain* or *universe* of $\mathbf{M}$). For $\sigma \in L$, $\sigma^{\mathbf{M}}$ is the *interpretation* of σ in $\mathbf{M}$ - a relation or function on M with the appropriate arity, or an element of M when σ is a constant symbol. We generally drop the superscript and write σ for $\sigma^{\mathbf{M}}$ if confusion is unlikely. We will talk about a *structure* if the signature L is clear from the context.

When referring to structures we will generally use boldface letters (e.g. $\mathbf{M}$, $\mathbf{N}$, etc.), whilst standard lettering will be used for the associated domains of structures (e.g. M, N, etc.). We will associate a structure's domain with itself allowing us to write $m \in \mathbf{M}$ instead of the more correct $m \in M$. If $L = R \uplus F$ we abuse our notation somewhat and write $\mathbf{M} = \langle M, R^{\mathbf{M}}, F^{\mathbf{M}} \rangle$ instead of the more correct $\mathbf{M} = \langle M, \langle r^{\mathbf{M}} \in: r \in R \rangle, \langle f^{\mathbf{M}} : f \in F \rangle \rangle$.

A *relational* L-*structure* is an L-structure where L is a relational signature. An L-*algebra* is an L-structure $\mathbf{A} = \langle A, L^{\mathbf{A}} \rangle$ with L functional where, for each $\sigma \in L$ and each $ar(\sigma)$-tuple $\underline{a}$, $\sigma(\underline{a})$ is defined. We also refer to an L-algebra as an *algebra of type* L, similarly for relational L-structures. We generally use $\mathbf{U}$, $\mathbf{V}$ for relational structures and $\mathbf{A}$, $\mathbf{B}$ for algebras. Where we refer to classes of structures we use script lettering, e.g. $\mathcal{K}$.

Given an L-algebra $\mathbf{A}$ we can define an associated relational L'-structure $\mathbf{U}$, where $L' = \{R_f : f \in L\}$, with

$$R_f^{\mathbf{U}}(x_0, x_1, \ldots, x_{n-1}, y) \text{ iff } f^{\mathbf{A}}(x_0, x_1, \ldots, x_{n-1}) = y,$$

where $ar(f) = n$, i.e. $R_f^{\mathbf{U}}$ is the *graph* of f. We also occasionally write $r(\underline{x})$ for $r(x_0, \ldots, x_n)$ and $r(\underline{x}, y)$ for $r(x_0, \ldots, x_{n-1}, y)$ where $ar(r) = n + 1$. (Note that where convenient we will use the abbreviation "iff" for the phrase "if, and only if".)

Structures $\mathbf{M}$ and $\mathbf{N}$ are said to be *similar* if they have the same signature. If L is a *sub-signature* of L', i.e. $L \subseteq L'$, then we write $\mathbf{M}\restriction_L$ for the L-*reduct* of $\mathbf{M}$ obtained by forgetting the interpretations of symbols of $L' \setminus L$. An *expansion* of an L-structure

$\mathbf{M}$ is a structure $\mathbf{N}$ in a larger signature, of which $\mathbf{M}$ is the L-reduct. Note that $\mathbf{M}$ and $\mathbf{N}$ have the same domain. For any set $Y \subseteq \mathbf{M}$ let $L(Y)$ be the signature with constant symbols $\overline{m}$ added to L, where $m \in Y$. We denote by $\mathbf{M}_Y$ the obvious expansion of $\mathbf{M}$ to this new signature $L(Y)$, where $\overline{m}^{\mathbf{M}} = m$. When required we may write $\langle \mathbf{M}, m_0, \ldots, m_\lambda, \ldots \rangle$ for the expansion $\mathbf{M}_Y$, where $Y = \{m_0, \ldots, m_\lambda, \ldots\}$.

2.2.3. Models and validity.

For an L-formula $\phi(\underline{x})$ with $\underline{x}$ an n-tuple, an L-structure $\mathbf{M}$, and $\underline{a} \in M^n$, we write $\mathbf{M} \models \phi(\underline{a})$ and say that $\underline{a}$ *satisfies* ϕ in $\mathbf{M}$, if ϕ is true in $\mathbf{M}$ under the assignment h, where $h(x_i) = a_i$ for $i < n$. By $\Phi = \Phi(x_0, x_1, \ldots, x_{n-1})$ we generally mean a set of formulas Φ with free variables among $x_0, x_1, \ldots, x_{n-1}$. We write $\mathbf{M} \models \Phi$ if, and only if, for all assignments h each formula $\phi \in \Phi$ is true in $\mathbf{M}$ under h.

By a *theory* T we mean a set of sentences in a language L. For an L-structure $\mathbf{M}$, we say $\mathbf{M}$ is a *model* of T if $\mathbf{M} \models T$. The set of all sentences σ such that $\mathbf{M} \models \sigma$ is called the *theory of* $\mathbf{M}$ and written $\operatorname{Th}\mathbf{M}$.

THEOREM 2.2.1 (Compactness Theorem). *A theory T has a model if, and only if, every finite subset of T has a model.*

For a proof of this theorem refer to [**ChK77**] Theorem 1.3.22.

For a set $\Phi = \Phi(x_0, x_1, \ldots, x_{n-1})$ of L-formulas we say $\mathbf{M}$ *realises* Φ if, and only if, there is some $\underline{a} \in M^n$ so that $\mathbf{M} \models \phi(\underline{a})$ for every $\phi \in \Phi$. Φ is *satisfiable* in $\mathbf{M}$ exactly when $\mathbf{M}$ realises Φ. We say Φ is *finitely satisfiable* in $\mathbf{M}$ if each finite subset of Φ is satisfiable in $\mathbf{M}$.

We say Φ is *consistent* if Φ is satisfiable by some structure. A formula ϕ is *consistent with a theory* T if, and only if, there is a model $\mathbf{M}$ of T which realises $\{\phi\}$. We say Φ is *consistent with* T if, and only if, T has a model which realises Φ. A set of formulas Φ is *satisfied* in T exactly when Φ is consistent with T. Accordingly a set of formulas Φ of L consistent with a theory T is satisfiable in $\mathbf{M}$ when $\mathbf{M} \models T \cup \Phi$.

PROPOSITION 2.2.2. *Let T be a theory and let $\Phi = \Phi(x_0, x_1, \ldots, x_{n-1})$. T has a model which satisfies Φ if, and only if, every finite subset of Φ is satisfied in T.*

PROOF. The forward direction is trivial. So let us assume that each finite subset of Φ is satisfiable in some model of T. Consider an expansion of the signature with constant symbols $c_0, \ldots, c_{n-1}$ and let $S = T \cup \Phi(c_0, \ldots, c_{n-1})$. Since any finite subset of $T \cup \Phi(x_0, \ldots, x_{n-1})$ has a model, every finite subset of S must have a model in the expanded signature. Hence, by compactness, S has a model, and the reduct of this model to our original signature realises $T \cup \Phi$. □

2.2.4. Derived structures and morphisms.

Let $\mathbf{M}$ and $\mathbf{N}$ be similar structures with signature $L = R \uplus F$.

DEFINITION 2.2.3. By a *homomorphism* h from $\mathbf{M}$ to $\mathbf{N}$ we mean a function h, where $h : \mathbf{M} \longrightarrow \mathbf{N}$, with the following properties.

(i) For each $r \in R$ with $ar(r) = n + 1$ and $m_0, \ldots, m_n \in \mathbf{M}$ if $r^{\mathbf{M}}(m_0, \ldots, m_n)$ then $r^{\mathbf{N}}(h(m_0), \ldots, h(m_n))$.
(ii) $h(f^{\mathbf{M}}(m_0, \ldots, m_{n-1})) = f^{\mathbf{N}}(h(m_0), \ldots, h(m_{n-1}))$, for each $f \in F$, with $ar(f) = n$ and $m_0, \ldots, m_{n-1} \in \mathbf{M}$.

DEFINITION 2.2.4. By an *embedding* of $\mathbf{M}$ into $\mathbf{N}$ we mean an injective homomorphism $h : \mathbf{M} \longrightarrow \mathbf{N}$ which satisfies the following strengthening of (i).

(i)' For each $r \in R$ with $ar(r) = n + 1$ and $m_0, \ldots, m_n \in \mathbf{M}$ then $r^{\mathbf{M}}(m_0, \ldots, m_n)$ if, and only if, $r^{\mathbf{N}}(h(m_0), \ldots, h(m_n))$.

If $M \subseteq N$ and h is the inclusion map, we say that $\mathbf{M}$ is a *substructure* of $\mathbf{N}$, or that $\mathbf{N}$ is an *extension* of $\mathbf{M}$, written $\mathbf{M} \leq \mathbf{N}$.

An *isomorphism* is a surjective embedding. $\mathbf{M}$ and $\mathbf{N}$ are said to be *isomorphic*, written $\mathbf{M} \cong \mathbf{N}$, if there exists an isomorphism between $\mathbf{M}$ and $\mathbf{N}$.

Note that for algebras the notion of substructures agrees with that of *subalgebras*, defined in the normal way.

2.2.5. Ideals, filters, ultraproducts and ultrapowers.

We use $L_{\mathsf{BA}} = \{\vee, \sim, \mathbb{0}\}$ as the signature for Boolean algebras where $\mathbf{B}$ is a Boolean algebra if it is an L_{BA}-algebra which satisfies the standard Boolean identities. We take $\mathbb{1} = \sim\mathbb{0}$ and define $\wedge$, the boolean conjunction, in the usual way.

DEFINITION 2.2.5. Let $\mathbf{B} = \langle B, \vee, \sim, \mathbb{0}\rangle$ be a Boolean algebra.

- A *filter* $\mathcal{F}$ over $\mathbf{B}$ is a subset $\mathcal{F}$ of B such that $\mathcal{F}$ is *closed upwards* (i.e. if $t \geq s \in \mathcal{F}$ then $t \in \mathcal{F}$) and *closed under* $\wedge$ (if $s, t \in \mathcal{F}$ then $s \wedge t \in \mathcal{F}$).
- A filter, $\mathcal{F}$ is said to be *proper* if $\mathcal{F} \neq B$ and *non-trivial* if $\mathcal{F} \supsetneq \{\mathbb{1}\}$.
- For any $b \in B$ let $[b) = \{c \in B : c \geq b\}$. $[b)$ is called the *principal filter generated* by b. Any filter not of this form is called *non-principal.*
- An *ultrafilter* is a filter that is not strictly contained in any proper filter.

The dual concept to that of a filter is an ideal and is defined as follows.

DEFINITION 2.2.6. Given a Boolean algebra $\mathbf{B} = \langle B, \vee, \sim, \mathbb{0}\rangle$.

- An *ideal* $\mathcal{I}$ over $\mathbf{B}$ is a non-empty subset $\mathcal{I}$ of B such that $\mathcal{I}$ is *closed downwards* (if $t \leq s \in \mathcal{I}$ then $t \in \mathcal{I}$) and *closed under* $\vee$ (if $s, t \in \mathcal{I}$ then $s \vee t \in \mathcal{F}$).
- For any element $b \in \mathbf{B}$ the ideal $(b] = \{c \in \mathbf{B} : c \leq b\}$ is called the *principal ideal generated* by b.

We generally use script letters, such as $\mathcal{F}$ or $\mathcal{G}$, when referring to filters and $\mathcal{I}$ when referring to ideals.

It is easily seen that by complementing elements of a filter we get an ideal and vice versa.

DEFINITION 2.2.7. Let $\mathbf{B} = \langle B, \vee, \sim, \mathbb{0}\rangle$ be a Boolean algebra and $G \subseteq \mathbf{B}$. We say G has the *finite intersection property* if for any finite subset X of G we have $\bigwedge X \neq \mathbb{0}$.

By taking the upward closure of finite meets of elements of a set with the finite intersection property we then make such a set into a proper filter. Formally this gives us the following theorem.

THEOREM 2.2.8. *Let* $\mathbf{B} = \langle B, \vee, \sim, \mathbb{0}\rangle$ *be a Boolean algebra and* $G \subseteq \mathbf{B}$ *such that* G *has the finite intersection property. Then there is a proper filter* $\mathcal{G}$ *over* $\mathbf{B}$ *such that* $G \subseteq \mathcal{G}$.

We say a filter $\mathcal{F}$ *extends a filter* $\mathcal{G}$ if $\mathcal{G} \subseteq \mathcal{F}$. We refer the reader to [**Jec78**] Theorem 12 for a proof of the following result. (Note that this result requires the use of Zorn's Lemma.)

THEOREM 2.2.9. *Every proper filter over* $\mathbf{B}$ *can be extended to an ultrafilter.*

There are several ways to characterise ultrafilters, the most important of which are listed in the theorem below.

THEOREM 2.2.10. *Let* γ *be a filter over the Boolean algebra* $\mathbf{B}$. *Then* γ *is an ultrafilter over* $\mathbf{B}$ *if, and only if, one of the following conditions hold:*

(i) *For any* $a \in \mathbf{B}$ *exactly one of* a *and* $\sim a$ *belongs to* γ, *or*
(ii) $\mathbb{C} \notin \gamma$ *and for any* $a, b \in \mathbf{B}$ $a \vee b \in \gamma$ *if, and only if,* $a \in \gamma$ *or* $b \in \gamma$

We refer the reader to [**BuS81**] Theorem 3.12 and Corollary 3.13 for the proof of this result.

PROPOSITION 2.2.11. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be Boolean algebras and* $h : \mathbf{A} \longrightarrow \mathbf{B}$ *a homomorphism. If* $\mathcal{F}$ *is an ultrafilter over* $\mathbf{B}$ *then* $h^{-1}[\mathcal{F}]$ *is an ultrafilter over* $\mathbf{A}$.

PROOF. Let $\mathcal{F}$ be an ultrafilter over $\mathbf{B}$.

First we show that $h^{-1}[\mathcal{F}]$ is a filter over $\mathbf{A}$. For any $a, b \in \mathcal{F}$ if $b \geq a \in h^{-1}[\mathcal{F}]$ then $a \cdot b = a$. Since h is a homomorphism $h(a) \cdot h(b) = h(a \cdot b) = h(a)$. Whence $h(b) \geq h(a) \in \mathcal{F}$ and so, since $\mathcal{F}$ is a filter, $b \in h^{-1}[\mathcal{F}]$ as required. So consider $a, b \in h^{-1}[\mathcal{F}]$ then clearly $h(a), h(b) \in \mathcal{F}$. Hence $h(a \cdot b) = h(a) \cdot h(b) \in \mathcal{F}$. But then $a \cdot b \in h^{-1}[\mathcal{F}]$

Now let $a \in \mathbf{A}$. Since $\mathcal{F}$ is an ultrafilter over $\mathbf{B}$ either $h(a) \in \mathcal{F}$ or $\sim h(a) \in \mathcal{F}$. If $h(a) \in \mathcal{F}$, then $a \in h^{-1}[\mathcal{F}]$. Otherwise if $h(\sim a) = \sim h(a) \in \mathcal{F}$, then $\sim a \in h^{-1}[\mathcal{F}]$. So assume that $a, \sim a \in h^{-1}[\mathcal{F}]$, then $\mathbb{C} = h(a) \wedge h(\sim a) \in \mathcal{F}$ contradicting the fact that $\mathcal{F}$ is a proper filter. □

Let Λ be any non-empty index set and let $\mathcal{F}$ be an ultrafilter over the Boolean algebra $\langle \mathcal{P}(\Lambda), \cup, -, \emptyset, \Lambda \rangle$. We say $\mathcal{F}$ is an *ultrafilter over* Λ.

DEFINITION 2.2.12. Let $L = F \uplus R$ be a signature and $\mathbf{M}_\lambda$, $\lambda \in \Lambda$, be L-structures with (possibly different) domains M_λ. We can define an equivalence relation '$\cong$' over the product $M = \prod_{\lambda \in \Lambda} M_\lambda$, by

$$\underline{a} \cong \underline{b} \text{ iff } \{\lambda \in \Lambda : a_\lambda = b_\lambda\} \in \mathcal{F}$$

where $\underline{a}, \underline{b} \in M$. Let $U = \{\underline{a}/\mathcal{F} : \underline{a} \in M\}$ be the set of all equivalence classes in M modulo $\cong$. We can now define an L-structure $\mathbf{U}$ with domain U, called the *ultraproduct* of the $\mathbf{M}_\lambda$ over $\mathcal{F}$, by interpreting the L-symbols in the following way.

- For $f \in F$, we let $f^{\mathbf{U}}(a_0/\mathcal{F}, a_1/\mathcal{F}, \ldots, a_{n-1}/\mathcal{F}) = f^{\mathbf{M}}(a_0, a_1, \ldots, a_{n-1})/\mathcal{F}$,
- and for $r \in R$, we specify that $r^{\mathbf{U}}(a_0/\mathcal{F}, a_1/\mathcal{F}, \ldots, a_m/\mathcal{F})$ if, and only if, $\{\lambda \in \Lambda : r^{\mathbf{M}_\lambda}(a_0(\lambda), a_1(\lambda), \ldots, a_m(\lambda))\} \in \mathcal{F}$,

where $ar(f) = n$, $ar(r) = m + 1$ and $a_i \in \mathbf{M}$ for all i. We write $(\prod_{\lambda \in \Lambda} \mathbf{M}_\lambda)/\mathcal{F}$ or simply $\prod_{\mathcal{F}} \mathbf{M}_\lambda$ for such an ultraproduct.

If all the $\mathbf{M}_\lambda$ are isomorphic to some structure $\mathbf{M}$, then the ultraproduct over $\mathcal{F}$ is called the *ultrapower* of $\mathbf{M}$ over $\mathcal{F}$ and written $\prod_{\mathcal{F}} \mathbf{M}$ or just $\mathbf{M}^\Lambda/\mathcal{F}$.

PROPOSITION 2.2.13. *Let* $\mathbf{M}$ *be an* L*-structure and* $\mathbf{M}^\Lambda/\mathcal{F}$ *an ultrapower of* $\mathbf{M}$. *Then the map* $h : \mathbf{M} \longrightarrow \mathbf{M}^\Lambda/\mathcal{F}$ *defined by* $m \mapsto \underline{m}/\mathcal{F}$, *where* $\underline{m} = \langle m, \ldots, m, \ldots \rangle$, *is an embedding.*

The embedding h in the above proposition is often referred to as the *diagonal embedding.*

DEFINITION 2.2.14. Let $\mathcal{K}$ be any class of L-structures. The class of all ultraproducts of members of $\mathcal{K}$ is denoted $\mathbf{P_u}\mathcal{K}$, while the class of all ultrapowers of members of $\mathcal{K}$ is denoted by $\mathbf{P_w}\mathcal{K}$.

2.2.6. Elementary classes. L-structures $\mathbf{M}$ and $\mathbf{N}$ are said to be *elementarily equivalent*, written $\mathbf{M} \equiv \mathbf{N}$, if

$$\mathbf{M} \models \sigma \text{ iff } \mathbf{N} \models \sigma$$

for all L-sentences σ.

The following theorem relating ultraproducts to first-order formulas turns out to be fundamental to the solution of many problems in model theory, universal algebra and algebraic logic.

THEOREM 2.2.15 (Łos). *Let* $\mathbf{M}_\lambda$, *for* $\lambda \in \Lambda$, *be a collection of* L*-structures,* $\mathcal{F}$ *be an ultrafilter over* Λ *and* $\phi = \phi(x_0, \ldots, x_{n-1})$ *a formula of* L. *Then, for any* $a_0, \ldots, a_{n-1} \in \prod_\mathcal{F} \mathbf{M}_\lambda$,

$$\prod\nolimits_\mathcal{F} \mathbf{M}_\lambda \models \phi(a_0/\mathcal{F}, \ldots, a_{n-1}/\mathcal{F})$$
$$\textit{iff}$$
$$\{\lambda \in \Lambda : \mathbf{M}_\lambda \models \phi(a_0(\lambda), \ldots, a_{n-1}(\lambda))\} \in \mathcal{F}$$

The reader is referred to [**Hod94**] Theorem 9.5.1 for a proof of this result. We then get the following corollary.

COROLLARY 2.2.16. *For any structure* $\mathbf{M}$ *and any ultrapower* $\mathbf{M}^\Lambda/\mathcal{F}$ *of* $\mathbf{M}$

$$\mathbf{M} \equiv \mathbf{M}^\Lambda/\mathcal{F}.$$

DEFINITION 2.2.17. A class $\mathcal{K}$ of L-structures is called an *elementary class* if, and only if, there exists a theory T such that $\mathcal{K}$ is exactly the class of all models of T.

The reader might wonder how elementary equivalence and elementary classes match. By the use of the Łos Theorem we get the following characterisation of elementary classes. (For a proof of this result we refer the reader to [**ChK77**] Theorem 4.1.12(i).)

THEOREM 2.2.18. $\mathcal{K}$ *is an elementary class if, and only if,* $\mathcal{K}$ *is closed under ultraproducts and elementary equivalence.*

This characterisation gives us a link between algebraic properties and syntactic properties, we can however give a purely algebraic characterisation of elementary classes. This result, generally referred to as the Keisler-Shelah Theorem, is presented in Chapter 4 (c.f. Corollary 4.2.5 p. 79).

DEFINITION 2.2.19. An *elementary map* from M to N is a (possibly partial) map $e : \mathbf{M} \longrightarrow \mathbf{N}$ such that for every L-formula $\phi(x_0, \ldots, x_{n-1})$ and n-tuple $\underline{a} \in M^n$, we have $\mathbf{M} \models \phi(\underline{a})$ iff $N \models \phi(e(\underline{a}))$. The map e is said to be an *elementary embedding* if it is total.

DEFINITION 2.2.20. $\mathbf{N}$ is an *elementary extension* of $\mathbf{M}$, written $\mathbf{M} \preceq \mathbf{N}$, if $M \subseteq N$ and the inclusion map is an elementary embedding. $\mathbf{M}$ is referred to as an *elementary substructure* of $\mathbf{N}$.

It is easy to see that $\mathbf{M} \preceq \mathbf{N}$ implies $\mathbf{M} \leq \mathbf{N}$ and $\mathbf{M} \equiv \mathbf{N}$. In fact it turns out that for any infinite cardinal at least the cardinality of the signature of $\mathbf{N}$ there exists an elementary submodel of $\mathbf{N}$ of that power. (For a proof of this famous model theoretic result we refer the reader to [**ChK77**] Theorem 3.1.6. and [**Hod94**] Corollary 3.1 5.)

THEOREM 2.2.21 (Downward Löwenheim-Skolem Theorem). *Let L be a first order language and $\mathbf{A}$ an L-structure. Then for any set X of elements of $\mathbf{A}$ and cardinal λ such that $|L| + |X| \leq \lambda \leq |\mathbf{A}|$ there exists an elementary substructure $\mathbf{B}$ of $\mathbf{A}$ such that X is a subset of the universe of $\mathbf{B}$ and $\mathbf{B}$ is of cardinality λ.*

Clearly it follows that any infinite algebra $\mathbf{A}$ has subalgebras of all infinite powers less than $|\mathbf{A}|$. where the signature of $\mathbf{A}$ is countable.

2.3. Relational structures

We now delve a bit more into the theory of relational structures. In particular we focus on some natural ways of mapping between relational structures, and of constructing new relational structures from old ones.

DEFINITION 2.3.1. Suppose $\mathbf{U} = \langle U, R^{\mathbf{U}} \rangle$ and $\mathbf{V} = \langle V, R^{\mathbf{V}} \rangle$ are relational structures. A map $\gamma : \mathbf{U} \longrightarrow \mathbf{V}$ is called a *bounded morphism* from $\mathbf{U}$ to $\mathbf{V}$ if for all $r \in R$ with $ar(r) = n + 1$ the following conditions hold.

zig: $r^{\mathbf{U}}(x_0, \ldots, x_n)$ implies $r^{\mathbf{V}}(\gamma(x_0), \ldots, \gamma(x_n))$ for all $x_0, \ldots, x_n \in U$, and
zag: $r^{\mathbf{V}}(y_0, \ldots, y_{n-1}, \gamma(u))$ implies there exists $x_0, \ldots, x_{n-1} \in U$ such that $r^{\mathbf{U}}(x_0, \ldots, x_{n-1}, u)$ and $\gamma(x_i) = y_i$ where $y_i \in \mathrm{ran}(\gamma)$ for all $i < n$.

For any class $\mathcal{K}$ of relational structures, we denote by $\mathbf{H_b}\mathcal{K}$ the class of all structures isomorphic to bounded morphic images of members of $\mathcal{K}$.

The denotations **zig** and **zag** refer to the forwards and backwards nature of these conditions. Together they form a zig-zag between the two relational structures.

DEFINITION 2.3.2. Let $\mathbf{U} = \langle U, R^{\mathbf{U}} \rangle$ and $\mathbf{V} = \langle V, R^{\mathbf{V}} \rangle$ be relational structures with $U \subseteq V$. We say $\mathbf{U}$ is an *inner substructure* of $\mathbf{V}$ if the inclusion map from U to V is a bounded morphism.

Observe that to prove that $\mathbf{U}$ is an inner substructure of $\mathbf{V}$ it is enough to show that the inclusion map satisfies **zag**, since **zig** will be satisfied by default.

Unfortunately the use of the term *inner* substructure has become standard. In leau of the previous definition of bounded morphisms a more natural term would be *bounded* substructures, which motivates the notation $\mathbf{S_b}$ defined below.

DEFINITION 2.3.3. For any class $\mathcal{K}$ of relational structures, we denote by $\mathbf{S_b}\mathcal{K}$ the class of all structures isomorphic to inner substructures of $\mathcal{K}$.

As the reader may have noted the notation used above suggests that these definitions are in some way dual to the (algebraic) constructions of homomorphisms **H** and subalgebras **S**. The question is thus what relational construction is dual to the (direct) product of algebras. As we will later demonstrate the dual idea is that of disjoint unions of structures. (These "dualities" will be made precise in Lemma 3.1.16 p. 38.)

DEFINITION 2.3.4. Let Λ be some index set and $\{\mathbf{U}_\lambda : \lambda \in \Lambda\}$ a collection of relational structures. The *disjoint union* $\mathbf{U}$ *of the structures* $\mathbf{U}_\lambda$ has as universe $U = \biguplus_{\lambda\in\Lambda} U_\lambda$ and relations defined by

$$r^{\mathbf{U}}((u_0, \lambda), \ldots, (u_n, \lambda)) \text{ iff } r^{\mathbf{U}_\lambda}(u_0, \ldots, u_n)$$

where $\lambda \in \Lambda$, $ar(r) = n + 1$ and $u_i \in U_\lambda$ for $i \leq n$. In particular $r^{\mathbf{U}}(u_0, \ldots, u_n) \neq \emptyset$ iff, for all $i \leq n$, $u_i \in \mathbf{U}_\lambda$ for a fixed λ.

For any class $\mathcal{K}$ of relational structures, we denote by $\mathbf{U_d}\mathcal{K}$ the class of all structures isomorphic to disjoint unions of members of $\mathcal{K}$.

2.4. Algebras

Similarly to the definitions presented above we can define an extension of the idea of homomorphisms for algebras.

DEFINITION 2.4.1. Suppose $\mathbf{A} = \langle A, F^{\mathbf{A}}\rangle$ and $\mathbf{B} = \langle B, F^{\mathbf{B}}\rangle$ are algebras. A map $\gamma : \mathbf{A} \longrightarrow \mathbf{B}$ is called a *bounded morphism* from $\mathbf{A}$ to $\mathbf{B}$ if for all $f \in F$ with $ar(r) = n$ the following conditions hold.

zig: f is a homomorphism, and
zag: $f^{\mathbf{B}}(b_0, \ldots, b_{n-1}) = \gamma(a)$ implies there exists $a_0, \ldots, a_{n-1} \in A$ such that $f^{\mathbf{A}}(a_0, \ldots, a_{n-1}) = a$ and $\gamma(a_i) = b_i$ where $b_i \in \mathrm{ran}(\gamma)$ for all $i < n$.

For any class $\mathcal{K}$ of algebras, we denote by $\mathbf{H_b}\mathcal{K}$ the class of all algebras isomorphic to bounded morphic images of members of $\mathcal{K}$.

DEFINITION 2.4.2. Let $\mathbf{A} = \langle A, F^{\mathbf{A}}\rangle$ and $\mathbf{B} = \langle B, F^{\mathbf{B}}\rangle$ be algebras with $A \subseteq B$. We say $\mathbf{A}$ is an *inner subalgebra* of $\mathbf{B}$ if the inclusion map from A to B is a bounded morphism.

Note that removing the **zag** condition from the above definition of inner subalgebras would leave us with the standard definition of a subalgebra.

2.4.1. Term Algebras. One fundamental way to construct an algebra is by taking a set of variables and then using the language of the algebra to generate all "legitimate" strings that can be formed using the symbols in this language. This so-called term algebra together with congruences, the algebraic counterpart of equivalence relations, play an important role in capturing the essence of classes of algebras, as will become clear in the sequel.

DEFINITION 2.4.3. Let X be a set of variables and L a functional signature. The *set of terms of type* L *over* X, denoted $\mathrm{Term}(X)$, is the smallest set such that

(i) $X \cup \{f \in L : ar(f) = 0\} \subseteq \mathrm{Term}(X)$, and
(ii) if $f \in L$, with $ar(f) > 0$, and $x_0, \ldots, x_{ar(f)-1} \in \mathrm{Term}(X)$ then the "string" $f(x_0, \ldots, x_{ar(f)-1}) \in \mathrm{Term}(X)$.

Note that a more precise notation would be $\mathrm{Term}_L(X)$, but in general the language L will be clear from the context.

DEFINITION 2.4.4. Given a functional signature L and a set or variables X, if the set of terms over X is non-empty we define $\mathbf{Term}(X) = \langle \mathrm{Term}(X), L^{\mathbf{Term}(X)} \rangle$, the *term algebra of type L over X*, by

$$f^{\mathbf{Term}(X)}(x_0, \ldots, x_{n-1}) = f(x_0, \ldots, x_{n-1})$$

for $f \in L$, $ar(f) = n$ and $x_0, \ldots, x_{n-1} \in X$. $\mathbf{Term}(X)$ is sometimes referred to as the *absolutely free algebra of type L over X*.

2.4.2. Congruences on algebras. Another natural way to form new algebras is from old ones using the quotient algebra construction. In such an algebra the universe is made up by dividing our original algebra into separate classes using an equivalence relation. As we shall see, for this construction to make sense, we need these equivalence relations to respect the fundamental operations of our algebra. This type of equivalence relation is called a congruence and defined as follows.

DEFINITION 2.4.5. Let $\mathbf{A}$ be an L-algebra and let θ be an equivalence relation on $\mathbf{A}$. Then θ is a *congruence* on $\mathbf{A}$ if θ satisfies the following property for each $f \in L$, with $ar(f) = n$ and elements $a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1} \in \mathbf{A}$,

$$\langle a_i, b_i \rangle \in \theta \text{ for } i < n \text{ implies } \langle f^{\mathbf{A}}(a_0, \ldots, a_{n-1}), f^{\mathbf{A}}(b_0, \ldots, b_{n-1}) \rangle \in \theta.$$

The *set of all congruences on an algebra* $\mathbf{A}$ is denoted $\mathrm{Con}\mathbf{A}$. For $\theta \in \mathrm{Con}\mathbf{A}$, A/θ is the set of all the equivalence classes $a/\theta = \{b : \langle b, a \rangle \in \theta\}$, for $a \in A$.

It can be shown that $\mathrm{Con}\mathbf{A}$ forms a complete lattice (c.f. [**BuS81**] Theorem 5.3). Thus we sometimes refer to $\mathrm{Con}\mathbf{A}$ as the *congruence lattice* of $\mathbf{A}$.

To construct our quotient algebra we then only need to specify how the functions of our language are interpreted over the equivalence classes of our original algebra.

DEFINITION 2.4.6. Let $\mathbf{A}$ be an L-algebra and $\theta \in \mathrm{Con}\mathbf{A}$. The *quotient algebra of $\mathbf{A}$ by θ*, denoted $\mathbf{A}/\theta$, is the algebra with universe A/θ and for each $f \in L$, with $ar(f) = n$,

$$f^{\mathbf{A}/\theta}(a_0/\theta, \ldots, a_{n-1}/\theta) = f^{\mathbf{A}}(a_0, \ldots, a_{n-1})/\theta$$

where $a_0, \ldots, a_{n-1} \in \mathbf{A}$.

We conclude this section with a result from [**BuS81**] (Theorem 3.5).

THEOREM 2.4.7. *Let $\mathbf{B}$ be a Boolean algebras and θ a binary relation on $\mathbf{B}$. θ is a congruence if, and only if,*

(i) *$\mathbb{C}/\theta$ is an ideal, and*
(ii) *$a\theta b$ iff $a \vee b \in \mathbb{C}/\theta$.*

For this reason we call an ideal $\mathcal{I}$ a *congruence ideal* over, a Boolean algebra, $\mathbf{B}$ if, for some congruence θ over $\mathbf{B}$, $\mathcal{I} = \mathbb{0}/\theta$.

2.4.3. Subdirectly irreducible algebras. Given an algebra $\mathbf{A}$ we may ask if there are some "foundational" algebras we can deconstruct $\mathbf{A}$ into. The quest for these general building blocks in the field of universal algebra led Birkhoff to consider certain special products.

DEFINITION 2.4.8. We say call an algebra $\mathbf{A}$ the *subdirect product* of a set of algebras $\{\mathbf{A}_\lambda : \lambda \in \Lambda\}$ if the following conditions hold.

(i) $\mathbf{A}$ is a subalgebra of $\prod \mathbf{A}_\lambda$, and
(ii) $\pi_\lambda(\mathbf{A}) = \mathbf{A}_\lambda$,

where π_λ is the λth projection function.

An embedding $h : \mathbf{A} \longrightarrow \prod \mathbf{A}_\lambda$ is called a *subdirect embedding* if $h(\mathbf{A})$ is a subdirect product of the $\mathbf{A}_\lambda$.

We are then ready to define these "building blocks". (We denote the *composition* of two maps $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ by $g \circ f$.)

DEFINITION 2.4.9. An algebra $\mathbf{A}$ is *subdirectly irreducible* if for every subdirect embedding $h : \mathbf{A} \longrightarrow \prod \mathbf{A}_\lambda$ there exists a $\lambda \in \Lambda$ such that $\pi_\lambda \circ h : \mathbf{A} \longrightarrow \mathbf{A}_\lambda$ is an isomorphism.

Before we get to clarifying why these algebras form our required building blocks we give a characterisation of subdirectly irreducible algebras in terms of Con$\mathbf{A}$. This characterisation ends up being one of the most useful criteria of distinguishing subdirectly irreducible algebras. (A proof of the following result can be found in [**BuS81**] Theorem 8.4.)

THEOREM 2.4.10. *An algebra* $\mathbf{A}$ *is subdirectly irreducible if, and only if,* $\mathbf{A}$ *is trivial or there is a minimum congruence in* Con$\mathbf{A} \setminus \{\mathbb{0}\}$, *where* $\mathbb{0}$ *is the trivial congruence.*

In the case where $\mathbf{A}$ is a nontrivial subdirectly irreducible algebra it turns out that this minimum congruence is unique and all nontrivial congruences lie above it. We refer to these minimum congruences of subdirectly irreducible algebras as *monoliths.* We say an algebra $\mathbf{A}$ is *simple* if the monolith is equal to the top element of Con$\mathbf{A}$, i.e. $|\text{Con}\mathbf{A}| = 2$.

The next theorem is due to G. Birkhoff (c.f. [**Bir44**]). For a proof of this theorem we refer the reader to [**MMT87**] Theorem 4.44.

THEOREM 2.4.11 (Subdirect Representation Theorem). *Every algebra* $\mathbf{A}$ *is isomorphic to a subdirect product of subdirectly irreducible algebras.*

2.4.4. Free algebras. As subdirectly irreducible algebras provide us with the fundamental structures to construct algebras from so are free algebras the fundamental structures that capture the equational properties of algebras, as will be made precise in the sequel.

DEFINITION 2.4.12. Let $\mathcal{K}$ be a class of algebras. $\mathbf{B}$ an algebra and X a subset of $\mathbf{B}$ that generates $\mathbf{B}$. We say that $\mathbf{B}$ has the *universal mapping property over* $\mathcal{K}$ *for* X if, and only if, for any $\mathbf{A} \in \mathcal{K}$ and every mapping $f : X \longrightarrow \mathbf{A}$, there is a homomorphism $h : \mathbf{B} \longrightarrow \mathbf{A}$ that extends f (i.e. $f(x) = h(x)$ for any $x \in X$). $\mathbf{B}$ is *free over* $\mathcal{K}$ *for* X if $\mathbf{B}$ has the universal mapping property over $\mathcal{K}$ for X. $\mathbf{B}$ is *free in* $\mathcal{K}$ *for* X if additionally we have $\mathbf{B} \in \mathcal{K}$.

If an algebra $\mathbf{B}$ is free in a class $\mathcal{K}$ for X then we sometimes refer to $\mathbf{B}$ as being *freely generated by X in $\mathcal{K}$*. We say $\mathbf{B}$ is an *infinitely-generated free algebra in $\mathcal{K}$* if $|X| \geq \omega$.

THEOREM 2.4.13. *Let $\mathcal{K}$ be the class of all algebras of a language L, and X a set of variables. Then $\mathbf{Term}(X)$ is the free algebra in $\mathcal{K}$ over X.*

We refer the reader to [**MMT87**] Lemma 4.116 and Theorem 4.117 for a proof of this result.

DEFINITION 2.4.14. Let $\mathbf{A}$ be an algebra and $\theta \in \mathrm{Con}\mathbf{A}$. θ is *fully invariant* if for any endomorphism $h : \mathbf{A} \longrightarrow \mathbf{A}$, $\langle a, b\rangle \in \theta$ implies $\langle h(a), h(b)\rangle \in \theta$.

LEMMA 2.4.15. *Let $\mathcal{K}$ be a family of algebras and X a set of variables. The congruence $\theta_{\mathcal{K}}(X)$ on $\mathbf{Term}(X)$, defined by*

$$\theta_{\mathcal{K}}(X) = \bigcap\{\theta \in \mathrm{Con}(\mathbf{Term}(X)) : \mathbf{Term}(X)/\theta \in \mathbf{S}\mathcal{K}\},$$

is fully invariant.

LEMMA 2.4.16. *Let X be a set of variables X and θ a fully invariant congruence on $\mathbf{Term}(X)$. Then for any $p, q \in \mathbf{Term}(A)$*

$$\mathbf{Term}(X)/\theta \models p = q \textit{ iff } \langle p, q\rangle \in \theta.$$

For proofs of the lemmas above we refer the reader to [**BuS81**] Lemma 14.4 and Lemma 14.7.

2.5. Classes of algebras

2.5.1. Class operators and varieties.

DEFINITION 2.5.1. A class operator is any map defined on all classes of algebras or all classes of relational structures. For a class operator $\mathbf{O}$, we say that a class $\mathcal{K}$ is *closed under* $\mathbf{O}$ if $\mathbf{O}\mathcal{K} \subseteq \mathcal{K}$.

To distinguish class operators from other morphisms we generally write them in in boldface. So far we have already come across a few class operators, such as $\mathbf{P_u}$, $\mathbf{P_w}$, $\mathbf{H_b}$, $\mathbf{S_b}$ and $\mathbf{U_d}$. In the sequel we will also need the following class operators.

DEFINITION 2.5.2. Let $\mathcal{K}$ be a class of algebras.
(i) By $\mathbf{H}\mathcal{K}$ we denote the class of all *homomorphic* images of $\mathcal{K}$.
(ii) By $\mathbf{S}\mathcal{K}$ we denote the class of all isomorphic copies of *subalgebras* of $\mathcal{K}$.
(iii) By $\mathbf{P}\mathcal{K}$ we denote the class of all isomorphic copies of (direct) *products* of $\mathcal{K}$.

Note that for any class $\mathcal{K}$ we always include the empty product in $\mathbf{P}\mathcal{K}$ and hence $\mathbf{P}\mathcal{K}$ will always contain the trivial algebra.

DEFINITION 2.5.3. A *variety* is a class $\mathcal{K}$ of algebras, such that $\mathcal{K}$ is closed under $\mathbf{H}$, $\mathbf{S}$ and $\mathbf{P}$.

It should be clear that the intersection of a class of varieties is again a variety, and that the class of all algebras of type L forms a variety. Consequently it follows that for any class $\mathcal{K}$ of L-algebras there is a smallest variety containing $\mathcal{K}$, i.e. the intersection of all varieties containing $\mathcal{K}$.

DEFINITION 2.5.4. Let $\mathcal{K}$ be a class of algebras. By the *variety generated by* $\mathcal{K}$, $\mathrm{V}(\mathcal{K})$ in symbols, we mean the smallest variety containing the class $\mathcal{K}$. A variety $\mathcal{V}$ is finitely generated if $\mathcal{V} = \mathrm{V}(\mathcal{K})$ for some finite class $\mathcal{K}$ of finite algebras.

The following important result was first proved by A. Tarski in [**Tar46**].

THEOREM 2.5.5 (HSP Theorem). *Let $\mathcal{K}$ be some class of algebras. Then*

$$\mathrm{V}(\mathcal{K}) = \mathbf{HSP}\mathcal{K}.$$

We refer the reader to the proofs in [**BuS81**] (Theorem 9.5) and [**MMT87**] (Theorem 4.131).

But clearly using this result and Theorem 2.4.11 the subdirectly irreducible members of a variety characterise the variety.

COROLLARY 2.5.6. *Let $\mathcal{V}$ be a variety and $\mathrm{Si}(\mathcal{V})$ be all the subdirectly irreducible members of $\mathcal{V}$. Then $\mathcal{V} = \mathrm{V}(\mathrm{Si}(\mathcal{V}))$.*

2.5.2. Equational Classes, Varieties and Free Algebras.

DEFINITION 2.5.7. Let $\mathcal{K}$ be a class of algebras and X a set of variables. $\mathbf{F}_{\mathcal{K}}(X)$ denotes the free algebra in $\mathrm{V}(\mathcal{K})$ with free generating set X.

Earlier we mentioned how the term algebra and congruence constructions in some way capture the essence of classes of algebras. We now have the tools at hand to make this comment more precise. First we link free algebras and quotients of the term algebra.

THEOREM 2.5.8. *Let X be a set of variables and $\mathcal{K}$ a class of algebras. Then $\mathbf{F}_{\mathcal{K}}(X) \in \mathbf{SP}\mathcal{K}$ and $\mathbf{F}_{\mathrm{V}(\mathcal{K})}(X) \cong \mathbf{F}_{\mathcal{K}}(X) \cong \mathbf{Term}(X)/\theta_{\mathcal{K}}(X)$.*

We refer the reader to [**MMT87**] Corollary 4.119 for a proof of the above theorem.

In the study of universal algebra equalities have always formed an important part of the study of classes of algebras. Hence the following definition.

DEFINITION 2.5.9. We define an *identity* is to be an expression of the form $p = q$, where $p, q \in \mathbf{Term}(X)$.

A class $\mathcal{V}$ of algebras is an *equational class* if there is a set of identities Σ such that $\mathcal{V} = \mathcal{K}$, where $\mathcal{K}$ is the class of algebras satisfying Σ.

DEFINITION 2.5.10. We say that an equational class $\mathcal{V}$ is *finitely based* if there is a finite set of identities Σ such that $\mathcal{V}$ is the class of algebras satisfying Σ.

We now show how identities, free algebras and congruences relate.

THEOREM 2.5.11. *Let $\mathcal{K}$ be a class of algebras and $p, q \in \mathbf{Term}(X)$. Then the following are equivalent:*

(i) $\mathcal{K} \models p = q$,
(ii) $\mathbf{F}_{\mathcal{K}}(X) \models p = q$, *and*
(iii) $(p, q) \in \theta_{\mathcal{K}}(X)$.

For a proof of this theorem we refer the reader to [**BuS81**] Theorem 11.4 and [**MMT87**] Theorem 4.127.

Thus the free algebras are closely linked with the equational theory of a language. In fact G. Birkhoff (c.f. [**Bir35**]) showed how that equational classes and varieties are different view of the same concept. (A proof of the following result can be found in [**BuS81**] Theorem 11.9.)

THEOREM 2.5.12 (Birkhoff). *$\mathcal{K}$ is an equational class if, and only if, $\mathcal{K}$ is a variety.*

As we would now expect free algebras and varieties are closely linked. (For a class of algebras $\mathcal{K}$ we denote by $\mathrm{Eq}(\mathcal{K})$ the class of all equations satisfied by $\mathcal{K}$.)

THEOREM 2.5.13. *Let $\mathcal{V}$ be a variety and X a set of variables, with $|X| \geq \omega$. Then*

$$\mathcal{V} = \mathrm{V}(\mathbf{F}_{\mathcal{V}}(X)).$$

PROOF. By Theorem 2.5.11 $\mathrm{Eq}(\mathcal{V}) = \mathrm{Eq}(\mathbf{F}_{\mathcal{V}}(X))$. Thus by the HSP Theorem $\mathcal{V} = \mathbf{HSP}\mathcal{V} = \mathbf{HSPF}_{\mathcal{V}}(X)$. □

2.5.3. Quasivarieties. In many cases, especially while studying representations of algebras, we will relax the homomorphism requirement needed for varieties. Such classes are called quasivarieties and defined as follows.

DEFINITION 2.5.14. A *quasivariety* is a class of algebras closed under $\mathbf{S}$, $\mathbf{P}$ and $\mathbf{P_u}$.

DEFINITION 2.5.15. Let p and q be terms. A *quasi-identity* is a formula of the form $(p_0 = q_0 \wedge \ldots p_{n-1} = q_{n-1}) \to (p = q)$, where p_i and q_i are terms, for $i < n$.

A class of algebras $\mathcal{K}$ is called *quasi-equational* if it can be axiomatised by quasi-identities.

THEOREM 2.5.16 (Maltsev). *$\mathcal{K}$ is a quasi-equational class if, and only if, $\mathcal{K}$ is a quasivariety.*

THEOREM 2.5.17. *$\mathcal{K}$ is a quasivariety if, and only if, $\mathcal{K} = \mathbf{SPP_u}\mathcal{K}$.*

We refer the reader to [**BuS81**] Theorem 2.25 for a proof of the previous two theorems.

2.5.4. Universal classes. We say a formula is in *prenex* form if it consists of a string of quantifiers followed by a quantifier free formula. A formula is called *universal* if it is in prenex form and it only contains universal quantifiers.

DEFINITION 2.5.18. An elementary class of algebras is called a *universal class* if it can be axiomatised by universal formulas.

A proof of the following result can be found in [**BuS81**] (Theorem 2.20).

THEOREM 2.5.19. *Let $\mathcal{K}$ be a class of algebras. Then the following are equivalent:*

(i) *$\mathcal{K}$ is a universal class,*
(ii) *$\mathcal{K}$ is closed under $\mathbf{S}$ and $\mathbf{P_u}$, and*
(iii) *$\mathcal{K} = \mathbf{SP_u}\mathcal{K}'$, for some class of algebras $\mathcal{K}'$.*

2.6. Categories

In this dissertation we also use some basic notions of category theory. The idea of categories was first introduced by Eilenberg and Mac Lane as a way of relating systems of algebraic structures and systems of topological spaces in algebraic topology. Through the work of modern day mathematicians such as Jónsson, Goldblatt and Davey these techniques have found a wide range of application in algebraic logic. (We refer the reader to [**McL95**] for an accessible text on category theory.)

DEFINITION 2.6.1. Let f be a morphism $A \longrightarrow B$, for some sets A and B. We call A the *domain* of f, written $\mathrm{Dom}(f)$, and B the *codomain* of f, written $\mathrm{Cod}(f)$.

A *category* is an object of the form $\mathsf{C} = \langle \mathrm{Obj}, \mathrm{Arr} \rangle$ where Obj is a non-empty class, called the *objects* of C, and Arr is non-empty collection of morphisms between elements of Obj, called the *arrows* of C. Furthermore Arr must satisfy the following conditions.

- For all $f, g \in \mathrm{Arr}$, if $\mathrm{Cod}(f) = \mathrm{Dom}(g)$ then there must exist some arrow h in C where $h : \mathrm{Dom}(f) \longrightarrow \mathrm{Cod}(g)$ and $h = g \circ f$.
- For all $f, g, h \in \mathrm{Arr}$, where $\mathrm{Cod}(f) = \mathrm{Dom}(g)$ and $\mathrm{Cod}(g) = \mathrm{Dom}(h)$, we have $(f \circ g) \circ h = f \circ (g \circ h)$.
- For each object A there must exist an identity morphism $\mathrm{Id}_A : A \longrightarrow A$, with $\mathrm{Id}_A \in \mathrm{Arr}$.

Let $\mathsf{A} = \langle \mathrm{Obj}_\mathsf{A}, \mathrm{Arr}_\mathsf{A} \rangle$ and $\mathsf{B} = \langle \mathrm{Obj}_\mathsf{B}, \mathrm{Arr}_\mathsf{B} \rangle$ be two categories. A *functor* $\mathbf{F}$ from category A to category B, written $\mathbf{F} : \mathsf{A} \longrightarrow \mathsf{B}$, assigns to each $A \in \mathrm{Obj}_\mathsf{A}$ an object $\mathbf{F}A \in \mathrm{Obj}_\mathsf{B}$ and to each arrow $f \in \mathrm{Arr}_\mathsf{A}$ an arrow $\mathbf{F}f \in \mathrm{Arr}_\mathsf{B}$, satisfying the following conditions.

(i) Given $f : A \longrightarrow B$ we have $\mathbf{F}f : \mathbf{F}A \longrightarrow \mathbf{F}B$, i.e. $\mathbf{F}$ *preserves domains and codomains.*
(ii) For any A, $\mathbf{F}\mathrm{Id}_A = \mathrm{Id}_{\mathbf{F}A}$, i.e. $\mathbf{F}$ *preserves identities.*
(iii) If $\mathrm{Cod}(f) = \mathrm{Dom}(g)$ then $\mathbf{F}(g \circ f) = \mathbf{F}g \circ \mathbf{F}f$, i.e. $\mathbf{F}$ *preserves composition.*

By a *contravariant functor* we mean a functor which reverses arrows, i.e. a morphism between two categories A and B as above, but which satisfies condition (ii) and the following modified versions of conditions (i) and (iii).

(i)' Given $f : A \longrightarrow B$ we have $\mathbf{F}f : \mathbf{F}B \longrightarrow \mathbf{F}A$, i.e. $\mathbf{F}$ *reverses domains and codomains.*
(iii)' If $\mathrm{Cod}(f) = \mathrm{Dom}(g)$ then $\mathbf{F}(g \circ f) = \mathbf{F}f \circ \mathbf{F}g$, i.e. $\mathbf{F}$ *reverses composition.*

As with class operators we use boldface to distinguish functors from other morphisms.

CHAPTER 3

The Paradigm Triangle for Complex Algebras

From any structure $\mathbf{U}$ we can construct the complex algebra $\mathrm{Cm}\mathbf{U}$ (defined below), by taking the power set of $\mathbf{U}$ as its universe and then lifting the relations and functions on $\mathbf{U}$ to functions on subsets of $\mathbf{U}$. So complex algebras are simply expansions of Boolean algebras. In fact, as we shall see in the sequel, they form part of a very interesting class of expansions of Boolean algebras called Boolean algebras with operators (BAOs).

For a countable propositional language the traditional semantics assigns truth values (true or false) to each propositional variable, structurally we can thus see the semantics as being represented by the countable Boolean algebra 2^ω. In this chapter we will reverse this process and study the logics associated with BAO structures and then describe the relational and algebraic semantics associated with them. In particular we will see that complex algebras give us the concrete algebraic semantics of such a logic and are related to the relational semantics in a well behaved manner. If a logic is "well behaved", i.e. is characterised by, its relational and algebraic semantics, then we get the following "paradigm" triangle (c.f. [**BGO95**] for the introduction of this concept). (The arrows denote the generic names for the fields studying the relationship between these three views of logic.)

$$
\begin{tikzcd}
\boxed{\text{Relational Semantics}} \arrow[rr, leftrightarrow, "\textit{Duality}", "\textit{Theory}"'] \arrow[dr, leftrightarrow, "\textit{Correspondence Theory}"'] & & \boxed{\text{Algebraic Semantics}} \arrow[dl, leftrightarrow, "\textit{Algebraic Logic}"] \\
& \boxed{\text{Logic}} &
\end{tikzcd}
$$

FIGURE 1. The Paradigm Triangle

3.1. Boolean Algebras with Operators

Since the perspective in this dissertation is more algebraically orientated we first formally introduce the class of algebraic structures that we are interested in and then work backwards to the (modal) logics associated with them. This particular class of structures was first introduced by Jónsson and Tarski in [**JoT51**], and is currently one of the most widely studied expansions of the class of Boolean algebras.

3.1.1. Preliminaries.

Recall that an *atom* of a Boolean algebra $\mathbf{B}$ is an element $a \in \mathbf{B} \setminus \{\mathbb{C}\}$ such that $b \not< a$ for all $b \in \mathbf{B} \setminus \{\mathbb{0}\}$. Intuitively we can say that the atoms of a Boolean algebra are the smallest elements in the algebra.

DEFINITION 3.1.1. Let $\mathbf{B} = \langle B, \vee, \sim, \mathbb{0}\rangle$ be a Boolean algebra.

- An *operator* f, with $ar(f) = n \in \omega$ is a function $f : B^n \longrightarrow B$ such that:
  (i) f is *normal*: for any $b_0, \ldots, b_{i-1}, b_{i+1}, \ldots, b_{n-1} \in B$
  $$f(b_0, \ldots, b_{i-1}, \mathbb{0}, b_{i+1}, b_{n-1}) = \mathbb{C},$$
  (ii) f is *additive*: for any $b, b', b_0, \ldots, b_{n-1} \in B$
  $$\begin{aligned} f(b_0, \ldots, b_{i-1}, b \vee b', b_{i+1}, b_{n-1}) &= f(b_0, \ldots, b_{i-1}, b, b_{i+1}, b_{n-1}) \\ &\quad \vee f(b_0, \ldots, b_{i-1}, b', b_{i+1}, b_{n-1}) \end{aligned}$$
- A *boolean algebra with operators* (BAO) is an expansion $\langle B, \vee, \sim, \mathbb{0}, F\rangle$ of $\mathbf{B}$ with signature $L_{\mathsf{BAO}} = L_{\mathsf{BA}} \cup F$ where F is a set of operators on $\mathbf{B}$. We call $\mathbf{B}$ the *Boolean reduct* or *Boolean part* of $\langle B, \vee, \sim, \mathbb{C}, F\rangle$.
- We denote the subclass of BAO whose Boolean reducts are complete and atomic by BAO^{ca}.
- We call f *completely additive* if for any $B_0, \ldots, B_{n-1} \subseteq \mathbf{B}$
  $$\bigvee\{f(b_0, \ldots, b_{n-1}) : b_i \in B_i\} = f(\bigvee B_0, \ldots, \bigvee B_{n-1}),$$
  where $ar(f) = n$.

Note that for a given signature L_{BAO} the associated class of algebras form a variety. Also observe that such a variety is finitely based if, and only if, the set F of operators is finite. However, the subclass BAO^{ca} does not form a subvariety of BAO.

PROPOSITION 3.1.2. *Let f be an operator on $\mathbf{B}$, with $ar(f) = n$, and $b_i \in \mathbf{B}$, for $i < n$. Then, for any $i < n$,*
$$\text{if } b_i \leq b_i', \text{ then } f(b_0, \ldots, b_i, \ldots, b_{n-1}) \leq f(b_0, \ldots, b_i', \ldots, b_{n-1}),$$
where $b_i' \in \mathbf{B}$.

LEMMA 3.1.3. *Let $\mathbf{B} \in \mathsf{BAO}^{ca}$ and f, with $ar(f) = n$, an operator in the signature of $\mathbf{B}$. Then the following are equivalent:*

(i) *Let $a \in \mathrm{At}(\mathbf{B})$ and $b_0, \ldots, b_{n-1} \in \mathbf{B}$ such that $f(b_0, \ldots, b_{n-1}) \geq a$. Then there exist $a_0, \ldots, a_{n-1} \in \mathrm{At}(\mathbf{B})$ such that $f(a_0, \ldots, a_{n-1}) \geq a$ and $a_i \leq b_i$ for $i < n$.*
(ii) *f is completely additive.*

PROOF. $(i) \rightarrow (ii)$: Let $B_0, \ldots B_{n-1} \subseteq \mathbf{B}$.

If for any $i < n$ $B_i = \emptyset$ then $\bigvee B_i = \mathbb{C}$. Hence $f(\bigvee B_0, \ldots, \bigvee B_{n-1}) = \mathbb{C}$ since f is normal. Now $\bigvee\{f(b_0, \ldots, b_{n-1})\} = \bigvee\emptyset = \mathbb{C}$. So we can assume $B_i \neq \emptyset$ for $i < n$.

Let $\bigvee B_i = c_i$. We need to show that $\bigvee\{f(b_0, \ldots, b_{n-1}) : b_i \in B_i\} = f(c_0, \ldots, c_{n-1})$. Now $b_i \leq c_i$ for all $b_i \in B_i$. Hence $f(b_0, \ldots, b_{n-1}) \leq f(c_0, \ldots, c_{n-1})$ and so
$$\bigvee\{f(b_0, \ldots, b_{n-1}) : b_i \in B_i\} \leq f(\bigvee B_0, \ldots, \bigvee B_{n-1}).$$

From the remark above we know there exists a unique set $C \subseteq \mathrm{At}(\mathbf{B})$ such that $f(\bigvee B_0, \ldots, \bigvee B_{n-1}) = \bigvee C$. Let $c \in C$ then $c \leq f(\bigvee B_0, \ldots \bigvee B_{n-1})$. From (i) it follows that for any $i < n$ there exists $c_i \in \mathrm{At}(\mathbf{B})$ such that $c_i \leq \bigvee B_i$ and

$f(c_0, \ldots, c_{n-1}) \geq c$. Since $c_i \in \mathrm{At}(B)$ and the represention of $\bigvee B_i$ by atoms is unique there exists some $b_i \in B_i$ such that $c_i \leq b_i$ for $i < n$. Thus there exist $b_i \in B_i$ such that $c \leq f(c_0, \ldots, c_{n-1}) \leq f(b_0, \ldots, b_{n-1})$. It follows that

$$f(\bigvee B_0, \ldots, \bigvee B_{n-1}) \leq \bigvee\{f(b_0, \ldots, b_{n-1}) : b_i \in B_i\}.$$

$(ii) \to (i)$: Let $a \in \mathrm{At}(\mathbf{B})$ and $b_0, \ldots, b_{n-1} \in \mathbf{B}$ such that $f(b_0, \ldots, b_{n-1}) \geq a$.

For every b_i there exists $A_i \subseteq \mathrm{At}(\mathbf{B})$ such that $\bigvee A_i = b_i$. Assume that for any $a_i \in A_i$ $f(a_0, \ldots, a_{n-1}) \not\geq a$. Since f is completely additive $f(b_0, \ldots, b_{n-1}) = \bigvee\{f(a_0, \ldots, a_{n-1}) : a_i \in A_i\}$. We let

$$A = \{a' \in \mathrm{At}(\mathbf{B}) : \text{ there exist } a_i \text{ such that } a' \leq f(a_0, \ldots, a_{n-1} \text{ for } a_i \in A_i)\}.$$

Then $\bigvee A = f(b_0, \ldots, b_{n-1}) \geq a$. But by assumption $a \notin A$ contradicting the fact that $f(b_0, \ldots, b_{n-1})$ has a unique representation as a join of atoms. □

Given a signature L_{BAO} we define L_{Str} to be the relational signature associated with the non-boolean symbols in L_{BAO}, i.e. $L_{\mathsf{Str}} = \{R_f : f \in L_{\mathsf{BAO}} \setminus L_{\mathsf{BA}}\}$, and Str to be the class of L_{Str}-structures, where $ar(R_f) = ar(f) + 1$.

3.1.2. Mapping between classes of structures.

In this section and the rest of the chapter we will let L_{BAO} be some fixed language. Here we consider different ways of mapping between BAO, Str and BAO^{ca}. The proper setting for such a study is with the use of elementary category theory. We will assume that the reader is familiar with the concepts of a category and functors between categories as introduced in Section 2.6.

Note that BAO forms a category under BAO-homomorphisms, Str forms a category under bounded morphisms and BAO^{ca} forms a category under complete BAO-homomorphisms (i.e. homomorphisms which preserve complete joins).

From BAO *to* Str. Here we introduce a way of defining a morphism from BAO to Str or, as we shall later see, how to generate the relational semantics given a logic (c.f. the "Correspondence Theory" arrow in Figure 1).

DEFINITION 3.1.4. We define a map $\mathbf{Uf}$ between the categories BAO and Str in the following way.

- For $\mathbf{B} = \langle B, \vee, \sim, \mathbb{0}, F\rangle$ a BAO where $F = L_{\mathsf{BAO}} \setminus L_{\mathsf{BA}}$. We define a L_{Str}-structure $\mathrm{Uf}(\mathbf{B}) = \langle \mathrm{Uf}(B), \langle R_f^{\mathrm{Uf}(\mathbf{B})} : f \in F\rangle\rangle$ where $\mathrm{Uf}(B)$ is the set of all ultrafilters on the Boolean reduct of $\mathbf{B}$ and for each $f \in F$, with $ar(f) = n$,
$$R_f^{\mathrm{Uf}(\mathbf{B})}(\mathcal{G}_0, \ldots, \mathcal{G}_{n-1}, \mathcal{F}) \text{ iff } \{f(b_0, \ldots, b_{n-1}) : b_i \in \mathcal{G}_i \text{ for } i < n\} \subseteq \mathcal{F},$$
where $\mathcal{G}_i \in \mathrm{Uf}(\mathbf{B})$ for $i < n$ and $\mathcal{F} \in \mathrm{Uf}(\mathbf{B})$.
- For $\mathbf{A}, \mathbf{B} \in \mathsf{BAO}$ and $h : \mathbf{A} \longrightarrow \mathbf{B}$ a BAO-homomorphism we define the map $\mathrm{Uf}(h) : \mathrm{Uf}(\mathbf{B}) \longrightarrow \mathrm{Uf}(\mathbf{A})$ by
$$\mathrm{Uf}(h)(\mathcal{G}) = h^{-1}[\mathcal{G}],$$
where $\mathcal{G} \in \mathrm{Uf}(B)$.

We call $\mathrm{Uf}(\mathbf{B})$ the *ultrafilter extension* of $\mathbf{B}$.

For a class $\mathcal{K}$ of BAOs we define $\mathbf{Uf}\mathcal{K} = \{\mathrm{Uf}(\mathbf{B}) : \mathbf{B} \in \mathcal{K}\}$ and for a class $\mathcal{H}$ of BAO-homomorphisms we define $\mathbf{Uf}\mathcal{H} = \{\mathrm{Uf}(h) : h \in \mathcal{H}\}$.

Van Bentham [**vBe79**] calls UfCmU the ultrafilter extension of a Kripke frame $\mathbf{U}$. Here we will use this term in a more general sense to apply to any algebra, not only those of the form CmU (defined below). In [**Gol89**] Goldblatt refers to $\mathrm{Uf}(\mathbf{B})$ as the canonical structure of $\mathbf{B}$.

PROPOSITION 3.1.5. Uf *is a contravariant functor from* BAO *to* Str

PROOF. It should be clear that $\mathrm{Uf}(\mathbf{B}) \in \mathsf{Str}$ for any $\mathbf{B} \in \mathsf{BAO}$.

Let $\mathcal{G} \in \mathrm{Uf}(\mathbf{B})$ and $h : \mathbf{A} \longrightarrow \mathbf{B}$ a BAO-homomorphsm. We need to show that $h^{-1}[\mathcal{G}]$ is an ultrafilter. Since each ultrafilter of a Boolean Algebra is determined by a homomorphism into the 2-element chain, there exists a homomorphism $g : \mathbf{B} \longrightarrow 2$ such that $\mathcal{G} = h^{-1}[\{1\}]$. Then

$$\mathrm{Uf}(h)(\mathcal{G}) = h^{-1}[\mathcal{G}] = h^{-1}g^{-1}[\{1\}] = (g \circ h)^{-1}[\{1\}].$$

Hence $\mathrm{Uf}(h)(\mathcal{G})$ is the ultrafilter determined by the composite homomorphism $g \circ h$.

We still need to show that, for any BAO-homomorphism h, $\mathrm{Uf}(h)$ is a bounded morphism*. To that end let $\mathbf{A}, \mathbf{B} \in \mathsf{BAO}$ and $h : \mathbf{A} \longrightarrow \mathbf{B}$ be a BAO-homomorphism and $f \in F$ with $ar(f) = n$.

zig: We assume that $R_f^{\mathrm{Uf}(\mathbf{B})}(\mathcal{G}_0, \dots, \mathcal{G}_{n-1}, \mathcal{F})$ holds and that it is not the case that $R_f^{\mathrm{Uf}(\mathbf{A})}(\mathrm{Uf}(h)(\mathcal{G}_0), \dots, \mathrm{Uf}(h)(\mathcal{G}_{n-1}), \mathrm{Uf}(h)(\mathcal{F}))$. Thus we can see that there must exist $a_i \in \mathrm{Uf}(h)(\mathcal{G}_i)$, for $i < n$, such that $f(a_0, \dots, a_{n-1}) \notin \mathrm{Uf}(h)(\mathcal{F})$. Then, by the definition of $\mathrm{Uf}(h)$, $h(f(a_0, \dots, a_{n-1})) \notin \mathcal{F}$. Hence $f(h(a_0), \dots, h(a_{n-1})) \notin \mathcal{F}$, where $h(a_i) \in \mathcal{G}_i$, contradicting our assumption on $R_f^{\mathrm{Uf}(\mathbf{B})}$.

zag: Let $R_f^{\mathrm{Uf}(\mathbf{A})}(\mathcal{G}_0, \dots, \mathcal{G}_{n-1}, \mathrm{Uf}(h)(\mathcal{F}))$, for $\mathcal{G}_i \in \mathrm{Uf}(\mathbf{A})$, $i < n$, and $\mathcal{F} \in \mathrm{Uf}(\mathbf{B})$. By assumption $\mathcal{G}_0, \dots, \mathcal{G}_{n-1} \in \mathrm{ran}(\mathrm{Uf}(h))$ it follows that there exist some $\mathcal{G}'_i \in \mathrm{Uf}(\mathbf{B})$ such that $\mathrm{Uf}(h)(\mathcal{G}_i) = \mathcal{G}'_i$, for $i < n$. By Proposition 2.2.11 it follows that $h^{-1}[\mathcal{G}'_i]$ is an ultrafilter. Since $\mathcal{G}_i \subseteq h^{-1}[\mathcal{G}'_i]$ and $\mathcal{G}_i$ is a maximal filter, it follows that $h^{-1}[\mathcal{G}'_i] = \mathcal{G}_i$, for $i < n$.

Consider any n-tuple $\langle b_0, \dots, b_{n-1}\rangle$ with $b_i \in \mathcal{G}'_i$, for $i < n$. Note that $b \in h[\mathcal{G}_i]$, since $\mathcal{G}' \subseteq h(h^{-1}[\mathcal{G}'_i]) = h[\mathcal{G}_i]$. Then $h^{-1}[\{b_i\}] \subseteq \mathcal{G}_i$ for each $i < n$. Hence $\{f(a_0, \dots, a_{n-1}) : h(a_i) = b_i\} \subseteq \{f(a_0, \dots, a_{n-1}) : a_i \in \mathcal{G}_i\}$. Consequently

$$\{h(f(a_0, \dots, a_{n-1})) : h(a_i) = b_i\} \subseteq \{h(f(a_0, \dots, a_{n-1})) : a_i \in \mathcal{G}_i\}.$$

But h is a homomorphism, whence $h(f(a_0, \dots, a_{n-1})) = f(h(a_0), \dots, h(a_{n-1}))$ and so $f(b_0, \dots, b_{n-1}) \in \{h(f(a_0, \dots, a_{n-1})) : a_i \in \mathcal{G}_i\}$. By our assumption on $R_f^{\mathrm{Uf}(A)}$ it follows that $\{h(f(a_0, \dots, a_{n-1})) : a_i \in \mathcal{G}_i\} \subseteq \mathcal{F}$. Thus $f(b_0, \dots, b_{n-1}) \in \mathcal{F}$, and so $R_f^{\mathrm{Uf}(\mathbf{B})}(\mathcal{G}'_0, \dots, \mathcal{G}_{n-1}, \mathcal{F})$ as required.

This proves that Uf is a well defined map between BAO and Str. To see that it is in fact a contravariant functor we first take a look at the identity map $\mathrm{Id} : \mathbf{B} \longrightarrow \mathbf{B}$, for $\mathbf{B} \in \mathsf{BAO}$. Since $\{\mathrm{Id}(b) : b \in \mathcal{G}\} = \mathcal{G}$, by definition $\mathrm{Uf}(\mathrm{Id})(\mathcal{G}) = \mathcal{G}$.

To conclude we need to show that for $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathsf{BAO}$ and BAO-homomorphisms $h : \mathbf{A} \longrightarrow \mathbf{B}$ and $g : \mathbf{B} \longrightarrow \mathbf{C}$ we have $\mathrm{Uf}(g \circ h) = \mathrm{Uf}(h) \circ \mathrm{Uf}(g)$. Hence consider the following calculation

$$\begin{aligned} \mathrm{Uf}(h) \circ \mathrm{Uf}(g)(\mathcal{G}) = \mathcal{F} &\text{ iff } \{h(a) : a \in \mathcal{F}\} \subseteq \mathrm{Uf}(g)(\mathcal{G}) \\ &\text{ iff } \{g \circ h(a) : a \in \mathcal{F}\} \subseteq \mathcal{G} \\ &\text{ iff } \mathrm{Uf}(g \circ h)(\mathcal{G}) = \mathcal{F}. \end{aligned}$$

□

*C.f Definition 2.3.1 (p. 20) for the definitions of a bounded morphism, **zig** and **zag**.

From Str *to* BAO^{ca}. From Str we can now define a morphism to complete and atomic BAOs, thus setting up the framework for generating algebraic semantics from relational semantics (c.f. the "Duality Theory" arrow in Figure 1).

DEFINITION 3.1.6. We define a map **Cm** between the categories Str and BAO^{ca} in the following way.

- Let $\mathbf{U} = \langle U, \langle r^{\mathbf{U}} : r \in L_{\mathsf{Str}}\rangle\rangle$ be in Str. For $L_{\mathsf{BAO}} = L_{\mathsf{BA}} \uplus \{f_r : r \in L_{\mathsf{Str}}\}$, we define an L_{BAO}-structure $\mathrm{Cm}\mathbf{U} = \langle \mathcal{P}(U), \vee, \sim, \complement, \langle f_r^{\mathrm{Cm}\mathbf{U}} : r \in L_{\mathsf{Str}}\rangle\rangle$ where $\mathcal{P}(U)$ is the set of all subsets of U and for each $r \in L_{\mathsf{Str}}$, with $ar(r) = n + 1$ and $ar(f_r) = n$,
$$f_r^{\mathrm{Cm}\mathbf{U}}(\underline{X}) = \{y \in \mathbf{U} : r^{\mathbf{U}}(x_0, \ldots, x_{n-1}, y) \text{ with } x_i \in X_i\}$$
where $\underline{X} = (X_0, X_1, \ldots, X_{n-1}) \in \mathcal{P}(U)^n$ and $i < n$.
- For $\mathbf{U}, \mathbf{V} \in$ Str and a bounded morphism $\gamma : \mathbf{U} \longrightarrow \mathbf{V}$ we define $\mathrm{Cm}(\gamma) : \mathrm{Cm}\mathbf{V} \longrightarrow \mathrm{Cm}\mathbf{U}$ by:
$$\mathrm{Cm}(\gamma)(X) = \gamma^{-1}[X]$$
where $X \in \mathcal{P}(V)$.

$\mathrm{Cm}\mathbf{U}$ is called the *full complex algebra* of $\mathbf{U}$. A subalgebra of any full complex algebra is referred to as a *complex algebra.*

For a class $\mathcal{K}$ of structures we define $\mathbf{Cm}\mathcal{K} = \{\mathrm{Cm}\mathbf{U} : \mathbf{U} \in \mathcal{K}\}$ and for a class $\mathcal{H}$ of bounded morphisms we define $\mathbf{Cm}\mathcal{H} = \{\mathrm{Cm}(\gamma) : \gamma \in \mathcal{H}\}$.

Note that where confusion is unlikely we will write $r^{\mathrm{Cm}\mathbf{U}}$ instead of $f_r^{\mathrm{Cm}\mathbf{U}}$.

PROPOSITION 3.1.7. *Cm is a contravariant functor from* Str *to* BAO^{ca}.

PROOF. It is easy to see that for any set U any Boolean algebra of the form $\mathbf{B} = \langle \mathcal{P}(U), \vee, \sim, \complement\rangle$ is complete and atomic. Hence any expansion of $\mathbf{B}$, in particular a BAO with Boolean reduct $\mathbf{B}$, will be complete and atomic. Before we can say that Cm maps to BAO^{ca} we first need to show that $r^{\mathrm{Cm}\mathbf{U}}$ is an operator.

Normal: By definition
$$r^{\mathrm{Cm}\mathbf{U}}(X_0, \ldots, X_{n-1}) = \{x_n : r^{\mathbf{U}}(x_0, \ldots, x_n) \text{ where } x_i \in X_i \text{ for } i < n\}.$$
Clearly if $X_i = \emptyset$ for some $i < n$ then $r^{\mathrm{Cm}\mathbf{U}}(X_0, \ldots, X_{n-1}) = \emptyset$.

Additive: Let $x_n \in r^{\mathrm{Cm}\mathbf{U}}(X_0, \ldots, X \cup X', \ldots, X_{n-1})$, for $X, X', X_i \subseteq U$, where $i < n$. Then there exists an $x \in X \cup X'$ such that $r^{\mathbf{U}}(x_0, \ldots, x, \ldots, x_n)$, where $x_i \in X_i$ for $i < n$. Consequently we either have $x_n \in r^{\mathrm{Cm}\mathbf{U}}(X_0, \ldots, X, \ldots, X_{n-1})$ or $x_n \in r^{\mathrm{Cm}\mathbf{U}}(X_0, \ldots, X', \ldots, X_{n-1})$, whence
$$x_n \in r^{\mathrm{Cm}\mathbf{U}}(X_0, \ldots, X, \ldots, X_{n-1}) \cup r^{\mathrm{Cm}\mathbf{U}}(X_0, \ldots, X', \ldots, X_{n-1}).$$
The converse follows by a similar argument. Hence
$$\begin{aligned} r^{\mathrm{Cm}\mathbf{U}}(X_0, \ldots, X \cup X', \ldots, X_{n-1}) &= r^{\mathrm{Cm}\mathbf{U}}(X_0, \ldots, X, \ldots, X_{n-1}) \\ &\quad \cup r^{\mathrm{Cm}\mathbf{U}}(X_0, \ldots, X', \ldots, X_{n-1}). \end{aligned}$$

Next we prove that, for any bounded morphism $\gamma : \mathbf{U} \longrightarrow \mathbf{V}$, $\mathrm{Cm}(\gamma)$ is a BAO-homomorphism, i.e. we prove that
$$\mathrm{Cm}(\gamma)(r^{\mathrm{Cm}\mathbf{V}}(V_0, \ldots, V_{n-1})) = r^{\mathrm{Cm}\mathbf{U}}(\mathrm{Cm}(\gamma)(V_0), \ldots, \mathrm{Cm}(\gamma)(V_{n-1}))$$
where $r \in$ Str and $ar(r) = n + 1$.

Now

$$\begin{aligned}\mathrm{Cm}(\gamma)(r^{\mathrm{Cm}\mathbf{V}}(V_0,\ldots,V_{n-1})) &= \gamma^{-1}[r^{\mathrm{Cm}\mathbf{V}}(V_0,\ldots,V_{n-1})] \\ &= \{u \in U : r^{\mathbf{V}}(v_0,\ldots,v_{n-1},\gamma(u)) \text{ for } v_i \in V_i\}\end{aligned}$$

and

$$\begin{aligned}r^{\mathrm{Cm}\mathbf{U}}(\mathrm{Cm}(\gamma)(V_0),\ldots,\mathrm{Cm}(\gamma)(V_{n-1})) &= r^{\mathrm{Cm}\mathbf{U}}(\gamma^{-1}[V_0],\ldots,\gamma^{-1}[V_{n-1}]) \\ &= \{u : r^{\mathbf{U}}(u_0,\ldots,u_{n-1},u) \text{ for } u_i \in \gamma^{-1}(V_i)\}.\end{aligned}$$

Note that $u_i \in \gamma^{-1}[V_i]$ implies that $\gamma(u_i) \in V_i$. So if $r^{\mathbf{U}}(u_0,\ldots,u_{n-1},u)$ as above then by **zig** it follows that $r^{\mathbf{V}}(\gamma(u_0),\ldots,\gamma(u_{n-1}),\gamma(u))$. Which then gives us the right to left inclusion.

For the left to right inclusion observe that if $r^{\mathbf{V}}(v_0,\ldots,v_{n-1},\gamma(u))$ then by **zag** we can deduce that there must exist $u_0,\ldots,u_{n-1} \in U$ such that $r^{\mathbf{U}}(u_0,\ldots,u_{n-1},u)$, where $\gamma(u_i) = v_i$. Thus the left to right inclusion must hold as well.

The following calculation shows that Cm preserves complement.

$$\begin{aligned}\{u : \gamma(u) \in {\sim}X\} &= \{u : \gamma(u) \notin X\} \\ &= {\sim}\{u : \gamma(u) \in X\}\end{aligned}$$

To show that Cm preserves arbitrary joins, and hence is a complete BAO-homomorphism, we note that the inverse image of a function preserves arbitrary unions. I.e. for a function $\gamma : U \longrightarrow V$ with $X \subseteq \mathcal{P}(V)$

$$\gamma^{-1}[\bigcup X] = \bigcup(\gamma^{-1}[X]).$$

Thus Cm is a well defined map from Str to BAO^{ca}.

It is easy to see that for any set X, $\mathrm{Id}^{-1}[X] = X$ and hence $\mathrm{Cm}(\mathrm{Id})(X) = X$. To show that Cm reverses composition of morphisms let $\mathbf{U},\mathbf{V},\mathbf{W} \in \mathsf{Str}$ and let $\gamma : \mathbf{U} \longrightarrow \mathbf{V}$ and $\delta : \mathbf{V} \longrightarrow \mathbf{W}$. Then

$$\begin{aligned}\mathrm{Cm}(\gamma) \circ \mathrm{Cm}(\delta)(X) &= \gamma^{-1}[\delta^{-1}[X]] \\ &= \{u : \gamma(u) \in \delta^{-1}[X]\} \\ &= \{u : \delta(\gamma(u)) \in X\} \\ &= (\delta \circ \gamma)^{-1}[X] \\ &= \mathrm{Cm}(\delta \circ \gamma)(X).\end{aligned}$$

Completely Additive: Let $a \in \mathrm{At}(\mathbf{B})$ and $b_0,\ldots,b_{n-1} \in \mathbf{B}$. By definition $a = \{y\}$ for some $y \in \mathbf{U}$ and $b_i \subseteq \mathbf{U}$ for $i < n$. So $y \in f_r^{\mathrm{Cm}\mathbf{U}}(b_0,\ldots,b_{n-1})$ and thus by definition there exist $x_i \in b_i$, $i < n$, such that $r^{\mathbf{U}}(x_0,\ldots,x_{n-1},y)$. So $y \in f_r^{\mathrm{Cm}\mathbf{U}}(\{x_0\},\ldots,\{x_{n-1}\})$ where $x_i \in \mathrm{At}(\mathbf{B})$, for $i < n$. Hence

$$f_r^{\mathrm{Cm}\mathbf{U}}(\{x_0\},\ldots,\{x_{n-1}\}) \geq \{y\}$$

and so by Proposition 3.1.3 $\mathbf{B}$ is completely additive.

□

We can however also make sense of the complex algebra construction over classes of algebras. This is done by considering a class of L-algebras as a class of L_{Str}-structures, where $L_{\mathsf{Str}} = \{R_f : f \in L\}$, with $ar(R_f) = ar(f) + 1$ and for each algebra $\mathbf{A}$,

$$R_f^{\mathbf{A}}(a_0,\ldots,a_n) \text{ iff } f^{\mathbf{A}}(a_0,\ldots,a_{n-1}) = a_n$$

(i.e. $R_f^{\mathbf{A}}$ is the graph of $f^{\mathbf{A}}$). The following definition accomplishes the same thing without the detour of viewing the algebras as structures.

DEFINITION 3.1.8. Let $\mathcal{K}$ be any class of L-algebras, where $L = F$. We define a map **Cm** on $\mathcal{K}$ in the following way.

- Let $\mathbf{A} = \langle A, F^{\mathbf{A}}\rangle$ be in $\mathcal{K}$. For $L_{\mathsf{BAO}} = L_{\mathsf{BA}} \uplus F$ we define an L_{BAO}-structure $\mathrm{Cm}\mathbf{A} = \langle \mathcal{P}(A), \vee, \sim, \mathbb{0}, F^{\mathrm{Cm}\mathbf{A}}\rangle$ where $\mathcal{P}(A)$ is the set of all subsets of A and for each $f \in F$, with $ar(f) = n$,
$$f^{\mathrm{Cm}\mathbf{A}}(\underline{X}) = \{y \in A : f^{\mathbf{A}}(x_0, \ldots, x_{n-1}) = y \text{ with } x_i \in X_i \text{ for } i < n\},$$
where $\underline{X} = (X_0, X_1, \ldots, X_{n-1}) \subseteq \mathcal{P}(A)^n$.
- For $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ and a bounded morphism $h : \mathbf{A} \longrightarrow \mathbf{B}$ we define the BAO-homomorphism $\mathrm{Cm}(h) : \mathrm{Cm}\mathbf{B} \longrightarrow \mathrm{Cm}\mathbf{A}$ by
$$\mathrm{Cm}(h)(X) = h^{-1}[X],$$
where $X \in \mathcal{P}(B)$.

$\mathrm{Cm}\mathbf{A}$ is called the *full complex algebra* of $\mathbf{A}$.

For a class $\mathcal{K}$ of algebras we define $\mathbf{Cm}\mathcal{K} = \{\mathrm{Cm}\mathbf{A} : \mathbf{A} \in \mathcal{K}\}$ and for any class $\mathcal{H}$ of homomorphisms we define $\mathbf{Cm}\mathcal{H} = \{\mathrm{Cm}(h) : h \in \mathcal{H}\}$.

Clearly the proof above easily extends to classes of algebras.

PROPOSITION 3.1.9. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be algebras. If* γ *is a bounded morphism from* $\mathbf{A}$ *to* $\mathbf{B}$ *then* $\mathrm{Cm}(h) : \mathrm{Cm}\mathbf{B} \longrightarrow \mathrm{Cm}\mathbf{A}$ *is a* BAO^{ca} *homomorphism.*

From BAO^{ca} *to* Str. Lastly we define a morphism that we will later use to relate algebraic semantics back to relational semantics (c.f. the "Duality Theory" arrow in Figure 1).

DEFINITION 3.1.10. We define a map **At** between the categories BAO^{ca} and Str in the following way:

- For $\mathbf{B} = \langle B, \vee, \sim, \mathbb{0}, F\rangle$ a BAO^{ca} where $F = L_{\mathsf{BAO}} \setminus L_{\mathsf{BA}}$. We define a L_{Str}-structure $\mathrm{At}(\mathbf{B}) = \langle \mathrm{At}(B), \langle R_f^{\mathrm{At}(\mathbf{B})} : f \in F\rangle\rangle$ where $\mathrm{At}(B)$ is the set of all atoms of the Boolean reduct of $\mathbf{B}$ and for each $f \in F$, with $ar(f) = n$,
$$R_f^{\mathrm{At}(\mathbf{B})}(\underline{b}, c) \text{ iff } f(\underline{b}) \geq c,$$
where $\underline{b} \in \mathrm{At}(B)^n$ and $c \in \mathrm{At}(B)$.
- For $\mathbf{A}, \mathbf{B} \in \mathsf{BAO}^{ca}$ and $h : \mathbf{A} \longrightarrow \mathbf{B}$ a complete BAO-homomorphism we define $\mathrm{At}(h) : \mathrm{At}(\mathbf{B}) \longrightarrow \mathrm{At}(\mathbf{A})$ by
$$\mathrm{At}(h)(b) = a \text{ iff } h(a) \geq b,$$
where $b \in \mathrm{At}(\mathbf{B})$ and $a \in \mathrm{At}(\mathbf{A})$.

We call $\mathrm{At}(\mathbf{B})$ the *atom structure* of $\mathbf{B}$.

Given a class $\mathcal{K}$ of complete and atomic BAOs we define $\mathbf{At}\mathcal{K} = \{\mathrm{At}(\mathbf{B}) : \mathbf{B} \in \mathcal{K}\}$ and for a class $\mathcal{H}$ of BAO^{ca}-homomorphisms $\mathbf{At}\mathcal{H} = \{\mathrm{At}(h) : h \in \mathcal{H}\}$.

LEMMA 3.1.11. *Let* $\mathbf{A}, \mathbf{B} \in \mathsf{BA}$, $h : \mathbf{A} \longrightarrow \mathbf{B}$ *a Boolean homomorphism,* $a \in \mathrm{At}(\mathbf{A})$, $b \in \mathrm{At}(\mathbf{B})$ *and* $h(a) \geq b$. *Then, for any* $x \in \mathbf{A}$, $h(x) \geq b$ *implies* $x \geq a$.

PROOF. Assume that $h(a) \geq b$, $h(x) \geq b$ and $x \not\geq a$. Now a is an atom of $\mathbf{A}$ and thus $x \wedge a = \mathbb{0}^{\mathbf{A}}$. But h is a homomorphism and hence $h(x) \wedge h(a) = h(x \wedge a) = \mathbb{0}^{\mathbf{B}}$. By assumption $h(x) \geq b$ and $h(a) \geq b$ so $\mathbb{0}^{\mathbf{B}} = h(x) \wedge h(a) \geq b$ contradicting the fact that b is an atom of $\mathbf{B}$. $\square$

PROPOSITION 3.1.12. At *is a contravariant functor from* BAO^{ca} *to* Str.

Proof. Clearly, for any $\mathbf{B} \in \mathsf{BAO}^{ca}$, it follows that $\mathrm{At}(\mathbf{B}) \in \mathsf{Str}$. Now let $\mathbf{A}, \mathbf{B} \in \mathsf{BAO}^{ca}$ and $h : \mathbf{A} \longrightarrow \mathbf{B}$ be a complete BAO-homomorphism and $f \in F$ with $ar(f) = n$.

To show that $\mathrm{At}(h)(b)$ is defined for any $b \in \mathrm{At}(\mathbf{B})$, note that since $\bigvee \mathrm{At}(\mathbf{A}) = \mathbb{1}$ and h is complete, we have $\bigvee\{h(a) : a \in \mathrm{At}(\mathbf{A})\} = \mathbb{1} \geq b$. However b is an atom, so it follows by complete distributivity that $b \leq h(a)$ for some $a \in \mathrm{At}(\mathbf{A})$.

Let $a, a' \in \mathrm{At}(\mathbf{A})$ and $b \in \mathrm{At}(B)$. By the lemma above if $h(a) \geq b$ and $h(a') \geq b$ then $a = a'$. Hence $\mathrm{At}(h)$ is a well defined function.

However we still need to show that, for any BAO^{ca}-homomorphism h, $\mathrm{At}(h)$ is a bounded morphism. Note that for any $b \in \mathrm{At}(\mathbf{B})$ it follows that $h(\mathrm{At}(h)(b)) \geq b$.

zig: Let $R_f^{\mathrm{At}(\mathbf{B})}(b_0, \ldots, b_{n-1}, y)$, for some $b_0, \ldots, b_{n-1} \in \mathrm{At}(\mathbf{B})$ and $y \in \mathrm{At}(\mathbf{B})$. From the remark above and the fact that f is order preserving it follows that

$$f^{\mathbf{B}}\big(h(\mathrm{At}(h)(b_0)), \ldots, h(\mathrm{At}(h)(b_{n-1})))\big) \geq f^{\mathbf{B}}(b_0, \ldots, b_{n-1})$$

By assumption $f^{\mathbf{B}}(b_0, \ldots, b_{n-1}) \geq y$ so

$$h\big(f^{\mathbf{A}}(\mathrm{At}(h)(b_0), \ldots, \mathrm{At}(h)(b_{n-1}))\big) = f^{\mathbf{B}}\big(h(\mathrm{At}(h)(b_0), \ldots, h(\mathrm{At}(h))(b_{n-1}))\big) \geq y.$$

But by the definition of $\mathrm{At}(h)$ this implies that

$$f^{\mathbf{A}}\big(\mathrm{At}(h)(b_0), \ldots, \mathrm{At}(h)(b_{n-1})\big) = \mathrm{At}(h)(y).$$

Hence $R_f^{\mathrm{At}(\mathbf{A})}(\mathrm{At}(h)(b_0), \ldots, \mathrm{At}(h)(b_{n-1}), \mathrm{At}(h)(y))$ as required.

zag: Let $g = \mathrm{At}(h)$ and assume that $R_f^{\mathrm{At}(\mathbf{A})}(a_0, \ldots, a_{n-1}, g(b))$, for $b \in \mathrm{At}(\mathbf{B})$ and $a_0, \ldots, a_{n-1} \in \mathrm{ran}(g)$. We will show that there exist $b_0, \ldots, b_{n-1} \in \mathrm{At}(\mathbf{B})$ such that $g(b_i) = a_i$ for $i < n$ and $R_f^{\mathrm{At}(\mathbf{B})}(b_0, \ldots, b_{n-1}, b)$.

By definition $R_f(a_0, \ldots, a_{n-1}, g(b))$ is equivalent to $f(a_0, \ldots, a_{n-1}) \geq g(b)$ and since h is a homomorphism it follows that

$$f(h(a_0), \ldots, h(a_{n-1})) = h(f(a_0, \ldots, a_{n-1})) \geq h(g(b)) \geq b.$$

Hence by Lemma 3.1.3 (p. 30) we can find $b_i \leq h(a_i)$, $b_i \in \mathrm{At}(\mathbf{B})$, for $i < n$ such that $f(b_0, \ldots, b_{n-1}) \geq b$. But this is equivalent to $g(b_i) = a_i$ and $R_f^{\mathrm{At}(\mathbf{B})}(b_0, \ldots, b_{n-1}, b)$.

To conclude we show that At preserves the identity map and reverses composition. Let $\mathbf{B} \in \mathsf{BAO}^{ca}$ and consider the identity map $\mathrm{Id} : \mathbf{B} \longrightarrow \mathbf{B}$. Clearly $\mathrm{Id}(b) \geq b$, for any $b \in \mathrm{At}(\mathbf{B})$, and so $\mathrm{At}(\mathrm{Id})(b) = b$.

Assume $\mathbf{A}$ $\mathbf{B}.\mathbf{C} \in \mathsf{BAO}^{ca}$ and $h : \mathbf{A} \longrightarrow \mathbf{B}$ and $g : \mathbf{B} \longrightarrow \mathbf{C}$ are complete BAO-homomorphisms. We need to show that $\mathrm{At}(h \circ g) = \mathrm{At}(g) \circ \mathrm{At}(h)$. But

$$\begin{aligned} \mathrm{At}(h) \circ \mathrm{At}(g)(c) = a &\text{ iff } h(a) \geq \mathrm{At}(g)(c) \\ &\text{ iff } g(h(a)) \geq c \\ &\text{ iff } \mathrm{At}(g \circ h)(c) = a, \end{aligned}$$

where the second line follows by the fact that BAO-homomorphisms are order preserving. □

Observe that Uf restricted to principal ultrafilters on complete and atomic BAOs is equivalent to At. Showing us that the At construction from BAO^{ca} to Str is related to the Uf construction from BAO to Str.

Completing the picture. This now allows us to find a morphism that will relate to the "Algebraic Logic" arrow in Figure 1. I.e. we can now construct a complete and atomic BAO $\mathrm{CmUf}\mathbf{B}$ if we are given some $\mathbf{B} \in \mathsf{BAO}$. Throughout the rest of this chapter, and in fact the next, we will spend a lot of effort analysing this construction.

DEFINITION 3.1.13. Let $\mathbf{B}$ be some Boolean algebra with operators. We define $\mathrm{Em} = \mathrm{CmUf}$ and call $\mathrm{Em}\mathbf{B}$ the *canonical embedding algebra* of $\mathbf{B}$. Let $\mathcal{K}$ be a class of BAOs, we define $\mathbf{Em}\mathcal{K} = \{\mathrm{Em}\mathbf{B} : \mathbf{B} \in \mathcal{K}\}$ and for a class $\mathcal{H}$ of BAO-homomorphisms $\mathbf{Em}\mathcal{H} = \{\mathrm{Em}(h) : h \in \mathcal{H}\}$.

We follow [**Gol89**] in referring to $\mathrm{Em}\mathbf{B}$ as the canonical embedding algebra.

Clearly $\mathbf{Em} = \mathbf{CmUf}$ and hence $\mathbf{Em}$ is a composition of functors. Then $\mathbf{Em}$ is itself a functor between the categories BAO and BAO^{ca}. (For more results on functors and the category of functors we refer the reader to [**McL95**] Chapter 8.) Thus we get the following commutative diagram, which, in the sequel, we will relate back to the paradigm triangle introduced at the beginning of this chapter.

$$\begin{tikzcd} \mathsf{Str} \arrow[rr, shift left, "\mathbf{Cm}"] & & \mathsf{BAO}^{ca} \arrow[ll, shift left, "\mathbf{At}"] \\ & \mathsf{BAO} \arrow[ul, "\mathbf{Uf}"] \arrow[ur, "\mathbf{Em}"'] & \end{tikzcd}$$

FIGURE 2. A categorical view of the Paradigm Triangle

Along the way we have also seen how we can turn any BAO into a complex algebra. However there is much more to this picture, as Jónsson and Tarski showed in their seminal work [**JoT51**].

THEOREM 3.1.14 (Jónsson and Tarski Representation Theorem). *Any Boolean algebra with operators can be embedded into a complex algebra. In particular, given a* BAO $\mathbf{B}$, *the representation function* $\mathrm{rep} : \mathbf{B} \longrightarrow \mathrm{CmUf}(\mathbf{B})$ *defined by*

$$\mathrm{rep}(a) = \{\gamma \in \mathrm{Uf}(\mathbf{A}) : a \in \gamma\}$$

is an embedding of $\mathbf{B}$ *into* $\mathrm{Em}\mathbf{B}$.

We refer the reader to the original proof in [**JoT51**] Theorem 3.10 and the more direct proof in [**BDV01**].

We now return to the dualities mentioned in Section 2.3. To that end we require the following lemma.

LEMMA 3.1.15. *Let* $\mathbf{U}, \mathbf{U}_\lambda, \mathbf{V}, \in \mathsf{Str}$ *and* $\mathbf{A}, \mathbf{A}_\lambda, \mathbf{B} \in \mathsf{BAO}$, *where* $\lambda \in \Lambda$.
- (i) *If* $\mathbf{U}$ *is an inner substructure of* $\mathbf{V}$, *then* $\mathrm{Cm}\mathbf{U}$ *is a homomorphic image of* $\mathrm{Cm}\mathbf{V}$.
- (ii) *If* $\mathbf{V}$ *is a bounded morphic image of* $\mathbf{U}$, *then* $\mathrm{Cm}\mathbf{V}$ *is a subalgebra of* $\mathrm{Cm}\mathbf{U}$.
- (iii) *If* $\mathbf{A}$ *is a subalgebra of* $\mathbf{B}$, *then* $\mathrm{Uf}(\mathbf{A})$ *is a bounded morphic image of* $\mathrm{Uf}(\mathbf{B})$.
- (iv) *If* $\mathbf{B}$ *is a homomorphic image of* $\mathbf{A}$, *then* $\mathrm{Uf}(\mathbf{B})$ *is a inner substructure of* $\mathrm{Uf}(\mathbf{A})$.
- (v) $\mathrm{Cm}(\biguplus_{\lambda\in\Lambda} \mathbf{U}_\lambda) \cong \prod_{\lambda\in\Lambda} (\mathrm{Cm}\mathbf{U}_\lambda)$.

PROOF. For the proofs of (i) to (iv) we refer the reader to the proof of Corollary 3.2.5. in [**Gol89**].

For (v) let γ_ι be the map $\mathbf{U}_\iota \longrightarrow \biguplus_{\lambda\in\Lambda} \mathbf{U}_\lambda$ defined by

$$\gamma_\iota(u_\iota) = \langle u_\iota, \iota\rangle,$$

where $\iota \in \Lambda$ and $u_\iota \in \mathbf{U}_\iota$. Clearly γ_ι is a bounded morphism and hence, by (i), $\mathrm{Cm}\gamma_\iota : \mathrm{Cm}\biguplus_{\lambda\in\Lambda} \mathbf{U}_\lambda \longrightarrow \mathrm{Cm}\mathbf{U}_\iota$ is a homomorphism. Using the γ_ι we can now define a homomorphism $h : \mathrm{Cm}(\biguplus_{\lambda\in\Lambda} \mathbf{U}_\lambda) \longrightarrow \prod_{\lambda\in\Lambda} \mathrm{Cm}\mathbf{U}_\lambda$ by letting

$$(*) \qquad h(X)(\lambda) = \mathrm{Cm}(\gamma_\lambda)(X) = \gamma_\lambda^{-1}(X).$$

To show that h is surjective first observe that, since $\mathrm{Cm}(\gamma)$ is itself surjective, for any $Y_\lambda \in \mathrm{Cm}\mathbf{U}_\lambda$ we can find a $Z_\lambda \in \mathrm{Cm}(\biguplus_{\lambda\in\Lambda} \mathbf{U}_\lambda)$ such that $\mathrm{Cm}(\gamma)(Z_\lambda) = Y_\lambda$. Let $Y_\lambda \in \mathrm{Cm}\mathbf{U}_\lambda$ for all $\lambda \in \Lambda$ then clearly there exist Z_λ such that

$$h(\bigcup_{\lambda\in\Lambda} Z_\lambda) = \langle \mathrm{Cm}(\gamma_0)(Z_0), \ldots, \mathrm{Cm}(\gamma_\lambda)(Z_\lambda), \ldots\rangle = \langle Y_0, \ldots, Y_\lambda, \ldots\rangle.$$

Now assume that $h(Y) = h(Z)$ then, for any $\langle x, \lambda\rangle \in Y$, it follows from $(*)$ that $x \in \mathrm{Cm}(\gamma_\lambda)(Y) = \mathrm{Cm}(\gamma_\lambda)(Z) = \gamma_\lambda^{-1}(Z)$. Hence $\langle x, \lambda\rangle \in Z$ showing that $Y \subseteq Z$. By a similar argument we get $Z \subseteq Y$ and hence h is injective. □

As a reference to the reader we collect the most important results on class operators below. For two class operators $\mathbf{O}_0$ and $\mathbf{O}_1$ and a class $\mathcal{K}$ we write $\mathbf{O}_0 \leq \mathbf{O}_1$ if $\mathbf{O}_0\mathcal{K} \subseteq \mathbf{O}_1\mathcal{K}$.

LEMMA 3.1.16.

(i) $\mathbf{P_w} \leq \mathbf{P_u}$.
(ii) $\mathbf{P_w U_d} \leq \mathbf{P_u U_d} \leq \mathbf{H_b U_d P_u}$.
(iii) $\mathbf{CmS_b} \leq \mathbf{HCm}$.
(iv) $\mathbf{CmH_b} \leq \mathbf{SCm}$.
(v) $\mathbf{CmU_d} = \mathbf{PCm}$.
(vi) $\mathbf{UfH} \leq \mathbf{S_bUf}$.
(vii) $\mathbf{UfS} \leq \mathbf{H_bUf}$.
(viii) $\mathbf{P_uCm} \leq \mathbf{SCmP_u}$.
(ix) $\mathbf{P_wCm} \leq \mathbf{SCmP_w}$.

The first result (i) follows directly from the definition of $\mathbf{P_w}$. (ii) follows by (i) and Lemma 4.1.24 (p. 76). Results (iii) through (vii) follow directly from Lemma 3.1.15. For (viii) and (ix) we refer the reader to Theorem 4.3.4 (p. 80).

3.2. Polymodal Logic

We now have the basic concepts we require to start looking at the language and logic associated with a class of BAOs. In essence this section introduces the "Logic" part of the Paradigm Triangle, c.f. Figure 1. The definitions and results in this section rely heavily on those originally presented in [**Gol99**], but have been adapted to our current notation. (We recommend [**Gol99**] to the reader as an excellent survey of the field of polymodal logics.) In the literature polymodal logics are also referred to as multimodal logics.

3.2.1. Language.

A natural way to extend propositional languages is to add so called *modalities* to the signature. These modalities are used to express concepts like *possibility/necessity*, *eventually/henceforth* or *it is permissible/it ought to be*. Normally the symbols $\Diamond$ and $\Box$ are used in formal languages to denote modalities and referred to as the *diamond* and *box modalities*. The concepts are also assumed to be dual to each other in the following way

$$\Box = \neg\Diamond\neg \text{ and } \Diamond = \neg\Box\neg.$$

Eventually we require languages that have multiple modalities. We will thus index our modalities with ordinals β and present them in the form $\langle\beta\rangle$ and $[\beta]$ with the understanding that they are dual, i.e.

$$[\beta] = \neg\langle\beta\rangle\neg \text{ and } \langle\beta\rangle = \neg[\beta]\neg.$$

In standard propositional languages we are normally limited to a countable set of propositional variables. We will however consider a more general scenario here in that we assume a (possibly uncountable) infinite set of propositional variables p_η, where $\eta < \xi$ for some infinite ξ. We denote the set of variables associated with ξ by

$$\Phi_\xi = \{p_\eta : \eta < \xi\}.$$

Let α and ξ be ordinals. The modal language $L_\xi(\alpha)$ is generated by Φ_ξ, the usual boolean connectives and the diamond modalities $\{\langle\beta\rangle : \beta < \alpha\}$. The set of formulas ψ of $L_\xi(\alpha)$ is given by

$$\psi ::= p_\eta \mid \bot \mid \neg\psi \mid \psi_0 \vee \psi_1 \mid \langle\beta\rangle\psi \quad ^\ddagger$$

where η ranges over ordinals less than ξ and β over ordinals less than α. The other boolean connectives $\wedge$, $\to$, $\top$ and $\leftrightarrow$ are taken as the usual abbreviations, with $[\beta]$ defined by $[\beta]\psi = \neg\langle\beta\rangle\neg\psi$.

Note that the definition of ψ above restricts the modalities $\langle\beta\rangle$ to only one argument. We thus call the languages $L_\xi(\alpha)$ unary modal languages. The standard modal language is then $L_\omega(1)$ and modal languages with two modalities, e.g. temporal logics, are $L_\omega(2)$. For a comprehensive survey of modal languages we refer the reader to the recent book [**BDV01**] by Blackburn, de Rijke and Venema.

We will see later that these modalities are related to operators in the language of BAOs. However we have not limited ourselves to unary operators for BAOs and hence it can be expected that we will have to consider modalities of arbitrary finite arity. Let Σ be a set of symbols and let each $\sigma \in \Sigma$ have an associated arity, $ar(\sigma) = n$. We then define the polymodal language $L_\xi(\Sigma)$ associated with some ordinal ξ. The formulas ψ of $L_\xi(\Sigma)$ are specified by

$$\psi ::= p_\eta \mid \bot \mid \neg\psi \mid \psi_0 \vee \psi_1 \mid \langle\sigma\rangle(\psi_0, \ldots, \psi_{ar(\sigma)-1})$$

for $\eta < \xi$ and $\sigma \in \Sigma$. The box modality of arity $ar(\sigma)$ dual to $\langle\sigma\rangle$ is defined by

$$[\sigma](\psi_0, \ldots, \psi_{ar(\sigma)-1}) = \neg\langle\sigma\rangle(\neg\psi_0, \ldots, \neg\psi_{ar(\sigma)-1}).$$

‡E.g. given formulas ϕ and χ you can construct a new formula ψ where $\psi = \langle\beta\rangle(\phi \vee \chi)$.

3.2.2. Logic.

Generally when we look at formal languages we are interested in special subsets of such languages. In particular we are interested in subsets that only contain theorems, i.e. subsets that only contain "true" formulas. Such a subset will be referred to as a logic and is formally defined as follows.

DEFINITION 3.2.1. Let ξ be some ordinal and Σ a set of functional symbols. A subset Λ of $L_\xi(\Sigma)$-formulas is called a *logic* if the following conditions hold:

(i) Λ contains all propositional tautologies of the language $L_\xi(\Sigma)$ and
(ii) Λ is closed under the inference rule of *Modus Ponens*, i.e.

$$\text{if } \psi, \psi \to \phi \in \Lambda \text{ then } \phi \in \Lambda.$$

The members of a logic Λ are called *theorems* and we write $\vdash_\Lambda \psi$ to denote that ψ is a theorem of Λ, i.e. $\psi \in \Lambda$.

For the following sections we will assume that Σ and ξ are fixed and that we are working with some particular logic Λ in $L_\xi(\Sigma)$, with its own associated propositional calculus. All formulas will be assumed to come from the language $L_\xi(\Sigma)$, unless otherwise stated. In general we will just write $L(\Sigma)$ instead of the more correct $L_\xi(\Sigma)$. If the logic Λ is clear from the context we will drop the superscript and write $\vdash_\Lambda$ simply as $\vdash$.

We now know what theorems are. If we however wish to know what "truths" we can deduce from some given set of formulas this idea is insufficient. We thus extend the use of $\vdash$.

DEFINITION 3.2.2. If $\Gamma \cup \{\psi\}$ is a set of formulas, then ψ is Λ-*deducible* from Γ, denoted $\Gamma \vdash \psi$, if there exist finitely many $\phi_0, \ldots, \phi_{n-1} \in \Gamma$ such that

$$\vdash (\phi_0 \wedge \cdots \wedge \phi_{n-1}) \to \psi,$$

if $n = 0$ this reduces to $\vdash \psi$. On the other hand ψ not being Λ-deducible from Γ is denoted by $\Gamma \nvdash \psi$.

This now allows us the tools to describe whether a set of formulas in a formal language is compatible with a logic. I.e. we want to know when a set of formulas does not lead to a contradiction (i.e. $\bot$). Such sets are called consistent and are formally defined as follows.

DEFINITION 3.2.3. Let Γ be a set of $L(\Sigma)$-formulas. Γ is said to be Λ-*consistent* if $\Gamma \nvdash \bot$, while a formula ψ is Λ-consistent if $\{\psi\}$ is Λ-consistent.

Γ is said to be Λ-*maximal* if it is Λ-consistent and any set of formulas containing Γ is Λ-inconsistent.

An example of a Λ-consistent set is the empty set $\emptyset$. Note however that $\emptyset$ is not Λ-maximal since $\emptyset \subset \{\top\}$.

Before continuing we first make two trivial observations about Λ-consistent sets.

PROPOSITION 3.2.4. *Let Γ be a Λ-consistent set of formulas. For any formula ψ*

(i) *if ψ is Λ-deducible from Γ then $\Gamma \cup \{\psi\}$ is Λ- consistent, and*
(ii) *if $\Gamma \neq \emptyset$ then either $\Gamma \cup \{\psi\}$ or $\Gamma \cup \{\neg\psi\}$ is Λ-consistent, but not both.*

PROOF. We will only prove the first part since the second follows easily from (i). Let Γ be a Λ-consistent set. Assume that ψ is Λ-deducible from Γ and that $\Gamma \cup \{\psi\}$ is Λ-inconsistent. Since Γ is consistent it follows that there exists a finite (possibly empty) sequence of formulas $\phi_0, \ldots, \phi_{n-1} \in \Gamma$ such that

$$\vdash (\phi_0 \wedge \ldots \wedge \phi_{n-1} \wedge \psi) \rightarrow \bot.$$

But since ψ is Λ-deducible from Γ there exists another finite (possibly distinct) sequence of formulas $\chi_0, \ldots, \chi_{m-1} \in \Gamma$ such that

$$\vdash (\chi_0 \wedge \ldots \wedge \chi_{m-1}) \rightarrow \psi.$$

From which we can deduce, by using the propositional calculus, that

$$\vdash (\chi_0 \wedge \ldots \wedge \chi_{m-1} \wedge \neg\psi) \rightarrow \bot$$

and so

$$\vdash (\chi_0 \wedge \ldots \wedge \chi_{m-1} \wedge \phi_0 \wedge \ldots \phi_{n-1}) \rightarrow \bot.$$

Thus contradicting our assumption that Γ is Λ-consistent. □

We will soon see that Λ-maximal sets of formulas play a very important role in later constructions since they encode all the "relevant" properties of a given logic. We now take a look at some specific properties satisfied by these maximally consistent sets.

PROPOSITION 3.2.5. *Let Λ be a logic and Γ a Λ-maximal set of formulas then:*

(i) *Γ is closed under modus ponens,*
(ii) *$\Lambda \subseteq \Gamma$,*
(iii) *for all formulas ψ, either $\psi \in \Gamma$ or $\neg\psi \in \Gamma$, and*
(iv) *for all formulas ψ and ϕ, $\psi \wedge \phi \in \Gamma$ if, and only if, $\psi \in \Gamma$ and $\phi \in \Gamma$.*

PROOF. Let Γ be a Λ-maximal set of formulas.

(i): Assume that ψ and $\psi \rightarrow \phi$ are elements of Γ. Then by definition ϕ is Λ-deducible from Γ and hence by Proposition 3.2.4 $\Gamma \cup \{\phi\}$ is Λ-consistent but since Γ is Λ-maximal $\Gamma \cup \{\phi\} \subseteq \Gamma$.

(ii): By definition all $\psi \in \Lambda$ are Λ-deducible from Γ and so by a similar argument as above $\Gamma \cup \{\psi\} \subseteq \Gamma$ and hence $\Lambda \subseteq \Gamma$.

(iii): Let ψ be some formula. As was shown in Proposition 3.2.4 either $\Gamma \cup \{\psi\}$ or $\Gamma \cup \{\neg\psi\}$ are Λ-consistent. Hence by the maximality of Γ either $\Gamma \cup \{\psi\}$ or $\Gamma \cup \{\neg\psi\}$ is a subset of Γ.

(iv): For the forward direction we assume that for some formulas ψ and ϕ that $\psi \wedge \phi \in \Gamma$. Since Γ is Λ-consistent we know that

$$\nvdash (\psi \wedge \phi) \rightarrow \bot.$$

Λ contains all propositional tautologies hence $(\psi \wedge \phi) \rightarrow \psi \in \Lambda \subseteq \Gamma$. Thud by (i) $\psi \in \Gamma$. Similarly $\phi \in \Gamma$.

For the backward direction we assume that $\psi, \phi \in \Gamma$. Hence $\nvdash \psi \wedge \phi \rightarrow \bot$. So $\vdash \psi \wedge \phi$ from which the results follows by (ii).

□

In essence this proposition shows us what we have already suspected, i.e. that a Λ-consistent set of formulas locally mimics the behavior of the logic. In some sense it is "closed" under the logic.

It is however not obvious whether we always have such maximal sets. The following lemma gives us an important clue as to when we can expect Λ-maximal sets to exist. (Observe that if $\nvdash \bot$ there will at least exist some Λ-consistent sets.)

LEMMA 3.2.6 (Lindenbaum's Lemma). *Let Γ be a Λ-consistent set of formulas then there exists a Λ-maximal set Γ' such that $\Gamma \subseteq \Gamma'$.*

PROOF. We will prove this lemma by constructing a transfinite ascending chain of Λ-consistent sets and then show that the limit of this chain is in fact Λ-maximal. Let Γ be a Λ-consistent set of formulas and

$$\psi_0, \psi_1, \ldots, \psi_\eta, \ldots, \eta < \xi$$

be a transfinite enumeration of all formulas in the language $L_\xi(\Sigma)$. We define an ascending chain of sets of formulas by

$$\begin{aligned}
\Gamma_0 &= \Gamma \\
\Gamma_{\eta+1} &= \begin{cases} \Gamma_\eta \cup \{\psi_\eta\} & \text{if this is } \Lambda\text{-consistent} \\ \Gamma_\eta \cup \{\neg\psi_\eta\} & \text{otherwise} \end{cases} \\
\Gamma_\eta &= \bigcup_{\lambda<\eta} \Gamma_\lambda \text{ if } \eta \text{ is a limit ordinal}
\end{aligned}$$

Finally we let $\Gamma' = \bigcup_{\eta<\xi} \Gamma_\eta$.

We first prove, by transfinite induction, that each Γ_η is Λ-consistent and hence Γ' is consistent. The base case follows from the fact that Γ is Λ-consistent. If Γ_η is consistent it follows directly from Proposition 3.2.4 that $\Gamma_{\eta+1}$ must be Λ-consistent.

Let us assume that η is a limit ordinal and that Γ_η is inconsistent. Then there exists a finite sequence of formulas $\psi_0, \ldots, \psi_{n-1}$ in Γ_η such that

$$\vdash (\psi_0 \wedge \ldots \wedge \psi_{n-1}) \to \bot.$$

By the construction of Γ_η and the fact that we are dealing with an ascending chain of sets there must exist a Γ_λ, with $\lambda < \eta$, such that $\psi_0, \ldots, \psi_{n-1} \in \Gamma_\lambda$. But this would mean that Γ_λ is inconsistent and hence contradict our induction hypothesis.

Claim: For each formula ψ exactly one of ψ or $\neg\psi$ is in Γ'.

Let ψ be some formula, then there exists some $\eta < \xi$ such that $\psi = \psi_\eta$. Thus by construction either $\psi \in \Gamma_{\eta+1}$ or $\neg\psi \in \Gamma_{\eta+1}$. Now assume that both ψ and $\neg\psi$ are in Γ' then from propositional calculus we know that $\vdash (\psi \wedge \neg\psi) \to \bot$ which would make Γ' inconsistent and thus lead to a contradiction.

To conclude we show that any set of formulas that properly contains Γ' must be Λ-inconsistent. Let Ψ be a set of formulas that properly contains Γ. Then there must exist some formula $\psi \in \Psi$ such that $\psi \notin \Gamma$. But then by our previous claim $\neg\psi \in \Gamma \subset \Psi$ and $\vdash (\psi \wedge \neg\psi) \to \bot$ which proves that Ψ must be Λ-inconsistent. □

Well-behaved logics. When studying logics, and modal logics in particular, it is important to know how such a logic responds to substituting some formulas for other formulas in a particular theorem.

DEFINITION 3.2.7. Let ψ and ϕ be formulas and p_η be a propositional variable. We denote the formula obtained from ψ by uniform replacement of p_η by ϕ as $\psi[p_\eta \mapsto \phi]$.

When convenient we will sometimes use the shorthand $\psi[\underline{p} \mapsto \underline{\phi}]$ for the formula $\psi[p_{\eta_0} \mapsto \phi_0] \cdots [p_{\eta_{n-1}} \mapsto \phi_{n-1}]$, where $\underline{p} = \langle p_{\eta_0}, \ldots, p_{\eta_{n-1}} \rangle$ and $\underline{\phi} = \langle \phi_{\eta_0}, \ldots, \phi_{\eta_{n-1}} \rangle$.

A logic where substitution is well-behaved is called uniform, formally defined as follows.

DEFINITION 3.2.8. A logic Λ is called *uniform* if it is closed under the rule of uniform substitution, i.e. if $\psi \in \Lambda$ and $\chi = \psi[\underline{p} \mapsto \underline{\phi}]$, then $\chi \in \Lambda$, where $\underline{p} = \langle p_{\eta_0}, \ldots, p_{\eta_{n-1}} \rangle$ is a sequence of propositional variables and $\underline{\phi} = \langle \phi_0, \ldots, \phi_{n-1} \rangle$ is a sequence of formulas.

Note that in contrast to the definition above some authors only consider a set of formulas to be a logic if it is already closed under uniform substitution (c.f. [**BDV01**]).

It should be clear from the definition above that uniform replacement is well behaved with regards to the logical connectives $\perp$, $\neg$, $\vee$ and the modality $\langle \sigma \rangle$.

PROPOSITION 3.2.9. *Let $\underline{p} = \langle p_{\eta_0}, \ldots, p_{\eta_{n-1}} \rangle$ be a sequence of propositional variables, $\underline{\phi} = \langle \phi_0, \ldots, \phi_{n-1} \rangle$ a sequence of formulas and $\sigma \in \Sigma$, with $ar(\sigma) = n$. Then for any formulas $\psi, \chi, \psi_0, \ldots, \psi_{n-1}$:*

(i) $p_{\eta_i}[\underline{p} \mapsto \underline{\phi}] = \phi_i$, *where* $i < n$,
(ii) $\perp[\underline{p} \mapsto \underline{\phi}] = \perp$,
(iii) $\neg(\psi[\underline{p} \mapsto \underline{\phi}]) = (\neg\psi)[\underline{p} \mapsto \underline{\phi}]$,
(iv) $(\phi \vee \chi)[\underline{p} \mapsto \underline{\phi}] = \phi[\underline{p} \mapsto \underline{\phi}] \vee \chi[\underline{p} \mapsto \underline{\phi}]$, *and*
(v) $\langle \sigma \rangle(\psi_0, \ldots, \psi_{n-1})[\underline{p} \mapsto \underline{\phi}] = \langle \sigma \rangle(\psi_0[\underline{p} \mapsto \underline{\phi}], \ldots, \psi_{n-1}[\underline{p} \mapsto \underline{\phi}])$.

Lastly we define two particular classes of well-behaved logics that will become important later in this chapter. Note how conditions **K** and **N** bear a particular resemblance to the operator conditions of additivity and normality.

DEFINITION 3.2.10. A logic Λ is called *normal* if it contains the following schema

K: $\langle \sigma \rangle(\psi_0, \ldots, \phi \vee \chi, \ldots, \psi_{ar(\sigma)-1}) \leftrightarrow$
$\langle \sigma \rangle(\psi_0, \ldots, \phi, \ldots, \psi_{ar(\sigma)-1}) \vee \langle \sigma \rangle(\psi_0, \ldots, \chi, \ldots, \psi_{ar(\sigma)-1})$,
N: $\langle \sigma \rangle(\psi_0, \ldots, \perp, \ldots, \psi_{ar(\sigma)-1}) \to \perp$,

and satisfies the *Monotonicity* rule, i.e.

Mono: if $\phi \to \chi \in \Lambda$, then
$\langle \sigma \rangle(\psi_0, \ldots, \phi, \ldots, \psi_{ar(\sigma)-1}) \to \langle \sigma \rangle(\psi_0, \ldots, \chi, \ldots, \psi_{ar(\sigma)-1}) \in \Lambda$.

3.3. Relational Semantics

Before we continue we would just like to make the reader aware that we are about to introduce two specialised concepts of satisfiability. To differentiate them from the standard notion of satisfiability ($\models$), defined in Section 2.2.3, we will use the symbol $\Vdash$ when referring to satisfiability in this context.

As mentioned at the beginning of this chapter each logic Λ has an associated relational semantics, c.f. Figure 1. The introduction of these semantics for unary modal languages is attributed to S. Kripke (c.f. [**Kri59**]) and accordingly quite often referred to in the literature as Kripke semantics. (Note that there are strong antecedents of Kripke's work in [**JoT51**].)

This original semantics of a unary modal language associates "possible world" models with such a language. A formula $\Diamond p$ is considered *true at a particular world* if there is a world, reachable from the current one, where p is true, i.e. p is currently possible. Let us analyse this idea from an algebraic perspective. There seem to be two main concepts underlying this type of semantics. Firstly we can view the set of all worlds as the universe of a relational structure where its single (binary) relation encodes the idea of which worlds can be reached from other worlds. Hence the reference to the relational semantics of modal logics. (The relations we are talking about are often referred to as *accessibility relations.*) Secondly each propositional variable has a related set of worlds at which it is true, thus we have a mapping, called a *valuation*, that to each propositional variable assigns the set of worlds at which it is true. single (binary) relation encodes the idea of which worlds can be reached from other worlds. (Such relations are sometimes referred to as accessibility relations.) Hence the reference to the relational semantics of modal logics.

We will now proceed to extend the above mentioned concept of relational models to polymodal languages in the style of [**Gol99**].

DEFINITION 3.3.1. A *model* for a polymodal language $L_\xi(\Sigma)$ is a pair $\mathbf{M} = \langle \mathbf{F}, V\rangle$ where $\mathbf{F} = \langle W, \langle R^{\mathbf{F}}_\sigma : \sigma \in \Sigma\rangle\rangle$ is a relational structure (also called a *frame*), with $ar(R_\sigma) = ar(\sigma) + 1$, and

$$V : \Phi_\xi \longrightarrow \mathcal{P}(W)$$

is a *valuation* function assigning a subset $V(p_\eta)$ of W to each propositional variable p_η, where $\eta < \xi$. We say $\mathbf{M} = \langle \mathbf{F}, V\rangle$ is a *model on the frame* $\mathbf{F}$.

As mentioned $V(p_\eta)$ is thought of as the set of worlds where p_η is true. For convenience we will write $R^{\mathbf{F}}_\Sigma$ for $\langle R^{\mathbf{F}}_\sigma : \sigma \in \Sigma\rangle$. By a model $\mathbf{M} = \langle W, R^{\mathbf{M}}_\Sigma, V\rangle$ we mean the model $\mathbf{M} = \langle \mathbf{F}, V\rangle$ on the frame $\mathbf{F} = \langle W, R^{\mathbf{F}}_\Sigma\rangle$.

DEFINITION 3.3.2. Let $\mathbf{M} = \langle W, R^{\mathbf{M}}_\Sigma, V\rangle$ be a model. We then define the *satisfaction relation* "ψ is true at a world w in $\mathbf{M}$", denoted $\mathbf{M}, w \Vdash \psi$, inductively as follows

$$\begin{array}{rcl} \mathbf{M}, w \Vdash p_\eta & \text{iff} & w \in V(p_\eta) \\ \mathbf{M}, w \nVdash \bot & & \\ \mathbf{M}, w \Vdash \neg\psi & \text{iff} & \mathbf{M}, w \nVdash \psi \\ \mathbf{M}, w \Vdash \psi \vee \phi & \text{iff} & \mathbf{M}, w \Vdash \psi \text{ or } \mathbf{M}, w \Vdash \phi \\ \mathbf{M}, w \Vdash \langle\sigma\rangle(\psi_0, \ldots, \psi_{n-1}) & \text{iff} & \text{for some } w_0, \ldots, w_{n-1} \in W, \\ & & R^{\mathbf{F}}_\sigma(w_0, \ldots, w_{n-1}, w) \text{ and} \\ & & \mathbf{M}, w_i \Vdash \psi_i \text{ for all } i < n, \end{array}$$

where $n = ar(\sigma)$.

DEFINITION 3.3.3. Let $\mathbf{M} = \langle W, R^{\mathbf{M}}_\Sigma, V\rangle$ be a model. A formula ψ is *true in the model* $\mathbf{M}$, denoted $\mathbf{M} \Vdash \psi$, if it is true at all the worlds of $\mathbf{M}$, i.e.

$$\mathbf{M} \Vdash \psi \text{ iff } \mathbf{M}, w \Vdash \psi \text{ for all } w \in W.$$

A set of formulas Γ is true in the model $\mathbf{M}$, in symbols $\mathbf{M} \Vdash \Gamma$, if for each formula $\psi \in \Gamma$, $\mathbf{M} \Vdash \psi$.

DEFINITION 3.3.4. Let $\mathbf{F} = \langle W, R_\Sigma^{\mathbf{F}} \rangle$ be a frame. A formula ψ is *valid in the frame* $\mathbf{F}$, written $\mathbf{F} \Vdash \psi$, if

$$\mathbf{M} \Vdash \psi \text{ for all models } \mathbf{M} = \langle \mathbf{F}, V \rangle \text{ on the frame } \mathbf{F}.$$

Similarly to the definition for models, a set of formulas Γ is valid in the frame $\mathbf{F}$, denoted $\mathbf{F} \Vdash \Gamma$, if for each formula $\psi \in \Gamma$, $\mathbf{F} \Vdash \psi$.

DEFINITION 3.3.5. Let $\mathcal{K}$ be a class of models (or frames) and $\Gamma \cup \{\psi\}$ a set of formulas. We say a formula ψ is a *semantic consequence* of Γ over $\mathcal{K}$, denoted $\mathcal{K} \Vdash_\Gamma \psi$, if for all models $\mathbf{M}$ from $\mathcal{K}$ and worlds w in $\mathbf{M}$ whenever $\mathbf{M}, w \Vdash \Gamma$ we have that $\mathbf{M}, w \Vdash \psi$. If $\Gamma = \emptyset$ we simply write $\mathcal{K} \Vdash \psi$.

We are particularly interested in when a class of frames (or models) matches with some logic, i.e. given a logic is there a class of frames in which precisely all the formulas in the logic are valid. (This subject is often referred to as *correspondence theory* in the literature, c.f. [**vBe84**].) Hence the following two definitions.

DEFINITION 3.3.6. **(Soundness)** Let $\mathcal{K}$ be a class of frames (or models). A logic Λ is *sound* with respect to $\mathcal{K}$ if $\vdash \psi$ implies $\mathbf{M} \Vdash \psi$ for all $\mathbf{M}$ from $\mathcal{K}$.

Equivalently we could have required that $\vdash \psi$ implies $\mathcal{K} \Vdash \psi$.

DEFINITION 3.3.7. **(Completeness)** Let $\mathcal{K}$ be a class of frames (or models). A logic Λ is *strongly complete* with respect to $\mathcal{K}$ if for any set $\Gamma \cup \{\psi\}$ of formulas, $\mathcal{K} \Vdash_\Gamma \psi$ implies that $\Gamma \vdash \psi$. A logic Λ is *(weakly) complete* with respect to $\mathcal{K}$ if $\mathcal{K} \Vdash \psi$ implies that $\vdash \psi$.

Note that (weak) completeness is the special case of strong completeness where the set Γ is empty. Hence strong completeness implies (weak) completeness.

To get back to more syntactic issues for a while let us look back at Λ-consistent sets of formulas. In some respect we would expect consistent sets of formulas to at least contain formulas that are satisfiable, thus providing us with another link between a logic and its relational semantics. In fact we get much more than that.

LEMMA 3.3.8. *A logic Λ is strongly complete with respect to a class of structures $\mathcal{K}$ if, and only if, every Λ- consistent set of formulas is satisfiable in some $\mathbf{U} \in \mathcal{K}$. Λ is complete with respect to $\mathcal{K}$ if, and only if, every Λ-consistent formula is satisfiable in some $\mathbf{U} \in \mathcal{K}$.*

PROOF. From the remark above it is only necessary to prove the result for strong completeness since the result for completeness will follow by letting $\Gamma = \emptyset$.

Suppose that Λ is not strongly complete with respect to $\mathcal{K}$. Then there is a set of formulas $\Gamma \cup \{\psi\}$ such that $\mathcal{K} \Vdash_\Gamma \psi$ and $\Gamma \nvdash \psi$. But then $\Gamma \cup \{\neg\psi\}$ is Λ- consistent, but not satisfiable in $\mathcal{K}$.

We assume that $\Gamma \cup \{\psi\}$ is Λ-consistent, but not satisfiable in $\mathcal{K}$. It then follows from the definition of $\Vdash_\Gamma$ that $\mathcal{K} \Vdash_\Gamma \neg\psi$. Hence by strong completeness $\Gamma \vdash \neg\psi$. But then $\Gamma \cup \{\psi\}$ is Λ-inconsistent contradicting our assumption. □

3.3.1. Characterising logics by relational structures.

Now we know what it would mean for a class $\mathcal{K}$ of models (or frames) to *characterise* a logic Λ, the logic would have to be sound and complete with respect to

$\mathcal{K}$. Can we however, given a logic Λ, construct some canonical class of models (or frames) to characterise Λ? We will see that the question for models has a definite answer but not so for frames.

Canonical models and frames. To construct such a model it is sufficient to show that each Λ- consistent set of formulas Γ is satisfiable (c.f. Lemma 3.3.8).

We first restrict ourselves to a language $L(\Sigma)$ without any modalities (i.e. $\Sigma = \emptyset$). Let Γ be Λ-consistent then, as was shown in Lindenbaum's Lemma (c.f. 3.2.6), Γ is contained in some Λ-maximal set Γ'. (Note that by definition this Γ' is unique.) Could we now construct a model which has Λ-maximal sets as its worlds, i.e. the set of worlds $W = \{\Gamma' : \Gamma' \text{ is } \Lambda\text{-maximal}\}$, and where Γ would be satisfiable at Γ'? For this to be true each formula $\psi \in \Gamma$ must be satisfiable at the world Γ'. As was shown in Proposition 3.2.5 we already know that Γ' would respect all the boolean connectives. If we thus just take care of which propositional variables are true at a world, we could make Γ satisfiable at the world Γ'. In particular we want all $p_\eta \in \Gamma$ true at Γ'. As we have seen this is the functionality provided by a valuation, it tells us which propositional variables are true where. So, for the set of worlds W consisting of all Λ-maximal sets, we define

$$V(p_\eta) = \{\Gamma' \in W : p_\eta \in \Gamma'\},$$

where $\eta < \xi$. Note that we have made it equivalent to say that a formula ψ is an element of Γ' and that ψ is satisfiable at Γ'.

However this is not enough as we want to deal with languages that include modalities, i.e. languages where $\Sigma \neq \emptyset$. So let us assume that $\psi = \langle\sigma\rangle(\psi_0, \ldots, \psi_{ar(\sigma)-1})$, for $\sigma \in \Sigma$, is an element of some Λ-consistent Γ. Then ψ is obviously an element of Γ', the Λ- maximal set containing Γ, and we would expect it to be satisfiable at the world Γ'. The question is now which, if any, worlds should be related to Γ' so that ψ is satisfied. If we take a look back at the definition of satisfiability we see that Γ' must be related to those worlds in which the ψ_i are satisfiable. By construction we would expect ψ_i to be satisfiable at a world Ψ_i' if $\psi_i \in \Psi_i'$. Hence $R_\sigma(\Psi_0, \ldots, \Psi_{ar(\sigma)-1}, \Gamma)$ should hold if, and only if, we know that $\langle\sigma\rangle(\psi_0, \ldots, \psi_{ar(\sigma)-1}) \in \Gamma$, for $\psi_i \in \Psi_i$ and $i < ar(\sigma)$.

We conclude this section with a number of formal definitions and results which aim to make concise what we have been discussing.

DEFINITION 3.3.9. Given a logic Λ in the language $L_\xi(\Sigma)$, we define the *canonical Λ-frame* to be the frame, $\mathrm{CanF}_\xi(\Lambda) = \langle \mathrm{CanF}_\xi(\Lambda), R_\Sigma^{\mathrm{CanF}_\xi(\Lambda)}\rangle$, with universe the set of all Λ-maximal sets and, for each $\sigma \in \Sigma$ with $ar(\sigma) = n$,

$$R_\sigma^{\mathrm{CanF}_\xi(\Lambda)}(\Psi_0, \ldots, \Psi_{n-1}, \Gamma) \text{ iff } \{\langle\sigma\rangle(\psi_0, \ldots, \psi_{n-1}) : \psi_i \in \Psi_i \text{ for all } i < n\} \subseteq \Gamma,$$

where $\Psi_0, \ldots, \Psi_{n-1} \in \mathrm{CanF}_\xi(\Lambda)$. When ξ is clear from the context we write $\mathrm{CanF}(\Lambda)$ instead of $\mathrm{CanF}_\xi(\Lambda)$

By the *canonical Λ-valuation* we mean a valuation V on $\mathrm{CanF}(\Lambda)$ such that

$$V(p_\eta) = \{w \in \mathrm{CanF}(\Lambda) : p_\eta \in w\},$$

for $\eta < \xi$.

Finally we define the *canonical Λ-model*, denoted $\mathrm{CanM}(\Lambda)$, to be the model $\langle \mathrm{CanF}(\Lambda), V\rangle$, where V is the canonical Λ-valuation.

Observe that we follow the algebraic tradition here in using direct images to define R_σ. This definition is convenient for our current approach in that it mirrors the definition of **Uf** presented earlier. For an alternative definition of R_σ using inverse images we refer the reader to [**Gol99**].

LEMMA 3.3.10. *Let Λ be a normal modal logic and $\sigma \in L_{\mathsf{BAO}} \setminus L_{\mathsf{BA}}$ with $ar(\sigma) = n$. For any $w \in \mathrm{CanF}(\Lambda)$ we have that:*

(i) *For any worlds $w_0, \ldots, w_{n-1} \in \mathrm{CanF}(\Lambda)$, $R_\sigma(w_0, \ldots, w_{n-1}, w)$ if, and only if, for all formulas $\psi_0, \ldots, \psi_{n-1}$, $[\sigma](\psi_0, \ldots, \psi_{n-1}) \in w$ implies $\psi_i \in w_i$ for some $i < n$.*

(ii) *Suppose $\langle\sigma\rangle(\psi_0, \ldots, \psi_{ar(\sigma)-1}) \in w$. Then there exist $w_0, \ldots, w_{n-1} \in \mathrm{CanF}(\Lambda)$ such that $R_\sigma(w_0, \ldots, w_{n-1}, w)$ and $\psi_i \in w_i$ for all $i < n$.*

PROOF. We will prove the forward implication of the first part. For the second part we refer the reader to [**BDV01**].

Suppose $R_\sigma(w_0, \ldots, w_{n-1}, w)$ and that $[\sigma](\psi_0, \ldots, \psi_{n-1}) \in w$. To arrive at a contradiction we assume that $\psi_i \notin w_i$ for all $i < n$. Since each w_i is Λ-maximal, by Proposition 3.2.5, we have $\neg\psi_i \in w_i$ for each $i < n$. As $R_\sigma(w_0, \ldots, w_{n-1}, w)$, it follows that $\langle\sigma\rangle(\neg\psi_0, \ldots, \neg\psi_{n-1}) \in w$. But w is also Λ-maximal, which implies that $[\sigma](\psi_0, \ldots, \psi_{n-1}) = \neg\langle\sigma\rangle(\neg\psi_0, \ldots, \neg\psi_{n-1}) \notin w$. □

We can now make concise the notion that membership in a world is directly related to satisfiability at that world.

LEMMA 3.3.11. *Let Λ be a normal logic and $w \in \mathrm{CanF}(\Lambda)$. Then for any formula ψ*

$$\mathrm{CanM}(\Lambda), w \Vdash \psi \textit{ iff } \psi \in w.$$

PROOF. Let $\mathbf{M} = \langle W, R_\Sigma, V\rangle$ be the canonical model $\mathrm{CanM}(\Lambda)$. We prove this lemma by induction on the form of ψ. If $\psi = p_\eta$ (the base case) the result follows by the definition of the canonical valuation. The Boolean cases follow from Proposition 3.2.5. So we assume that $\psi = \langle\sigma\rangle(\psi_0, \ldots, \psi_{n-1})$.

For the forward direction observe that $\mathbf{M}, w \Vdash \psi$ if, and only if, for all $i < n$ there exist w_i such that $\mathrm{R}_\sigma(w_0, \ldots, w_{n-1}, w)$ and $\mathbf{M}, w_i \Vdash \psi_i$. By the induction hypothesis $\mathrm{R}_\sigma(w_0, \ldots, w_{n-1}, w)$ and $\psi_i \in w_i$, for $i < n$, which, by the definition of R_σ, implies that $\psi \in w$.

For the backward direction assume $\psi \in w$. We want to show that $\mathbf{M}, w \Vdash \psi$. Thus we must find worlds w_i such that $R_\sigma(w_0, \ldots, w_{n-1}, w)$ and $\mathbf{M}, w_i \Vdash \psi_i$, for $i < n$. By the induction hypothesis it is enough to show that there exist w_i such that $R_\sigma(w_0, \ldots, w_{n-1}, w)$ and $\psi_i \in w_i$, for $i < n$, but this is exactly what the second part of Lemma 3.3.10 guarantees. □

We were looking to show that Λ would be complete with respect to its canonical model, we however get even more than that.

THEOREM 3.3.12. *Every normal modal logic Λ is strongly complete with respect to its canonical model $\mathrm{CanM}(\Lambda)$, i.e. for any set of formulas $\Gamma \cup \{\psi\}$*

$$\mathrm{CanM}(\Lambda) \Vdash_\Gamma \psi \textit{ implies } \Gamma \vdash \psi.$$

PROOF. By Lemma 3.3.8 it is enough to show that each Λ-consistent set is satisfiable in $\mathrm{CanM}(\Lambda)$. Let Γ be any Λ-consistent set of formulas. By Lindenbaum's Lemma (c.f. 3.2.6) there exists a Λ-maximal $\Gamma' \supseteq \Gamma$ and from the previous lemma $\mathrm{CanM}(\Lambda), \Gamma' \Vdash \Gamma$. □

Finally it is an easy step to show the canonicity of this construction.

COROLLARY 3.3.13. *Any normal logic* Λ *is characterised by its canonical model* $\mathrm{CanM}(\Lambda)$, *i.e. for any formula* ψ

$$\mathrm{CanM}(\Lambda) \Vdash \psi \textit{ iff } \vdash \psi.$$

It immediately follows that, for any normal logic Λ and formula ψ,

$$\mathrm{CanF}(\Lambda) \Vdash \psi \text{ implies } \vdash \psi,$$

however the converse does not hold (c.f. [**Gol99**] Theorem 5.3.5 for a counter example).

The corollary above gives us a precise way of relating logics to models. Coming back to the Paradigm Triangle, it shows us how the "Correspondence Theory" arrow from the "Logic" block to the "Relational Semantics" block works.

Logics generated by relational structures. To complete this section we take a look at how to define a logic given a class of frames (or models). set of formulas that are valid (or true) in this class of frames (or models).

DEFINITION 3.3.14. For any model (or frame) $\mathbf{M}$ we define the *logic generated by the model (or frame)* $\mathbf{M}$ to be $\Lambda(\mathbf{M}) = \{\psi : \mathbf{M} \Vdash \psi\}$.

This definition is in fact jumping the gun. We seem to already be assuming that such a set of formulas is a logic. This is in fact true, but we get even more.

PROPOSITION 3.3.15. *Let* $\mathbf{M}$ *be a model and* $\mathbf{F}$ *be a frame then*

(i) $\Lambda(\mathbf{M})$ *is a normal logic and*
(ii) $\Lambda(\mathbf{F})$ *is a normal uniform logic.*

PROOF. Let $\sigma \in \Sigma$ with $ar(\sigma) = n$.

(i): The fact that all propositional tautologies are in $\Lambda(\mathbf{M})$ follows immediately, since the satisfaction relation respects $\bot$, $\neg$ and $\vee$.

Modes ponens: To see that $\Lambda(\mathbf{M})$ is closed under modes ponens, let ψ and $\psi \to \phi$ be in $\Lambda(\mathbf{M})$ and $\phi \notin \Lambda(\mathbf{M})$. Then there exists a world $w \in \mathbf{M}$ such that $\mathbf{M}, w \nVdash \phi$ and, since $\psi \to \phi \in \Lambda(\mathbf{M})$, $\mathbf{M}, w \Vdash \neg\psi \vee \phi$. Hence $\mathbf{M}, w \Vdash \neg\psi$ contradicting $\psi \in \Lambda(\mathbf{M})$.

K: To arrive at a contradiction we assume that there exists a world w such that

$$\mathbf{M}, w \nVdash \langle\sigma\rangle(\psi_0, \dots, \phi \vee \chi, \dots, \psi_{n-1}) \to \langle\sigma\rangle(\psi_0, \dots, \phi, \dots, \psi_{n-1}) \vee \langle\sigma\rangle(\psi_0, \dots, \chi, \dots, \psi_{n-1}).$$

Consequently it must follow that $\mathbf{M}, w \Vdash \langle\sigma\rangle(\psi_0, \dots, \phi \vee \chi, \dots, \psi_{n-1})$ and that $\mathbf{M}, w \nVdash \langle\sigma\rangle(\psi_0, \dots, \phi, \dots, \psi_{n-1}) \vee \langle\sigma\rangle(\psi_0, \dots, \chi, \dots, \psi_{n-1})$. Thus there exist worlds $w_0, \dots, w_{n-1}$ such that $R_\sigma(w_0, \dots, w_{n-1})$ and $\mathbf{M}, w_i \Vdash \phi \vee \chi$. From the consequent it follows that for all $w'_0, \dots, w'_{n-1}$ where $R_\sigma(w'_0, \dots, w'_{n-1})$ $\mathbf{M}, w'_i \nVdash \phi$ and $\mathbf{M}, w'_i \nVdash \chi$. Hence $\mathbf{M}, w'_i \nVdash \phi \vee \chi$ leading to a contradiction.

The backward implication follows from a similar line of reasoning.

N: The proof follows an argument similar to that presented for **K** above.

Mono: Let ϕ and χ be formulas such that $\phi \to \chi \in \Lambda(\mathbf{M})$. Assume that

$$\mathbf{M} \nVdash \langle\sigma\rangle(\psi_0, \dots, \phi, \dots, \psi_{n-1}) \to \langle\sigma\rangle(\psi_0, \dots, \chi, \dots, \psi_{n-1}).$$

Then by a similar argument as for $\mathbf{K}$ there must exist a world w_i such that $\mathbf{M}, w_i \Vdash \phi$ and $\mathbf{M}, w_i \nVdash \chi$, contradicting $\mathbf{M}, w_i \Vdash \phi \to \chi$.

(ii): Let $\mathbf{F} = \langle W, R^{\mathbf{F}}_{\Sigma}\rangle$ be a frame. The proof that $\Lambda(\mathbf{F})$ is normal is in essence the same argument as for $\Lambda(\mathbf{M})$. We thus only prove the uniformity of $\Lambda(\mathbf{F})$.

Uniform: Let $\psi \in \Lambda(\mathbf{F})$, V be a valuation and let $\chi = \phi[p_\eta \mapsto \psi]$. We need to show that $\chi \in \Lambda(\mathbf{F})$, i.e. $\mathbf{F} \Vdash \chi$. We do this by defining a new valuation V' such that $V'(p_\eta) = V(p_\xi[p_\eta \mapsto \psi])$. Since V is only defined for propositional variables p_η we need to extend V to arbitrary formulas ψ. We do this by inductively defining V as follows.

- If $\psi = \phi \vee \chi$ then $V(\psi) = V(\phi) \cup V(\chi)$.
- If $\psi = \perp$ then $V(\psi) = \emptyset$.
- If $\psi = \neg\phi$ then $V(\psi) = W \setminus V(\phi)$.
- If $\psi = \langle\sigma\rangle(\phi_0, \dots, \phi_{ar(\sigma)-1})$ then
  $V(\psi) := \{w : R^{\mathbf{F}}_{\sigma}(w_0, \dots, w_{ar(\sigma)-1}, w)$ and $w_i \in V(\psi_i)$ for all $i < ar(\sigma)\}$.

We will show that for $\mathbf{M} = \langle W, R^{\mathbf{F}}_{\sigma}, V\rangle$ and $\mathbf{M}' = \langle W, R^{\mathbf{F}}_{\Sigma}, V'\rangle$

$$(*) \qquad \mathbf{M}' \Vdash \phi \text{ implies } \mathbf{M} \Vdash \phi[p_\eta \mapsto \psi].$$

Then since the antecedent holds by assumption, $\Lambda(\mathbf{F})$ will be uniform.

Claim: $\mathbf{M}', w \Vdash p_\eta$ implies $\mathbf{M}, w \Vdash p_\eta[p_\eta \mapsto \psi]$ for any formula ψ.

The claim can be proven by induction on the form of ψ. Here we will only prove the case where $\psi = \langle\sigma\rangle(\psi_0, \dots, \psi_{ar(\sigma)-1})$. Assume that $\mathbf{M}', w \vdash p_\eta$. Note that $\psi = p_\eta[p_\eta \mapsto \psi]$ and hence $w \in V'(p_\eta) = V(p_\eta[p_\eta \mapsto \psi]) = V(\psi)$. Thus from our extension to the definition of V there exist $w_i \in W$, for each $i < ar(\sigma)$, such that $\mathbf{M}, w_i \Vdash \psi_i$ and so $\mathbf{M}, w \Vdash \psi$ as required.

Now we are ready to prove $(*)$. This proof proceeds by induction on the form of ϕ. The claim above establishes the base case.

So let $\phi = \langle\sigma\rangle(\phi_0, \dots, \phi_{n-1})$, where $ar(\sigma) = n$, and assume that $\mathbf{M}', w \Vdash \phi$ for any $w \in W$. Hence there exist $w_i \in W$, for $i < n$, such that $R^{\mathbf{F}}_{\sigma}(w_0, \dots, w_{n-1}, w)$ and $\mathbf{M}', w_i \Vdash \phi_i$. Thus by the induction hypothesis $\mathbf{M}, w_i \Vdash \phi_i[p_\eta \mapsto \psi]$ for each $i < n$. Since $\phi[p_\eta \mapsto \psi] = \langle\sigma\rangle(\phi_0[p_\eta \mapsto \psi], \dots, \phi_{n-1}[p_\eta \mapsto \psi])$, with the use of Proposition 3.2.9, it follows that $\mathbf{M}, w \Vdash \phi[p_\eta \mapsto \psi]$.

The proofs for $\phi = \phi_0 \vee \phi_1$ and $\phi = \perp$ follow along similar lines as that used below and will not be presented here.

Lastly consider $\phi = \neg\phi_0$ and assume $\mathbf{M}', w \Vdash \phi$. Then $\mathbf{M}', w \nVdash \phi_0$, whence $w \notin V'(\phi_0)$. By definition $V'(\phi_0) = V(\phi_0[p_\eta \mapsto \psi])$, so $w \notin V(\phi_0[p_\eta \mapsto \psi])$. Hence $\mathbf{M}, w \nVdash \phi_0[p_\eta \mapsto \psi]$ and so $\mathbf{M}, w \Vdash \neg\phi[p_\eta \mapsto \psi]$. Thus the result follows. ☐

Clearly any normal logic Λ is sound and complete with regards to a class $\mathcal{K}$ if $\Lambda(\mathcal{K}) = \Lambda$. This then completes the bi-directionality of the "Correspondence Theory" arrow of the Paradigm Triangle (c.f. 1).

3.4. Algebraic Semantics

We now turn to the arrow of the Paradigm Triangle where we try to find an algebraic structure $\mathbf{B}$, or class of structures, that closely corresponds with a logic Λ. In extension of the work of Boole relating propositional logic to Boolean algebras, this section demonstrates how to relate a modal language to some BAO, following the examples of [**Gol99**] and [**BDV01**]. The intuitive idea is that the underlying order in the Boolean part will relate to the underlying truth ordering of the logic, each modality will correspond to a different operator and propositional variables will behave somewhat like projections.

Let $\mathbf{B} = \langle B, \vee, \sim, \mathbb{C}, \Sigma\rangle$ be a BAO and ψ be an $L_\xi(\Sigma)$-formula with variables $p_{\eta_0}, p_{\eta_1}, \dots, p_{\eta_{n-1}}$. The formula ψ then induces an n-ary operation $\psi^{\mathbf{B}}$ on $\mathbf{B}$ defined inductively as follows.

$$\begin{aligned}
p_{\eta_i}^{\mathbf{B}}(a_0, \dots, a_{n-1}) &= a_i,\\
\perp^{\mathbf{B}}(a_0, \dots, a_{n-1}) &= \mathbb{0},\\
(\neg\psi)^{\mathbf{B}}(a_0, \dots, a_{n-1}) &= {\sim}\psi^{\mathbf{B}}(a_0, \dots, a_{n-1}),\\
(\psi \vee \phi)^{\mathbf{B}}(a_0, \dots, a_{n-1}) &= \psi^{\mathbf{B}}(a_0, \dots, a_{n-1}) \vee \phi^{\mathbf{B}}(a_0, \dots, a_{n-1}),
\end{aligned}$$

and, for $\sigma \in \Sigma$,

$$\begin{aligned}
&(\langle\sigma\rangle(\psi_0, \dots, \psi_{ar(\sigma)-1}))^{\mathbf{B}}(a_0, \dots, a_{n-1}) =\\
&\qquad\qquad \sigma(\psi_0^{\mathbf{B}}(a_0, \dots, a_{n-1}), \dots, \psi_{ar(\sigma)-1}^{\mathbf{B}}(a_0, \dots, a_{n-1})).
\end{aligned}$$

DEFINITION 3.4.1. A formula ψ is *valid* in $\mathbf{B}$, written $\mathbf{B} \Vdash \psi$, if the function $\psi^{\mathbf{B}}$ is constantly equal to $\mathbb{1}$. If $\mathcal{K}$ is a class of BAOs, then $\mathcal{K} \Vdash \psi$ if $\mathbf{B} \Vdash \psi$ for all $\mathbf{B} \in \mathcal{K}$. If Γ is a set of formulas, then $\mathbf{B} \Vdash \Gamma$ if $\mathbf{B} \Vdash \psi$ for all $\psi \in \Gamma$.

If we treat the propositional variables in a formula ψ as variables ranging over elements of $\mathbf{B}$, and the logical symbols $\vee$, $\neg$, $\perp$ and $\langle\sigma\rangle$ as naming the BAO symbols $\vee$, $\sim$, $\mathbb{0}$ and σ, ψ can be considered a term in the language of the BAO $\mathbf{B}$. This induces an equivalence between logical formulas and BAO equations. A formula ψ is valid in $\mathbf{B}$ if, and only if, $\mathbf{B}$ satisfies the equation "$\psi = \mathbb{1}$", in symbols

$$\mathbf{B} \Vdash \psi \text{ iff } \mathbf{B} \models (\psi = \mathbb{1}).$$

Each BAO equation is of the form $\psi = \phi$ and is satisfied in $\mathbf{B}$ if $\mathbf{B} \Vdash \psi \leftrightarrow \phi$. We can thus see that for a set of formulas Γ the class of algebras

$$\{\mathbf{B} : \mathbf{B} \Vdash \Gamma\}$$

forms an equational class, denoted $\mathrm{V}(\Gamma)$, also called the *variety generated by* Γ (c.f. Theorem 2.5.12). Using the HSP Theorem (c.f. p. 25) we can see that $\mathrm{V}(\Gamma)$ is closed under $\mathbf{H}$, $\mathbf{S}$ and $\mathbf{P}$, i.e. these operations preserve validity of formulas.

3.4.1. Characterising logics by algebras.

Similarly to Section 3.3.1 we want to find some "canonical" algebra, or class of algebras, that characterise a logic. I.e. we want an algebra $\mathbf{B}$ such that $\mathbf{B} \Vdash \psi$ if, and only if, $\vdash \psi$. But before we do this we introduce an algebra related to the whole language $L_\xi(\Sigma)$.

DEFINITION 3.4.2. Let ξ be an ordinal and take Σ to be a set of functional symbols. The *formula algebra over* Φ_ξ is defined to be the algebra $\langle \mathrm{Form}(\Phi_\xi), \vee, \sim, \mathbb{C}, \Sigma\rangle$, denoted $\mathbf{Form}(\Phi_\xi)$, where $\mathrm{Form}(\Phi_\xi)$ is the set of all formulas of $L_\xi(\Sigma)$. The interpretation of the symbols $\vee$, $\sim$ and $\mathbb{C}$ are defined by

$$\begin{aligned} \sim^{\mathbf{F}}\psi &= \neg\psi, \\ \psi \vee^{\mathbf{F}} \phi &= \psi \vee \phi \text{ and} \\ \mathbb{0}^{\mathbf{F}} &= \bot, \end{aligned}$$

while for each $\sigma \in \Sigma$, with $ar(\sigma) = n$, we define

$$\sigma^{\mathbf{F}}(\psi_0, \ldots, \psi_{n-1}) = \langle\sigma\rangle(\psi_0, \ldots, \psi_{n-1})$$

where $\psi, \phi, \psi_0, \ldots, \psi_{n-1} \in \mathrm{Form}(\Phi_\xi)$ and $\mathbf{F} = \mathbf{Form}(\Phi_\xi)$.

It is easily seen that $\mathbf{Form}(\Phi_\xi)$ is isomorphic to $\mathbf{Term}(\Phi_\xi)$, as defined in Definition 2.4.4 (p. 22).

The Lindenbaum-Tarski Construction. Let Λ be some logic in the language $L_\xi(\Sigma)$. We are looking to construct an algebra $\mathbf{B}$ such that, for any formula $\psi \in L_\xi(\Sigma)$, $\vdash \psi$ if, and only if, $\mathbf{B} \Vdash \psi$. From our definition of (algebraic) validity ψ is valid in $\mathbf{B}$ if, and only if, $\psi^{\mathbf{B}}$ is constantly equal to $\mathbb{1}$. There might of course be several such formulas. We would however expect that $\top^{\mathbf{B}}$ should always satisfy this requirement. Is there then some way in which we can identify all the formulas in Λ with $\top$? From propositional calculus we know that $\vdash \psi$ if, and only if, $\vdash \psi \leftrightarrow \top$. But $\leftrightarrow$ also induces an equivalence relation $\cong$ over formulas from $L_\xi(\Sigma)$ in the following way

$$\psi \cong \phi \text{ iff } \vdash \psi \leftrightarrow \phi$$

where $\psi, \phi \in L_\xi(\Sigma)$. The reflexivity of $\cong$ follows from the fact that $\phi \leftrightarrow \phi$ is a propositional tautology, symmetry from the fact that $\vdash \phi \leftrightarrow \psi$ if, and only if, $\vdash \psi \leftrightarrow \phi$ and transitivity from the fact that if $\vdash \phi \leftrightarrow \psi$ and $\vdash \psi \leftrightarrow \chi$ then $\vdash \phi \leftrightarrow \chi$.

If $\cong$ turns out to be a congruence then $\mathbf{Form}(\Phi_\xi)/\cong$ will be an algebra with universe consisting of equivalence classes modulo $\cong$. Looking at the equivalence classes modulo $\cong$, i.e. $\|\psi\|$ for $\psi \in \mathrm{Form}(\Phi_\xi)$, we see that

$$\psi \in \|\top\| \text{ iff } \vdash \psi \leftrightarrow \top$$

and hence

$$\psi \in \|\top\| \text{ iff } \vdash \psi.$$

Clearly we would want $\|\bot\| = \mathbb{C}^{\mathbf{B}}$. If we assume that the universe of $\mathbf{B}$ should be $\mathrm{Form}(\Phi_\xi)/\cong$ how should the other Boolean operations be defined? The intuitive choices for $\vee^{\mathbf{B}}$ and $\sim^{\mathbf{B}}$ should be clear, $\|\psi\| \vee^{\mathbf{B}} \|\phi\| = \|\psi \vee \phi\|$ and $\sim^{\mathbf{B}}\|\psi\| = \|\neg\psi\|$. It would be nice if, for each $\sigma \in \Sigma$, we could define $\sigma(\|\psi_0\|, \ldots, \|\psi_{ar(\sigma)-1}\|)$ to be $\|\langle\sigma\rangle(\psi_0, \ldots, \psi_{ar(\sigma)-1})\|$, but σ has to be normal and additive. In general this will not be the case.

Let us take a look at what conditions are necessary to make σ normal, where $ar(\sigma) = n$. For σ to be normal we require

$$\sigma(\|\psi_0\|, \ldots, \|\psi_{i-1}\|, \mathbb{C}, \|\psi_{i+1}\|, \ldots, \|\psi_{n-1}\|) = \mathbb{0}.$$

Thus far our definition then implies that

$$\begin{aligned}\sigma(\|\psi_0\|, \ldots, \|\psi_{i-1}\|, \mathbb{0}, \|\psi_{i+1}\|, \ldots, \|\psi_{n-1}\|) \\ &= \sigma(\|\psi_0\|, \ldots, \|\psi_{i-1}\|, \|\bot\|, \|\psi_{i+1}\|, \ldots, \|\psi_{n-1}\|) \\ &= \|\langle\sigma\rangle(\psi_0, \ldots, \psi_{i-1}, \bot, \psi_{i+1}, \ldots, \psi_{n-1})\|.\end{aligned}$$

Hence if $\vdash \langle\sigma\rangle(\psi_0, \ldots, \psi_{i-1}, \bot, \psi_{i+1}, \ldots, \psi_{ar(\sigma)-1}) \leftrightarrow \bot$ the result would follow. But this is exactly what **N** guarantees (c.f. Definition 3.2.10). Similarly **K** guarantees the additivity of σ and hence motivates us to make the following definition.

DEFINITION 3.4.3. Let Λ be a normal modal logic. We define a binary relation $\cong_\Lambda$ between formulas by

$$\psi \cong_\Lambda \phi \text{ iff } \vdash_\Lambda \psi \leftrightarrow \phi$$

and say the ψ and ϕ are *equivalent modulo* Λ.

We then show that $\cong_\Lambda$ is not only an equivalence relation but also a congruence:

PROPOSITION 3.4.4. *Let Λ be a normal modal logic. For any set Φ_ξ of propositional letters $\cong_\Lambda$ is a congruence on* $\mathbf{Form}(\Phi_\xi)$.

PROOF. We have to prove that the equivalence relation $\cong_\Lambda$ satisfies

(1) $$\phi \cong_\Lambda \psi \text{ implies } \neg\phi \cong_\Lambda \neg\psi,$$

(2) $$\phi_0 \cong_\Lambda \psi_0 \text{ and } \phi_1 \cong_\Lambda \psi_1 \text{ implies } \phi_0 \vee \phi_1 \cong_\Lambda \psi_0 \vee \psi_1,$$

and

(3) $$\phi_0 \cong_\Lambda \psi_0, \ldots, \phi_{ar(\sigma)-1} \cong_\Lambda \psi_{ar(\sigma)-1} \text{ implies } \langle\sigma\rangle(\phi_0, \ldots, \phi_{ar(\sigma)-1}) \cong_\Lambda \langle\sigma\rangle(\psi_0, \ldots, \psi_{ar(\sigma)-1}).$$

We prove (2) and (3), the proof of (1) is quite similar to (2) and can be found in [**BDV01**].5.12.

(2): Suppose that $\phi_0 \cong_\Lambda \psi_0$ and $\phi_1 \cong_\Lambda \psi_1$, then by definition $\vdash \phi_0 \leftrightarrow \psi_0$ and $\vdash \phi_1 \leftrightarrow \psi_1$. From the propositional calculus it follows that $\vdash \phi_0 \vee \phi_1 \leftrightarrow \psi_0 \vee \psi_1$ or equivalently that $\phi_0 \vee \phi_1 \cong_\Lambda \psi_0 \vee \psi_1$.

(3): We will demonstrate this fact for the case where $ar(\sigma) = 2$, the general result can be proven by a similar argument. Suppose that $\phi_0 \cong_\Lambda \psi_0$ and $\phi_1 \cong_\Lambda \psi_1$. Thus, by the definition of $\cong_\Lambda$, $\vdash \phi_0 \leftrightarrow \psi_0$ and $\vdash \phi_1 \leftrightarrow \psi_1$. Using **K** we can see that

$$\vdash \langle\sigma\rangle(\phi_0 \vee \psi_0, \phi_1 \vee \psi_1) \leftrightarrow \big(\langle\sigma\rangle(\phi_0, \phi_1) \vee \langle\sigma\rangle(\psi_0, \phi_1) \vee \langle\sigma\rangle(\phi_0, \psi_1) \vee \langle\sigma\rangle(\psi_0, \psi_1)\big).$$

Using propositional calculus we can deduce that

(*) $$\vdash \langle\sigma\rangle(\psi_0, \psi_1) \rightarrow \langle\sigma\rangle(\phi_0 \vee \psi_0, \phi_1 \vee \psi_1).$$

By assumption $\vdash \psi_0 \rightarrow \phi_0$ and $\vdash \psi_1 \rightarrow \phi_1$. Hence we can show that $\vdash \phi_0 \vee \psi_0 \rightarrow \phi_0$ and $\vdash \phi_1 \vee \psi \rightarrow \phi_1$. Then, using **Mono**,

$$\vdash \langle\sigma\rangle(\phi_0 \vee \psi_0, \phi_1 \vee \psi_1) \rightarrow \langle\sigma\rangle(\phi_0, \phi_1 \vee \psi_1)$$

and

$$\vdash \langle\sigma\rangle(\phi_0, \phi_1 \vee \psi_1) \rightarrow \langle\sigma\rangle(\phi_0, \phi_1).$$

Hence it follows that

(**) $$\vdash \langle\sigma\rangle(\phi_0 \vee \psi_0, \phi_1 \vee \psi_1) \rightarrow \langle\sigma\rangle(\phi_0, \phi_1).$$

Putting (*) and (**) together we get $\vdash \langle\sigma\rangle(\psi_0, \psi_1) \to \langle\sigma\rangle(\phi_0, \phi_1)$. By an analogous argument $\vdash \langle\sigma\rangle(\phi_0, \phi_1) \to \langle\sigma\rangle(\psi_0, \psi_1)$, whence

$$\vdash \langle\sigma\rangle(\phi_0, \phi_1) \leftrightarrow \langle\sigma\rangle(\psi_0, \psi_1).$$

Giving us $\langle\sigma\rangle(\phi_0, \phi_1) \cong_\Lambda \langle\sigma\rangle(\psi_0, \psi_1)$ as required. □

We are now ready to define our so-called "canonical" algebras.

DEFINITION 3.4.5. Let Λ be a modal logic and Φ_ξ the set of propositional variables in $L_\xi(\Sigma)$. The *Lindenbaum-Tarski algebra of* Λ, denoted $\mathrm{LT}_\xi(\Lambda)$, is defined to be the algebra $\langle \mathrm{Form}(\Phi_\xi)/\cong_\Lambda, \vee, \sim, \mathbb{C}, \Sigma\rangle$ with operations $\vee$, $\sim$, $\mathbb{C}$ and $\sigma \in \Sigma$ defined by

$$\begin{aligned} \|\psi\| \vee \|\phi\| &= \|\psi \vee \phi\| \\ \sim\|\psi\| &= \|\neg\psi\| \\ \mathbb{C} &= \|\bot\| \\ \sigma(\|\psi_0\|, \ldots, \|\psi_{ar(\sigma)-1}\|) &= \|\langle\sigma\rangle(\psi_0, \ldots, \psi_{ar(\sigma)-1})\|. \end{aligned}$$

When ξ is clear from the context we write $\mathrm{LT}(\Lambda)$ instead of $\mathrm{LT}_\xi(\Lambda)$

Before we can prove a characterisation theorem for normal uniform logics using the Lindenbaum-Tarski construction we need the following lemma.

LEMMA 3.4.6. *Let $\sigma \in \Sigma$, $\psi, \phi_0, \ldots, \phi_{n-1}$ be Σ-formulas and let $p_{\eta_0}, \ldots, p_{\eta_{n-1}} \in \Phi_\xi$, where $ar(\sigma) = n$. Then for any logic Λ*

$$\psi^{\mathrm{LT}(\Lambda)}(\|\phi_0\|, \ldots, \|\phi_{n-1}\|) = \|\psi[\underline{p} \mapsto \underline{\phi}]\|$$

where $\underline{p} = \langle p_{\eta_0}, \ldots, p_{\eta_{n-1}}\rangle$ and $\underline{\phi} = \langle \phi_0, \ldots, \phi_{n-1}\rangle$.

PROOF. We do this proof by induction on the form of ψ. The base cases where $\psi = p_\eta$ and $\psi = \bot$ follow directly from Proposition 3.2.9. for $\vee$ follows in a similar fashion to $\neg$.

The following calculation proves the case where $\psi = \neg\chi$.

$$\begin{aligned} \psi^{\mathrm{LT}(\Lambda)}(\|\phi_0\|, \ldots, \|\phi_{n-1}\|) &= \sim\chi^{\mathrm{LT}(\Lambda)}(\|\phi_0\|, \ldots, \|\phi_{n-1}\|) \\ &= \sim\|\chi[\underline{p} \mapsto \underline{\phi}]\| && \text{(induction hypothesis)} \\ &= \|\neg(\chi[\underline{p} \mapsto \underline{\phi}])\| \\ &= \|\psi[\underline{p} \mapsto \underline{\phi}]\| && \text{(by Prop. 3.2.9)} \end{aligned}$$

The cases for $\vee$ and $\bot$ follow similar arguments. For the final case we let $\sigma \in \Sigma$, with $ar(\sigma) = n$ and $\psi = \langle\sigma\rangle(\chi_0, \ldots, \chi_{m-1})$ then

$$\begin{aligned} &\psi^{\mathrm{LT}(\Lambda)}(\|\phi_0\|, \ldots, \|\phi_{n-1}\|) \\ &\quad= (\langle\sigma\rangle(\chi_0, \ldots, \chi_{m-1}))^{\mathrm{LT}(\Lambda)}(\|\phi_0\|, \ldots, \|\phi_{n-1}\|) \\ &\quad= \sigma(\chi_0^{\mathrm{LT}(\Lambda)}(\|\phi_0\|, \ldots, \|\phi_{n-1}\|), \ldots, \chi_{m-1}^{\mathrm{LT}(\Lambda)}(\|\phi_0\|, \ldots, \|\phi_{n-1}\|)) \\ &\quad= \sigma(\|\chi_0[\underline{p} \mapsto \underline{\phi}]\|, \ldots, \|\chi_{m-1}[\underline{p} \mapsto \underline{\phi}]\|) && \text{(induction hypothesis)} \\ &\quad= \|\langle\sigma\rangle(\chi_0[\underline{p} \mapsto \underline{\phi}], \ldots, \chi_{m-1}[\underline{p} \mapsto \underline{\phi}])\| \\ &\quad= \|\psi[\underline{p} \mapsto \underline{\phi}]\|, \end{aligned}$$

where the last line follows from Proposition 3.2.9. □

We are now ready to prove the main result of this section, which relates a logic Λ to a particular algebraic semantics, c.f. the "Algebraic Logic" arrow in Figure 1.

THEOREM 3.4.7. *Let ψ be a formula and Λ a normal uniform logic then*

$$\vdash_\Lambda \psi \text{ iff } \mathrm{LT}(\Lambda) \Vdash \psi.$$

PROOF. Let $\underline{p}$ and $\underline{\phi}$ be as in Lemma 3.4.6 and let ψ be some formula. For the forward direction assume that $\vdash_\Lambda \psi$. Since Λ is closed under uniform substitution we have $\vdash_\Lambda \psi[\underline{p} \mapsto \underline{\phi}]$ and so by Lemma 3.4.6

$$\psi^{\mathrm{LT}(\Lambda)}(\|\phi_0,\ldots,\phi_{n-1}\|) = \|\psi[\underline{p} \mapsto \underline{\phi}]\| = \|\psi\| = \mathbb{1}.$$

For the backward direction we assume $\nvdash_\Lambda \psi$. Since by Lemma 3.4.6

$$\psi^{\mathrm{LT}(\Lambda)}(\|p_{\eta_0}\|,\ldots,\|p_{\eta_{n-1}}\|) = \|\psi[\underline{p} \mapsto \underline{p}]\| = \|\psi\|$$

and $\|\psi\| \neq 1$ we have $\mathrm{LT}(\Lambda) \nVdash_\Lambda \psi$. □

Logics generated by algebras. As with relational semantics we now reverse direction and take a look at logics generated by algebras.

PROPOSITION 3.4.8. *Let $\mathcal{K}$ be a class of* BAO*s then $\{\psi : \mathcal{K} \Vdash \psi\}$ is a normal uniform logic.*

PROOF. Let $\Lambda = \{\psi : \mathcal{K} \Vdash \psi\}$, i.e. $\psi \in \Lambda$ if $\mathbf{B} \Vdash \psi$ for all $\mathbf{B} \in \mathcal{K}$. It should be clear that, since the validity relation interprets $\perp$, $\neg$ and $\vee$ by connecting them with their related Boolean symbols, all propositional tautologies are contained in Λ.

In the sequel we assume ϕ, χ, ψ and ψ_i to be formulas with variables $p_{\eta_0},\ldots,p_{\eta_{n-1}}$ and that $\mathbf{B}$ is any algebra in $\mathcal{K}$. For the remainder of this proof we will use the shorthand $\underline{a}$ for the sequence $\langle a_0,\ldots,a_{n-1}\rangle$.

Modes Ponens: Assume that $\mathbf{B} \Vdash \phi$ and $\mathbf{B} \Vdash \phi \to \psi$. Then for any $a_i \in \mathbf{B}$, for $i < n$, $\phi(\underline{a}) = \mathbb{1}$ and $\sim\phi(\underline{a}) \vee \psi(\underline{a}) = \mathbb{1}$. Since

$$\sim\phi(\underline{a}) \vee \psi(\underline{a}) = \mathbb{0} \vee \psi(\underline{a}) = \psi(\underline{a})$$

it follows that $\psi(\underline{a}) = \mathbb{1}$ and so $\mathbf{B} \Vdash \psi$.

K: Consider any $a_i \in \mathbf{B}$ with $i < n$. Since σ, with $ar(\sigma) = m$, is an operator it follows that

$$\begin{aligned}\sigma(\psi_0^{\mathbf{B}}(\underline{a}),\ldots,\phi^{\mathbf{B}}(\underline{a}) \vee \chi^{\mathbf{B}}(\underline{a}),\ldots,\psi_{m-1}^{\mathbf{B}}(\underline{a})) &= \\ \sigma(\psi_0^{\mathbf{B}}(\underline{a}),\ldots,\phi^{\mathbf{B}}(\underline{a}),\ldots,\psi_{m-1}^{\mathbf{B}}(\underline{a})) &\vee \sigma(\psi_0^{\mathbf{B}}(\underline{a}),\ldots,\chi^{\mathbf{B}}(\underline{a}),\ldots,\psi_{m-1}^{\mathbf{B}}(\underline{a})).\end{aligned}$$

Thus, using the interpretation of $\vee$ and $\neg$ in $\mathbf{B}$ as defined before,

$$\begin{aligned}&\big(\langle\sigma\rangle(\psi_0,\ldots,\phi \vee \chi,\ldots,\psi_{m-1}) \to \\ &\qquad \langle\sigma\rangle(\psi_0,\ldots,\phi,\ldots,\psi_{m-1}) \vee \langle\sigma\rangle(\psi_0,\ldots,\chi,\ldots,\psi_{m-1})\big)^{\mathbf{B}}(\underline{a}) = \mathbb{1}\end{aligned}$$

and

$$\begin{aligned}&\big(\langle\sigma\rangle(\psi_0,\ldots,\phi,\ldots,\psi_{m-1}) \vee \langle\sigma\rangle(\psi_0,\ldots,\chi,\ldots,\psi_{m-1}) \to \\ &\qquad \langle\sigma\rangle(\psi_0,\ldots,\phi \vee \chi,\ldots,\psi_{m-1})\big)^{\mathbf{B}}(\underline{a}) = \mathbb{1}.\end{aligned}$$

Combining the above two results it follows that

$$\begin{aligned}&\mathbf{B} \Vdash \langle\sigma\rangle(\psi_0,\ldots,\phi \vee \chi,\ldots,\psi_{m-1}) \leftrightarrow \\ &\qquad \langle\sigma\rangle(\psi_0,\ldots,\phi,\ldots,\psi_{m-1}) \vee \langle\sigma\rangle(\psi_0,\ldots,\chi,\ldots,\psi_{m-1}).\end{aligned}$$

N: Similarly to the case for **K**, **N** follows from the fact that for any operator σ, with $ar(\sigma) = n$,

$$\sigma(\psi_0^{\mathbf{B}}(\underline{a}), \ldots, \mathbb{0}, \ldots, \psi_{m-1}^{\mathbf{B}}(\underline{a})) = \mathbb{0}.$$

Mono: Assume $\mathbf{B} \Vdash \phi \to \chi$ and $ar(\sigma) = m$. Thus for any $a_i \in \mathbf{B}$, $i < n$, $\sim\phi(\underline{a}) \vee \chi(\underline{a}) = \mathbb{1}$. Consequently $\phi(\underline{a}) \leq \chi(\underline{a})$. By Proposition 3.1.2 it follows that

$$\sigma\big(\psi_0(\underline{a}), \ldots, \phi(\underline{a}), \ldots, \psi_{m-1}(\underline{a})\big) \leq \sigma\big(\psi_0(\underline{a}), \ldots, \chi(\underline{a}), \ldots, \psi_{m-1}(\underline{a})\big).$$

Thus $\sim\sigma(\psi_0, \ldots, \phi, \ldots, \psi_{m-1})(\underline{a}) \vee \sigma(\psi_0, \ldots, \chi, \ldots, \psi_{m-1})(\underline{a}) = \mathbb{1}$, from which it follows that $\mathbf{B} \Vdash \langle\sigma\rangle(\psi_0, \ldots, \phi, \ldots, \psi_{m-1}) \to \langle\sigma\rangle(\psi_0, \ldots, \chi, \ldots, \psi_{m-1})$.

Uniform: We need to show that if $\phi^{\mathbf{B}}(\underline{a}) = \mathbb{1}$ for all $a_0, \ldots, a_{n-1} \in \mathbf{B}$ then $\phi[p_\eta \mapsto \psi](\underline{a}) = \mathbb{1}$ for all $a_0, \ldots, a_{n-1} \in \mathbf{B}$. We will only prove this for the $L_1(\Sigma)$ case, i.e. where our language only contains one variable.

To prove this we show that

$$(*) \qquad \phi[p \mapsto \psi]^{\mathbf{B}}(a) = \phi^{\mathbf{B}}(\psi^{\mathbf{B}}(a)),$$

for any $a \in \mathbf{B}$. Then, if we assume that $\phi^{\mathbf{B}}(a) = \mathbb{1}$ for all $a \in \mathbf{B}$, the result follows. (Note that in the general case $(*)$ becomes

$$(\phi[p_{\eta_0} \mapsto \psi_0], \ldots, [p_{\eta_{n-1}} \mapsto \psi_{n-1}])^{\mathbf{B}}(a) = \phi^{\mathbf{B}}(\psi_0^{\mathbf{B}}(a), \ldots, \psi_{n-1}^{\mathbf{B}}(a)),$$

where the formula ϕ has n variables.)

We prove $(*)$ by induction on the form of ϕ. The base case follows easily from Proposition 3.2.9 and the fact that $p^{\mathbf{B}}(a) = a$.

Let $\phi = \neg\phi_0$. Consider the following calculation

$$\begin{aligned} ((\neg\phi_0)[p \mapsto \psi])^{\mathbf{B}}(a) &= (\neg(\phi_0[p \mapsto \psi]))^{\mathbf{B}}(a) \\ &= \sim(\phi_0[p \mapsto \psi]^{\mathbf{B}}(a)) \\ &= (\sim(\phi(\psi^{\mathbf{B}}(a)))) \\ &= (\neg\phi(\psi))^{\mathbf{B}}(a), \end{aligned}$$

where the first line follows by Proposition 3.2.9 and the third line follows from our induction hypothesis. The cases where $\psi = \bot$, $\phi = \phi_0 \vee \phi_1$ and $\phi = \langle\sigma\rangle(\phi_0, \ldots, \phi_{n-1})$ follow similar arguments. Thus $(*)$ follows and hence the closure of Λ under uniform substitution. □

Observe that the above proof can be used to show that the set $\{\psi : \mathbf{B} \Vdash \psi\}$ is a normal uniform logic for any BAO $\mathbf{B}$.

DEFINITION 3.4.9. For any BAO $\mathbf{B}$ we define the *logic generated by the* BAO $\mathbf{B}$ to be $\Lambda(\mathbf{B}) = \{\psi : \mathbf{B} \Vdash \psi\}$. For a class of BAOs $\mathcal{K}$ we define $\Lambda(\mathcal{K}) = \{\psi : \mathcal{K} \Vdash \psi\}$.

In fact, using Theorem 3.4.7, it is easy to see that each normal uniform logic Λ is of the form $\Lambda(\mathbf{B})$.

To complete this section recall that earlier we noted the isomorphism between $\mathbf{Form}(\Phi_\xi)$ and $\mathbf{Term}(\Phi_\xi)$, the absolutely free algebra of type Σ over Φ_ξ. Consequently the Lindenbaum-Tarski algebra $\mathrm{LT}(\Lambda)$ turns out to be a very important algebra in the variety $\mathrm{V}(\Lambda)$: logic Λ is the free algebra in $\mathrm{V}(\Lambda)$.

THEOREM 3.4.10. *In the variety* $\mathrm{V}(\Lambda)$, $\mathrm{LT}(\Lambda)$ *is the free algebra on the set of generators*

$$\|\Phi_\xi\| = \{\|p_\eta\| : \eta < \xi\}.$$

PROOF. Clearly the algebra $\mathrm{LT}(\Lambda)$, as defined above, is generated by the set $\|\Phi_\xi\|$. Let $\mathbf{A}$ be an algebra from $\mathrm{V}(\Lambda)$ and f a function $\|\Phi_\xi\| \longrightarrow \mathbf{A}$. We define the map $h : \mathrm{LT}(\Lambda) \longrightarrow \mathbf{A}$ by

$$h(\|\psi\|) = \psi^{\mathbf{A}}(f(\|p_{\eta_0}\|), \dots, f(\|p_{\eta_{n-1}}\|)),$$

where p_{η_i}, for $i < n$, are the propositional variables occurring in ψ. From the definition of h it should be clear that $h(\|p_{\eta_i}\|) = f(\|p_{\eta_i}\|)$. Now if $\|\psi\| = \|\phi\|$ then $\vdash \psi \leftrightarrow \phi$. Since $\mathbf{A} \in \mathrm{V}(\Lambda)$, $\mathbf{A} \Vdash \psi \leftrightarrow \phi$ and so $\psi^{\mathbf{A}} = \phi^{\mathbf{A}}$ from which we can infer that h is well-defined.

We will show that h respects all operators $\sigma \in \Sigma$. The cases for the Boolean operations can be proved using similar arguments. Consider any operator σ, with $ar(\sigma) = m$, then

$$\begin{aligned}
&h(\sigma(\|\psi_0\|, \dots, \|\psi_{m-1}\|)) \\
&\qquad = h(\|\langle\sigma\rangle(\psi_0, \dots, \psi_{m-1})\|) \\
&\qquad = \big(\langle\sigma\rangle(\psi_0, \dots, \psi_{m-1})\big)^{\mathbf{A}}\big(f(\|p_{\eta_0}\|, \dots, f(\|p_{\eta_{n-1}}\|))\big) \\
&\qquad = \sigma\big(\psi_0^{\mathbf{A}}(f(\|p_{\eta_0}\|, \dots, f\|p_{\eta_{n-1}}\|)), \dots, \psi_{m-1}^{\mathbf{A}}(f(\|p_{\eta_0}\|, \dots f\|p_{\eta_{n-1}}\|))\big) \\
&\qquad = \sigma(h(\|\psi_0\|), \dots, h(\|\psi_{n-1}\|)).
\end{aligned}$$

So we can conclude that h is a homomorphism and hence $\mathrm{LT}(\Lambda)$ is the free algebra required. $\square$

This result and its proof is modelled on [**Gol99**] Theorem 5.2.1.

3.5. From relational to algebraic semantics and back again

To complete our triangle we only have to show how relational and algebraic semantics interact. From our preliminary studies in Section 3.1.2 we already know how to construct BAOs from relational structures, and vice versa. The question is just whether, if at all, these mappings, i.e. Cm and Uf, preserve validity. First we show how Cm does in fact preserve validity and then go on to show how Uf preserves canonical constructions.

3.5.1. Preserving validity.

We are interested in seeing which formulas are preserved by Cm, i.e. given a frame, $\mathbf{F} = \langle W, R_\Sigma^{\mathbf{F}}\rangle$, for which ψ does $\mathbf{F} \Vdash \psi$ coincide with $\mathrm{Cm}\mathbf{F} \Vdash \psi$? To productively look at this question we first reformulate the definition of satisfaction.

Recall that the universe of $\mathrm{Cm}\mathbf{F}$ consists of subsets of W. Is there some way in which we can associate subsets of W with our relational semantics? Given a formula ψ the obvious subset to look at first would be the subset of worlds at which ψ is true, also called the *truth set* of ψ, formally defined as follows.

DEFINITION 3.5.1. Let $\mathbf{M} = \langle W, R_\Sigma^{\mathbf{M}}, V\rangle$ be a model and ψ some formula. The *truth set* of ψ in $\mathbf{M}$, denoted $\psi^{\mathbf{M}}$, is defined by

$$\psi^{\mathbf{M}} = \{w \in W : \mathbf{M}, w \Vdash \psi\}.$$

The satisfaction relation for models then amounts to the following properties of truth sets

$$
\begin{aligned}
p_\eta^{\mathbf{M}} &= V(p_\eta),\\
\bot^{\mathbf{M}} &= \emptyset,\\
(\neg\psi)^{\mathbf{M}} &= W \setminus \psi^{\mathbf{M}},\\
(\psi \vee \phi)^{\mathbf{M}} &= \psi^{\mathbf{M}} \cup \phi^{\mathbf{M}} \text{ and}\\
(\langle\sigma\rangle(\psi_0, \dots, \psi_{ar(\sigma)-1}))^{\mathbf{M}} &= \sigma^{\mathrm{Cm}\mathbf{F}}(\psi_0^{\mathbf{M}}, \dots, \psi_{ar(\sigma)-1}^{\mathbf{M}}),
\end{aligned}
\tag{3.1}
$$

where $\mathbf{M}$ is a model on $\mathbf{F}$.

Hence ψ is valid in a frame $\mathbf{F}$ if, and only if, $\psi^{\mathbf{M}} = W = \top^{\mathbf{M}}$ for any model $\mathbf{M}$ based on that frame. It should come as no surprise then that the validity of ψ in $\mathbf{F}$ can be checked by looking at the term function $(\psi)^{\mathrm{Cm}\mathbf{F}}$.

THEOREM 3.5.2. *Let ψ be a formula with variables among $p_{\eta_0}, \dots, p_{\eta_{n-1}} \in \Phi_\xi$ and let $\mathbf{F}$ be some frame. Then for any model $\mathbf{M}$ on $\mathbf{F}$,*

$$(\psi)^{\mathrm{Cm}\mathbf{F}}(p_{\eta_0}^{\mathbf{M}}, \dots, p_{\eta_{n-1}}^{\mathbf{M}}) = \psi^{\mathbf{M}}.$$

PROOF. The proof follows by a straightforward induction on the form of ψ. As demonstration we will do the case for $\vee$. Let $\mathbf{F}$, $\mathbf{M}$ and $p_{\eta_0}, \dots, p_{\eta_{n-1}}$ be as in the statement of the theorem and $\psi = \phi \vee \chi$. Then

$$
\begin{aligned}
(\phi \vee \chi)^{\mathrm{Cm}\mathbf{F}}(p_{\eta_0}^{\mathbf{M}}, \dots, p_{\eta_{n-1}}^{\mathbf{M}}) &= (\phi)^{\mathrm{Cm}\mathbf{F}}(p_{\eta_0}^{\mathbf{M}}, \dots, p_{\eta_{n-1}}^{\mathbf{M}}) \cup (\chi)^C m[\mathbf{F}](p_{\eta_0}^{\mathbf{M}}, \dots, p_{\eta_{n-1}}^{\mathbf{M}})\\
&= \phi^{\mathbf{M}} \cup \chi^{\mathbf{M}} \qquad \text{(induction hypothesis)}\\
&= (\phi \vee \chi)^{\mathbf{M}} \qquad \text{(fourth line in (3.1)).}
\end{aligned}
$$

□

From here it is easy to prove that the validity of any formula is preserved by Cm. (The following corollary and its proof is modelled on that presented in [**Gol99**] Corollary 5.3.2.)

COROLLARY 3.5.3. *For any formula ψ and frame $\mathbf{F}$*

$$\mathbf{F} \Vdash \psi \textit{ iff } \mathrm{Cm}\mathbf{F} \Vdash \psi.$$

PROOF. Let ψ be a formula and $\mathbf{F} = \langle W, R\rangle$ some frame. If $\mathbf{F} \nVdash \psi$ then $\mathbf{M} \nVdash \psi$ for some model $\mathbf{M}$ on $\mathbf{F}$, hence $\psi^{\mathbf{M}} \neq W$. By Theorem 3.5.2 it follows that $\mathrm{Cm}\mathbf{F}(\psi)$ is not constantly equal to $\mathbb{1}$, so $\mathrm{Cm}\mathbf{F} \nVdash \psi$.

Conversely, if ψ is not valid in $\mathrm{Cm}\mathbf{F}$ then for some $X_0, \dots, X_{n-1} \in \mathcal{P}(W)$

$$\mathrm{Cm}\mathbf{F}(\psi)(X_0, \dots, X_{n-1}) \neq W.$$

So we let $\mathbf{M}$ be the model on $\mathbf{F}$ having $p_{\eta_i}^{\mathbf{M}} = X_i$, for $i < n$. Then Theorem 3.5.2 implies that $\psi^{\mathbf{M}} \neq W$ and hence $\mathbf{F} \nVdash \psi$. □

In essence this corollary shows us that complex algebras capture the idea of frame validity. But complex algebras are also important from the perspective that they provide us with concrete algebraic structures related to a logic. On the abstract side we have proved (c.f. Theorem 3.4.7) that a logic Λ is complete with respect to its Lindenbaum-Tarski algebra $\mathrm{LT}(\Lambda)$. If we can represent $\mathrm{LT}(\Lambda)$ concretely, as a complex algebra, then this would turn Theorem 3.4.7 into a completeness theorem with regards to complex algebras. In fact we already have the tools to prove such a result, i.e. the Jónsson-Tarski Representation Theorem (c.f. Theorem 3.1.14).

3.5.2. Canonical frames and "canonical" algebras.

We know that the Lindenbaum-Tarski algebra (i.e. the "canonical" algebra) characterises the logic associated with it (c.f. Theorem 3.4.7). However the same cannot be said of the canonical frame associated with a logic. We have also seen that the map Uf generates a frame (or relational structure) from any algebra. What is then the significance, if any, of the frame $\mathrm{Uf}(\mathrm{LT}(\Lambda))$ for a logic Λ?

If we look back at Proposition 3.2.5 we notice that in a sense Λ-maximal sets satisfy all the properties required of filters. In fact if $\Lambda = \mathrm{Form}(\Phi_\xi)$ we might even expect a Λ-maximal set to be a filter over the formula algebra $\mathbf{Form}(\Phi_\xi)$. Hence the following result.

THEOREM 3.5.4. *Let* Λ *be a logic. Then* $\mathrm{CanF}(\Lambda) \cong \mathrm{Uf}(\mathrm{LT}(\Lambda))$.

PROOF. Consider the function $h : \mathrm{CanF}(\Lambda) \longrightarrow \mathrm{Uf}(\mathrm{Form}(\Phi_\xi)/\cong_\Lambda)$, defined by

$$h(\Gamma) = \{\|\phi\| : \phi \in \Gamma\}.$$

We will prove that h is an isomorphism between $\mathrm{CanF}(\Lambda)$ and $\mathrm{Uf}(\mathrm{LT}(\Lambda))$.

First we show that h is well defined, i.e. for any Λ-maximal set Γ, $h(\Gamma)$ is an ultrafilter.

Filter: Let $\|\psi\| \in h(\Gamma)$ and $\|\psi\| \leq \|\phi\|$. Take any $\chi \in \Gamma$ such that $\|\psi\| = \|\chi\|$. It is easy to show that

$$\|\chi\| \leq \|\phi\| \text{ iff } \vdash \chi \rightarrow \phi.$$

Hence, by Proposition 3.2.5(ii), $\chi \rightarrow \phi \in \Gamma$. So by 3.2.5(i) $\phi \in \Gamma$. Thus $\|\phi\| \in h(\Gamma)$. Similarly, by 3.2.5(iv), it follows that if $\|\psi\|, \|\phi\| \in \Gamma$ then $\|\psi \wedge \phi\| \in h(\Gamma)$ and hence $\|\psi\| \wedge \|\phi\| \in h(\Gamma)$.

Ultrafilter: To see that $\mathcal{G}$ is a proper filter we observe that

$$\|\psi\| = \mathbb{C} \text{ iff } \vdash \psi \leftrightarrow \bot.$$

If $\mathbb{C} \in h(\Gamma)$ there exists some $\psi \in \Gamma$ such that $\|\psi\| = \mathbb{C}$. Hence $\bot$ is Λ-deducible from Γ which contradicts the consistency of Γ. From 3.2.5(iii) it easily follows that, for any ψ, either $\|\psi\| \in h(\Gamma)$ or $\sim\|\psi\| \in h(\Gamma)$, but not both.

Next we demonstrate that h is in fact bijective.

Surjectivity: Let $\mathcal{G} \in \mathrm{Uf}(\mathrm{LT}(\Lambda))$ and define

$$\Gamma_\mathcal{G} = \{\psi : \|\psi\| \in \mathcal{G}\}.$$

It is easily seen that $h(\Gamma_\mathcal{G}) = \mathcal{G}$, but is $\Gamma_\mathcal{G}$ Λ-maximal? First we show that $\Gamma_\mathcal{G}$ is Λ-consistent. Assume that $\Gamma_\mathcal{G}$ is not Λ-consistent, then there exist $\psi_0, \ldots, \psi_{n-1} \in \Gamma_\mathcal{G}$ such that

$$\vdash (\psi_0 \wedge \ldots \wedge \psi_{n-1}) \rightarrow \bot.$$

Since $\|\psi_0\|, \ldots, \|\psi_{n-1}\| \in \mathcal{G}$ and $\mathcal{G}$ is a filter $\|\psi_0\| \wedge \ldots \wedge \|\psi_{n-1}\| \in \mathcal{G}$. But

$$\|\psi_0\| \wedge \ldots \wedge \|\psi_{n-1}\| = \|\psi_0 \wedge \ldots \wedge \psi_{n-1}\| = \|\bot\|$$

contradicting $\mathcal{G}$ being a proper filter. Hence $\Gamma_\mathcal{G}$ is Λ-consistent.

To see that it is maximal assume that there exists some $\psi \notin \Gamma_\mathcal{G}$ such that $\Gamma_\mathcal{G} \cup \{\psi\}$ is consistent. Since $\psi \notin \Gamma_\mathcal{G}$ it is easily seen that $\|\psi\| \notin \mathcal{G}$. Hence $\sim\|\psi\| \in \mathcal{G}$ and thus $\|\neg\psi\| \in \mathcal{G}$, so by definition $\neg\psi \in \Gamma_\mathcal{G}$. But $\vdash \psi \wedge \neg\psi \rightarrow \bot$ contradicting the consistency of $\Gamma_\mathcal{G} \cup \{\psi\}$.

Injectivity: Now take $\Gamma, \Psi \in \mathrm{CanF}(\Lambda)$ such that there exists some $\psi \in \Psi$ with $\psi \notin \Gamma$. If $h(\Gamma) = h(\Psi)$ then $\|\psi\| \in h(\Gamma)$ and, since $\psi \notin \Gamma$, it must be that $\neg\psi \in \Gamma$. So $\|\neg\psi\| = \sim \|\psi\| \in h(\Gamma)$, contradicting $h(\Gamma)$ being an ultrafilter.

To conclude we show that h is a homomorphism. Let $\sigma \in \Sigma$ with $ar(\sigma) = n$.

Assume that $R_\sigma^{\mathrm{CanF}(\Lambda)}(\Psi_0, \ldots, \Psi_n)$ for $\Psi_0, \ldots, \Psi_n \in \mathrm{CanF}(\Lambda)$. We need to show that $R_\sigma^{\mathrm{Uf}(\mathrm{LT}(\Lambda))}(h(\Psi_0), \ldots, h(\Psi_{n-1}))$, i.e.

$$\{\sigma^{\mathrm{LT}(\Lambda)}(\|\psi_0\|, \ldots, \|\psi_{n-1}\|) : \|\psi_i\| \in h(\Psi_i)\} \subseteq h(\Psi_n).$$

So we look at some $\sigma^{\mathrm{LT}(\Lambda)}(\|\psi_0\|, \ldots, \|\psi_{n-1}\|)$, where $\|\psi_i\| \in h(\Psi_i)$ for $i < n$. Observe that from $\|\psi_i\| \in h(\Psi_i)$ it follows that $\psi_i \in \Psi_i$. Thus by our assumption above $\langle\sigma\rangle(\psi_0, \ldots, \psi_{n-1}) \in \Psi_n$. The required result then follows from the fact that $\sigma^{\mathrm{LT}(\Lambda)}(\|\psi_0\|, \ldots, \|\psi_{n-1}\|) = \|\langle\sigma\rangle(\psi_0, \ldots, \psi_{n-1})\|$.

□

3.6. Canonical and complete logics and their characterisation

In conclusion we introduce the classes of canonical, complete and strongly complete logics and use the techniques we have derived earlier in this chapter to find algebraic criteria to characterise them by. For more on the classes of algebras associated with these logics we refer the reader to Section 4.3.

3.6.1. Canonical logics.

As was mentioned earlier, c.f. Section 3.3.1, not all the formulas of a logic are validated in its canonical frame. We do however know that the Lindenbaum-Tarski algebra does not suffer from this malady. Can we then use the previous result linking algebras and frames to eliminate this discrepancy?

DEFINITION 3.6.1. A logic Λ in a language $L_\xi(\Sigma)$ is said to be ξ-*canonical* if Λ is valid in $\mathrm{CanF}_\xi(\Lambda)$. Λ is said to be canonical if Λ is ξ-canonical for all $\xi \geq \omega$.

So we can see that Λ is ξ-canonical if, and only if, $\mathrm{CmUf}(\mathrm{LT}_\xi(\Lambda)) \Vdash \Lambda$ if, and only if, $\mathrm{EmLT}_\xi(\Lambda) \Vdash \Lambda$. But if a variety were to contain all the canonical embedding algebras of $\mathrm{LT}_\xi(\Lambda)$, for $\xi \geq \omega$, then we can translate this logical question into a question about algebras. In fact the criteria we are talking about is even more powerful than we might expect.

THEOREM 3.6.2. *A variety $\mathcal{V}$ contains all the canonical embedding algebras of its infinitely-generated free algebras if, and only if, the variety is closed under canonical embedding algebras, i.e.* $\mathbf{Em}\mathcal{V} \subseteq \mathcal{V}$.

PROOF. Given an algebra $\mathbf{A} \in \mathcal{V}$, let $\mathbf{F}$ be the free algebra in $\mathcal{V}$ generated by a set X of cardinality $|\mathbf{A}| + \omega$ (the existence of such an $\mathbf{F}$ is due to Birkhoff; c.f. [**BuS81**] Theorem 10.12). Then any onto map from X to $\mathbf{A}$ can be extended to a homomorphism from $\mathbf{F}$ onto $\mathbf{A}$. But then from Lemma 3.1.15 it follows that $\mathrm{Uf}(\mathbf{F})$ is an inner substructure of $\mathrm{Uf}(\mathbf{A})$. Hence, again by Lemma 3.1.15, we know that $\mathrm{CmUf}(\mathbf{A}) = \mathrm{Em}\mathbf{A}$ is a homomorphic image of $\mathrm{CmUf}(\mathbf{F}) = \mathrm{Em}\mathbf{F}$. So if $\mathrm{Em}\mathbf{F} \in \mathcal{V}$ then $\mathcal{V}$ will contain $\mathrm{Em}\mathbf{A}$ by the closure of varieties under homomorphic images. □

Thus we can say that the discrepancy between the Lindenbaum-Tarski algebra and canonical frames does not exist if the logic is canonical.

(The proof of this result is modelled on that presented in [**Gol99**] Theorem 4.1.2.) With the use of this result we can thus restate the question about canonical logics algebraically.

COROLLARY 3.6.3. *A logic Λ is canonical if, and only if, the variety* $\mathrm{V}(\Lambda)$ *is closed under canonical embedding algebras.*

This also motivates the following definition.

DEFINITION 3.6.4. Let $\mathcal{V}$ be a variety. We call $\mathcal{V}$ a *canonical variety* if $\mathcal{V}$ is closed under canonical embedding algebras.

In the next chapter we will continue looking at closure of classes under canonical embedding algebras and extract criteria for when a variety is canonical. Such criteria on $\mathrm{V}(\Lambda)$ would then capture when the formulas of Λ are valid in its canonical frame.

3.6.2. Complete logics.

We define the class $\mathrm{Str}(\Lambda)$ of all structures that validate a logic Λ by

$$\mathrm{Str}(\Lambda) = \{\mathbf{F} : \mathbf{F} \Vdash \Lambda\}.$$

We say a logic Λ is *complete* if it is characterised by some class $\mathcal{K}$ of structures. Since $\mathcal{K} \subseteq \mathrm{Str}(\Lambda)$ we can restate this definition in the following way, a logic Λ is complete if, and only if, $\mathrm{Str}(\Lambda)$ characterises Λ. As in the previous section we wish to obtain an algebraic characterisation of complete logics.

THEOREM 3.6.5. *A logic Λ is complete if, and only if,* $\mathrm{V}(\Lambda) = \mathrm{V}(\mathbf{Cm}\mathrm{Str}(\Lambda))$.

PROOF. Let Λ be a complete logic. By Corollary 3.5.3 $\mathbf{Cm}\mathrm{Str}(\Lambda) \subseteq \mathrm{V}(\Lambda)$, since for any $\phi \in \Lambda$ if $\mathbf{F} \Vdash \phi$ then $\mathrm{Cm}\mathbf{F} \Vdash \phi$ and hence $\mathrm{Cm}\mathbf{F} \in \mathrm{V}(\Lambda)$. Consider any ψ and ϕ such that $\mathbf{Cm}\mathrm{Str}(\Lambda) \models (\psi = \phi)$ then, by the equivalence of equations and formulas, $\mathbf{Cm}\mathrm{Str}(\Lambda) \Vdash \psi \leftrightarrow \phi$. It follows from Corollary 3.5.3 that $\mathrm{Str}(\Lambda) \Vdash \psi \leftrightarrow \phi$. Since Λ is a complete logic $\Lambda \vdash \psi \leftrightarrow \phi$, whence $\mathrm{V}(\Lambda) \models (\psi = \phi)$.

Assume $\mathrm{V}(\Lambda) = \mathrm{V}(\mathbf{Cm}\mathrm{Str}(\Lambda))$. If $\vdash \psi$ then $\mathrm{V}(\Lambda) \models (\psi = \mathbb{1})$. It follows that $\mathbf{Cm}\mathrm{Str}(\Lambda) \models (\psi = \mathbb{1})$ and so, by Corollary 3.5.3, $\mathrm{Str}(\Lambda) \Vdash \psi$. To conclude we assume that $\mathrm{Str}(\Lambda) \Vdash \psi$. Then $\mathbf{Cm}\mathrm{Str}(\Lambda) \Vdash \psi$. Hence $\mathrm{V}(\Lambda) \models (\psi = \mathbb{1})$ and so $\vdash \psi$. □

The above theorem then motivates the following definition.

DEFINITION 3.6.6. Let $\mathcal{V}$ be a variety. We call $\mathcal{V}$ a *complete variety* if there exists a class $\mathcal{K}$ of structures such that $\mathcal{V} = \mathrm{V}(\mathrm{Cm}\mathcal{K})$.

Not all logics are complete. As was shown by S. K. Thomason, in [**Tho72**], there exists a temporal logic such that $\mathsf{Str}\Lambda = \emptyset$ for which $\mathrm{V}(\Lambda)$ is non-trivial. It then follows from the above theorem that such a logic is incomplete. (Temporal logics are logics used to describe discrete flows of time which include two unary model operators to denote this discreet flow of time.)

3.6.3. Strongly complete logics.

We call a logic Λ *strongly complete* if there exists some class of structures $\mathcal{K}$ such that Λ is strongly complete with respect to $\mathcal{K}$ and $\mathcal{K} \subseteq \mathsf{Str}\Lambda$.

As in the previous sections we wish to relate these logics to algebraic properties in such a way that we get "strongly complete" varieties. The varieties associated with strongly complete logics have historically been referred to as complex varieties and are defined as follows.

DEFINITION 3.6.7. A class of algebras $\mathcal{W}$ is called *complex* if there exists a class of structures $\mathcal{K}$ such that $\mathcal{W} = \mathbf{SCm}\mathcal{K}$. Consequently $\mathcal{V}$ is termed a *complex variety* if $\mathcal{V}$ is both a variety and complex.

We then get the following theorem. (For a proof of this result we refer the reader to [**Gol99**] Theorem 5.6.1.)

THEOREM 3.6.8. *Let $\mathcal{W}$ be a quasivariety.*

(i) *If $\mathcal{W}$ is complex, then its associated logic is strongly complete.*
(ii) *If the logic associated with $\mathcal{W}$ is strongly complete then $\mathbf{H}\mathcal{W}$ is complex.*

Thus we get the following characterisation of strongly complete logics.

COROLLARY 3.6.9. *A logic Λ is strongly complete if, and only if, $\mathrm{V}(\Lambda)$ is complex.*

3.7. Logics and categories

Throughout this chapter we described the standard way of navigating through the Paradigm Triangle using certain categorical constructions. However we never showed how to view a particular logic from a categorical perspective. This is part of the subject matter of categorical logic, on which we will not elaborate further in this dissertation.

CHAPTER 4

Complex and canonical varieties

We now focus on the algebraic view of the paradigm triangle. In the first section we focus on canonical embedding algebras in search of criteria under which a class of algebras will be closed under Em. In the next section we then apply some of the results that arise from this search to characterise elementary classes. In Section 4.3 we conclude by presenting several results pertaining to the varieties related to canonical and complete logics (c.f. Section 3.6).

4.1. Saturation, Good Ultrafilters and Canonical Extensions

In Corollary 3.6.3 we saw that a logic is canonical if, and only if, the variety generated by this logic is closed under canonical extensions. Hence it would be useful if we could find a characterisation of the closure of a class of algebras under canonical embedding algebras.

DEFINITION 4.1.1. For a structure $\mathbf{M}$ we define the *canonical extension* of $\mathbf{M}$ to be the structure $\mathrm{Uf}(\mathrm{Cm}\mathbf{M})$.

It turns out that we can find criteria on the relational semantic level to give us insight into the closure under canonical embedding algebras. As a criteria we present a beautiful result by Fine, van Benthem and Goldblatt (c.f. Theorem 4.1.25) which's proof uses the fact that the canonical extension of a structure $\mathbf{M}$ is a bounded morphic image of an ultrapower of $\mathbf{M}$ (i.e. $\mathrm{Uf}(\mathrm{Cm}\mathbf{M}) \in \mathbf{H_bP_w}(\{\mathbf{M}\})$).

The above mentioned theorem relies quite heavily on some deep model theoretic results. We will carefully work our way backwards from the theorem introducing these results as they become necessary. For a "road map" to the main results that arise in this process we refer the reader to Figure 3.given in Figure 3. (The arrows in this figure are to be interpreted as implications connecting the criteria listed in each box.)

If we take a look back at Lemma 3.1.15(i) (p. 37) we can see that the above mentioned theorem can be reduced to the question of whether there exists a bounded morphism $\psi : \mathbf{M}^\Lambda/\mathcal{F} \longrightarrow \mathrm{Uf}(\mathbf{M})$, with $\mathcal{F}$ an ultrafilter over Λ. Firstly we will consider what criteria we need to make ψ bounded and afterwards show that we can make the domain of ψ an ultrapower of $\mathbf{M}$.

Let $\mathbf{M} = \langle M, R^{\mathbf{M}}, F^{\mathbf{M}}\rangle$ be an L-structure. We require a map $\psi : \mathbf{U} \longrightarrow \mathrm{Uf}(\mathbf{M})$, where $\mathbf{U} \in \mathbf{P_w}(\{\mathbf{M}\})$. For every $u \in U$, ψ needs to map u to a set of subsets of $\mathbf{M}$. To capture the idea of a subset being in $\psi(u)$ we introduce unary predicates $\overline{Y}$

$$\begin{tikzcd}
\boxed{\text{Lemma 4.1.18}} \arrow[d] & \\
\boxed{\text{Theorem 4.1.21}} \arrow[d] & \\
\boxed{\text{Theorem 4.1.13}} \arrow[d] & \\
\boxed{\text{Lemma 4.1.7}} \arrow[d] & \boxed{\text{Lemma 4.1.24}} \arrow[dl] \\
\boxed{\text{Theorem 4.1.25}} &
\end{tikzcd}$$

FIGURE 3. Results leading to the main theorem

for each $Y \subseteq M$. Intuitively the idea is that $\overline{Y}(u)$ is true if, and only if, $Y \in \psi(u)$. Then for $\psi(u)$ to be an ultrafilter we'd expect the following sentences to hold:

$$\begin{gathered}
\forall u \overline{M}(u), \qquad \neg\exists u \overline{\emptyset}(u), \\
\forall u \overline{X \cap Y}(u) \text{ iff } \overline{X}(u) \text{ and } \overline{Y}(u), \\
\forall u \overline{X \cup Y}(u) \text{ iff } \overline{X}(u) \text{ or } \overline{Y}(u).
\end{gathered} \tag{4.1}$$

Consider the expansion $\mathbf{M}'$ of $\mathbf{M}$ obtained by adding unary predicates $\overline{Y}$ to L for each subset Y of M, i.e.

$$\mathbf{M}' = \langle M, R^{\mathbf{M}}, F^{\mathbf{M}}, \langle \overline{Y}^{\mathbf{M}'} : Y \subseteq M \rangle\rangle. \tag{4.2}$$

It should be clear that (4.1) holds in $\mathbf{M}'$. Since $\mathbf{M}'$ shares its universe with $\mathbf{M}$ we try to find a structure

$$\mathbf{U} = \langle U, R^{\mathbf{U}}, F^{\mathbf{U}}, \langle \overline{Y}^{\mathbf{U}} : Y \subseteq M \rangle\rangle \tag{4.3}$$

such that $\langle U, R^{\mathbf{U}}, F^{\mathbf{U}} \rangle$ is an ultrapower of $\mathbf{M}$ and $\mathbf{U}$ is also an elementary extension of $\mathbf{M}'$. Then for any element $u \in U$ we can define the function ψ as follows

$$\psi(u) = \{Y \subseteq M : \overline{Y}^{\mathbf{U}}(u)\}. \tag{4.4}$$

The function ψ then maps from the L-structure $\mathbf{V}$ to $\mathrm{Uf}(\mathbf{M})$, where $\mathbf{V}$ is defined by

$$\mathbf{V} = \langle U, R^{\mathbf{U}}, F^{\mathbf{U}} \rangle. \tag{4.5}$$

To see that $\psi(u)$ is in fact an ultrafilter over $\mathbf{M}$ observe that from (4.1) we can easily deduce that:

$$\begin{gathered}
M \in \psi(u), \qquad \emptyset \notin \psi(u), \\
X \cap Y \in \psi(u) \text{ iff } X \in \psi(u) \text{ and } Y \in \psi(u), \\
X \cup Y \in \psi(u) \text{ iff } X \in \psi(u) \text{ or } Y \in \psi(u).
\end{gathered}$$

However this map needs to be onto and bounded. Let us consider an ultrafilter $\mathcal{G}$ over $\mathbf{M}$, for ψ to be onto we need a $u \in U$ such that $\psi(u) = \mathcal{G}$. I.e. for every $Y \in M$ if $Y \in \mathcal{G}$ then $Y \in \psi(u)$ and if $Y \notin \mathcal{G}$ then $Y \notin \psi(u)$. Accordingly we let

$$\Gamma_0(x) = \{\overline{Y}(x) : Y \in \mathcal{G}\} \text{ and } \Gamma_1(x) = \{\neg\overline{Y}(x) : Y \notin \mathcal{G}\}. \tag{4.6}$$

Thus if we have a u that satisfies $\Gamma(x) = \Gamma_0(x) \cup \Gamma_1(x)$ in $\mathbf{U}$ then $\psi(u) = \mathcal{G}$. It is easy to find a $u \in V$ that finitely satisfies $\Gamma = \Gamma(x)$. I.e. for $\Delta_0 \subseteq_\omega \Gamma_0$ and $\Delta_1 \subseteq_\omega \Gamma_1$, u satisfies Δ_0 and Δ_1 in $\mathbf{U}$. To see this consider any $I, J \subseteq_\omega \mathcal{P}(M)$ with $I \subseteq \mathcal{G}$ and $J \subseteq M \setminus \mathcal{G}$ where $\Delta_0 = \{\overline{Y}(x) : Y \in I\}$ and $\Delta_1 = \{\neg\overline{Y}(x) : Y \in J\}$. Since $\mathcal{G}$ is an ultrafilter we know that $\bigcap I \in \mathcal{G}$ and $\bigcup J \notin \mathcal{G}$, so there exists $y \in \bigcap I$ with $y \notin \bigcup J$ which satisfies Δ_0 and Δ_1. If we can say that Γ being finitely satisfiable in $\mathbf{U}$ implies Γ satisfiable in $\mathbf{U}$ then there must exist a $u \in U$ such that u satisfies Γ and hence $\psi(u) = \mathcal{G}$. However finite satisfiability is very closely linked to the theory of saturated models as we shall see in the sequel.

We have thus far only discussed ontoness, as we shall see in the sequel, boundedness can be dealt with in a similar fashion.

4.1.1. Saturation.

DEFINITION 4.1.2. Let κ be a cardinal. A structure $\mathbf{M}$ is said to be *κ-saturated* if, and only if, for every set $Y \subseteq M$ with $|Y| < \kappa$, every set of formulas $\Phi(x)$ of $L(Y)$ consistent with $\operatorname{Th}\mathbf{M}_Y$ is satisfiable in $\mathbf{M}_Y$.

(Refer to p 16 for the definition of $\mathbf{M}_Y$ the expansion $\mathbf{M}$.)

PROPOSITION 4.1.3. *Let $\mathbf{M}$ be an L-structure and $\Phi = \Phi(x_0, \ldots, x_{n-1})$ a set of L-formulas. Φ is finitely satisfiable in $\mathbf{M}$ if, and only if, Φ is consistent with* $\operatorname{Th}\mathbf{M}$.

PROOF. First we assume that every finite subset of Φ is satisfiable in $\mathbf{M}$. Let $\Delta \subseteq_\omega \Phi$ then there exists $\underline{a} \in \mathbf{M}$ such that $\mathbf{M} \models (\Delta \cup \operatorname{Th}\mathbf{M})(\underline{a})$. By Proposition 2.2.2 (p. 16) there exists a structure $\mathbf{N}$ such that $\mathbf{N} \models (\Phi \cup \operatorname{Th}\mathbf{M})(\underline{a})$.

We will prove the backwards implication by induction on the number of free variables in Φ, where Φ is assumed to be consistent with $\operatorname{Th}\mathbf{M}$.

For the case $n = 1$ let $\Delta(x_0) \subseteq_\omega \Phi(x_0)$ such that there is no $a_0 \in \mathbf{M}$ with $\mathbf{M} \models \Delta(a_0)$. Let $\delta = \neg\exists x_0 \bigwedge \Delta(x_0)$ then $\mathbf{M} \models \delta$ so $\delta \in \operatorname{Th}\mathbf{M}$. But this leads to a contradiction since by the consistency of Φ with $\operatorname{Th}\mathbf{M}$ there exists an L-structure $\mathbf{N}$ and $b \in \mathbf{N}$ such that $\mathbf{N} \models \Phi(b)$ and $\mathbf{N} \models \delta$ even though δ is not consistent with Φ.

Assume the result holds for n and let $\Phi = \Phi(x_0, \ldots, x_n)$. We consider any $\Delta \subseteq_\omega \Phi$. By assumption there exists an L-structure $\mathbf{N}$ such that $\mathbf{N}$ satisfies Φ and $\operatorname{Th}\mathbf{M}$. Thus there exist $b_0, \ldots, b_n \in N$ such that $\mathbf{N} \models \Delta(b_0, \ldots, b_n)$. We let $\delta(x_0, \ldots, x_{n-1}) = \exists x_n \bigwedge \Delta(x_0, \ldots, x_n)$. Then $\mathbf{N} \models \delta(b_0, \ldots, b_{n-1})$ so by the inductive hypothesis δ must be finitely satisfiable in $\mathbf{M}$. In particular there must exist $a_0, \ldots, a_{n-1} \in \mathbf{M}$ such that $\mathbf{M} \models \delta(a_0, \ldots, a_{n-1})$. But, by construction, $\delta(x_0, \ldots, x_{n-1}) = \exists x_n \bigwedge \Delta(x_0, \ldots, x_n)$. Hence there exists some $a_n \in M$ such that $\mathbf{M} \models \Delta(a_0, \ldots, a_n)$. $\square$

The proposition below is a generalisation of a result from **[ChK77]** (c.f. Proposition 2.3.6 p. 98).

PROPOSITION 4.1.4. *Let* $\mathbf{M} = \langle M, R^{\mathbf{M}}, F^{\mathbf{M}} \rangle$ *be an* L*-structure and let* κ *be some infinite cardinal.* $\mathbf{M}$ *is* κ*-saturated if, and only if, for every* $Y \subseteq M$ *with* $|Y| < \kappa$, *each set of formulas* $\Phi(x_0, \ldots, x_{n-1})$ *of* $L(Y)$ *consistent with* $\mathrm{Th}\,\mathbf{M}_Y$ *is satisfiable in* $\mathbf{M}_Y$ *the expansion of* $\mathbf{M}$.

PROOF. We prove the forward direction by induction on the number of free variables in Φ. By definition the result holds for $n = 1$. Assume the result holds for n and let $\Phi = \Phi(x_0, \ldots, x_n)$ be consistent with $\mathrm{Th}\,\mathbf{M}_Y$. Since a set of formulas is consistent with a theory Γ if, and only if, its closure under finite conjunctions is consistent with Γ, we may assume that Φ is closed under finite conjunctions. Let

$$\Phi'(x_0, \ldots, x_{n-1}) = \{\exists x_n \phi(x_0, \ldots, x_n) : \phi \in \Phi\}.$$

By the inductive hypothesis Φ' is consistent with $\mathrm{Th}\,\mathbf{M}_Y$. Thus there is an n-tuple $a_0, \ldots, a_{n-1}$ which satisfies Φ' in $\mathbf{M}_Y$. We let $Y' = Y \cup \{a_0, \ldots, a_{n-1}\}$. Then Y' is still of power $< \kappa$. For each $\phi_0, \ldots, \phi_{m-1} \in \Phi$, $(\exists x_n)(\phi_0 \wedge \ldots \wedge \phi_{m-1}) \in \Phi'$ so $\Phi(\overline{a_0}, \ldots, \overline{a_{n-1}}, x_n)$ is consistent with $\mathrm{Th}\,\mathbf{M}_{Y'}$. Since $\mathbf{M}$ is κ-saturated, there exists $a_n \in M$ that satisfies $\Phi(\overline{a_0}, \ldots, \overline{a_{n-1}}, x_n)$ in $\mathbf{M}_{Y'}$. Thus $a_0, \ldots, a_n$ satisfy Φ in $\mathbf{M}_Y$.

For the backward direction we just take $n = 1$ and the result follows trivially. □

The following corollary, which follows directly from Proposition 4.1.3 and 4.1.4, now provides us the characterisation of finite satisfiability that we were looking for, for the domain of our map ψ.

COROLLARY 4.1.5. *Let* κ *be some infinite cardinal,* $\mathbf{M}$ *an* L*-structure.* $\mathbf{M}$ *is* κ*-saturated if, and only if, for each* $Y \subseteq M$ $(|Y| < \kappa)$ *each set* $\Phi(x_1, \ldots, x_{n-1})$ *of* $L(Y)$*-formulas that are finitely satisfiable in* $\mathbf{M}_Y$ *is itself satisfiable in* $\mathbf{M}_Y$.

PROPOSITION 4.1.6. *Let* $\mathbf{M}$ *be an* L*-structure.* $\mathbf{M}$ *is* ω*-saturated if, and only if,* $\mathbf{M}$ *is* n*-saturated for all* $n \in \omega$.

PROOF. The forward direction is trivial. So assume $\mathbf{M}$ is n-saturated for all $n \in \omega$ and let $Y \subseteq M$ with $|Y| < \omega$ then there exists some $m \in \omega$ such that $|Y| < m$. But $\mathbf{M}$ is $m + 1$-saturated from which the result follows easily. □

Note that the above result can be extended to arbitrary limit cardinals (c.f. [**ChK77**] Proposition 5.1.1 for more details).

It should be clear that to make the map ψ, as defined in (4.4), onto it will suffice to require that the domain of ψ be ω-saturated, since ontoness only requires 0-saturation.

LEMMA 4.1.7. *Let* $\mathbf{M} = \langle M, R^{\mathbf{M}} \rangle$ *be a relational structure and* $\mathbf{M}'$, $\mathbf{U}$, $\mathbf{V}$ *and* ψ *be as in (4.2)-(4.4), with* $\mathbf{U}$ *an elementary extension of* $\mathbf{M}'$. *If* $\mathbf{U}$ *is* ω*- saturated then* ψ *is a bounded morphism from* $\mathbf{V}$ *onto* $\mathrm{Uf}(\mathbf{M})$.

PROOF. Let us just quickly remind ourselves of the definition of ψ

$$\psi(u) = \{Y \subseteq M : \overline{Y}^{\mathbf{U}}(u)\}. \tag{4.4}$$

First we show that ψ is onto. Let $\mathcal{G}$ be a member of $\mathrm{Uf}(\mathbf{M})$, $\Gamma_0(x)$ and $\Gamma_1(x)$ be as in (4.6) and $\Gamma = \Gamma_0(x) \cup \Gamma_1(x)$. As shown earlier Γ is then finitely satisfiable in $\mathbf{V}$. From the ω-saturation of $\mathbf{U}$ we know that there exists a $u \in \mathbf{V}$ which satisfies Γ and hence $\psi(u) = \mathcal{G}$.

We now show that ψ is a bounded morphism. Let $r \in R$ with $ar(r) = n + 1$.

zig: Assume there exist $u_0, \ldots, u_n \in U$ such that $r^{\mathbf{U}}(u_0, \ldots, u_n)$. Let $Y_i \in \psi(u_i)$ for $i < n$. If $Y_n = \{y_n : r^{\mathbf{M}}(y_0, \ldots, y_n)$ where $y_i \in Y_i$ for $i < n\}$ then

$$\forall x_0, \ldots, \forall x_n(\overline{Y_0}(x_0) \wedge \ldots \wedge \overline{Y_{n-1}}(x_{n-1}) \wedge r(x_0, \ldots, x_n) \rightarrow \overline{Y_n}(x_n))$$

holds in $\mathbf{M}'$ and thus in $\mathbf{U}$. Since $\overline{Y_i}^{\mathbf{U}}(u_i)$, for $i < n$, this implies that $\overline{Y_n}^{\mathbf{U}}(u_n)$. Hence, by the definition of ψ, $Y_n \in \psi(u_n)$. Thus $r^{\mathrm{Uf}(\mathbf{M})}(\psi(u_0), \ldots, \psi(u_n))$.

zag: Suppose that $r^{\mathrm{Uf}(\mathbf{M})}(\mathcal{G}_0, \ldots, \mathcal{G}_{n-1}, \psi(u))$ for $u \in U$ and $\mathcal{G}_i \in \mathrm{Uf}(M)$, where $i < n$. Then we must find $u_i \in U$ such that $r^{\mathbf{U}}(u_0, \ldots, u_{n-1}, u)$ and $\psi(u_i) = \mathcal{G}_i$. Let $\Gamma = \Gamma(x_0, \ldots, x_{n-1})$ be the set of formulas defined by

$$\Gamma(x_0, \ldots, x_{n-1}, x) = \\ \{r(x_0, \ldots, x_{n-1}, x)\} \cup \{\overline{Y}(x_0) : Y \in \mathcal{G}_0\} \cup \ldots \cup \{\overline{Y}(x_{n-1}) : Y \in \mathcal{G}_{n-1}\}.$$

If $u_0, \ldots, u_{n-1}, u$ satisfies Γ in $\mathbf{U}$, where each variable x_i is interpreted as u_i, then $r^{\mathbf{U}}(u_0, \ldots, u_{n-1}, u)$ and, for each $i < n$, $\overline{Y}^{\mathbf{U}}(u_i)$, for all $Y \in \mathcal{G}_i$, so that $\mathcal{G}_i \subseteq \psi(u_i)$. Since each $\mathcal{G}_i$ is maximal this gives us $\psi(u_i) = \mathcal{G}_i$ as desired.

To show that Γ is satisfiable in $\mathbf{U}$, by ω-saturation and Corollary 4.1.5, it is enough to show that Γ is finitely satisfiable in $\mathbf{U}$. Since each $\mathcal{G}_i$ is closed under finite intersection, all we need to show is that if $Y_i \in \mathcal{G}_i$ and $i < n$ then

$$\Delta = \{r(x_0, \ldots, x_{n-1}, x), \overline{Y_0}(x_0), \ldots, \overline{Y_{n-1}}(x_{n-1})\}$$

is satisfiable in $\mathbf{U}$. But given such Y_i, since $r^{\mathrm{Uf}(\mathbf{M})}(\mathcal{G}_0, \ldots, \mathcal{G}_{n-1}, \psi(u))$, it follows that $Y = \{y : r^{\mathbf{M}}(y_0, \ldots, y_{n-1}, y)$ with $y_i \in Y_i$ for $i < n-1\} \in \psi(u)$. Hence, by the definition of ψ, we can see that $\overline{Y}^{\mathbf{U}}(u)$. But for this Y the sentence

$$\forall x(\overline{Y}(x) \rightarrow \exists x_0 \ldots \exists x_{n-1}(\overline{Y_0}(x_0) \wedge \ldots \wedge \overline{Y_{n-1}}(x_{n-1}) \wedge r(x_0, \ldots, x_{n-1}, x)))$$

is true in $\mathbf{M}'$ and hence in $\mathbf{U}$. So there exist $u_0, \ldots, u_{n-1} \in U$ such that $\overline{Y_i}^{\mathbf{U}}(u_i)$, for all $i < n$, and $r^{\mathbf{U}}(u_0, \ldots, u_{n-1}, u)$. Then $u_0, \ldots, u_{n-1}$ satisfies Δ in $\mathbf{U}$ and so, by ω-saturation, Γ is also satisfiable in $\mathbf{U}$. □

For more on the applications of saturated models in model theory we refer the reader to Chapter 5 of [**ChK77**] and Chapter 10 of [**Hod94**].

4.1.2. Ultraproducts and saturation.

We have just seen that we would like an ω-saturated ultrapower $\mathbf{U} = \mathbf{M}^{\Lambda}/\mathcal{F}$. The question now is what conditions on an ultrafilter $\mathcal{F}$ over Λ will facilitate this. We naively look for conditions on $\mathbf{U}$ such that if a set of formulas $\Phi = \Phi(x)$ in one free variable is finitely satisfiable in $\mathbf{U}$ then Φ must be satisfiable in $\mathbf{U}$.

Given a particular Λ the only freedom we are allowed is in our choice of ultrafilter. To find conditions so that $\mathcal{F}$ will fulfill our requirements, we need to see how Φ and $\mathcal{F}$ are related. For finite Φ the Los Theorem (p. 19) gives us

$$\mathbf{U} \models \Phi(\underline{a}) \text{ iff } \{\lambda \in \Lambda : \mathbf{U}_\lambda \models \Phi(a_\lambda)\} \in \mathcal{F}$$

where $\underline{a} = \langle a_\lambda : \lambda \in \Lambda\rangle \in \prod \mathbf{U}_\lambda$ and $\mathbf{U} = \prod_{\mathcal{F}} \mathbf{U}_\lambda$. Giving us some direction in which to hunt.

We are particularly interested in finitely satisfiable subsets Δ of Φ, so we define a map $f : \mathcal{P}_\omega(\Phi) \longrightarrow \mathcal{P}(\Lambda)$ as follows

$$f(\Delta) = \{\lambda \in \Lambda : \Delta \text{ is satisfiable in } \mathbf{U}_\lambda\}. \tag{4.7}$$

Then $f(\Delta) \in \mathcal{F}$ and f is order reversing. For any function f we define

$$\Phi_f(\lambda) = \{\phi \in \Phi : \lambda \in f(\{\phi\})\} \tag{4.8}$$

which provides us with the formulas associated with f in each coordinate. Note that if $\Phi_f(\lambda)$ is satisfiable in $\mathbf{U}_\lambda$ for each $\lambda \in \Lambda$ then Φ is satisfiable in $\mathbf{U}$. To see this we choose elements $a_\lambda \in \mathbf{U}_\lambda$ for each λ such that a_λ satisfies $\Phi_f(\lambda)$ in $\mathbf{U}_\lambda$. Then for each $\phi \in \Phi$

$$\{\lambda \in \Lambda : \mathbf{U}_\lambda \models \phi(a_\lambda)\} \supseteq \{\lambda \in \Lambda : \phi \in \Phi_f(\lambda)\} = f(\{\phi\}) \in \mathcal{F}$$

and therefore $\mathbf{U} \models \phi(\underline{a}/\mathcal{F})$ where $\underline{a} = \langle a_\lambda : \lambda \in \Lambda \rangle$.

However there is no reason why each $\Phi_f(\lambda)$ need be satisfiable, or even finitely satisfiable, in $\mathbf{U}_\lambda$. This set might very well contain too many elements of Φ. So we look for a function $h : \mathcal{P}_\omega(\Phi) \longrightarrow \mathcal{P}(\Lambda)$ such that $\Phi_h(\lambda) \subseteq \Phi_f(\lambda)$. Note that this condition follows if we have $h \leq f$, i.e. $h(\Delta) \subseteq f(\Delta)$ for any $\Delta \in \mathcal{P}_\omega(\Phi)$.

We also want each $\Phi_h(\lambda)$ to be finitely satisfiable in $\mathbf{U}_\lambda$. Let Δ be a finite subset of $\Phi_h(\lambda)$ then for each $\phi \in \Delta$ we know $\lambda \in h(\{\phi\})$ and we want $\lambda \in f(\Delta)$, c.f. (4.7). We have $\lambda \in \bigcap_{\phi\in\Delta} h(\{\phi\})$ and all we need is $\lambda \in h(\bigcup_{\phi\in\Delta}\{\phi\}) = h(\Delta) \subseteq f(\Delta)$. This leads us to the following definition.

Definition 4.1.8. A function $h : \mathcal{P}(M) \longrightarrow \mathcal{P}(N)$, with M and N sets, is called an $(\cup, \cap)$-*morphism* if for any $X \subseteq_\omega \mathcal{P}(M)$ we have that $h(\bigcup X) = \bigcap h(X)$.

Thus we require the following condition on $\mathcal{F}$.

(4.9) For every order reversing map $g : \mathcal{P}_\omega(\Phi) \longrightarrow \mathcal{F}$ there must exist an $(\cup, \cap)$-morphism $h : \mathcal{P}_\omega(\Phi) \longrightarrow \mathcal{F}$ such that $h \leq g$.

If we can have $\Phi_h(\lambda)$ finite then $\Phi_h(\lambda)$ is satisfiable. Thus for each $\Delta \in \mathcal{P}_\omega(\Phi)$ we want some way of forcing $h(\Delta)$ to be small. Let us assume we have a countable decreasing chain $\Lambda = \Lambda_0 \supseteq \Lambda_1 \supseteq \dots$ in $\mathcal{F}$. We let

$$g(\Delta) = f(\Delta) \cap \Lambda_n, \text{ where } n = |\Delta|. \tag{4.10}$$

Now assume h satisfies condition (4.9). For each $\lambda \in \Lambda$ we let $\phi_0, \dots, \phi_{n-1}$ be distinct members of $\Phi_h(\lambda)$ and $\Delta = \{\phi_0, \dots, \phi_{n-1}\}$ then

$$\lambda \in h(\Delta) \subseteq g(\Delta) \subseteq \Lambda_n. \tag{4.11}$$

If the intersection of the chain is empty there is an m such that $\lambda \notin \Lambda_m$ and hence $\Phi_h(\lambda)$ has less than m elements. This leads us to the following definition.

Definition 4.1.9. A filter $\mathcal{F}$ is said to be κ-*complete*, where κ is a cardinal, if for every family $\mathcal{K} \subseteq \mathcal{F}$, $|\mathcal{K}| < \kappa$ implies $\bigcap \mathcal{K} \in \mathcal{F}$, otherwise $\mathcal{F}$ is said to be κ-*incomplete*.

Proposition 4.1.10. *Let Λ be a non-empty index set. An ultrafilter $\mathcal{F}$ over Λ is κ-incomplete if, and only if, for some $\mathcal{K} \subseteq \mathcal{F}$, $|\mathcal{K}| < \kappa$ and $\bigcap \mathcal{K} = \emptyset$.*

Proof. Let $\mathcal{F}$ be an ultrafilter and $\mathcal{K} \subseteq \mathcal{F}$, with $|\mathcal{K}| < \kappa$.

For the backward direction assume that $\mathcal{F}$ is κ-complete. Then by assumption $\bigcap \mathcal{K} \in \mathcal{F}$ and since $\mathcal{F}$ is an ultrafilter $\bigcap \mathcal{K} \neq \emptyset$, by Theorem 2.2.10 (p. 18).

Conversely let $\mathcal{F}$ be κ-incomplete and take $\mathcal{K} \subseteq \mathcal{F}$ witnessing this, i.e. $\bigcap \mathcal{K} \notin \mathcal{F}$. Let $\mathcal{G} = \{\bigcap \mathcal{K} \cap X : X \in \mathcal{K}\}$. Then $\mathcal{G} \subseteq \mathcal{F}$, $|\mathcal{G}| = |\mathcal{K}|$ and $\bigcap \mathcal{G} = \emptyset$. □

PROPOSITION 4.1.11. *An ultrafilter $\mathcal{F}$ over an index set Λ is ω-incomplete if, and only if, $\mathcal{F}$ contains a countable descending chain $\mathcal{I}$ such that $\bigcap \mathcal{I} = \emptyset$.*

PROOF. The backwards direction follows directly from the proposition above.

Conversely let $\mathcal{F}$ be ω-incomplete and take $\mathcal{K} \subset \mathcal{F}$ witnessing this. We enumerate the elements of $\mathcal{K}$ as follows

$$X_0, X_1, \ldots, X_n, \ldots, \text{ where } n < \omega$$

Now define a descending chain $\mathcal{I} = \Lambda_0 \supseteq \Lambda_1 \supseteq \ldots$ as follows

$$\Lambda_0 = X_0 \setminus \bigcap \mathcal{K} \text{ and}$$

$$\Lambda_{n+1} = \Lambda_n \cap X_{n+1}.$$

Then $\mathcal{I} \subseteq \mathcal{F}$ and $\bigcap \mathcal{I} = \emptyset$. □

Note that the condition we imposed on $\mathbf{U}$ at the beginning of this section limits the size of Φ to $\omega + |L|$. However if we, as here, want to get ω-saturation we will need the following more general property on $\mathcal{F}$, c.f. (4.9).

DEFINITION 4.1.12. An ultrafilter $\mathcal{F}$ on a set Λ is said to be *κ-good* if, for every set V with $|V| < \kappa$, and for every order-reversing map $g : \mathcal{P}_\omega(V) \longrightarrow \mathcal{F}$, there exists a $(\cup, \cap)$-morphism $h : \mathcal{P}_\omega(V) \longrightarrow \mathcal{F}$ with $h \leq g$.

THEOREM 4.1.13. *Suppose κ is an infinite cardinal, and suppose $\mathbf{U}_\lambda$, $\lambda \in \Lambda$, are L-structures with $|L| < \kappa$. If $\mathcal{F}$ is an ω-incomplete κ-good ultrafilter over Λ, then $\prod_{\mathcal{F}} \mathbf{U}_\lambda$ is κ-saturated*

PROOF. Let $Y \subseteq \mathbf{M}$, with $|Y| < \kappa$. (For ease of writing we assume that we are working with the extensions of all structures to this language, unless explicitly stated otherwise.) By Corollary 4.1.5 it is sufficient to prove:

(1) For every set $\Phi = \Phi(x_0, \ldots, x_{n-1})$ of formulas of $L(Y)$, if every finite subset of Φ is satisfiable in $\prod_{\mathcal{F}} \mathbf{U}_\lambda$, then Φ is itself satisfiable in $\prod_{\mathcal{F}} \mathbf{U}_\lambda$.

To simplify presentation we will only consider the case where $\Phi = \Phi(x)$.

Suppose $\mathcal{F}$ is ω-incomplete and that every finite subset of Φ is satisfiable in $\prod_{\mathcal{F}} \mathbf{U}_\lambda$. By Proposition 4.1.11 it follows that $\mathcal{F}$ contains a countable descending chain

$$\Lambda = \Lambda_0 \supseteq \Lambda_1 \supseteq \cdots$$

such that $\bigcap \Lambda_i = \emptyset$.

So we define $f, g : \mathcal{P}_\omega(\Phi) \longrightarrow \mathcal{F}$ as in (4.7) and (4.10), where $g(\emptyset) = \Lambda$. Since each $\Delta \in \mathcal{P}_\omega(\Phi)$ is finite, by assumption, it is satisfiable in $\prod_{\mathcal{F}} \mathbf{U}_\lambda$. By the Łos Theorem $g(\Delta) \in \mathcal{F}$. Let $\Delta, \Delta' \in \mathcal{P}_\omega(\Phi)$ so that $\Delta \subseteq \Delta'$ then $\Lambda_{|\Delta|} \supseteq \Lambda_{|\Delta'|}$ and $\vdash \exists x \bigwedge \Delta' \rightarrow \exists x \bigwedge \Delta$. Thus $g(\Delta) \supseteq g(\Delta')$ and hence g is order reversing.

Note that since $|L(Y)| < \kappa$ it follows that $|\Phi| < \kappa$ [†]. Thus, considering that $\mathcal{F}$ is κ-good, there exists a $(\cup, \cap)$-morphism $h : \mathcal{P}_\omega(\Phi) \longrightarrow \mathcal{F}$ such that $h \leq g$.

Now define $\Phi_f(\lambda)$ as in (4.8). Since $\lambda \in \Lambda$ and $\bigcap \Lambda_i = \emptyset$ it follows that there exists some least $n \in \omega$ such that $\lambda \notin \Lambda_n$. Assume that $\Phi_h(\lambda)$ has n distinct elements $\phi_0, \ldots, \phi_{n-1}$ then

$$\lambda \in \bigcap_{i<n} h(\{\phi_i\}) = h(\bigcup_{i<n} \{\phi_i\}) \subseteq g(\bigcup_{i<n} \{\phi_i\}) \subseteq \Lambda_n.$$

[†] C.f. Lemma 2.1.2 p. 14

However this contradicts the fact that $\lambda \notin \Lambda_n$. Consequently $|\Phi_h(\lambda)| < n$, i.e. for any $\lambda \in \Lambda$ the set $\Phi_h(\lambda)$ is finite.

We proceed by constructing an element $a_{\mathcal{F}}$ which satisfies $\Phi(x)$ in $\prod_{\mathcal{F}} \mathbf{U}_\lambda$. From the definition of $\Phi_f(\lambda)$, the fact that $\Phi_h(\lambda)$ is finite, and since h is a $(\cup, \cap)$-morphism it follows that

$$\lambda \in \bigcap \{h(\{\phi\}) : \phi \in \Phi_h(\lambda)\} = h(\Phi_h(\lambda)) \subseteq g(\Phi_h(\lambda)).$$

By the definition of g we can choose $a_\lambda \in \mathbf{U}_\lambda$ such that

$$(*) \qquad \mathbf{U}_\lambda \models \bigwedge \Phi_h(\lambda)(a_\lambda).$$

Let $a_{\mathcal{F}} = \langle a_0, \ldots, a_\lambda, \ldots \rangle$. It should now be clear that if $\phi \in \Phi$ and $\lambda \in g(\{\phi\})$ then $\phi \in \Phi_h(\lambda)$ and thus, by $(*)$, $\mathbf{U}_\lambda \models \phi(a_\lambda)$. However $h(\{\phi\}) \in \mathcal{F}$ so by the Łos Theorem $\prod_{\mathcal{F}} \mathbf{U}_\lambda \models \phi(a_{\mathcal{F}})$ for all $\phi \in \Phi$, i.e. $a_{\mathcal{F}}$ satisfies $\Phi(x)$ in $\prod_{\mathcal{F}} \mathbf{U}_\lambda$. □

4.1.3. Good ultrafilters.

We now have a reasonable way of producing the ultraproducts we require. All we need is to demonstrate the existence of good ultrafilters.

A naive way of constructing an ultrafilter over Λ is to look at all subsets of Λ and for each subset either choose it or its complement to be in our filter (c.f. Theorem 2.2.10 p. 18). So let us assume we have a transfinite enumeration of $\mathcal{P}(\Lambda)$

$$\Lambda_0, \Lambda_1, \ldots, \Lambda_\xi, \ldots, \quad \xi < 2^\kappa \text{ where } \kappa = |\Lambda|.$$

We start off with some proper filter $\mathcal{F}_0$ and iteratively process all the Λ_ξ's in the way mentioned above to construct an increasing sequence of proper filters $\mathcal{F}_\xi$. For a limit ordinal we just take the union of all earlier filters. Such a process will however just provide us with an ultrafilter. To generate a good ultrafilter we need to also consider all order reversing maps from $\mathcal{P}_\omega(\Lambda)$ to $\mathcal{P}(\Lambda)$. Note that there are also 2^κ such maps†, so we arrange these into a similar sequence

$$(*) \qquad g_0, g_1, \ldots, g_\xi, \ldots, \quad \xi < 2^\kappa$$

Provided we have $\mathcal{F}_\xi$ as above and some corresponding g_ξ, if the image of g_ξ is in $\mathcal{F}_\xi$ we try to extend $\mathcal{F}_\xi$ to a filter $\mathcal{G}_\xi$ such that there exists a $(\cup, \cap)$-morphism $h_\xi : \mathcal{P}_\omega(\Lambda) \longrightarrow \mathcal{G}_\xi$ with $h_\xi \leq g_\xi$. We however need each g_ξ to satisfy this condition for every Λ_η, $\eta < 2^\kappa$. So we let each g_ξ occur 2^κ times in our sequence of maps $(*)$. (Note that, by Lemma 2.1.2 p. 14, this does not increase the cardinality of the sequence.)

Let us consider the construction of an appropriate h, for some map g from $(*)$, in isolation. We are looking for an extension $\mathcal{G}$ of a proper filter $\mathcal{F}$, such that $\mathrm{ran}(h) \subseteq \mathcal{G}$ and $\mathcal{G}$ is also a proper filter‡. Note that if we take $\mathcal{G}$ to be the filter generated by $\mathcal{F} \cup \mathrm{ran}(h)$ then we easily satisfy the first condition. However $\mathcal{G}$ should also be proper. By assumption $\mathcal{F}$ is proper and since h should be a $(\cup, \cap)$-morphism $\mathrm{ran}(h)$ is closed under intersection. So to ensure the finite intersection property (c.f. Theorem 2.2.8 p. 17) we are left with ensuring that individual elements of $\mathcal{F}$ are closed under intersection with individual elements of $\mathrm{ran}(h)$. I.e. for $X \in \mathcal{F}$ and $\Delta \in \mathcal{P}_\omega(\Delta)$ we require that $X \cap h(\Delta) \neq \emptyset$. If $h \leq g$ as required then $h(\Delta) \subseteq g(\Delta)$

† $|\mathcal{P}_\omega(\Lambda)| = \kappa$ and $|\mathcal{P}(\Lambda)| = 2^\kappa$ hence the cardinality of the sequence of order reversing maps is $(2^\kappa)^\kappa = 2^{\kappa \cdot \kappa} = 2^\kappa$ (c.f. Lemma 2.1.2 p. 14).

‡ For our purposes here we define $\mathrm{ran}(h) = \{h(X) : X \text{ is in the domain of } h\}$.

and so $X \cap h(\Delta) \subseteq X \cap g(\Delta) \neq \emptyset$. Thus for each Λ we need to look for a set Y such that

$$X \cap h(\Delta) \supseteq X \cap g(\Delta) \cap Y \neq \emptyset. \tag{4.12}$$

So we list all members of $\mathcal{P}_\omega(\Lambda)$ in a transfinite sequence $\Delta_0, \Delta_1, \ldots, \Delta_\eta, \ldots$ and assume that we have a transfinite sequence $Y_0, Y_1, \ldots, Y_\eta, \ldots$ of members of $\mathcal{P}(\Lambda)$, where each Y_η is related to Δ_η, for $\eta < \kappa$. We proceed by trying to find properties on the Y_η that will facilitate our iterative construction.

Observe that, since $X \cap g(\Delta_\eta) \in \mathcal{F}$, to satisfy (4.12) it is enough to require that each Y_η has non-empty intersection with any member of $\mathcal{F}$. Let $Y' = \Lambda \setminus \bigcup Y_\eta$, since we cannot assume that no member of $\mathcal{F}$ is contained in Y', the Y_η's must in fact cover Λ. However we still need to ensure that h is a $(\cup, \cap)$-morphism, in fact we need to define h as well.

Note that if $h(\Gamma) \subseteq g(\Gamma')$ for $\Gamma \subseteq \Gamma' \in \mathcal{P}_\omega(\Lambda)$ then since g is order reversing $h(\Gamma) \subseteq g(\Gamma)$, but for each $\Gamma \in \mathcal{P}_\omega(\Lambda)$ there are several Δ_η's such that $\Gamma \subseteq \Delta_\eta$ so we stipulate that

$$\lambda \in h(\Gamma) \text{ iff } \Gamma \subseteq \Delta_\eta \text{ and } \lambda \in g(\Delta_\eta). \tag{4.13$'$}$$

However this does not necessarily give us a consistent definition. Consider the case, for some $\eta, \eta' < \kappa$, with $\Gamma \subseteq \Delta_\eta$ and $\Gamma \subseteq \Delta_{\eta'}$ where $\lambda \notin g(\Delta_\eta) \cap g(\Delta_{\eta'})$. But each Δ_η has an Y_η associated with it, so maybe we can use these Y_η to choose which Δ_η should be used. In fact we can get rid of the mentioned ambivalence by requiring that all the Y_η's are disjoint and then modifying the definition (??$'$) in the following way

$$\text{for } \lambda \in Y_\eta \text{ let } \lambda \in h(\Gamma) \text{ iff } \Gamma \subseteq \Delta_\eta \text{ and } \lambda \in g(\Delta_\eta). \tag{4.13}$$

From the above definition of h it easily follows that for $\Gamma, \Gamma' \in \mathcal{P}_\omega(\Lambda)$ and $\eta < \kappa$

$$h(\Gamma \cup \Gamma') \cap Y_\eta = h(\Gamma) \cap h(\Gamma') \cap Y_\eta.$$

Since we require that $\bigcup_{\eta<\kappa} Y_\eta = \Lambda$ it then easily follows that h will in fact be a $(\cup, \cap)$-morphism.

To summarise, we require the Y_η to form a partition P of Λ, and that each member of P has non- empty intersection with every member of $\mathcal{F}$. We say the partition P *is consistent with* $\mathcal{F}$.

REMARK 4.1.14. Note that P might not be consistent with the filter generated by $\mathcal{F} \cup \operatorname{ran}(h)$ so we will possibly need a large collection of different partitions when constructing the G_ξ.

We will now give more precise definitions as well as a systematic construction of these consistent partitions.

DEFINITION 4.1.15. For a filter $\mathcal{F}$ on Λ and a family Π of partitions of Λ we say that $(\mathcal{F}, \Pi)$ is *consistent* if given any $X \in \mathcal{F}$ and any $X_0, \ldots, X_{n-1}$, with each X_i belonging to a distinct partition $P_i \in \Pi$ we have that $X \cap \bigcap_{i<n} X_i \neq \emptyset$.

That is $(\mathcal{F}, \Pi)$ is consistent if the intersection of a member of $\mathcal{F}$ and finitely many blocks from different members of Π is never empty.

DEFINITION 4.1.16. A filter $\mathcal{F}$ on a set Λ is said to be *uniform* if, for all $X \in \mathcal{F}$, $|X| = |\Lambda|$

Observe that this definition implies that for $\Lambda \neq \emptyset$ each uniform filter over Λ is proper.

LEMMA 4.1.17. *Let κ be an infinite cardinal. Suppose $\mathcal{K}$ is a family of κ sets each of cardinality κ. Then it is possible to associate with each member X of $\mathcal{K}$ a set X' such that:*

(i) $X' \subseteq X$,
(ii) $|X'| = \kappa$ *and*
(iii) *if* $X \neq Y$, *then* $X' \cap Y' = \emptyset$ *for* $X, Y \in \mathcal{K}$.

PROOF. We arrange the members of $\mathcal{K}$ into a transfinite sequence

$$X_0, X_1, \ldots, X_\xi, \ldots \text{ where } \xi < \kappa.$$

Let R_η be the set

$$R_\eta = \{\langle \xi, \delta \rangle : \xi \leq \delta \text{ and } \delta < \eta\}.$$

Note that R_η is a subset of $\eta \times \eta$. Since κ is a limit ordinal, we have $R_\kappa = \bigcup_{\eta<\kappa} R_\eta$. We shall find an injective function f with domain R_κ such that

$$(*) \qquad \text{whenever } \xi \leq \delta < \kappa,\ f(\xi, \delta) \in X_\xi.$$

Once such a function f is found, we can define a family of sets

$$(**) \qquad X'_\xi = \{f(\xi, \delta) : \xi \leq \delta < \kappa\}.$$

Then property (i) follows from $(*)$, by the definition of X'_ξ in $(**)$ and the injectivity of f it follows that property (ii) will by satisfied (since κ is a cardinal), and the injectivity of f implies that (iii) will hold.

We define this function by transfinite induction. I.e. we construct a chain of functions

$$f_0 \subseteq f_1 \subseteq \ldots f_\eta \subseteq \ldots, \text{ where } \eta < \kappa$$

such that each f_η has domain R_η, is injective and satisfies $(*)$. Since $R_0 = \emptyset$ it is trivial to construct f_0 in such a way as to satisfy these criteria.

So assume we have constructed an injective function f_η that has domain R_η and satisfies $(*)$. Now for each $\xi \leq \eta$ we choose a value $f_{\eta+1}(\xi, \eta) \in X_\xi$ which is different from all previously chosen values for $f_{\eta+1}$. (Such elements exist since, for all $\eta < \kappa$ and $\xi < \kappa$, $|R_\eta| < \kappa = |X_\xi|$.) Then $(*)$ holds for $f_{\eta+1}$, and $f_{\eta+1}$ is obviously injective. At limit ordinals λ we define $f_\lambda = \bigcup_{\eta<\lambda} f_\eta$. To complete the proof we let $f = \bigcup_{\eta<\kappa} f_\eta$, which by construction is injective and satisfies $(*)$. $\square$

In other words this lemma shows that any family of κ sets of cardinality κ can be refined to a family of κ disjoint sets of cardinality κ.

For the following lemmas we assume Λ is infinite, and let $\kappa = |\Lambda|$.

LEMMA 4.1.18. *If $\mathcal{F}$ is a uniform filter over Λ generated by a family of cardinality at most κ, then there exists a family Π of 2^κ distinct partitions of Λ such that $(\mathcal{F}, \Pi)$ is consistent, and such that each partition $\mathrm{P} \in \Pi$ has κ blocks, with each block consisting of κ elements.*

PROOF. Let $\mathcal{K}$ be a family of order at most κ that generates $\mathcal{F}$. By Lemma 4.1.17 there exist pairwise disjoint sets $X' \subseteq X$ for $X \in \mathcal{K}$, all of cardinality κ. We may assume that the union of the sets X' is Λ, if not we can just add the missing elements to one of the X'. Let

$$B = \{\langle F, f\rangle : F \in \mathcal{P}_\omega(\Lambda) \text{ and } f : \mathcal{P}(F) \longrightarrow \Lambda\}.$$

Since $|B| = \kappa$ we can, for each $X \subseteq \Lambda$, choose an indexing such that

$$B = \{\langle F_\lambda, f_\lambda\rangle : \lambda \in X'\}.$$

Note that $\langle F_\lambda, f_\lambda\rangle$ is thus defined for all $\lambda \in \Lambda$. For $I \subseteq \Lambda$, we define a map $g_I : \Lambda \longrightarrow \Lambda$ by

$$g_I(\lambda) = f_\lambda(F_\lambda \cap I).$$

Finally, let

$$\mathrm{P}(I) = \{g_I^{-1}(\eta) : \eta \in \Lambda\} \text{ and}$$

$$\Pi = \{\mathrm{P}(I) : I \subseteq \Lambda\}.$$

Clearly if each of the $\mathrm{P}(I)$ are distinct then $|\Pi| = |\mathcal{P}(\Lambda)| = 2^\kappa$.

We claim that each $\mathrm{P}(I)$ is a partitioning of Λ into κ blocks each of cardinality κ. To see that the $g_I^{-1}(\eta)$ are disjoint consider any $\eta_0, \eta_1 \in \Lambda$. If $g_I^{-1}(\eta_0) \cap g_I^{-1}(\eta_1) \neq \emptyset$ then there exists a λ such that $f_\lambda(F_\lambda \cap I) = \eta_0$ and $f_\lambda(F_\lambda \cap I) = \eta_1$ and hence $\eta_0 = \eta_1$. If we can show that, for each $I \subseteq \Lambda$ and $\eta \in \Lambda$, the equation $g_I(\lambda) = \eta$ has κ solutions it follows that each block is of cardinality κ. Consider any non-empty finite subset F of Λ. There are κ functions $f : \mathcal{P}(F) \longrightarrow \Lambda$ with $f(F \cap I) = \eta$, and if λ is any index with $\langle F, f\rangle = \langle F_\lambda, f_\lambda\rangle$, then $g_I(\lambda) = \eta$. Since the $g_I^{-1}(\eta)$ are thus disjoint and $|\Lambda| = \kappa$ there exist κ blocks.

We still need to show that, for each $I \subseteq \Lambda$, these 2^κ partitions $\mathrm{P}(I)$ are pairwise distinct, and that $(\mathcal{F}, \Pi)$ is consistent. Consider the following observation.

For any $X \in \mathcal{K}$, distinct subsets $I_0, \dots, I_{n-1}$ of Λ, and any $\eta_0, \dots, \eta_{n-1} \in \Lambda$, there exists $\lambda \in X'$ such that $g_{I_m}(\lambda) = \eta_m$, for $m < n$. To see this note that we can choose a finite subset F of Λ such that the sets $F \cap I_m$ are pairwise distinct, a function $f : \mathcal{P}(F) \longrightarrow \Lambda$ such that $f(F \cap I_m) = \eta_m$, for $m < n$, and then let λ be the member of X' such that $\langle F, f\rangle = \langle F_\lambda, f_\lambda\rangle$. For such a λ it follows that $\lambda \in X'$ and $\lambda \in g_{I_m}^{-1}(\eta_m)$, for $m < n$, whence it follows that $(\mathcal{F}, \Pi)$ is consistent.

Observe that for any distinct $I_m, I_n \subseteq \Lambda$ we can find two sets $F, F' \in \mathcal{P}_\omega(\Lambda)$ such that $F \cap I_m \neq F \cap I_n$ and $F' \cap I_m \neq F' \cap I_n$. Now we choose two functions $f : \mathcal{P}(F) \longrightarrow \Lambda$ and $f' : \mathcal{P}(F') \longrightarrow \lambda$ such that

$$f(F \cap I_m) = f'(F' \cap I_m) \text{ and}$$

$$f(F \cap I_n) \neq f'(F' \cap I_n).$$

Let λ be the member of X' such that $\langle F, f\rangle = \langle F_\lambda, f_\lambda\rangle$ and μ be the member of X' such that $\langle F', f'\rangle = \langle F_\mu, f_\mu\rangle$. Then there exists an $\eta \in \Lambda$ such that $\{\lambda, \mu\} \subseteq g_{I_m}^{-1}(\eta)$, but for all $\xi \in \Lambda$ we have $\{\lambda, \mu\} \not\subseteq g_{I_n}^{-1}(\xi)$. Hence it follows that all the partitions in Π are pairwise distinct.

□

For the following lemmas we assume that $|\Pi| = 2^\kappa$ and that each member of Π is also of cardinality κ. We also assume that each block of any partition has cardinality κ and that $|\Lambda| = \kappa$ (as usual).

LEMMA 4.1.19. *Suppose $\mathcal{F}$ is a filter over Λ, and Π is a family of partitions of Λ such that $(\mathcal{F}, \Pi)$ is consistent. For any subset I of Λ, either $(\mathcal{F}_I, \Pi)$ is consistent or else $(\mathcal{F}_{\Lambda\setminus I}, \Pi')$ is consistent for some co-finite subset Π' of Π, where $\mathcal{F}_X$ is the filter generated by $\mathcal{F} \cup \{X\}$.*

PROOF. Assume that $(\mathcal{F}_I, \Pi)$ is not consistent. Choose $\mathrm{P}_0, \dots, \mathrm{P}_{m-1}$ to be members of Π witnessing that $(\mathcal{F}_I, \Pi)$ is inconsistent. I.e. let $\mathrm{P}_i \in \Pi$, for $i < n$, be those partitions with blocks $X_i \in \mathrm{P}_i$ such that, for some $X \in \mathcal{F}$,

$$(*) \qquad I \cap X \cap X_0 \cap \dots \cap X_{n-1} = \emptyset.$$

We define $\Pi' = \Pi \setminus \{\mathrm{P}_0, \dots, \mathrm{P}_{n-1}\}$.

Claim: $(\mathcal{F}_{\Lambda\setminus I}, \Pi')$ is consistent.

Suppose there exist $Y_j \in \mathrm{P}_j$, where $\mathrm{P}_j \in \Pi'$ for $j < m$ and some $Y \in \mathcal{F}$ such that $(\Lambda \setminus I) \cap Y \cap Y_0 \cap \dots \cap Y_{m-1} = \emptyset$. Let $X' = X \cap X_0 \cap \dots \cap X_{n-1}$ and $Y' = Y \cap Y_0 \cap \dots \cap Y_{m-1}$ and consider any $x \in X' \cap Y'$. Then either $x \in I$ or $x \in \Lambda \setminus I$. Hence $I \cap X' = \emptyset$ or $(\Lambda \setminus I) \cap Y' = \emptyset$. But by assumption $(\Lambda \setminus I) \cap Y' = \emptyset$. Hence $I \cap X' \neq \emptyset$ contradicting $(*)$. □

We may assume that $\mathrm{P} = \{Y_\Delta : \Delta \in \mathcal{P}_\omega(\Lambda)\}$. Given $g : \mathcal{P}_\omega(\Lambda) \longrightarrow \mathcal{F}$, define $B_g = \bigcup_{\Delta\in\mathcal{P}_\omega(\Lambda)} (Y_\Delta \cap g(\Delta))$. Define h by

$$h(\Delta) = B_g \cap \underset{\Delta\subseteq\Delta'\in\mathcal{P}_\omega(\Delta)}{Y_{\Delta'}}.$$

Now it is straightforward to check that $h \le g$, h is a $(\cap, \cup)$-morphism, and the filter generated by $\mathcal{F} \cup \mathrm{ran}(h)$ is consistent with $\Pi \setminus \{\mathrm{P}\}$. We need to remove P because $h(\Delta) \cap Y_{\Delta\setminus\{d\}} = \emptyset$ if $d \in \Delta$, giving immediate inconsistency.

LEMMA 4.1.20. *Suppose $\mathcal{F}$ is a filter over Λ, and Π is a family of partitions of Λ such that $(\mathcal{F}, \Pi)$ is consistent. Let $g : \mathcal{P}_\omega(\Lambda) \longrightarrow \mathcal{F}$ be an order reversing map, and let $\mathrm{P} \in \Pi$. Then there exists a $(\cup, \cap)$-morphism $h : \mathcal{P}_\omega(\Lambda) \longrightarrow \mathcal{P}(\Lambda)$ with $h \le g$ such that the filter $\mathcal{G}$ generated by $\mathcal{F} \cup \mathrm{ran}(h)$ is a proper filter and $(\mathcal{G}, \Pi \setminus \{\mathrm{P}\})$ is consistent.*

PROOF. If we return to the discussion at the beginning of this section, it should be clear that the function h defined by (4.13) satisfies all the conditions of our lemma except for the last one (c.f. Remark 4.1.14). Note that from (4.12) it follows that $h(\Delta_\eta) \supseteq g(\Delta_\eta) \cap Y_\eta$, where $Y_\eta \in \mathrm{P}$. Let $X \in \mathcal{F}$, $\Gamma \in \mathcal{P}_\omega(\Lambda)$ and $X_i \in \mathrm{P}_i$ with $\mathrm{P}_i \in \Pi \setminus \{\mathrm{P}\}$ for $i < n$. But $\Gamma = \Delta_\eta$, for some $\eta < \kappa$, thus $g(\Gamma) = g(\Delta_\eta) \in \mathcal{F}$ and since $(\mathcal{F}, \Pi)$ is consistent

$$X \cap g(\Delta_\eta) \cap Y_\eta \cap \bigcap_{i<n} X_i \neq \emptyset.$$

Thus the filter $\mathcal{G}$ generated by $\mathcal{F} \cup \mathrm{ran}(h)$ is consistent with $\Pi \setminus \{\mathrm{P}\}$. □

THEOREM 4.1.21. *If $\mathcal{F}_0$ is a uniform filter over Λ, with $|\mathcal{F}_0| \le \kappa$ then there exists a κ^+-good ultrafilter $\mathcal{F}$ over Λ, such that $\mathcal{F}_0 \subseteq \mathcal{F}$.*

PROOF. Without loss of generality we may assume that $\Lambda = \kappa$. As in the exposition at the beginning of this section we arrange all the subsets of Λ into a transfinite sequence

$$\Lambda_0, \Lambda_1, \dots, \Lambda_\xi, \dots, \quad \xi < 2^\kappa \text{ where } \kappa = |\Lambda|$$

and all the order reversing maps from $\mathcal{P}_\omega(\Lambda)$ to $\mathcal{P}(\Lambda)$ into a transfinite sequence

$$(*) \qquad g_0, g_1, \ldots, g_\xi, \ldots, \quad \xi < 2^\kappa$$

in such a way that each order reversing map g occurs 2^κ times in the above sequence.

We will now proceed to construct two sequences of filters and a sequence of partitions of Λ such that

$$\mathcal{F}_0 \subseteq \mathcal{G}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{G}_1 \subseteq \ldots \mathcal{F}_\xi \subseteq \mathcal{G}_\xi \subseteq \ldots$$
$$\Pi_0 \supseteq \Pi_1 \supseteq \ldots \Pi_\xi \supseteq \ldots \text{ for } \xi < 2^\kappa$$

where each member of Π_0 has κ blocks, and (for $\xi < 2^\kappa$) the following holds:

(1) $(\mathcal{F}_\xi, \Pi_\xi)$ is consistent, and $|\Pi_\xi| = 2^\kappa$,
(2) either $\Lambda_\xi \in \mathcal{F}_{\xi+1}$ or $(\Lambda \setminus \Lambda_\xi) \in \mathcal{F}_{\xi+1}$, and
(3) if $\mathrm{ran}(g_\xi) \subseteq \mathcal{F}_\xi$, then there exists a $(\cup, \cap)$-morphism $h : \mathcal{P}_\omega(\Lambda) \longrightarrow \mathcal{P}(\Lambda)$ with $h \leq g_\xi$ and $\mathrm{ran}(h) \subseteq \mathcal{G}_\xi$.

Since $\mathcal{F}_0$ is uniform we can use Lemma 4.1.18 to obtain a family Π_0 of partitions of Λ such that (1) holds for $\xi = 0$.

Assuming that (1) for a given ξ, we apply Lemma 4.1.20 to obtain $\mathcal{G}_\xi$ and h satisfying (3) and a new partition Π'_ξ such that $(\mathcal{G}_\xi, \Pi'_\xi)$ is consistent and $|\Pi'_\xi| = |\Pi_\xi|$. We let $\mathcal{F}_{\xi+1}$ be the filter generated by $\mathcal{G}_\xi \cup \{\Lambda_\xi\}$ and $\Pi_{\xi+1} = \Pi'_\xi$ if they are consistent, otherwise we take $\mathcal{F}_{\xi+1}$ to be the filter generated by $\mathcal{G}_\xi \cup \{\Lambda \setminus \Lambda_\xi\}$ and apply Lemma 4.1.19 to obtain a co-finite subset $\Pi_{\xi+1}$ of Π'_ξ such that $(\mathcal{F}_{\xi+1}, \Pi_{\xi+1})$ is consistent. For any limit ordinal ξ we take

$$\mathcal{F}_\xi = \bigcup_{\eta<\xi} \mathcal{F}_\eta \text{ and } \Pi_\xi = \bigcap_{\eta<\xi} \Pi_\eta.$$

It should be clear that this construction preserves properties (1) through (3). By construction the union $\mathcal{F}$ of all the filters $\mathcal{F}_\xi$ is an ultrafilter over Λ.

To see that $\mathcal{F}$ is in fact κ^+-good recall that $\mathrm{cf}\, 2^\kappa > \kappa$ (c.f. Theorem 2.1.4 p. 14). For any order reversing map $g : \mathcal{P}_\omega(\Lambda) \longrightarrow \mathcal{F}$ the range of g has power at most κ. Therefore there must exist a $\xi < 2^\kappa$ such that the range of g is contained in $\mathcal{F}_\xi$. But by construction there are 2^κ ordinals δ such that $g = g_\delta$. Hence the set $\{g_\delta : g = g_\delta\}$ is unbounded in $(*)$. Thus there must exist an $\eta < 2^\kappa$ such that $\mathcal{F}_\xi \subseteq \mathcal{F}_\eta$ and $g = g_\eta$. Consequently, by (3), there exists a $\cup, \cap$-morphism $h : \mathcal{P}_\omega(\Lambda) \longrightarrow \mathcal{G}_\eta \subseteq \mathcal{F}$ with $h \leq g$. □

THEOREM 4.1.22. *There exists an ω-incomplete uniform filter, of cardinality κ, over Λ.*

PROOF. Note that since $|\Lambda| = \kappa$ it is enough to demonstrate that some set of cardinality κ has an ω-incomplete uniform filter. We will show that there exists such a filter over $\mathcal{P}_\omega(\Lambda)$. For each $\lambda \in \Lambda$ define

$$\Delta_\lambda = \{\Delta \in \mathcal{P}_\omega(\Lambda) : \lambda \in \Delta\}$$

and let

$$G = \{\Delta_\lambda : \lambda \in \Lambda\}.$$

Observe that $|G| = \kappa$ and $|\Delta_\lambda| = \kappa$, for each $\lambda \in \Lambda$. However each $\Delta \in \mathcal{P}_\omega(\Lambda)$ belongs to only finitely many $\Delta_\lambda \in G$, since Δ is finite and $\Delta \in \Delta_\lambda$ implies $\lambda \in \Delta$. Thus there exists a countably infinite subset $\mathcal{K}$ of G such that $\bigcap \mathcal{K} = \emptyset$. Now G has the finite intersection property since

$$\{\lambda_0, \ldots, \lambda_{n-1}\} \in \Delta_{\lambda_0} \cap \ldots \cap \Delta_{\lambda_{n-1}}.$$

Also observe that for $\Delta_{\lambda_0}, \Delta_{\lambda_1} \in G$ since $\Delta_{\lambda_0} \cap \Delta_{\lambda_1} = \{\Delta \in \mathcal{P}_\omega(\Lambda) : \{\lambda_0, \lambda_1\} \subseteq \Delta\}$ it follows that $|\Delta_{\lambda_0} \cap \Delta_{\lambda_1}| = \kappa$. Similarly for any finite number of elements of G the intersection of these sets has cardinality κ. Hence the filter $\mathcal{G}$ generated by G, using Theorem 2.2.8 (p. 17), will have power κ and $|X| = \kappa$ for each $X \in \mathcal{G}$. □

Now, if $\mathcal{F}_0$ in Theorem 4.1.21 is ω-incomplete then $\mathcal{F}$ will also be ω- incomplete. Thus we get the result we required to be able to apply Theorem 4.1.13.

COROLLARY 4.1.23. *There exits a κ^+-good ω-incomplete ultrafilter over Λ.*

The notion of a good ultrafilter, the above corollary and the result that good ultraproducts are saturated were first presented by H. J. Keisler, c.f. [**Kei64**], using GCH. Later in [**Kun72**] K. Kunen eliminated the need for GCH to prove the existence of good ultrafilters. The proof presented here is based on the work of Keisler as presented in [**ChK77**].

4.1.4. The Fine-van Benthem-Goldblatt Theorem.

All we need to complete the proof of the result by Fine, van Benthem and Goldblatt is the following lemma.

LEMMA 4.1.24. *For any class $\mathcal{K}$ of structures,* $\mathbf{P_u U_d}\mathcal{K} \subseteq \mathbf{H_b U_d P_u}\mathcal{K}$

PROOF. Let $\mathbf{U} \in \mathbf{P_u U_d}\mathcal{K}$, i.e. $\mathbf{U} = \prod_{\mathcal{F}} \mathbf{U}_\lambda$ where $\mathcal{F}$ is an ultrafilter over Λ and each $\mathbf{U}_\lambda$, $\lambda \in \Lambda$, is the disjoint union of structures $\mathbf{U}_{\lambda,i} \in \mathcal{K}$ with $i \in I_\lambda$. We let $J = \prod_{\lambda\in\Lambda} I_\lambda$, and for $j \in J$ let $\mathbf{V}_j = (\prod_{\lambda\in\Lambda} \mathbf{U}_{\lambda,j(\lambda)})/\mathcal{F}$, where $j(\lambda)$ is taken to be the λth coordinate of j. The natural embedding of $\prod_{\lambda\in\Lambda} \mathbf{U}_{\lambda,j(\lambda)}$ into $\prod_{\lambda\in\Lambda} \mathbf{U}_\lambda$ induces an embedding γ_j of $\mathbf{V}_j$ into $\mathbf{U}$.

Claim: The union of the maps γ_j is a bounded morphism γ from $\mathbf{V}$ onto $\mathbf{U}$, where $\mathbf{V} = \biguplus_{j\in J} \mathbf{V}_j$.

onto: Let $\underline{u}/\mathcal{F} \in \mathbf{U}$ where $\underline{u} = \langle\langle u_0, \langle 0, i_0\rangle\rangle, \dots, \langle u_\lambda, \langle \lambda, i_\lambda\rangle\rangle, \dots\rangle \in \prod_\Lambda \mathbf{U}_\lambda$, with $i_\lambda \in I_\lambda$ and $v_\lambda \in \mathbf{U}_{\lambda,i_\lambda}$ for each $\lambda \in \Lambda$. Take $j \in J$ to be the tuple $\langle i_0, \dots, i_\lambda, \dots\rangle$ and $\underline{v} = \langle u_0, \dots, u_\lambda, \dots\rangle$. Then $\underline{v}/\mathcal{F} \in \mathbf{V}_j$ and by the definition of γ it follows that $\gamma(\langle \underline{v}/\mathcal{F}, j\rangle) = \gamma_j(\underline{v}/\mathcal{F}) = \underline{u}/\mathcal{F}$.

Consider any relational symbol r in the language of $\mathcal{K}$, with $ar(r) = n + 1$.

zig: We assume that $r^{\mathbf{V}}(\langle v_0, j_0\rangle, \dots, \langle v_n, j_n\rangle)$, where $v_m \in \mathbf{V}_m$. We need to prove that $r^{\mathbf{U}}(\gamma(v_0), \dots, \gamma(v_n))$. By our assumption and the definition of the disjoint union of structures it follows that there exists a $j \in J$ such that $v_m \in \mathbf{V}_j$ for all $m \le n$. Since γ_j is a homomorphism we have $r^{\mathbf{U}}(\gamma_j(v_0), \dots, \gamma_j(v_n))$. The result follows from the observation that $\gamma(\langle v_m, j_m\rangle) = \gamma_j(v_m)$.

zag: Assume that $r^{\mathbf{U}}(u_0/\mathcal{F}, \dots, u_{n-1}/\mathcal{F}, \gamma(v))$, with $v \in V$ and $u_m/\mathcal{F} \in \operatorname{ran}(\gamma)$ for $m < n$. We need $\langle x_0, j_0\rangle, \dots, \langle x_{n-1}, j_{n-1}\rangle \in V$ such that $\gamma(\langle x_m, j_m\rangle) = u_m/\mathcal{F}$ and $r^{\mathbf{V}}(\langle x_0, j_0\rangle, \dots, \langle x_{n-1}, j_{n-1}\rangle, v)$, where $x_m \in V_{j_m}$ for $m < n$. Since $\mathbf{U}$ is an ultraproduct it follows that

$$(*) \qquad X = \{\lambda \in \Lambda : r^{\mathbf{U}_\lambda}(u_0(\lambda), \dots, u_{n-1}(\lambda), \gamma(v)(\lambda))\} \in \mathcal{F}.$$

We will define $j \in J$ and $x_m \in \prod_{\lambda\in\Lambda} \mathbf{U}_{\lambda,j(\lambda)}$ such that $\langle x_m/\mathcal{F}, j\rangle \in \mathbf{V}$ and $\gamma(\langle x_m/\mathcal{F}, j\rangle) = u_m/\mathcal{F}$.

For $\lambda \in X$ there exists an $i_\lambda \in I_\lambda$ such that $u_0(\lambda), \dots, u_{n-1}(\lambda), \gamma(v)(\lambda) \in \mathbf{U}_{\lambda,i_\lambda}$. Thus for $\lambda \in X$ we let $j(i) = i_\lambda$ and $x_m(\lambda) = u_m(\lambda)$. We recall that $u_m/\mathcal{F} \in \operatorname{ran}(\gamma)$.

Hence there exists $\langle x'_m/\mathcal{F}, j'\rangle \in \mathbf{V}$ such that $\gamma(\langle x'_m/\mathcal{F}, j'\rangle) = u_m/\mathcal{F}$. So for $\lambda \notin X$ we let $i_\lambda = j'(i)$ and $x_m(\lambda) = x'_m(\lambda)$.

Clearly $\langle x_m/\mathcal{F}, j\rangle \in \mathbf{V}$ and $\gamma(\langle x_m/\mathcal{F}, j\rangle) = u_m/\mathcal{F}$. for all $m < n$. Consequently it follows that $v = \langle v'/\mathcal{F}, j\rangle$, for some $v'/\mathcal{F} \in V_j$, with $v'(\lambda) = \gamma(v)(\lambda)$. Now $X \subseteq \{\lambda \in \Lambda : r^{\mathbf{U}_{\lambda, j(\lambda)}}(x_0(\lambda), \ldots, x_{n-1}(\lambda), v'(\lambda))\}$. Hence

$$\{\lambda \in \Lambda : r^{\mathbf{U}_{\lambda, j(\lambda)}}(x_0(\lambda), \ldots, x_{n-1}(\lambda), v'(\lambda))\} \in \mathcal{F}.$$

Therefore $r^{\mathbf{V}}(\langle x_0/\mathcal{F}, j\rangle, \ldots, \langle x_{n-1}/\mathcal{F}, j\rangle, v)$.

Thus we can conclude that $\mathbf{U} \in \mathbf{H_b U_d P_u}\mathcal{K}$ as required. □

THEOREM 4.1.25 (Fine-van Benthem-Goldblatt Theorem). *If a class $\mathcal{K}$ of structures is closed under ultraproducts then* Q($\mathbf{Cm}\mathcal{K}$) *is closed under canonical embedding algebras.*

PROOF. Let $\mathcal{V} = \mathrm{Q}(\mathbf{Cm}\mathcal{K}) = \mathbf{SPCm}\mathcal{K}$. From Lemma 3.1.16 it follows that $\mathbf{CmUfS} \leq \mathbf{CmH_bUf} \leq \mathbf{SCmUf}$. Hence if $\mathbf{N} \in \mathrm{S}\{\mathbf{M}\}$ then $\mathrm{Em}\mathbf{N} \in \mathrm{SEm}\{\mathbf{M}\}$. Thus it suffices to show $\mathrm{Em}\mathbf{M} \in \mathcal{V}$ whenever $\mathbf{M} \in \mathbf{PCm}\mathcal{K}$. Since $\mathbf{PCm} = \mathbf{CmU_d}$ we have $\mathbf{M} \cong \mathrm{Cm}\mathbf{V}$ where $\mathbf{V} \in \mathbf{U_d}\mathcal{K}$. Hence $\mathrm{Em}\mathbf{M} \cong \mathrm{EmCm}\mathbf{V} = \mathrm{CmUfCm}\mathbf{V}$ and by Lemma 4.1.7

$$\mathrm{UfCm}\mathbf{V} \in \mathbf{H_b P_w U_d}\mathcal{K} \subseteq \mathbf{H_b H_b U_d P_u}\mathcal{K} = \mathbf{H_b U_d}\mathcal{K}$$

which implies that $\mathrm{CmUfCm}\mathbf{V} \in \mathbf{SPCm}\mathcal{K} = \mathcal{V}$. Since $\mathcal{V}$ is closed under isomorphism it follows that $\mathrm{Em}\mathbf{M} \in \mathcal{V}$. □

Note that this is an extension of the original result. In general this theorem is used and presented as follows.

COROLLARY 4.1.26. *If a class $\mathcal{K}$ of structures is closed under ultraproducts then* V($\mathbf{Cm}\mathcal{K}$) *is closed under canonical embedding algebras.*

4.2. Ultrafilters and Elementary Classes

We now return to the question raised in Section 2.2.6 concerning how we can algebraically characterise a class of elementary structures. But first we take a little detour into the theory of saturated models.

Informally we can say that a saturated structure is a structure which can internally express all properties satisfied by its elements. Thus if two models are saturated and satisfy the same properties we might very well expect to be able to extend this equivalence to an isomorphism. (Note that for this to be possible $\mathbf{M}$ and $\mathbf{N}$ should also be of the same cardinality.) Let $\mathbf{M}$ and $\mathbf{N}$ be two such models and arrange the elements of $\mathbf{M}$ and $\mathbf{N}$ into transfinite sequences

$$m_0, \ldots, m_\xi, \ldots \text{ and } n_0, \ldots, n_\xi, \ldots.$$

If we could extend $\mathbf{M}$ and $\mathbf{N}$ in parallel such that $(\mathbf{M}, m_0, \ldots, m_\xi) \equiv (\mathbf{N}, n_0, \ldots, n_\xi)$ then by mapping m_η to n_η we could have a "partial" isomorphism between $\mathbf{M}$ and $\mathbf{N}$. If we wanted to do this for all elements of $\mathbf{M}$ and $\mathbf{N}$ we would be need to extend the language by κ elements, where $\kappa = |\mathbf{M}| = |\mathbf{N}|$. Thus $\mathbf{M}$ and $\mathbf{N}$ should be κ-saturated. Hence the following definition.

DEFINITION 4.2.1. $\mathbf{M}$ is said to be *saturated* if it is $|\mathbf{M}|$-saturated.

The following lemma captures the idea of iteratively extending two structures in the way described above.

LEMMA 4.2.2. *Let κ be an infinite cardinal. Suppose* $\mathbf{M}$ *and* $\mathbf{N}$ *are both κ-saturated, and* $\mathbf{M} \equiv \mathbf{N}$. *Let $\underline{m} \in \mathbf{M}^\kappa$ and $\underline{n} \in \mathbf{N}^\kappa$. Then there exists $\underline{x} \in \mathbf{M}^\kappa$ and $\underline{y} \in \mathbf{N}^\kappa$ such that*

$$\{m_\xi : m_\xi \in \underline{m}\} \subseteq \{x_\xi : x_\xi \in \underline{x}\}$$
$$\{n_\xi : n_\xi \in \underline{n}\} \subseteq \{y_\xi : y_\xi \in \underline{y}\}$$
$$\mathbf{M}_X \equiv \mathbf{N}_Y$$

where $X = \{x_\xi : x_\xi = \underline{x}(\xi) \text{ for } \xi < \kappa\}$ and $Y = \{y_\xi : y_\xi = \underline{y}(\xi) \text{ for } \xi < \kappa\}$.

PROOF. Note that each ordinal $\xi < \kappa$ can be represented uniquely by a sum $\xi = \lambda + n$, where λ is a limit ordinal and $n < \omega$. We say ξ is even if n is even, otherwise ξ is odd. We will iteratively construct two sequences $\underline{x} = (x_0, \ldots, x_\xi)$ and $\underline{y} = (y_0, \ldots, y_\xi)$, for $\xi < \kappa$, such that

(1) if $\eta = \lambda + 2n$ is even then $x_\eta = m_{\lambda+n}$,
(2) if $\eta = \lambda + (2n+1)$ is odd then $y_\eta = n_{\lambda+n}$ and
(3) $(\mathbf{M}, x_0, \ldots, x_\eta) \equiv (\mathbf{N}, y_0, \ldots, y_\eta)$

for $\eta < \xi$.

Suppose that for $\xi < \kappa$ we have $(x_0, \ldots, x_\eta)$ and $(y_0, \ldots, y_\eta)$, with $\eta < \xi$, such that all the above conditions hold for ξ. If $\xi = \lambda + 2n$ is even, let $x_\xi = m_{\lambda+n}$ and let $X = \{x_\eta : \eta < \xi\}$. Take $\Phi(z)$ to be the set of $L(X)$ formulas satisfied by x_ξ in $\mathbf{M}$. By (3) $\Phi(z)$ is consistent with $\mathrm{Th}\,\mathbf{N}_Y$ and hence, by the κ-saturation of $\mathbf{N}$, $\Phi(z)$ is satisfied by some element $y_\xi \in \mathbf{N}$. It follows that

$$(\mathbf{M}, x_0, \ldots, x_\xi) \equiv (\mathbf{N}, y_0, \ldots, y_\xi).$$

When $\xi = \lambda + (2n+1)$ is odd we let $y_\xi = n_{\lambda+n}$ and find a x_ξ in a similar fashion to that just described above. The sequences $\underline{x} \in \mathbf{M}^\kappa$ and $\underline{y} \in \mathbf{N}^\kappa$ constructed in this fashion clearly satisfy the lemma's requirements. □

Using this construction we can now get the isomorphism we were looking for between saturated models.

THEOREM 4.2.3. *Let* $\mathbf{M}$ *and* $\mathbf{N}$ *be elementarily equivalent saturated models of the same power. Then* $\mathbf{M} \cong \mathbf{N}$.

PROOF. Let $\kappa = |\mathbf{M}| = |\mathbf{N}|$ and let

$$m_0, \ldots, m_\xi, \ldots \text{ and } n_0, \ldots, n_\xi, \ldots \text{ for } \xi < \kappa$$

be enumerations of the elements of $\mathbf{M}$ and $\mathbf{N}$. Thus, using Lemma 4.2.2, we can find sequences $\underline{x} \in \mathbf{M}^\kappa$ and $\underline{y} \in \mathbf{N}^\kappa$ such that there exists an η and η' with $m_\xi = \underline{x}(\eta)$ and $n_\xi = \underline{y}(\eta')$ for each $\xi < \kappa$ where

$$(\mathbf{M}, \underline{x}) \equiv (\mathbf{N}, \underline{y}).$$

Since $\underline{x}$ and $\underline{y}$ still enumerate $\mathbf{M}$ and $\mathbf{N}$, the isomorphism follows. □

The reader might well ask what this uniqueness result on saturated models has to do with the question at hand. We wish to find algebraic criteria that force a class $\mathcal{K}$ to be closed under ultraproducts and elementary equivalence (c.f. Theorem 2.2.18 p. 19). Consider any class $\mathcal{K}$ that is closed under ultraproducts and elementary equivalence and let $\mathbf{M} \in \mathcal{K}$. (Note that since $\mathcal{K}$ is closed under elementary extension

$\mathcal{K}$ needs to be closed under isomorphism.) If there exists a structure $\mathbf{N}$ such that $\mathbf{M} \not\equiv \mathbf{N}$ then for any ultrapowers $\mathbf{M}^\Lambda/\mathcal{F}$, $\mathbf{N}^\Lambda/\mathcal{F}$ of $\mathbf{M}$ and $\mathbf{N}$, $\mathbf{M}^\Lambda/\mathcal{F} \not\equiv \mathbf{N}^\Lambda/\mathcal{F}$. Since $\mathbf{M}^\Lambda/\mathcal{F} \cong \mathbf{N}^\Lambda/\mathcal{F}$ implies $\mathbf{M}^\Lambda/\mathcal{F} \equiv \mathbf{N}^\Lambda/\mathcal{F}$, we assume that the complement of $\mathcal{K}$ must be closed under ultrapowers.

It should however be clear that the converse would follow if elementary equivalence implied isomorphism of ultrapowers. In the previous section we constructed ultrapowers using κ-good ω-incomplete ultrafilters and saw that this implied that these ultrapowers were κ-saturated. Now if these ultrapowers were in fact saturated we could use the uniqueness of saturated models to get our required result.

THEOREM 4.2.4. *Let $|L| \leq \kappa$ and $\mathbf{M}$ and $\mathbf{N}$ be models for L with $|\mathbf{M}|, |\mathbf{N}| \leq \kappa^+$ and assume* GCH. Let $\mathcal{F}$ be a κ^+-good ω-incomplete ultrafilter over a set Λ, where $|\Lambda| = \kappa$. Then

$$\mathbf{M} \equiv \mathbf{N} \text{ iff } \mathbf{M}^\Lambda/\mathcal{F} \cong \mathbf{N}^\Lambda/\mathcal{F}.$$

PROOF. Assume that $\mathbf{M} \equiv \mathbf{N}$. By Theorem 4.1.13 it follows that $\mathbf{M}^\Lambda/\mathcal{F}$ and $\mathbf{N}^\Lambda/\mathcal{F}$ are κ^+-saturated and $|\mathbf{M}^\Lambda/\mathcal{F}|, |\mathbf{N}^\Lambda/\mathcal{F}| \leq (\kappa^+)^\kappa = 2^\kappa$. By GCH $2^\kappa = \kappa^+$, thus both ultrapowers are saturated. Now by Corollary 2.2.16

$$\mathbf{M}^\Lambda/\mathcal{F} \equiv \mathbf{M} \equiv \mathbf{N} \equiv \mathbf{N}^\Lambda/\mathcal{F}.$$

Since saturated models are unique $\mathbf{M}^\Lambda/\mathcal{F} \cong \mathbf{N}^\Lambda/\mathcal{F}$.

The converse follows directly from Corollary 2.2.16 and the fact that isomorphism implies elementary equivalence. □

Using this result and Theorem 2.2.18 (p. 19) we get the required result.

COROLLARY 4.2.5 (Keisler-Shelah Theorem). *$\mathcal{K}$ is an elementary class if, and only if, $\mathcal{K}$ is closed under ultraproducts and isomorphisms, and the complement of $\mathcal{K}$ is closed under ultrapowers.*

The above result was first proven by Keisler in [**Kei61**] using GCH. Later, in [**She72**], S. Shelah eliminated the requirement for GCH from this proof. For a proof of the above result without GCH the reader is referred to the proof of Theorem 6.1.15 and Corollary 6.1.16(i) in [**ChK77**]. Theorem 6.1.15 proves that elementary equivalent structures have isomorphic ultrapowers, without the use of good ultrafilters.

We close of this section by quoting a result on the structures associated with a variety (c.f. [**Gol89**] Theorem 3.8.4).

THEOREM 4.2.6. *For any variety $\mathcal{V}$, the following are equivalent*

(i) $\{\mathbf{U} : \mathrm{Cm}\,\mathbf{U} \in \mathcal{V}\}$ *is elementary,*
(ii) $\{\mathbf{U} : \mathrm{Cm}\,\mathbf{U} \in \mathcal{V}\}$ *is closed under ultrapowers and*
(iii) $\{\mathbf{U} : \mathrm{Cm}\,\mathbf{U} \in \mathcal{V}\}$ *is closed under ultraproducts.*

4.3. Complex, Complete and Canonical varieties

As was motivated in the last section of chapter 3 the ideas of complex, complete and canonical varieties are closely tied up with the relational semantics of certain classes of polymodal logics. In this section we concentrate most of our effort on complex varieties, towards the end presenting some results relating directly to complete and canonical varieties.

Since we are also interested in the underlying structures associated with these varieties we are likely to find the following definition quite useful.

DEFINITION 4.3.1. Let $\mathcal{K}$ be any class of algebras. We define the *structures associated with* $\mathcal{K}$, denoted $\mathbf{Str}\mathcal{K}$, by $\mathbf{Str}\mathcal{K} = \{\mathbf{U} : \mathrm{Cm}\mathbf{U} \in \mathcal{K}\}$.

It turns out that canonical, complete and strongly complete logics form a natural hierarchy. The following theorem expresses this hierarchy in an algebraic way.

THEOREM 4.3.2. *Let $\mathcal{V}$ be a variety.*
(i) *If $\mathcal{V}$ is canonical then $\mathcal{V}$ is complex.*
(ii) *If $\mathcal{V}$ is complex then $\mathcal{V}$ is complete.*

PROOF.

(i): Assume that $\mathcal{V}$ is canonical, i.e. $\mathcal{V}$ is closed under canonical embedding algebras. We will show that $\mathcal{V} = \mathbf{SCmStr}\mathcal{V}$. By the definition of $\mathbf{Str}$ it follows that $\mathbf{SCmStr}\mathcal{V} \subseteq \mathcal{V}$.

For the reverse inclusion consider any $\mathbf{A} \in \mathcal{V}$. Since the variety $\mathcal{V}$ is canonical $\mathrm{CmUf}(\mathbf{A}) = \mathrm{Em}\mathbf{A} \in \mathcal{V}$. But, by Theorem 3.1.14 (p. 37), $\mathbf{A}$ is a subalgebra of $\mathrm{CmUf}(\mathbf{A})$. Now $\mathrm{Uf}(\mathbf{A}) \in \mathbf{Str}\mathcal{V}$, hence $\mathbf{A} \in \mathbf{SCmStr}\mathcal{V}$.

(ii): Recall that $\mathcal{V}$ is complete if $\mathcal{V} = \mathbf{HSPCm}\mathcal{K}$ for some class of structures $\mathcal{K}$. Since $\mathcal{V}$ is complex there exists a class of structures $\mathcal{K}$ such that $\mathcal{V} = \mathbf{SCm}\mathcal{K}$. Thus $\mathcal{V} = \mathbf{HSPSCm}\mathcal{K} = \mathbf{HSPCm}\mathcal{K}$. □

Throughout this chapter we have seen that the ultraproduct construction gives us an algebraic handle on a lot of logical questions. In fact if the class of structures of a variety is closed under ultrapowers it collapses the above mentioned hierarchy.

THEOREM 4.3.3. *For any variety $\mathcal{V}$, if $\mathbf{Str}\mathcal{V}$ is closed under ultrapowers then the following are equivalent:*
(i) *$\mathcal{V}$ is complete,*
(ii) *$\mathcal{V}$ is complex and*
(iii) *$\mathcal{V}$ is canonical.*

We refer the reader to [**Gol89**] Corollary 3.7.5 for a proof of this theorem.

Many of the classes of algebras studied in the field of algebraic logic are in fact of the form $\mathbf{SCm}\mathcal{K}$ for some class of relational structures $\mathcal{K}$. These classes include relational, cylindrical, modal and closure algebras. In fact for the mentioned classes of algebras the associated class of structures $\mathcal{K}$ is elementary. Hence $\mathcal{K}$ is closed under ultraproducts.

THEOREM 4.3.4. *Let $\mathcal{K}$ be a class of structures. Then $\mathbf{P_uCm}\mathcal{K} \subseteq \mathbf{SCmP_u}\mathcal{K}$ and $\mathbf{P_wCm}\mathcal{K} \subseteq \mathbf{SCmP_w}\mathcal{K}$.*

PROOF. Let $\{\mathbf{U}_\lambda : \lambda \in \Lambda\}$ be a collection of structures. We will show that, for any ultrafilter $\mathcal{F}$ over Λ, there exists an embedding $h : \prod_{\mathcal{F}} \mathrm{Cm}\mathbf{U}_\lambda \longrightarrow \mathrm{Cm}\prod_{\mathcal{F}} \mathbf{U}_\lambda$. Let $\mathbf{U} = \prod_{\mathcal{F}} \mathrm{Cm}\mathbf{U}_\lambda$ and $\mathbf{V} = \prod_{\mathcal{F}} \mathbf{U}_\lambda$.

For $\underline{X} \in \prod \mathrm{Cm}\mathbf{U}_\lambda$ we define h as follows

$$\underline{u}/\mathcal{F} \in h(\underline{X}/\mathcal{F}) \text{ iff } \{\lambda : u_\lambda \in X_\lambda\} \in \mathcal{F},$$

where $\underline{X} = \langle X_0, \ldots, X_\lambda, \ldots \rangle$ and $\underline{u} = \langle u_0, \ldots, u_\lambda, \ldots \rangle \in \prod \mathbf{U}_\lambda$. To see that h is well defined note that if $\underline{u}/\mathcal{F} \in h(\underline{X}/\mathcal{F})$ then $u_\lambda \in X_\lambda \in \mathrm{Cm}\mathbf{U}_\lambda$ and hence $u_\lambda \in \mathbf{U}_\lambda$. To see that h is independant of the choice of $\underline{X}$ assume $\underline{u} \in h(\underline{X}/\mathcal{F})$ and $\underline{X} \neq \underline{X}'$ with $\underline{X}/\mathcal{F} = \underline{X}'/\mathcal{F}$. Let

$$Z = \{\lambda : X_\lambda = X'_\lambda\},\ Y = \{\lambda : u_\lambda \in X_\lambda\} \text{ and } Y' = \{\lambda : u_\lambda \in X'_\lambda\}.$$

Now by assumption $Y \in \mathcal{F}$ and by definition $Z \in \mathcal{F}$, thus since $\mathcal{F}$ is a filter $Y \cap Z \in \mathcal{F}$. Clearly $Y \cap Z \subseteq Y'$. Since $\mathcal{F}$ is an ultrafilter it follows that $Y' \in \mathcal{F}$ and so $\underline{u}/\mathcal{F} \in h(\underline{X}'/\mathcal{F})$. A similar argument suffices to show that the definition of h is independant of the choice of $\underline{u}$.

We observe that $h(\underline{X}/\mathcal{F}) = \emptyset$ if, and only if, $\{\lambda : u_\lambda \in X_\lambda\} \notin \mathcal{F}$ for all $u \in \prod \mathbf{U}_\lambda$. Thus if $h(\underline{X}/\mathcal{F}) = \emptyset$ then $\underline{X}/\mathcal{F} = \mathbb{C}^{\mathbf{U}}$. Recall that a BA-homomorphism $g : \mathbf{A} \longrightarrow \mathbf{B}$ is an embedding if $g(a) = \mathbb{C}^{\mathbf{B}}$ implies $a = \mathbb{C}^{\mathbf{A}}$. By using the fact that $\mathcal{F}$ is a filter it can easily be checked that h preserves boolean operations. Consequently we only need to show that h is a homomorphism.

Observe that $\mathrm{At}(\mathrm{Cm}\mathbf{U}_\lambda) = \{\{u\} : u \in \mathbf{U}_\lambda\}$ and

$$R^{\mathrm{At}(\mathrm{Cm}\mathbf{U}_\lambda)}(\{x_0\}, \ldots, \{x_n\}) \text{ iff } R^{\mathbf{U}_\lambda}(x_0, \ldots, x_n).$$

Thus given f in the language of $\mathbf{U}$ the equivalent relation in $\mathbf{V}$ would be $R_f^{\mathrm{At}(\mathbf{U})}$ where we identify singleton sets with element of that set.

Consider any symbol f from the language of $\mathbf{U}$, where $ar(f) = n - 1$. The following calculation shows that h respects f.

$$u/\mathcal{F} \in h(f^{\mathbf{U}}(X_0/\mathcal{F}, \ldots, X_{n-1}/\mathcal{F}))$$
$$\text{iff}$$
$$\{\lambda : u(\lambda) \in f^{\mathrm{Cm}\mathbf{U}_\lambda}(X_0(\lambda), \ldots, X_{n-1}(\lambda))\} \in \mathcal{F}$$
$$\text{iff}$$
$$\{\lambda : R_f^{\mathbf{U}_\lambda}(x_0(\lambda), \ldots, x_{n-1}(\lambda), u(\lambda)) \text{ for some } x_i \in X_i\} \in \mathcal{F} \tag{4.14}$$
$$\text{iff}$$
$$\{\lambda : R_f^{\mathbf{U}_\lambda}(y_0(\lambda), \ldots, y_{n-1}(\lambda), u(\lambda)) \text{ for some } y_i/\mathcal{F} \in h(X_i/\mathcal{F})\} \in \mathcal{F} \tag{4.15}$$
$$\text{iff}$$
$$R_f^{\mathbf{V}}(y_0/\mathcal{F}, \ldots, y_{n-1}/\mathcal{F}, u/\mathcal{F}) \text{ for some } y_i/\mathcal{F} \in h(X_i/\mathcal{F})$$
$$\text{iff}$$
$$u/\mathcal{F} \in f^{\mathrm{Cm}\mathbf{V}}(h(X_0/\mathcal{F}, \ldots, X_{n-1}/\mathcal{F}))$$

Where (4.15) follows from (4.14) since $\{\lambda : x_i(\lambda) \in X_i\lambda\} = \Lambda \in \mathcal{F}$ whenever $x_i \in X_i$.

To see why (4.14) follows from (4.15) let $Y = \{\lambda : y_i(\lambda) \in X_i(\lambda)\}$. Now observe that if $Y \in \mathcal{F}$ we can find an $x_i \in X_i$ such that $x_i(\eta) = y_i(\eta)$ for each $\eta \in Y$, where $i < n$. It follows that

$$(*) \quad \{\lambda : R_f^{\mathbf{U}_\lambda}(y_0(\lambda), \ldots, y_{n-1}(\lambda), u(\lambda)) \text{ for some } y_i/\mathcal{F} \in h(X_i/\mathcal{F})\}$$
$$\subseteq \{\lambda : R_f(\mathbf{U}_\lambda)(x_0(\lambda), \ldots, x_{n-1}(\lambda), u(\lambda)) \text{ for some } x_i \in X_i\},$$

but from (4.15) we know that $(*)$ is an element of $\mathcal{F}$. But $\mathcal{F}$ is an ultrafilter filter and thus upwardly closed.

The result for ultrapowers is just a special case of the construction above where all the $U_\lambda = \mathbf{M}$ for some structure $\mathbf{M}$. □

Thus if $\mathcal{K}$ is elementary, hence closed under ultrapowers, and $\mathcal{V} = \mathrm{V}(\mathbf{Cm}\mathcal{K})$ then $\mathrm{Str}(\mathcal{V})$ is closed under ultrapowers, in fact if $\mathcal{V} = \mathbf{SCm}\mathcal{K}$ then $\mathcal{V}$ is closed under ultrapowers. Observe that $\mathbf{P_wS} \leq \mathbf{SP_w}$. Hence by the theorem above we get

$$\mathbf{P_wSCm} \leq \mathbf{SP_wCm} \leq \mathbf{SSCmP_w} = \mathbf{SCmP_w}.$$

Thus it follows that all the classes we mentioned above are complex. In fact we can strengthen this result.

THEOREM 4.3.5. *For any elementary class $\mathcal{K}$, $\mathbf{SCm}\mathcal{K}$ is an elementary class.*

PROOF. Similarly to the argument presented above it follows from $\mathbf{P_uS} \leq \mathbf{SP_u}$ that $\mathbf{P_uSCm} \leq \mathbf{SCmP_u}$. Now since $\mathcal{K}$ is closed under ultraproducts it then follows that $\mathbf{SCm}\mathcal{K}$ is also closed under ultraproducts. To see that the complement of $\mathbf{SCm}\mathcal{K}$ is closed under ultrapowers let $\mathbf{A}$ be an algebra such that $\mathbf{A}^A/\mathcal{F} \in \mathbf{SCm}\mathcal{K}$. Since $\mathbf{A}^A/\mathcal{F}$ is an ultrapower of $\mathbf{A}$, by Proposition 2.2.13 (p. 19), $\mathbf{A}$ can be embedded into $\mathbf{A}^A/\mathcal{F}$. Thus $\mathbf{A} \in \mathbf{SSCm}\mathcal{K} = \mathbf{SCm}\mathcal{K}$. □

4.3.1. Complex varieties.

We have defined a variety $\mathcal{V}$ to be complex if there exists $\mathcal{K}$ such that $\mathcal{V} = \mathbf{SCm}\mathcal{K}$. Observe that, by Lemma 3.1.16 (p. 38) we know that $\mathbf{PCm}\mathcal{K} = \mathbf{CmU_d}\mathcal{K}$. But then $\mathcal{V} \subseteq \mathbf{P}\mathcal{V} \subseteq \mathbf{PSCm}\mathcal{K} \subseteq \mathbf{SPCm}\mathcal{K} \subseteq \mathrm{V}(\mathbf{Cm}\mathcal{K}) = \mathcal{V}$. (Most of the results presented in this section are based on the work of P. Jipsen in [**Jip01**].)

PROPOSITION 4.3.6. *$\mathcal{V}$ is a complex variety if, and only if, there exists a class $\mathcal{K}$ of structures such that $\mathcal{V} = \mathbf{SPCm}\mathcal{K}$.*

PROOF. Let $\mathcal{K}' = \mathbf{U_d}\mathcal{K}$ then from the observation above $\mathcal{V} = \mathbf{SCm}\mathcal{K}'$ if, and only if, $\mathcal{V} = \mathbf{SPCm}\mathcal{K}$. □

We will now proceed to find criteria on $\mathcal{K}$ such that $\mathbf{SPCm}\mathcal{K}$ is a variety. To achieve this we first need to make a detour into the fields of congruences and subdirectly irreducible algebras. We start off by extending our definition of a congruence ideal to BAOs.

DEFINITION 4.3.7. Let θ be a BAO congruence. We call an ideal $\mathcal{I}$ a *congruence ideal* if $\mathcal{I} = \mathbb{C}/\theta$ for some BAO congruence θ, i.e. $\mathcal{I}$ is the $\mathbb{C}$-equivalence class modulo θ.

Let $\mathbf{B}$ be a Boolean algebra with operators. An element $b \in \mathbf{B}$ is called a *congruence element* of $\mathbf{B}$ if the principle ideal $(b]$ is a congruence ideal.

We can however get a very concrete description of these ideals in the case where BAO has a finite signature. But first we need the following definition.

DEFINITION 4.3.8. Let f be an n-ary operation on a Boolean algebra $\mathbf{B}$, and $\underline{b} \in \mathbf{B}^n$. We define a *unary section* of f by

$$f_{\underline{b},j}(x) = f(b_0, \ldots, b_{j-1}, x, b_{j+1}, \ldots, b_{n-1}).$$

Note that the following lemma does not require the set of operators F to be finite. (A proof of this lemma can be found in [**Jip93**] (c.f. Lemma 1).)

LEMMA 4.3.9. *Let* $\mathbf{B}$ *be a Boolean algebra with operators* F *and* $\mathcal{I}$ *an ideal over* $\mathbf{B}$. *Then the following are equivalent:*

(i) $\mathcal{I}$ *is a congruence ideal, and*
(ii) $x \in \mathcal{I}$ *implies* $f_{\underline{\mathbb{1}},j}(x) \in \mathcal{I}$, *for all* $f \in F$ *and* $j < n$, *where* $\underline{\mathbb{1}}$ *is an* n*-tuple of* $\mathbb{1}$*'s and* $ar(f) = n$.

Using the lemma above we then get the following result.

PROPOSITION 4.3.10. *Let the signature* L_{BAO} *be finite with operators* $\{f_0, \dots, f_{n-1}\}$, *where* $ar(f_i) = m_i$, *and let* τ *be the unary term defined by*

$$\tau(x) = x \vee \bigvee_{i<n, j<m_i} (f_i)_{\underline{\mathbb{1}},j}(x).$$

Consider any $\mathbf{B} \in \mathsf{BAO}$. *Then*

(i) *an ideal* $\mathcal{I}$ *of* $\mathbf{B}$ *is a congruence ideal if, and only if,* $x \in \mathcal{I}$ *implies* $\tau^{\mathrm{Cm}\mathbf{B}}(x) \in \mathcal{I}$,
(ii) *an element* $b \in \mathbf{B}$ *is a congruence element of* $\mathbf{B}$ *if, and only if,* $\tau(b) \leq b$.

Thus, for a finite signature L_{BAO}, if an ideal $\mathcal{I}$ only contains congruence elements then it follows that $\mathcal{I}$ must be a congruence ideal. Let $\mathbf{B} \in \mathsf{BAO}$, for any element $x \in \mathbf{B}$, we define the set

$$\mathrm{ci}(x) = \{y \in \mathbf{B} : y \leq \tau^n(x) \text{ for some } n \in \omega\},$$

where τ^n is the composition of n copies of τ. It follows that $\mathrm{ci}(x)$ is a congruence ideal.

Theorem 2.4.7 (p. 22) showed that for Boolean algebras congruence ideals are in bijective correspondence with congruence relations. Using Lemma 4.3.9 this result can be extended to BAOs.

PROPOSITION 4.3.11. *For any Boolean algebra with operators congruence ideals are in bijective correspondence to congruence relations.*

Observe that it follows from Theorem 2.4.10 (p. 23) that if $\mathbf{B}$ contains a minimum nontrivial congruence ideal then $\mathbf{B}$ must be subdirectly irreducible.

PROPOSITION 4.3.12. *Let* $\mathbf{U}$ *be a relational* L*-structure.* $S \in \mathrm{Cm}\mathbf{U}$ *is a congruence element if, and only if,* $\mathbf{U} \setminus S$ *is an inner substructure of* $\mathbf{U}$.

PROOF. We consider any $r \in L$. Take $S \in \mathrm{Cm}\mathbf{U}$ to be a congruence element and let $r^{\mathbf{U}}(y_0, \dots, y_{n-1}, x)$, with $x \in U \setminus S$. We will show that $y_i \in U \setminus S$, for $i < n$. Assume there exists some $i < n$ such that $y_i \in S$. Since $r^{\mathrm{Cm}\mathbf{U}}$ is an operator, it is order preserving and hence

$$r^{\mathrm{Cm}\mathbf{U}}(\{y_0\}, \dots, \{y_i\}, \dots, \{y_{n-1}\}) \subseteq r^{\mathrm{Cm}\mathbf{U}}(\mathbb{1}, \dots, S, \dots \mathbb{1}) \subseteq S,$$

where the last inclusion follows from the fact that S is a congruence element. But then, by the definition of $r^{\mathrm{Cm}\mathbf{U}}$, $x \in S$ contradicting our assumption.

For the converse we assume $\mathbf{U} \setminus S$ is an inner substructure of $\mathbf{U}$. Assume that $r^{\mathrm{Cm}\mathbf{U}}(\mathbb{1}, \dots, S, \dots, \mathbb{1}) \nsubseteq S$. Let $x \in U$ such that $x \in r^{\mathrm{Cm}\mathbf{U}}(\mathbb{1}, \dots, S, \dots, \mathbb{1}) \setminus S$. Thus there exist $y_0, \dots, y_{n-1} \in \mathbf{U}$ and $s \in S$ such that $r^{\mathrm{Cm}\mathbf{U}}(y_0, \dots, s, \dots, y_{n-1}, x)$. But since the inclusion map is a bounded morphism and $x \in U \setminus S$ it follows from **zag** that $s \in U \setminus S$. $\square$

Note that inner substructures are closed under intersection. Hence, given a subset X of a structure $\mathbf{U}$, we can generate the smallest inner substructure containing X by intersecting all inner substructures greater than X.

DEFINITION 4.3.13. Let $\mathbf{U}$ be a relational structure and $X \subseteq \mathbf{U}$. By $\mathrm{S}_{\mathrm{b}}^{\mathbf{U}}(X)$ we denote the intersection of all inner substructures of $\mathbf{U}$ that contain X. We refer to $\mathrm{S}_{\mathrm{b}}^{\mathbf{U}}(X)$ as the *inner substructure of* $\mathbf{U}$ *generated by* X.

If an inner substructure $\mathbf{V}$ of $\mathbf{U}$ is generated by a singleton set then we call $\mathbf{V}$ a *one-generated inner substructure* of $\mathbf{U}$.

Consider for a moment any $y \in \tau(X)$, where $X \in \mathrm{Cm}\mathbf{U}$, then either $y = x \in X$ or there exist $u_i \in \mathbb{1} = \mathbf{U}$ such that $r^{\mathbf{U}}(u_0, \ldots, u_{n-1}, y)$ and $u_j = x \in X$ for some $j < n$. Consequently we define a binary relation ρ, the relational "counterpart" of τ, in the following way

$$x\rho y \text{ iff } y = x \text{ or there exist } r \in \Sigma \text{ and } u_0, \ldots, u_{n-1} \in \mathbf{U} \text{ such that } r^{\mathbf{U}}(u_0, \ldots, u_{n-1}, y) \text{ where } u_j = x \text{ for some } j < n, \tag{4.16}$$

where Σ is the signature of $\mathbf{U}$.

Recall that $\rho^{\mathrm{Cm}\mathbf{U}}(X) = \{y : x\rho y \text{ for } x \in X\}$. Clearly $\rho^{\mathrm{Cm}\mathbf{U}}(X) = \tau(X)$ and $(\rho^n)^{\mathrm{Cm}\mathbf{U}}(X) = \tau^n(X)$, where ρ^n denotes the composition of n copies of ρ.

LEMMA 4.3.14. *Let* $\mathbf{V}$ *be an inner substructure of* $\mathbf{U}$, *where both these structures are of finite type. Then* $\mathbf{V}$ *is one-generated if, and only if,* $\mathrm{Cm}\mathbf{V}$ *is subdirectly irreducible.*

PROOF. Let ρ be defined as in (4.16), ρ^{-1} be the inverse of ρ and ρ^* the reflexive transitive closure of ρ. By the observation above it follows that

$$\mathrm{ci}(X) = \bigcup_{n<\omega} \mathcal{P}((\rho^n)^{\mathrm{Cm}\mathbf{U}}(X)).$$

Claim: $\mathrm{S}_{\mathrm{b}}^{\mathbf{U}}(X) = \sigma^{\mathrm{Cm}\mathbf{U}}(X)$, where $\sigma = \rho^{*-1}$.

Let $r^{\mathbf{U}}(u_0, \ldots, u_{n-1}, y)$, where $y \in \sigma^{\mathrm{Cm}\mathbf{U}}(X)$. Then by definition $u_i\rho y$ and hence $u_i \in \sigma^{\mathrm{Cm}\mathbf{U}}(X)$, for all $i < n$. Thus $\sigma^{\mathrm{Cm}\mathbf{U}}(X)$ is an inner substructure of $\mathbf{U}$.

Consider any inner substructure $\mathbf{W}$ of $\mathbf{U}$ such that $X \subseteq \mathbf{W} \subseteq \sigma^{\mathrm{Cm}\mathbf{U}}(X)$ and $u \in \sigma^{\mathrm{Cm}\mathbf{U}}(X)$. By definition there exists $x_i \in \mathbf{U}$, where $x_{n-1} \in X$, such that $x_{i-1}\rho x_i$, for $0 < i < n$, and $u\rho x_0$. Since $\mathbf{W}$ is an inner substructure of $\mathbf{U}$ it follows that each $x_i \in \mathbf{W}$ and hence $u \in \mathbf{W}$. Thus $\sigma^{\mathrm{Cm}\mathbf{U}}$ is the smallest inner substructure containing X.

So $\mathbf{W}$ is a one-generated inner substructure of $\mathbf{U}$ if, and only if, $\mathbf{W} = \sigma^{\mathrm{Cm}\mathbf{U}}(\{u\})$. We now observe that $\{u\} \in \mathrm{ci}(\{w\})$ if, and only if, $\mathrm{ci}(\{u\}) \subseteq \mathrm{ci}(\{w\})$. But then it follows that $\{u\} \subset \mathrm{ci}(\{w\})$ if, and only if, $w\rho^k u$ for some $k < \omega$ if, and only if, $w \in \sigma^{\mathrm{Cm}\mathbf{U}}(\{u\})$.

This shows that, for $u \in \mathbf{U}$, $\mathbf{W} = \mathrm{S}_{\mathrm{b}}^{\mathbf{U}}(\{u\})$ if, and only if, $\mathrm{ci}(\{u\})$ is a minimum nontrivial congruence ideal of $\mathrm{Cm}\mathbf{U}$, which is equivalent to saying that $\mathrm{Cm}\mathbf{U}$ is subdirectly irreducible (c.f. Proposition 4.3.11 and Theorem 2.4.10 p. 23). $\square$

Looking back at Section 2.4.3 it would not be too surprising if this lemma lead to some relational "counterpart" of the Subdirect Representation Theorem (p. 23).

LEMMA 4.3.15. *Any structure is a bounded morphic image of the disjoint union of its one-generated inner substructures, i.e.* $\mathbf{U} \in \mathbf{H_b U_d}\{\mathrm{S}_\mathrm{b}^\mathbf{U}(\{u\}) : u \in \mathbf{U}\}$.

PROOF. Assume $\mathbf{W} = \biguplus\{\mathrm{S}_\mathrm{b}^\mathbf{U}(\{u_\lambda\}) : u_\lambda \in \mathbf{U}\}$ and define a map $\gamma : \mathbf{W} \longrightarrow \mathbf{U}$ by

$$\gamma(\langle x, \lambda\rangle) = x.$$

By definition $r^\mathbf{W}(\langle x_0, \lambda_0\rangle, \ldots, \langle x_n, \lambda_n\rangle)$ if, and only if, $\lambda_j = \lambda_k$ for $j, k \leq n$ and $r^\mathbf{V}(x_0, \ldots, x_n)$, where $\mathbf{V} = \mathbf{U}_{\lambda_i} = \mathrm{S}_\mathrm{b}^\mathbf{U}(\{u_{\lambda_i}\})$ for all $i \leq n$. The forward implication shows that γ satisfies **zig**.

Let $r^\mathbf{U}(x_0, \ldots, x_{n-1}, \gamma(\langle x_n, \lambda\rangle))$, where $x_n \in \mathrm{S}_\mathrm{b}^\mathbf{U}(\{u_\lambda\})$. Since $\mathrm{S}_\mathrm{b}^\mathbf{U}(\{u_\lambda\})$ is an inner substructure of $\mathbf{U}$, it follows that $x_i \in \mathrm{S}_\mathrm{b}^\mathbf{U}(\{u_\lambda\})$, for $i < n$. Consequently $r^\mathbf{W}(\langle x_0, \lambda\rangle, \ldots, \langle x_n, \lambda\rangle)$ with $\gamma(\langle x_i, \lambda\rangle) = x_i$. □

From the lemma above and Lemma 3.1.16 (p. 38) we get the following result.

COROLLARY 4.3.16. *For any structure* $\mathbf{U}$, $\mathrm{Cm}\mathbf{U} \in \mathbf{SPCm}\{\mathrm{S}_\mathrm{b}^\mathbf{U}(\{u\}) : u \in \mathbf{U}\}$.

Using the Fine-van Bentham-Goldblatt Theorem (c.f. 4.1.25) we can now give sufficient conditions on a class $\mathcal{K}$ of structures such that the variety generated by $\mathcal{K}$ (i.e. $\mathrm{V}(\mathbf{Cm}\mathcal{K})$) is complex. (This result and the next first appeared in a preprint by P. Jipsen [**Jip01**].)

THEOREM 4.3.17. *Let $\mathcal{K}$ be a class of structures that is closed under ultraproducts and one-generated inner substructures. Then* $\mathrm{V}(\mathbf{Cm}\mathcal{K}) = \mathbf{SPCm}\mathcal{K}$.

PROOF. Let $\mathcal{K}$ be a class of structures such that $\mathbf{P_u}\mathcal{K} \subseteq \mathcal{K}$ and $\mathcal{K}$ is closed under one-generated inner substructures. Consider any $\mathbf{A} \in \mathbf{HSPCm}\mathcal{K} = \mathbf{HSCmU_d}\mathcal{K}$. Consequently there exists an algebra $\mathbf{B}$ such that $\mathbf{A} \in \mathbf{H}\{\mathbf{B}\}$, where $\mathbf{B}$ is a subalgebra of $\mathrm{Cm}\biguplus\mathrm{U}_\lambda$, for some $\mathbf{U}_\lambda \in \mathcal{K}$ with $\lambda \in \Lambda$. By Lemma 3.1.15 (p. 37) it follows that $\mathrm{Uf}(\mathbf{A}) \in \mathbf{S_b}\{\mathrm{Uf}(\mathbf{B})\}$ and $\mathrm{Uf}(\mathbf{B}) \in \mathbf{H_b}\mathrm{Uf}(\mathrm{Cm}\biguplus\mathbf{U}_\lambda)$. Thus, by Theorem 4.1.25, $\mathbf{W} = \mathrm{Uf}(\mathrm{Cm}\biguplus\mathbf{U}_\lambda)$ is a bounded morphic image of an ultrapower of $\biguplus\mathbf{U}_\lambda$. Then, by Lemma 4.1.24, $\mathbf{W} \in \mathbf{H_b U_d P_u}\{\mathrm{U}_\lambda : \lambda \in \Lambda\}$. Since $\mathcal{K}$ is closed under ultraproducts $\mathbf{W} \in \mathbf{H_b U_d}\mathcal{K}$ and so $\mathrm{Uf}(\mathbf{B}) \in \mathbf{H_b H_b U_d}\mathcal{K} = \mathbf{H_b U_d}\mathcal{K}$. Let γ be a bounded morphism such that $\gamma : \biguplus_{\upsilon\in\Upsilon}\mathbf{V}_\upsilon \longrightarrow \mathrm{Uf}(\mathbf{B})$, where $\mathbf{V}_\upsilon \in \mathcal{K}$.

We will show that $\mathrm{Uf}(\mathbf{A}) \in \mathbf{H_b U_d}X$, where X is a collection of one-generated inner substructures of the $\mathbf{V}_\upsilon$. Since $\mathcal{K}$ is closed under one-generated inner substructures $X \subseteq \mathcal{K}$ and hence $\mathrm{Uf}(\mathbf{A}) \in \mathbf{H_b U_d}\mathcal{K}$. Let

$$\mathbf{W}_\upsilon = \{w \in \mathbf{V}_\upsilon : \gamma(w) \in \mathrm{Uf}(\mathbf{A})\}.$$

Then $\biguplus_{\upsilon\in\Upsilon}\mathbf{W}_\upsilon = \gamma^{-1}[\mathrm{Uf}(\mathbf{A})]$. Since preimages of inner substructures and components of disjoint unions are again inner substructures, each $\mathbf{W}_\upsilon$ is an inner substructure of $\mathbf{V}_\upsilon$. By the preceding lemma $\mathbf{W}_\upsilon$ is a bounded morphic image of one-generated inner substructures. But then each of these one-generated inner substructures are inner substructure of the $\mathbf{V}_\upsilon \in \mathcal{K}$.

Applying Lemma 3.1.15 we get $\mathrm{CmUf}(\mathbf{A}) \in \mathbf{SPCm}\mathcal{K}$. Recall that $\mathbf{A}$ is a subalgebra of $\mathrm{Em}\mathbf{A} = \mathrm{CmUf}(\mathbf{A})$, whence $\mathbf{A} \in \mathbf{SPCm}\mathcal{K}$. □

If $\mathcal{K}$ is an universal class of algebras then we also get a necessary condition. (As remarked prior to Definition 3.1.8 p. 35, we can consider the algebras in the following theorem as relational structures and thus Theorem 4.3.17 applies for the "sufficient" condition.)

THEOREM 4.3.18. *Let $\mathcal{K}$ be a universal class of algebras. Then $\mathcal{K}$ is closed under one-generated inner substructures if, and only if,* $\mathrm{V}(\mathbf{Cm}\mathcal{K}) = \mathbf{SPCm}\mathcal{K}$.

PROOF. Assume that $\mathcal{K}$ is a universal class of algebras. Then by Theorem 2.5.19 (p. 26) it follows that $\mathbf{S}\mathcal{K} \subseteq \mathcal{K}$ and $\mathbf{P_u}\mathcal{K} \subseteq \mathcal{K}$.

Suppose $\mathrm{V}(\mathbf{Cm}\mathcal{K}) = \mathbf{SPCm}\mathcal{K}$. Take any $\mathbf{A} \in \mathcal{K}$, such that, for some $a \in \mathbf{A}$, $\mathbf{B} = \mathrm{S}_{\mathfrak{b}}^{\mathbf{A}}(\{a\}) \notin \mathcal{K}$. Since $\mathcal{K}$ is closed under subalgebras it follows that $\mathbf{B} \notin \mathbf{S}\mathcal{K}$. Hence $\mathbf{B}$ cannot be a subalgebra of $\mathbf{A}$. Clearly the universe of $\mathbf{B}$ is contained in that of $\mathbf{A}$, whence there must exist some functional symbol f and elements $b_0, \ldots, b_{n-1} \in \mathbf{B}$ such that

$$f^{\mathbf{A}}(b_0, \ldots, b_{n-1}) \neq b \text{ for any } b \in \mathbf{B},$$

where $ar(f) = n$.

As $\mathbf{B}$ is an inner substructure of $\mathbf{A}$, it follows, by Proposition 3.1.9 (p. 35), that $\mathrm{Cm}\mathbf{B} \in \mathbf{HCm}\{\mathbf{A}\} \subseteq \mathrm{V}(\mathbf{Cm}\mathcal{K}) = \mathbf{SPCm}\mathcal{K}$. Then $\mathrm{Cm}\mathbf{B} \in \mathbf{SCm}\mathcal{K}$, since by Lemma 4.3.14 $\mathrm{Cm}\mathbf{B}$ is subdirectly irreducible. But $\mathcal{K}$ is a class of algebras, thus the universal formula

$$X_0 \neq \mathbb{C}, \ldots, X_{n-1} \neq \mathbb{C} \rightarrow f(X_0, \ldots, X_{n-1}) \neq \mathbb{C}$$

is satisfiable in the class $\mathbf{SCm}\mathcal{K}$. Clearly if we take $X_i = \{b_i\}$ this formula does not hold in $\mathrm{Cm}\mathbf{B}$, contradicting the fact that $\mathrm{Cm}\mathbf{B} \in \mathbf{SCm}\mathcal{K}$.

The converse follows directly from the previous theorem. □

In Proposition 6.3.1 (p. 114) we show that the varieties of Boolean algebras and Boolean rings are trivially closed under one-generated substructures, since their members have no proper inner substructures. Similar results can be proven for the varieties of groups, rings and lattices. Hence the classes $\mathbf{SPCm}\mathsf{BA}$, $\mathbf{SPCm}\mathsf{BR}$, $\mathbf{SPCm}(\mathit{groups})$, $\mathbf{SPCm}(\mathit{rings})$ and $\mathbf{SPCm}(\mathit{lattices})$ are complex varieties. (The class $\mathbf{SPCm}(\mathit{groups})$ is referred to as the class of group relation algebras and has been studied intensively, c.f. [**McK70**], [**Com86**] and [**HiH99**].)

Discriminator varieties. Before the advent of the result proved above it was already known that for a class of discriminator varieties $\mathbf{HSPCm} = \mathbf{SPCm}$. But the class of semi-lattices is not a discriminator variety, it is however closed under one-generated inner substructures and hence the result above applies.

There are however many other reasons to study discriminator varieties not least of which being that they have very nice structural properties. We now make a slight detour into this field presenting some results that are useful in the study of many of the examples of complex varieties mentioned above. (Many of the results contained in this section were published in [**Jip93**].)

DEFINITION 4.3.19. A *discriminator algebra* $\mathbf{A}$ is an algebra for which there exists a ternary term t, called a *discriminator term*, such that for $x, y, z \in \mathbf{A}$

$$t^{\mathbf{A}}(x, x, z) = z \text{ and } t^{\mathbf{A}}(x, y, z) = x \text{ if } x \neq y.$$

A *discriminator variety* is a variety generated by a class of algebras which all have the same discriminator t.

In the case where we are dealing with varieties of complex algebras all these algebras have an underlying boolean structure. In this case a discriminator term in such a class of algebras can be expressed as a unary term.

DEFINITION 4.3.20. Let $\mathbf{B}$ be a Boolean algebra. We call c a *unary discriminator* over $\mathbf{B}$ if

$$c^{\mathbf{B}}(\mathbb{O}) = \mathbb{O} \text{ and } c^{\mathbf{B}}(x) = \mathbb{1} \text{ if } x > \mathbb{0}.$$

PROPOSITION 4.3.21. *Let $\mathbf{B}$ be a Boolean algebra. $\mathbf{B}$ is a discriminator algebra if, and only if, $\mathbf{B}$ has a unary discriminator term.*

PROOF. Let t be a discriminator term over $\mathbf{B}$ and define $c(x) = {\sim}t(\mathbb{0}, x, \mathbb{1})$. Clearly c is a unary discriminator term over $\mathbf{B}$. For the reverse implication we let $t(x, y, z) = \big(x \wedge c(x \oplus y)\big) \vee \big(z \wedge {\sim}c(x \oplus y)\big)$, where $x \oplus y = (x \wedge {\sim}y) \vee ({\sim}x \wedge y)$. $\square$

The first justification for our statement that discriminator varieties have a nice structure theory is the fact that for a BAO variety $\mathcal{V}$ the statement that a subdirectly irreducible member of $\mathcal{V}$ has a discriminator term can be characterised by equations.

THEOREM 4.3.22. *Let $\mathcal{V}$ be a variety of* BAO*s and let c be a unary term of $\mathcal{V}$. The following are equivalent:*

(i) *c is a unary discriminator in all subdirectly irreducible members of $\mathcal{V}$, and*
(ii) *$\mathcal{V}$ satisfies the equations $c(\mathbb{0}) = \mathbb{0}$, $x \leq c(x)$ and*

$$f_{\underline{\mathbb{1}},i}(c(x)) \leq c(x) \text{ and } f_{\underline{\mathbb{1}},i}({\sim}c(x)) \leq {\sim}c(x)$$

for each operator f and $i < ar(f)$.

For the proof of this result we refer the reader to [**Jip93**] Theorem 3.

Secondly a discriminator variety $\mathcal{V}$ has the nice property that any universal sentence can be translated into an equation such that the sentence holds in simple members of $\mathcal{V}$ if, and only if, $\mathcal{V}$ satisfies the corresponding equation.

DEFINITION 4.3.23. Let $\mathcal{V}$ be a BAO discriminator variety with unary discriminator $c(x)$. Then for any universal formula σ we define $\sigma^{=}$ inductively as follows:

(i) If σ is an atomic formula $s = t$, let $\sigma^{=} = {\sim}(s \oplus t)$,
(ii) if σ is a conjunction of two open formulas ψ and φ, let $\sigma^{=} = \psi^{=} \wedge \phi^{=}$, and
(iii) if σ is the negation of an open formula ψ, let $\sigma^{=} = c({\sim}\psi^{=})$.

As described in [**Jip93**] (also c.f. [**McK75**] Theorem 1.3 and [**Wer78**] Lemma 1.10) this then gives us our translation

$$\mathcal{V} \models (\sigma^{=} = \mathbb{1}) \text{ iff } \mathcal{S} \models \sigma, \tag{4.17}$$

where $\mathcal{S}$ is the class of all simple members of $\mathcal{V}$. Note that for any member $\mathbf{A}$ of a discriminator variety, with $|\mathbf{A}| \geq 2$, the concepts of an algebra being simple and being subdirectly irreducible coincide (c.f. [**Wer78**] Theorem 2.2).

Transferring equations. To conclude this section on complex varieties we turn to the transference of equations between varieties and their complex counterparts. Most of the results and definitions presented here where first published in [**Gau57**] (c.f. also [**Jip01**]).

DEFINITION 4.3.24. Given a term τ, we let $\mathrm{var}(\tau)$ denote the set of variables in τ. A term τ is said to be *linear* if each variable in τ occurs at most once in τ.

Note that the following results only make sense when considering terms constructed without the Boolean operations.

LEMMA 4.3.25. *If τ is a n-ary linear term in the similarity type of* R *then*

$$\tau^{\mathrm{CmR}}(X_1, \ldots, X_n) = \{\tau^{\mathrm{R}}(x_1, \ldots, x_n) : x_1 \in X_1, \ldots, x_n \in X_n\}.$$

We refer the reader to the proof of this lemma in [**Gau57**].

LEMMA 4.3.26. *Let $\sigma(\underline{x})$ and $\tau(\underline{x})$ be any terms and let $\mathcal{K}$ be a class of algebras. Then $\mathcal{K} \models \sigma(\underline{x}) = \tau(\underline{x})$ if, and only if,* $\mathbf{Cm}\mathcal{K} \models x_0 \neq \mathbb{0} \wedge \ldots \wedge x_{n-1} \neq \mathbb{0}$ *implies* $\sigma(\underline{x}) \cap \tau(\underline{x}) \neq \mathbb{0}$.

PROOF. Assume $\mathcal{K} \models \sigma(\underline{x}) = \tau(\underline{x})$ and let $X_0, \ldots, X_{n-1}$ be nonempty subsets of $\mathbf{A}$ for $\mathbf{A} \in \mathcal{K}$. Let $\underline{x}$ be some element of $\underline{X} = X_0 \times \cdots \times X_{n-1}$ then $\sigma(\underline{x}) \in \sigma(\underline{X})$ and $\tau(\underline{x}) \in \tau(\underline{X})$. But $\sigma(\underline{x}) = \tau(\underline{x})$, hence $\sigma(\underline{X}) \cap \tau(\underline{X}) \neq \emptyset$.

For the backwards direction assume that the implication in $\mathbf{Cm}\mathcal{K}$ holds, and let $\underline{x} \in \mathbf{A}^n$. Then the singleton sequence $\underline{X} = (\{x_0\}, \ldots, \{x_{n-1}\})$ satisfies the premise of this implication. Hence $\sigma(\underline{X}) \cap \tau(\underline{X}) \neq \emptyset$. But for singleton sequences $\sigma(\underline{X}) = \{\sigma(\underline{x})\}$ and similarly for τ, thus $\sigma(\underline{x}) = \tau(\underline{x})$. □

LEMMA 4.3.27. *Let σ and τ be terms, and $\underline{x} = (x_0, \ldots, x_{n-1})$ and let σ be linear.*

(i) $\mathcal{K} \models \sigma(\underline{x}) = \tau(\underline{x})$ *if, and only if,* $\mathbf{Cm}\mathcal{K} \models x_0 \neq \mathbb{0} \wedge \ldots \wedge x_{n-1} \neq \mathbb{0}$ *implies* $\sigma(\underline{x}) \leq \tau(\underline{x})$.

(ii) *If* $\mathrm{var}(\tau) \subseteq \mathrm{var}(\sigma)$ *then* $\mathcal{K} \models \sigma(\underline{x}) = \tau(\underline{x})$ *if, and only if,* $\mathbf{Cm}\mathcal{K} \models \sigma(\underline{x}) \leq \tau(\underline{x})$.

(iii) *If $\tau(\underline{x})$ is linear and* $\mathrm{var}(\tau) \subseteq \mathrm{var}(\sigma)$ *then* $\mathcal{K} \models \sigma(\underline{x}, y_0, \ldots, y_{m-1}) = \tau(\underline{x})$ *if, and only if,* $\mathbf{Cm}\mathcal{K} \models \sigma(\underline{x}, \mathbb{1}, \ldots, \mathbb{1}) = \tau(\underline{x})$.

(iv) *If $\tau(\underline{x})$ is linear, and* $\mathrm{var}(\sigma) = \mathrm{var}(\tau)$ *then* $\mathcal{K} \models \sigma(\underline{x}) = \tau(\underline{x})$ *if, and only if,* $\mathbf{Cm}\mathcal{K} \models \sigma(\underline{x}) = \tau(\underline{x})$.

PROOF. We will do the proofs of the forward implication of parts (i) and (ii) as examples. The converse direction of the proofs use similar arguments as in the lemma proved above.

(i): We assume $\sigma(\underline{x}) = \tau(\underline{x})$ holds in $\mathcal{K}$. Let $X_0, \ldots, X_{n-1}$ be nonempty subsets of $\mathbf{A}$ for some $\mathbf{A} \in \mathcal{K}$ and let $a \in \sigma(X_0, \ldots, X_{n-1})$. By Lemma 4.3.25 we know there exist $x_i \in X_i$, for $i < n$, such that $a = \sigma(\underline{x})$. Then $a = \tau(\underline{x}) \in \tau(X_0, \ldots, X_{n-1})$ as required.

(ii): If all the x_i's are non-zero then the proof of part (i) suffices. Hence we consider the case where for some $j < n$, $x_j \in \mathrm{var}(\sigma)$ and $x_j = \mathbb{0}$. Since σ is only constructed using operators it is easily seen that $\sigma(\underline{x}) = \mathbb{0} \subseteq \tau(\underline{x})$.

The proof of (iv) is due to Gautam (c.f. [**Gau57**]), who in addition showed that the restrictions on σ and τ are necessary in this case (see also [**GrW84**] and [**GrL88**]). □

4.3.2. Complete varieties.

PROPOSITION 4.3.28. *Let $\mathcal{V}$ be a variety. If there exists a class of structures $\mathcal{K}$ such that* $\mathrm{V}(\mathbf{Cm}\mathcal{K}) = \mathcal{V}$ *then* $\mathrm{V}(\mathbf{CmStr}\mathcal{V}) = \mathrm{V}(\mathbf{Cm}\mathcal{K})$.

PROOF. Recall that by definition $\mathbf{Str}\mathcal{V} = \{\mathcal{K} : \mathrm{Cm}\mathcal{K} \in \mathcal{V}\}$.

We assume that $\mathbf{Cm}\mathcal{K} \subseteq \mathcal{V}$. Consequently it follows that $\mathbf{Cm}\mathcal{K} \subseteq \mathbf{CmStr}\mathcal{V} \subseteq \mathcal{V}$. Hence $\mathrm{V}(\mathbf{CmStr}\mathcal{V}) = \mathrm{V}(\mathbf{Cm}\mathcal{K})$. □

Thus we can reformulate the definition of a complete variety as follows.

DEFINITION 4.3.29. A variety $\mathcal{V}$ is complete if, and only if, $\mathcal{V} = \mathrm{V}(\mathbf{CmStr}\mathcal{V})$.

Recall that in Theorem 4.3.2 we proved that every complex variety is complete. The question whether the converse holds, has been answered in the negative. In [**Gol89**] (Theorem 3.7.1) it is shown that the variety of diagonalisable algebras is complete, but not complex. (*Diagonalisable algebras* are Boolean algebras extended with one unary operator f, where f satisfies the inequality $f(x) \leq f(x - f(x))$.)

4.3.3. Canonical varieties.

To conclude we give two characterisations of canonical varieties without proofs. (We refer the reader to [**Gol89**] Theorem 3.5.5 and Theorem 3.6.7 for the proofs of these results).

THEOREM 4.3.30. *$\mathcal{V} = \mathrm{V}(\mathrm{Cm}K)$ is canonical if, and only if, $\mathbf{CmStr}\mathcal{V}$ is closed under canonical embedding algebras.*

THEOREM 4.3.31. *If $\mathcal{K}$ is closed under ultraproducts, then $\mathrm{V}(\mathbf{Cm}\mathcal{K})$ is canonical.*

This result is a strengthening of Theorem 4.3.4, since Theorem 4.3.4 only implies that $\mathrm{V}(\mathbf{Cm}\mathcal{K})$ will be complex.

We end this section by quoting some results proven in [**Jón95**] (c.f Theorem 3.11 and Theorem 4.1). To this end we require the following definition.

DEFINITION 4.3.32. Let $L = L_{\mathsf{BA}} \uplus F$ be a functional language. A term τ in the language L is called *strictly positive* if it does not involve the complementation, i.e. $\sim$, in its definition.

THEOREM 4.3.33. *Let $\mathbf{B}$ be a BAO and σ and τ be strictly positive terms in the language of $\mathbf{B}$. Then $\mathbf{B} \models \sigma = \tau$ implies $\mathrm{Em}\mathbf{B} \models \sigma = \tau$.*

This result is in fact a weaker one than that proven in [**Jón95**], but provides a simpler presentation than that of the original result published in [**JoT51**].

THEOREM 4.3.34. *Let $\mathbf{B}$ be a BAO and ρ, σ and τ be strictly positive terms. Then $\mathbf{B} \models \sigma \neq \mathbb{0} \rightarrow \rho = \tau$ implies $\mathrm{Em}\mathbf{B} \models \sigma \neq \mathbb{0} \rightarrow \rho = \tau$.*

4.4. Open problems

We conclude with some unanswered questions related to the theory we have covered in this chapter.

(1) The proof given of the Keisler-Shelah Theorem (Corollary 4.2.5) relies on Theorem 4.2.4, which uses the Generalised Continuum Hypothesis. As was mentioned it has been shown that GCH is not necessary to prove the Keisler-Shelah Theorem, however it is still unknown whether or not GCH is required for the proof of Theorem 4.2.4.
(2) As was mentioned in Section 4.3.2 there exist varieties that are complete but not complex. However it is still unknown whether or not there are any complex varieties that are not canonical.
(3) Another longstanding open problem is the question whether every canonical variety of BAOs is of the form $\mathrm{V}(\mathrm{Cm}\mathcal{K})$ for some class $\mathcal{K}$ closed under ultraproducts.

CHAPTER 5

Complex algebras and logical games

In [**Lyn50**] R. Lyndon used a step-by-step method to construct proper relation algebras from a class of finite relation algebras satisfying certain conditions, he then proved that these conditions were both necessary and sufficient. With the use of this result he proceeded to demonstrate the existence of non-representable relation algebras answering a question first raised by A. Tarski (c.f. [**Tar41**] for more on relation algebras). Other well- known examples of this kind include the construction of the representable relation algebras (RRA) using complex algebras of full pair arrow frames (FPAF), constructing modal algebras using complex algebras of Kripke frames and group relation algebras (GRA) using complex algebras of groups.

As mentioned in the introduction logical games have proved very useful in this regard and there is now an abundance of application of this technique to different classes of structures. In general they have been useful in answering the following generic questions for some class of structures $\mathbf{W} = \mathbf{SCm}\mathcal{K}$.

(i) Can we find axioms for $\mathbf{W}$?
(ii) Given some finite boolean algebra with operators how do we check if it is in $\mathbf{W}$?
(iii) When is the equational, quasi-equational, universal or first order theory of $\mathbf{W}$ decidable?

In this chapter we will only address the first two of these questions. Most of the results we present here were first demonstrated by Hodkinson, Mikulas and Venema (c.f. [**HMV99**]), we use their results to model our answers to these questions on. Most of the proofs in this chapter are then also based on those presented in [**HMV99**]. We conclude by discussing how their work can be extended to incorporate a bigger class of problems.

5.1. Networks and games

We now proceed to systematically introduce the tools required to play these logical games which we will use to answer (i) and (ii).

For the rest of the section we will be considering the language $L_{\mathsf{BAO}} = L_{\mathsf{BA}} \cup F$ where F is a finite set of operators. $\mathcal{V}$ will be some fixed variety of F-algebras. Due to the extensive use of the logical connectives $\vee$ (disjunction) and $\wedge$ (conjunction) in this chapter we will take $L_{\mathsf{BA}} = \langle +, -, 0\rangle$ to avoid any confusion. We consider 1 the usual abbreviation, i.e. $1 = -0$, and define the boolean conjunction '$\cdot$' in the usual manner.

5.1.1. Preliminaries.

DEFINITION 5.1.1. We say an algebra $\mathbf{B}$ is *representable over a structure* $\mathbf{N}$ if there exists an embedding rep : $\mathbf{B} \longrightarrow \mathrm{Cm}\mathbf{N}$. $\mathbf{B}$ is *representable over a class* $\mathcal{K}$ if it belongs to $\mathbf{SCm}\mathcal{K}$.

Note that $\mathbf{B}$ is representable over (a structure) $\mathbf{N}$ exactly when $\mathbf{B}$ is representable over the class $\{\mathbf{N}\}$.

We will use specific structures, called networks, to show whether an algebra is representable or not. But before we can define these "networks" we need to make a slight detour into the theory of partial algebras.

DEFINITION 5.1.2. A structure $\mathbf{N} = \langle N, F^{\mathbf{N}} \rangle$ is called a *partial F-algebra* if each symbol in $f \in F$ is functional.

Note that the only difference between an F-algebra and partial F-algebra is that the condition that f must be defined for all $ar(f)$-tuples, is dropped. Hence all F-algebras are partial F-algebras. As we will be working with a fixed set F we will generally refer to partial algebras instead of the more verbose partial F-algebras.

DEFINITION 5.1.3. Let $\mathbf{N}$ be a partial F-algebra and let σ and τ be terms in the language F. An equation $\sigma = \tau$ is said to be satisfied in $\mathbf{N}$, if for any assignment of the free variables of σ and τ to elements of $\mathbf{N}$, where both $\sigma^{\mathbf{N}}$ and $\tau^{\mathbf{N}}$ exist for this assignment, it implies that they are equal.

We extend this definition in the natural way to include formulas of type F.

DEFINITION 5.1.4. Let $\mathcal{V}$ be a variety and $\mathbf{N}$ a partial algebra. We say $\mathbf{N}$ is a partial $\mathcal{V}$-algebra if it satisfies all the equations of $\mathcal{V}$.

Consequently if such a partial algebra were a (total) algebra then it would be a member of $\mathcal{V}$.

PROPOSITION 5.1.5. *Let $\mathbf{N}$ be a partial $\mathcal{V}$-algebra. If $\mathbf{N}$ is an algebra then $\mathbf{N} \in \mathcal{V}$.*

Lastly we extend the notion of subalgebras to our current context.

DEFINITION 5.1.6. Let $\mathbf{M} = \langle M, F^{\mathbf{M}} \rangle$ and $\mathbf{N} = \langle N, F^{\mathbf{N}} \rangle$ be two partial algebras. We say $\mathbf{M}$ is a *partial subalgebra* of $\mathbf{N}$ if $M \subseteq N$, and for any $f \in F$ and $k_0, \ldots, k_{n-1} \in M$, if $f^{\mathbf{M}}(k_0, \ldots, k_{n-1})$ is defined, then $f^{\mathbf{N}}(k_0, \ldots, k_{n-1})$ is defined, and $f^{\mathbf{M}}(k_0, \ldots, k_{n-1}) = f^{\mathbf{N}}(k_0, \ldots, k_{n-1})$, where $ar(f) = n$.

5.1.2. Networks.

We are now ready to introduce the networks which, in the sequel, will form the playing boards for the logical games we are constructing.

DEFINITION 5.1.7. Given some L_{BAO}-structure $\mathbf{B} = \langle B, +, -, 0, F^{\mathbf{B}} \rangle$, a *network over* $\mathbf{B}$ is a structure $\mathbf{N} = \langle N, F^{\mathbf{N}}, \lambda \rangle$ such that $\langle N, F^{\mathbf{N}} \rangle$ is a partial algebra and λ a map $N \longrightarrow B$. Elements of N are called *nodes*, and λ is called the *labelling* of the network. The *empty network* $\langle \emptyset, \emptyset, \emptyset \rangle$ is denoted by $\mathbf{N}_{\emptyset}$.

A network $\mathbf{N} = \langle N, F^{\mathbf{N}}, \lambda \rangle$ is called a $\mathcal{V}$-network if $\langle N, F^{\mathbf{N}} \rangle$ is a partial $\mathcal{V}$-algebra. $\mathbf{N}$ is said to be *coherent* if $\lambda(k) \neq 0$ for each node $k \in N$, and in addition λ satisfies the following condition for each function symbol $f \in F$, where $ar(f) = n$, and all nodes $k_0, \ldots, k_{n-1}$, such that $f^{\mathbf{N}}(k_0, \ldots, k_{n-1})$ is defined,

$$\lambda(f^{\mathbf{N}}(k_0, \ldots, k_{n-1})) \cdot f^{\mathbf{B}}(\lambda(k_0), \ldots, \lambda(k_{n-1})) \neq 0. \tag{5.1}$$

We say *a network* $\mathbf{N} = \langle N, F^{\mathbf{N}}, \lambda \rangle$ *is finite* if the underlying set N is finite.

DEFINITION 5.1.8. A network $\mathbf{N}' = \langle N', F^{\mathbf{N}'}, \lambda' \rangle$ over an algebra $\mathbf{B}$ is said to *extend* or to be an *extension of* a network $\mathbf{N} = \langle N, F^{\mathbf{N}}, \lambda \rangle$, written $\mathbf{N} \lhd \mathbf{N}'$, if $\langle N, F^{\mathbf{N}} \rangle$ is a partial subalgebra of $\langle N', F^{\mathbf{N}'} \rangle$ and $\mathbf{B} \models \lambda'(k) \leq \lambda(k)$, for all $k \in N$. For such a λ and λ' we say λ' is a *tightening* of λ.

5.1.3. Games.

Informally a game will consist of us (the earthlings) bringing some BAO to the table with the intention of showing them (the aliens) that this algebra is representable over some network $\mathbf{N}$. Being the devil's advocates that all aliens are, they will ask us whether our network satisfies certain conditions and we will then have to either show them that our network $\mathbf{N}$ already satisfies it or we will have to extend our network to show that we can in fact satisfy their condition.

First, however, let us introduce some notation. Given two sets X and Y, a function $f : X^n \longrightarrow Y$ and $a_0, \ldots, a_{n-1} \in X$, $b \in Y$, we let $f_{[(a_0, \ldots, a_{n-1}) \mapsto b]}$ be the new function $f' : X \longrightarrow Y$ where:

$$f'(x_0, \ldots, x_{n-1}) = \begin{cases} b & \text{if } x_i = a_i \text{ for all } i < n, \\ f(x_0, \ldots, x_{n-1}) & \text{otherwise.} \end{cases}$$

In the case where $ar(f) = 1$ we simply write $f_{[a \mapsto b]}$. Where convenient we write $f_{\underline{a} \mapsto b}$, and assume that $\underline{a}$ is in the domain of f. By $f_{[\underline{a}_0 \mapsto b_0, \underline{a}_1 \mapsto b_1, \ldots, \underline{a}_{n-1} \mapsto b_{n-1}]}$ we mean $(\ldots(f_{[\underline{a}_0 \mapsto b_0]})_{[\underline{a}_1 \mapsto b_1]} \ldots)_{[\underline{a}_{n-1} \mapsto b_{n-1}]}$. If $ar(f) = 1$ we will abuse our notation somewhat and write $f_{[\underline{a} \mapsto \underline{b}]}$ for $f_{[a_0 \mapsto b_0, a_1 \mapsto b_1, \ldots, a_{n-1} \mapsto b_{n-1}]}$.

Especially for labelling functions it will be convenient to have the following abbreviation with similar conventions to those mentioned above. For a function $f : X^n \longrightarrow Y$ and $\underline{a} \in X^n$, $b \in Y$ we define $f_{[\underline{a} \looparrowright b]}$ to be the function $f_{[\underline{a} \mapsto f(\underline{a}) \cdot b]}$.

DEFINITION 5.1.9. Let $\mathbf{B}$ be some BAO with associated language $L_{\mathsf{BAO}} = L_{\mathsf{BA}} \cup F$ as before, let $\mathbf{N}$ be some network over $\mathbf{B}$ and let $\eta \leq \omega$ be an ordinal. We define a *game* $G_\eta(\mathbf{N}, \mathbf{B}, \mathcal{V})$ between two teams of players: $\forall$ (the aliens) and $\exists$ (the earthlings).

A *match* of the game consists of η *rounds* numbered $0, 1, \ldots, i, \ldots,$ for $i < \eta$. The match starts with the network $\mathbf{N}_0 = \mathbf{N}$, and each round then successively generates a sequence of networks $\mathbf{N}_0, \mathbf{N}_1, \ldots, \mathbf{N}_i, \ldots$ $(i \leq \eta)$, all networks over $\mathbf{B}$. A particular round consists of a *move* by $\forall$ and a *response move* by $\exists$. In the ith round the playing board consists of a network $\mathbf{N}_i$, the actions of the teams during this round will determine the playing board $\mathbf{N}_{i+1}$ for the next round; and so on.

During each round $\forall$ have a choice of four kinds of moves they can make, described below. In round i of the game they will propose to extend the network $\mathbf{N}_i$, forming the current playing board, in some way. $\exists$ will then have to either accept or reject their proposal and in doing so generate the playing board for the next round. At our (i.e. the earthlings') disposal we also have a fixed infinite set Q from which to draw new nodes if we wish to enlarge N_i. We extend the labelling λ to include Q by defining $\lambda(m) = 1^{\mathbf{B}}$, for any $m \in Q \setminus N_0$.

α: The first type of move consists of $\forall$ choosing a node k of the network $\mathbf{N}_i$ and an element $b \in \mathbf{B}$. If $\exists$ accept we let $N_{i+1} = \langle N_i, F^{\mathbf{N}_i}, \lambda_{[k \mapsto b]}\rangle$, otherwise we let $N_{i+1} = \langle N_i, F^{\mathbf{N}_i}, \lambda_{[k \mapsto -b]}\rangle$.

β: In the second type of move $\forall$ choose a node $k \in \mathbf{N}_i$, a symbol $f \in F$, where $ar(f) = n$, and $b_0, \ldots, b_{n-1} \in \mathbf{B}$. If $\exists$ reject this move then the board for the next round will be

$$\mathbf{N}_{i+1} = \langle N_i, F^{\mathbf{N}_i}, \lambda_{[k \mapsto -f^{\mathbf{B}}(b_0, \ldots, b_{n-1})]}\rangle.$$

If $\exists$ accept this move and there exist elements $m_0, \ldots, m_{n-1} \in N_i$ such that $f^{\mathbf{N}_{i+1}}(m_0, \ldots, m_{n-1}) = k$ we let $\mathbf{N}_{i+1} = \mathbf{N}_i$. Otherwise, if no such elements exist, we must choose $m_0, \ldots, m_{n-1} \in N_i \cup Q$ such that $f^{\mathbf{N}_i}(m_0, \ldots, m_{n-1})$ is undefined. We then let $f^{\mathbf{N}_{i+1}}(m_0, \ldots, m_{n-1}) = k$ and define

$$\mathbf{N}_{i+1} = \langle N_i \cup \{m_0, \ldots, m_{n-1}\}, H^{\mathbf{N}_i} \cup \{f^{\mathbf{N}_i}_{[\underline{m} \mapsto k]}\}, \lambda_{[\underline{m} \mapsto \underline{b}]}\rangle,$$

where $H = F \backslash \{f\}$, $\underline{m}$ is the n-tuple $\langle m_0, \ldots, m_{n-1}\rangle$ and $\underline{b}$ is the n-tuple $\langle b_0, \ldots, b_{n-1}\rangle$.

γ: The third kind of move consists of $\forall$ choosing a symbol $f \in F$, with $ar(f) = n$, and nodes $k_0, \ldots, k_{n-1} \in N_i$. We then have no other choice but to accept this move. If $f(\underline{k} = m)$ then we let $\mathbf{N}_{i+1} = \mathbf{N}_i$. Otherwise we have to choose $m \in Q \cup N_i$ and define

$$\mathbf{N}_{i+1} = \langle N_i \cup \{m\}, H^{\mathbf{N}_i} \cup \{f^{\mathbf{N}_i}_{[\underline{k} \mapsto m]}\}, \lambda\rangle,$$

where $H = F \backslash \{f\}$ and $\underline{k}$ is the n-tuple $\langle k_0, \ldots, k_{n-1}\rangle$.

δ: In the last kind of move $\forall$ choose $b \in \mathbf{B}$ with $b \neq 0$. Again $\exists$ have no choice but to accept and so provide a node $k \in Q \cup N_i$ and define the new playing board as follows

$$\mathbf{N}_{i+1} = \langle N_i \cup \{k\}, F^{\mathbf{N}_i}, \lambda_{[k \mapsto b]}\rangle.$$

$\exists$ are said to *win* the match if $\mathbf{N}_0$ and every $\mathbf{N}_{i+1}$, for $i < \eta$, is a coherent $\mathcal{V}$-network. If we do not win the match, then $\forall$ do.

As mentioned before the idea is that the aliens make a specific type of move which forces us to make our network satisfy certain conditions. We are now in a position to elaborate on this. $\forall$ will make a type (α) move if they want to know whether b or $-b$ 'holds' at some point k of the representation. Informally we say b holds at k if $b \geq \lambda(k)$. Move (β) ensures that if we have that $f^{\mathbf{B}}(b_0, \ldots, b_{n-1})$ holds at some point k, we have corresponding $m_0, \ldots, m_{n-1} \in \mathbf{N}_i$ such that $f^{\mathbf{N}_i}(\underline{m}) = k$ and $\underline{b}$ holds at $\underline{m}$ (note the homomorphism like quality of this condition). It is easy to see that moves of type (γ) ensure the totality of each operator. To see the purpose of moves of type (δ) we remind the reader that a BA-homomorphism h is injective if $h(b) \neq 0$, for every $b \neq 0$.

We need to be able to talk about whether or not $\exists$ actually have a representation. To this end we make the following definition.

DEFINITION 5.1.10. $\exists$ are said to have a *winning strategy* for the game $G_\eta(\mathbf{N}, \mathbf{B}, \mathcal{V})$ if there is a set of rules that tells us how to win any match. In other words, given a particular match the set of rules describe how we should respond to $\forall$ in each round, depending on play so far, such that if we follow these rules we will win the match.

5.2. Characterising the game

In this section it is our aim to prove the following characterisation of the game described above. As before we will assume a fixed language $L_{\mathsf{BAO}} = L_{\mathsf{BA}} \cup F$, where F is a set of operators. (This result was first proved in [**HMV99**] Theorem 4.1.)

THEOREM 5.2.1. *Let* $\mathbf{B}$ *be a countable* BAO. *Then* $\exists$ *have a winning strategy for the game* $G_\omega(\mathbf{N}_\emptyset, \mathbf{B}, \mathcal{V})$ *if, and only if,* $\mathbf{B}$ *belongs to* $\mathbf{SCm}\mathcal{V}$.

We will naturally break the proof up into two parts. First we show the left to right part (c.f. Theorem 5.2.2), also called the soundness direction. Then in Theorem 5.2.5 we show the converse, also referred to as the completeness direction.

5.2.1. Soundness.

Here we can drop the restriction of the countability of $\mathbf{B}$. The following result was first presented in [**HMV99**] Proposition 4.2. We here present a proof of this result including the arguments for moves of type (α), (γ) and (δ) which were omitted in [**HMV99**].

THEOREM 5.2.2. *Let* $\mathbf{B}$ *be an algebra in* $\mathbf{SCm}\mathcal{V}$. *Then* $\exists$ *have a winning strategy for* $G_\omega(\mathbf{N}_\emptyset, \mathbf{B}, \mathcal{V})$.

PROOF. If $\mathbf{B} = \langle B, +, -, 0, F^{\mathbf{B}} \rangle$ belongs to $\mathbf{SCm}\mathcal{V}$ then there exists an algebra $\mathbf{A} = \langle A, F^{\mathbf{A}} \rangle \in \mathcal{V}$ and a representation map $\text{rep} : \mathbf{B} \longrightarrow \mathcal{P}(\mathbf{A})$ which embeds $\mathbf{B}$ into $\text{Cm}\mathbf{A}$.

Now let $\mathbf{N} = \langle N, F^{\mathbf{N}}, \lambda \rangle$ be a network over $\mathbf{B}$. We say $\mathbf{N}$ *matches with* rep if $\mathbf{N} = \langle N, F^{\mathbf{N}} \rangle$ is a partial subalgebra of $\mathbf{A}$, while λ satisfies

$$(*) \qquad k \in \text{rep}(\lambda(k)), \text{ for all } k \in N.$$

I.e. $\mathbf{N}$ can be seen as an approximation for the representation of $\mathbf{B}$. We will take the universe of $\mathbf{A}$ to be the set of nodes Q available to $\exists$ during play.

We will presently show how $\exists$ can maintain the condition that the current network matches with rep.

Claim: If $\mathbf{N}$ matches with rep then $\mathbf{N}$ is a coherent $\mathcal{V}$-network.

By definition $\mathbf{N}$ is a partial $\mathcal{V}$-algebra. Since rep is injective and $\text{rep}(\lambda(k)) \neq \emptyset$ it follows that $\lambda(k) \neq 0$ for all $k \in N$.

Let $f \in F$, where $ar(f) = n$, and assume that $f^{\mathbf{N}}(k_0, \ldots, k_{n-1})$ is defined for $k_0, \ldots, k_{n-1} \in N$. We will show that

$$(**) \qquad \text{rep}\big(\lambda\big(f^{\mathbf{N}}(k_0, \ldots, k_{n-1})\big) \cdot f^{\mathbf{B}}\big(\lambda(k_0), \ldots, \lambda(k_{n-1})\big)\big) \neq \emptyset.$$

Then by the injectivity of rep it follows that

$$\lambda\big(f^{\mathbf{N}}(k_0, \ldots, k_{n-1})\big) \cdot f^{\mathbf{B}}\big(\lambda(k_0), \ldots, \lambda(k_{n-1})\big) \neq 0.$$

Since rep is a homomorphism

$$\begin{aligned} &\text{rep}\big(\lambda(f^{\mathbf{N}}(k_0, \ldots, k_{n-1})) \cdot f^{\mathbf{B}}(\lambda(k_0), \ldots, \lambda(k_{n-1}))\big) \\ &\qquad = \text{rep}\big(\lambda(f^{\mathbf{N}}(k_0, \ldots, k_{n-1}))\big) \cap f^{\text{Cm}\mathbf{A}}\big(\text{rep}(\lambda(k_0), \ldots, \lambda(k_{n-1}))\big). \end{aligned}$$

Now $\mathbf{N}$ matches with rep, whence $f^{\mathbf{N}}(k_0, \ldots, k_{n-1}) \in \text{rep}\big(\lambda(f^{\mathbf{N}}(k_0, \ldots, k_{n-1}))\big)$. But $\mathbf{N}$ is a partial subalgebra of $\mathbf{A}$ and $\{k_i\} \subseteq \text{rep}(\lambda(k_i))$, for $i < n$. Thus

$$f^{\mathbf{N}}(k_0, \ldots, k_{n-1}) \in f^{\text{Cm}\mathbf{A}}(\{k_0\}, \ldots, \{k_{n-1}\}) \subseteq f^{\text{Cm}\mathbf{A}}\big(\text{rep}(\lambda(k_0), \ldots, \lambda(k_{n-1}))\big).$$

Hence, by the existence of $f^{\mathbf{N}}(k_0, \ldots, k_{n-1})$, $(**)$ is satisfied.

We will now prove that if $\mathbf{N} = \langle N, F^{\mathbf{N}}, \lambda\rangle$ is a network that matches with rep then $\exists$ can counter any move of $\forall$ with a new network $\mathbf{N}' = \langle N', F^{\mathbf{N}'}, \lambda'\rangle$ which again matches with rep. Let us take a look at each type of move separately. (Where applicable we take $ar(f) = n$.)

α: In this kind of move $\forall$ picks $k \in N$ and $b \in B$. If $k \notin \text{rep}(b)$ then $\exists$ rejects the move and all that changes is that $\lambda' = \lambda_{[k \looparrowright -b]}$, i.e. $\lambda'(k) = \lambda(k) \cdot -b$. Thus $\text{rep}(\lambda'(k)) = \text{rep}(\lambda(k)) \setminus \text{rep}(b)$. But since $\mathbf{N}$ matches with rep $k \in \text{rep}(\lambda(k))$ and by assumption $k \notin \text{rep}(b)$, it follows that $k \in \text{rep}(\lambda'(k))$. The argument for acceptance is similar. Hence (α) type moves preserve $(*)$. Note that there is no change to the underlying partial algebra $\langle N, F^{\mathbf{N}}\rangle$. Consequently $\mathbf{N}'$ matches with rep.

β: For this kind of move $\forall$ chooses $f \in F$, $k \in N$ and $b_0, \ldots, b_{n-1} \in B$.

If $k \notin \text{rep}(f(\underline{b}))$ $\exists$ rejects the proposal and thus the new network $\mathbf{N}'$ will be $\langle N, F^{\mathbf{N}}, \lambda_{[k \looparrowright -f(\underline{b})]}\rangle$. Since the only difference between $\mathbf{N}$ and $\mathbf{N}'$ is in the labelling it suffices to check $(*)$ for k and λ'. But $\lambda'(k) = \lambda_{[k \looparrowright -f(\underline{b})]}(k) = \lambda(k) \cdot -f(\underline{b})$, whence $\text{rep}(\lambda'(k)) = \text{rep}(\lambda(k)) \setminus \text{rep}(f(\underline{b}))$. Thus, by the inductive hypothesis and a similar argument to (α) above, $k \in \text{rep}(\lambda'(k))$. Consequently $\mathbf{N}'$ matches with rep.

On the other hand assume $k \in \text{rep}(f(\underline{b})) \subseteq \mathbf{A}$. Since rep is a homomorphism it follows that

$$\text{rep}(f^{\mathbf{B}}(\underline{b})) = \text{rep}(f^{\mathbf{B}}(b_0, \ldots, b_{n-1})) = f^{\text{Cm}\mathbf{A}}(\text{rep}(b_0), \ldots \text{rep}(b_{n-1})).$$

Hence either $f^{\mathbf{N}}(\underline{a}) = k$ already or there exist elements $a_0, \ldots, a_{n-1} \in \mathbf{A}$ such that $k = f^{\mathbf{A}}(a_0, \ldots, a_{n-1})$ and $a_i \in \text{rep}(b_i)$ for each i. $\exists$ then defines $\mathbf{N}'$ as the network $\langle N \cup \{a_i : i < n\}, H^{\mathbf{N}} \cup \{f^{\mathbf{N}}_{[\underline{a} \mapsto k]}\}, \lambda_{[\underline{a} \looparrowright \underline{b}]}\rangle$, with $H = F \setminus \{f\}$. Since the labelling of $k \in N$ does not change it follows from the inductive hypothesis that $k \in \text{rep}(\lambda'(k))$. However we need to check that $a_i \in \text{rep}(\lambda'(a_i))$, where $\lambda'(a_i) = \lambda(a_i) \cdot b_i$. If $a_i \in N$ then by the inductive hypothesis $a_i \in \text{rep}(\lambda(a_i))$, otherwise $\lambda(a_i) = \text{rep}(1) = \mathbf{A}$. Consequently

$$a_i \in \text{rep}(\lambda(a_i)) \cdot \text{rep}(b_i) = \text{rep}(\lambda'(a_i)).$$

Clearly $N \cup \{a_i : i < n\} \subseteq \mathbf{A}$ and the only change in $F^{\mathbf{N}'}$ is to the interpretation of f. Note that by assumption $f^{\mathbf{A}}(a_0, \ldots, a_{n-1}) = k \in \mathbf{A}$. Hence the partial algebra $\langle N \cup \{a_i : i < n\}, H^{\mathbf{N}} \cup \{f^{\mathbf{N}}_{[\underline{a} \mapsto k]}\}\rangle$ is still a partial subalgebra of $\mathbf{A}$.

γ: In this type of move $\forall$ choose an $f \in F$ and $k_0, \ldots, k_{n-1} \in N$. $\exists$ can only accept. However the function $f^{\mathbf{A}}$ is total and hence there exists an $a \in A$ such that $f^{\mathbf{A}}(k_0, \ldots, k_{n-1}) = a$. Thus we take $\mathbf{N}' = \langle N \cup \{a\}, H^{\mathbf{N}} \cup \{f^{\mathbf{N}}_{[\underline{k} \mapsto a]}\}, \lambda\rangle$ to be the new network, where $H = F \setminus \{f\}$. We need to show that $a \in \text{rep}(\lambda(a))$. If $a \in N$ then this result follows from the inductive hypothesis. Otherwise $\lambda(a) = 1^{\mathbf{B}}$. But $a \in \mathbf{A} = \text{rep}(1^{\mathbf{B}}) = \text{rep}(\lambda(a))$.

Observe that $a \in A$ and that $f^{\mathbf{A}}(k_0, \ldots, k_{n-1}) = a$. Hence by a similar argument to that for acceptance of β moves $\langle N \cup \{a\}, H^{\mathbf{N}} \cup \{f^{\mathbf{N}}_{[\underline{k} \mapsto a]}\}\rangle$ is a partial subalgebra of $\mathbf{A}$.

δ: In the last kind of move $\forall$ chooses $b \in \mathbf{B}$ with $b \neq 0$. Again $\exists$ have no choice but to accept. Since rep is an embedding it follows that there exists some non-empty X such that $\text{rep}(b) = X \subseteq A$. Then $\exists$ picks any node $a \in X$ and lets $\mathbf{N}' = \langle N \cup \{a\}, F^{\mathbf{N}}, \lambda'\rangle$, where $\lambda' = \lambda_{[a \looparrowright b]}$. Clearly, whether or not $a \in N$, it follows that $a \in \text{rep}(\lambda(a)) \cdot \text{rep}(b) = \text{rep}(\lambda'(a))$.

Since $a \in A$ and there is no change to any of the interpretations of symbols in F it follows that $\langle N \cup \{a\}, F^{\mathbf{N}}\rangle$ is a partial subalgebra of $\mathbf{A}$.

We have thus shown that irrespective of which type of move $\forall$ chooses to make $\exists$ can respond in accordance with the requirements of such a type of move and the resulting network after such a move still matches with rep. By definition this means that $\exists$ will win any match of length i by following this strategy, where $i < \omega$. $\square$

5.2.2. Completeness.

Before proving the converse to Theorem 5.2.2 let us take an informal look at how we aim to achieve this objective. As has been intimated our aim is to show that we can use the countable game $G_\omega(\mathbf{N}_\emptyset, \mathbf{B}, \mathcal{V})$ to construct a network $\mathbf{N}$ such that $\mathbf{B}$ is representable over $\mathbf{N}$ if $\exists$ have a winning strategy. Each of the possible types of moves correlate to ensuring that the network is in fact a representation of $\mathbf{B}$.

In particular to ensure that our eventual network satisfies all the required properties we will consider the game where $\forall$ will list all possible moves that become possible during the match and then actually make each of these moves at some stage during the match. $\exists$ of course will play according to our winning strategy. However our game has at most ω rounds and thus we should be sure that $\forall$ can play all their moves before the end of the game. (For simplicity we will consider a signature F limited to one symbol f, i.e. $F = \{f\}$ with $ar(f) = n$.) Consider the following set

$$M = (Q \times B) \cup (B^n \times Q) \cup (Q^n) \cup B.$$

If the reader looks back on the definitions of moves (α) through (δ) it should be clear that M encodes all the possible moves that $\forall$ can make. Now by assumption B is countable, however we have not limited the size of Q. In fact $|Q|$ could be greater than ω. Clearly the generalisation of M to arbitrary finite F is still countable.

Considering the possible moves open to $\exists$, it should be clear that in no type of move do we require more than a finite number of elements from Q. Since there are at most a countable number of moves, we need a set Q of at most ω elements to choose from. Lastly recall that we require the set F of symbols to be finite. Hence $|M| \leq \omega$ and so $\forall$ will be able to play all their moves.

Since we are playing according to our winning strategy, $\exists$ will be busy constructing a chain of networks

$$\mathbf{N}_\emptyset \lhd \mathbf{N}_1 \lhd \cdots \mathbf{N}_i \lhd \cdots, \text{ for } i \in \omega.$$

We will take the universe of our new structure to be the limit of this chain, i.e.

$$N = \bigcup_{i \in \omega} N_i. \tag{5.2}$$

However we require some kind of algebraic structure on this universe N, i.e. if we take $\mathbf{N} = \langle N, f^{\mathbf{N}} \rangle$ we need to define the interpretation of f over N. As was mentioned earlier, moves of type (γ) are supposed to ensure the totality of our operators, since at any round i in our game f might only be a partial function. Note that if, at any round i and for any n-tuple $\underline{k}$, $f^{\mathbf{N}_i}(\underline{k})$ is defined then it follows that $f^{\mathbf{N}_i}(\underline{k}) = f^{\mathbf{N}_j}(\underline{k})$ for all rounds j at which $f^{\mathbf{N}_j}(\underline{k})$ is defined. Hence the following definition is unambiguous, where $\underline{k} \in N^n$,

$$f^{\mathbf{N}}(\underline{k}) = f^{\mathbf{N}_i}(\underline{k}) \text{ for any } i \text{ such that } f^{\mathbf{N}_i}(\underline{k}) \text{ is defined.} \tag{5.3}$$

Lastly we need to define the representation function rep : $\mathbf{B} \longrightarrow \mathrm{Cm}\mathbf{N}$. For the moment however we only have the labelling functions $\lambda_i : \mathbf{N}_i \longrightarrow \mathbf{B}$ to work from.

Observe that $\mathrm{At}(\mathrm{Cm}\mathbf{N}) \cong \mathbf{N}$. To get an idea of how to define rep, we take a look at the map $\mathrm{At}(\mathrm{rep}) : \mathrm{At}(\mathrm{Cm}\mathbf{N}) \longrightarrow \mathrm{At}(\mathbf{B})$. By definition $\mathrm{At}(\mathrm{rep})(\{k\}) = a$ if, and only if, $\mathrm{rep}(a) \geq \{k\}$. Or equivalently $k \in \mathrm{rep}(a)$. By (δ) we must have played a $k \in N_i \cup Q$ at some round i such that $a \geq \lambda_i(k)$. Thus we define rep as follows

$$\mathrm{rep}(a) = \{k : a \geq \lambda_i(k) \text{ for some } i \in \omega\}. \tag{5.5'}$$

For convenience we let

$$\lambda(k) = \{b \in \mathbf{B} : b \geq \lambda_i(k) \text{ for some } i \in \omega\} \tag{5.4}$$

and then we can more succinctly define rep by

$$\mathrm{rep}(a) = \{k \in N : a \in \lambda(k)\}. \tag{5.5}$$

First we prove that $\mathbf{N}$ is in fact the kind of algebra we are looking for.

LEMMA 5.2.3. *Let $G_\omega(\mathbf{N}_\emptyset, \mathbf{B}, \mathcal{V})$ be a game such that $\exists$ have a winning strategy. Then there exists a (total) algebra $\mathbf{N} = \langle \bigcup_{i\in\omega} N_i, F^{\mathbf{N}}\rangle$ such that $\mathbf{N}$ belongs to $\mathcal{V}$, where each $f \in F$ satisfies the following condition.*

If $f^{\mathbf{N}_i}(\underline{k})$ is defined, then $f^{\mathbf{N}}(\underline{k}) = f^{\mathbf{N}_i}(\underline{k})$, for any $\underline{k} \in N^{ar(f)}$.

PROOF. We take N to be defined as in equation (5.2). Let f be any symbol in F, with $ar(f) = n$, and take (5.3) to define $f^{\mathbf{N}}$. We will show that $f^{\mathbf{N}}$ is a total operation.

Consider any elements $k_0, \ldots, k_{n-1}$ of N. By construction there must exist some $\mathbf{N}_i$ such that $k_0, \ldots, k_{n-1} \in \mathbf{N}_i$. Since we assume $\forall$ will play according to the strategy mentioned above it should be clear that at some round $j > i$, $\forall$ will make a move of type (γ) in which they choose $k_0, \ldots, k_{n-1}$ and the symbol $f \in F$. But then by our winning strategy $f^{\mathbf{N}_{j+1}}(k_0, \ldots, k_{n-1})$ will be defined and hence $f^{\mathbf{N}}(k_0, \ldots, k_{n-1})$ will be defined.

Since each $\mathbf{N}_i$ is a coherent $\mathcal{V}$-network it follows that $\mathbf{N}$ is a partial $\mathcal{V}$-algebra. However we have just shown that $\mathbf{N}$ is total and hence $\mathbf{N} \in \mathcal{V}$. □

LEMMA 5.2.4. *Let $G_\omega(\mathbf{N}_\emptyset, \mathbf{B}, \mathcal{V})$ be a game such that $\exists$ have a winning strategy. Then the function λ defined in (5.4) maps from N to ultrafilters over $\mathbf{B}$, where $N = \bigcup_{i\in\omega} N_i$.*

PROOF. Let λ be the function defined in (5.4) we will show that $\lambda(k)$ is an ultrafilter for $k \in N$.

Note that, for any $k \in N$, $\lambda_i(k) \geq \lambda_j(k)$ where $i \leq j$. Clearly $\lambda(k)$ must be upwardly closed. Let $a, b \in \lambda(k)$ then there exist i, j such that $a \geq \lambda_i(k)$ and $b \geq \lambda_j(k)$. If $i \geq j$ then $a \cdot b \geq \lambda_i(k)$. Hence $a \cdot b \in \lambda(k)$. A similar argument suffices for $j \geq i$. Thus $\lambda(k)$ is a filter.

To see that $\lambda(k)$ is proper observe that each $\mathbf{N}_i$ forms a coherent $\mathcal{V}$-network and hence $\lambda_i(k) > 0$. Thus $0 \notin \lambda(k)$. To conclude we will show that for any $b \in \mathbf{B}$ either $b \in \lambda(k)$ or $-b \in \lambda(k)$. From our assumption on the strategy of $\forall$ it follows that at some round i they will play move (α) with the element b. If $\exists$'s winning strategy implies that we accept the move then it should be clear that $b \geq \lambda_{i+1}(k)$, hence $b \in \lambda(k)$. Otherwise if we reject the move then $-b \geq \lambda_{i+1}(k)$ and thus $-b \in \lambda(k)$. □

THEOREM 5.2.5. *Let $\mathbf{B} = \langle B, +, -, 0, F^{\mathbf{B}}\rangle$ be a countable Boolean algebra with operators F. If $\exists$ have a winning strategy in the game $G_\omega(\mathbf{N}_\emptyset, \mathbf{B}, \mathcal{V})$ then $\mathbf{B}$ is representable over $\mathcal{V}$.*

PROOF. Let $\mathbf{N} = \langle N, F^{\mathbf{N}}\rangle$ be the algebra in Lemma 5.2.3, λ to be the function from Lemma 5.2.4 and rep to be defined by (5.5). Thus all we need to do to complete the proof is to show that rep is in fact an embedding.

First we show that rep is a Boolean homomorphism. I.e. we need to prove that

(i) $\mathrm{rep}(a+b) = \mathrm{rep}(a) \cup \mathrm{rep}(b)$,
(ii) $\mathrm{rep}(-b) = B \setminus \mathrm{rep}(b)$ and
(iii) $\mathrm{rep}(0) = \emptyset$.

We here only present a proof for (i), the other conditions follow by similar arguments using the fact that $\lambda(k)$ is an ultrafilter.

Assume $k \in \mathrm{rep}(a+b)$, for $k \in N$ and $a, b \in B$. Then $a + b \in \lambda(k)$. Since $\lambda(k)$ is an ultrafilter it follows that either $a \in \lambda(k)$ or $b \in \lambda(k)$. Hence $k \in \mathrm{rep}(a) \cup \mathrm{rep}(b)$.

For the converse suppose that $k \in \mathrm{rep}(a) \cup \mathrm{rep}(b)$. Thus either $k \in \mathrm{rep}(a)$ or $k \in \mathrm{rep}(b)$. Consider the case where $k \in \mathrm{rep}(a)$. Since $a + b \geq a$ and $\lambda(k)$ is a filter it follows that $a + b \in \lambda(k)$. A similar argument suffices if $k \in \mathrm{rep}(b)$. Hence $k \in \mathrm{rep}(a+b)$.

Let $f \in F$ with $ar(f) = n$. To show that rep is a BAO-homomorphism we need to prove that for any $b_0, \ldots, b_{n-1} \in B$

$$\mathrm{rep}(f^{\mathbf{B}}(b_0, \ldots, b_{n-1})) = f^{\mathrm{Cm}\mathbf{N}}(\mathrm{rep}(b_0), \ldots, \mathrm{rep}(b_{n-1})).$$

Assume $k \in \mathrm{rep}(f^{\mathbf{B}}(b_0, \ldots, b_{n-1}))$. By the definition of λ there exists some i such that $f^{\mathbf{B}}(b_0, \ldots, b_{n-1}) \geq \lambda_i(k)$. By our assumption on $\forall$'s strategy, it follows that at some round j they make a move of type (β) choosing k, f and $b_0, \ldots, b_{n-1}$. But $\exists$ will accept, since $\lambda_j(k) \leq -f^{\mathbf{B}}(b_0, \ldots, b_{n-1})$ would contradict the fact that we are playing according to our winning strategy. Thus we can choose $k_0, \ldots, k_{n-1} \in N$ such that $f^{\mathbf{N}_j}(k_0, \ldots, k_{n-1}) = k$ and $\lambda_{j+1}(k_i) \leq b_i$. Consequently $f^{\mathbf{N}}(k_0, \ldots, k_{n-1}) = k$ and $k_i \in \mathrm{rep}(b_i)$ for $i < n$. Hence $k \in f^{\mathrm{Cm}\mathbf{N}}(\mathrm{rep}(b_0), \ldots, \mathrm{rep}(b_{n-1}))$.

Conversely let $k \in f^{\mathrm{Cm}\mathbf{N}}(\mathrm{rep}(b_0), \ldots, \mathrm{rep}(b_{n-1}))$. It follows that at some round j there exist nodes $k_0, \ldots, k_{n-1} \in N_j$ such that $f^{\mathbf{N}_j}(k_0, \ldots, k_{n-1}) = k$, $\lambda_j(k_i) \leq b_i$, and either $\lambda_j(k) \leq f^{\mathbf{B}}(b_0, \ldots, b_{n-1})$ or $\lambda_j(k) \leq -f^{\mathbf{B}}(b_0, \ldots, b_{n-1})$. Note that since $f^{\mathbf{B}}$ is an operator it follows that

$$f^{\mathbf{B}}(\lambda_j(k_0), \ldots, \lambda_j(k_{n-1})) \leq f^{\mathbf{B}}(b_0, \ldots, b_{n-1}).$$

Consequently, using the coherence of $\mathbf{N}_j$ and the fact that $f^{\mathbf{N}_j}(k_0, \ldots, k_{n-1}) = k$,

$$\lambda_j(k) \cdot f^{\mathbf{B}}(b_0, \ldots, b_{n-1}) \neq 0.$$

Thus $\lambda_j(k) \not\leq -f^{\mathbf{B}}(b_0, \ldots, b_{n-1})$, and so $\lambda_j(k) \leq f^{\mathbf{B}}(b_0, \ldots, b_{n-1})$, whence we see that $k \in \mathrm{rep}(f^{\mathbf{B}}(b_0, \ldots, b_{n-1}))$.

We conclude by showing that rep is injective. Since rep is a BA-homomorphism it is sufficient to show that $\mathrm{rep}(b) \neq \emptyset$ for any $b \neq 0$. But at some round i of the game $\forall$ will play a move of type (δ) choosing the element b. $\exists$ can only accept and in doing so will choose a $k \in N$ such that $\lambda_{i+1}(k) \leq b$. Hence $b \in \lambda(k)$ and so $k \in \mathrm{rep}(b)$. □

This theorem and its proof were first presented in [**HMV99**] Proposition 4.3.

5.3. Axiomatisations via games

So far we have shown how logical games can be used to see whether an algebra is representable or not. Let us now take an informal look at how these games work.

In our exposition of the types of possible moves we have tried to convey how each ensures that certain properties are satisfied by the networks we use to play these games. Thus at each round i of a game $G_\omega(\mathbf{N}, \mathbf{B}, \mathcal{V})$ the associated network $\mathbf{N}_i$ will satisfy these properties up to a limited extent. But even the language we have used thus far is suggestive of some underlying formulas that are satisfied or "hold" for each of these networks. In fact it will turn out that each of these moves can be "translated" into universal formulas. The hope is then that the set of all such translated formulas will in fact axiomatise the class of representable algebras under consideration.

Obviously we will be considering games for which $\exists$ have a winning strategy. So before we consider translating any of the types of moves we need to capture the concept of a network being coherent. We start off by associating a term with each node in our network. To be a bit more precise, given a partial algebra $\mathbf{N} = \langle N, F^{\mathbf{N}} \rangle$ we associate a map τ, called a *term labelling*, with each member of N, i.e.

$$\tau : N \longrightarrow \mathrm{Term}_{\mathsf{BA}}(X). \tag{5.6}$$

For our network $\mathbf{N}$ to be coherent we require the underlying structure $\langle N, F^{\mathbf{N}} \rangle$ to be a partial $\mathcal{V}$- algebra. Thus we define the following term

$$\pi(\mathbf{N}) = \begin{cases} \top & \text{if } \langle N, F^{\mathbf{N}} \rangle \text{ is a partial } \mathcal{V}\text{-algebra} \\ \bot & \text{otherwise.} \end{cases} \tag{5.7}$$

Furthermore we require that, for each node k, the label $\lambda(k)$ should be non-zero and $\lambda(m) \cdot f(\lambda(k_0), \ldots, \lambda(k_{n-1})) \neq 0$ for $f^{\mathbf{N}}(\underline{k}) = m$, where $\underline{k} = (k_0, \ldots, k_{n-1})$. Thus if we can find some way of mapping term labels into network labels the following term would capture these requirements,

$$\chi(\mathbf{N}) = \bigwedge_{k \in N} \tau(k) \neq 0 \wedge \bigwedge_{f \in F} \bigwedge_{f(\underline{k}) = m} \Big(\tau(m) \cdot f\big(\tau(k_0), \ldots, \tau(k_{ar(f)-1})\big) \Big), \tag{5.8}$$

where $\underline{k} = (k_0 \ \ldots, k_{ar(f)-1})$.

Bringing these two terms together should determine whether a network is indeed coherent. Hence we let

$$\psi_0(\mathbf{N}) = \pi(\mathbf{N}) \wedge \chi(\mathbf{N}). \tag{5.9}$$

Our intention is now to recursively define terms ψ_i to express whether or not $\exists$ have successfully managed to survive round i. To do this we first need to translate each of the possible types of moves into terms.

Recall that for moves of type (α) $\exists$ have the option of either accepting or rejecting the move. (These options will be reflected as disjuncts in the term we will define.) In this type of move $\forall$ will choose some new element of $b \in \mathbf{B}$ after which $\exists$ will change the labelling to satisfy b or $-b$ such that we end up surviving this particular round. Consequently the state of the board after the ith round has been played by the execution of a type (α) move could be encoded as follows

$$\alpha_{i+1}(\mathbf{N}) = \forall x \bigwedge_{k\in N} \Big(\psi_i\big(\langle N, F^{\mathbf{N}}, \tau_{[k\hookrightarrow x]}\rangle\big) \vee \psi_i\big(\langle N, F^{\mathbf{N}}, \tau_{[k\hookrightarrow \neg x]}\rangle\big)\Big), \tag{5.10}$$

where x is a new variable.

For the second type of move $\forall$ chooses a node k, a symbol f with arity n, and elements $b_0, \ldots, b_{n-1} \in \mathbf{B}$. If $\exists$ reject this move then the new board will be nothing but a label change and hence expressed as

$$\psi_i\big(\langle N, F^{\mathbf{N}}, \tau_{[k\hookrightarrow \neg f(\underline{x})]}\rangle\big) \tag{$*$}$$

If $\exists$ accept the move then we must choose $m_0, \ldots, m_{n-1} \in N_i \cup Q$ such that if $f(\underline{m})$ is undefined then $f^{\mathbf{N}_{i+1}}(\underline{m}) = k$ and change the labelling so that b_i holds at m_i, for each $i < n$. Thus the term corresponding to acceptance should be

$$\psi_i\big(\langle N, H^{\mathbf{N}} \cup \{f^{\mathbf{N}}_{[\underline{m}\mapsto k]}\}, \tau_{[\underline{m}\hookrightarrow \underline{x}]}\rangle\big). \tag{$**$}$$

However we need the term to hold for any of the possible choices of the n-tuple $\underline{m}$. Hence we will have a disjunction over possible choices of $\underline{m}$. This is problematic though, since $N_i \cup Q$ is infinite and thus would lead to an infinite disjunction and hence an infinite term.

To take care of this problem we need to take a careful look at how we will make choices for $\underline{m}$. Note that each N_i is finite and that we only need to add at most $ar(f)$ possible nodes from Q for the game to proceed. If we can thus have some canonical way of adding these $ar(f)$ new nodes we would be left with a finite disjunction. In the discussion at the beginning of Section 5.2.2 we mentioned that Q need only have at most ω nodes. Consequently we can easily enumerate all the members of Q with order type ω.

Let Q_i be the set containing all members of Q chosen up to round i. Note that $Q_i \subseteq N_i$. If we now need n new nodes we can use the enumeration mentioned above to choose the first n available nodes, i.e. the first n nodes in $Q \setminus Q_i$. Thus we define the set M_i^n which contains all current nodes and the n canonically chosen new nodes in the following way

$$M_i^n = N_i \cup \{q_0, \ldots, q_{n-1}\}, \tag{5.11}$$

where the q_i are the first n nodes in $Q \setminus Q_i$. By construction M_i^n is finite.

We are now ready to translate the type (β) moves using $(*)$ and $(**)$. We define the "translation" term in the following way

$$\beta_{i+1}(\mathbf{N}) = \bigwedge_{k\in N} \bigwedge_{f\in F} \forall \underline{x}\Big(\psi_i\big(\langle N, F^{\mathbf{N}}, \tau_{[k\hookrightarrow \neg f(\underline{x})]}\rangle\big) \vee \bigvee_{\underline{m}\in M_i^{ar(f)}} \psi_i\big(\langle N \cup \underline{m}, H^{\mathbf{N}} \cup \{f^{\mathbf{N}}_{[\underline{m}\mapsto k]}\}, \tau_{[\underline{m}\hookrightarrow \underline{x}]}\rangle\big)\Big), \tag{5.12}$$

where $\underline{x}$ is a $ar(f)$-tuple of new variables.

For moves of type (γ) $\forall$ choose a $f \in F$ and $ar(f)$-tuple $\underline{k}$ from $\mathbf{N}$. Then $\exists$ must add one node $m \in M_i^1$ such that $f(\underline{k}) = m$ and extend the labelling so that at least $1^{\mathbf{B}}$ holds at m. This option of choices of m is represented as the disjunct below.

But $\exists$ must be able to respond no matter which f and $\underline{k}$ are chosen, hence the two conjunctions. In fact this is not completely correct since if $f^{\mathbf{N}}(\underline{k})$ is already defined, $\exists$ could obviously just choose the previous value of $f^{\mathbf{N}}(\underline{k})$. Thus we should exclude this possibility from the term.

$$\gamma_{i+1}(\mathbf{N}) = \bigwedge_{f \in F} \bigwedge_{\substack{\underline{k} \in N^{ar(f)} \\ f^{\mathbf{N}}(\underline{k})\uparrow}} \bigvee_{m \in M_i^1} \psi_i\Big(\big\langle N \cup \{m\}, H^{\mathbf{N}} \cup \{f^{\mathbf{N}}_{[\underline{k} \mapsto m]}\}, \tau_{[m \hookrightarrow \underline{1}]}\big\rangle\Big), \tag{5.13}$$

where $f^{\mathbf{N}}(\underline{k}) \uparrow$ is defined to be true if $f^{\mathbf{N}}(\underline{k})$ is undefined, false otherwise.

Lastly, in moves of type (δ), $\exists$ must again be able to extend $\mathbf{N}$ by choosing a $m \in M_i^1$ such that the new network is coherent for any non-zero $b \in \mathbf{B}$ chosen by $\forall$, where m gets relabeled so that b holds at m.

$$\delta_{i+1}(\mathbf{N}) = \forall x\Big(x \neq 0 \rightarrow \bigvee_{m \in M_i^1} \psi_i\big(\langle N \cup \{m\}, F^{\mathbf{N}}, \tau_{[m \hookrightarrow x]}\rangle\big)\Big). \tag{5.14}$$

Finally we have the tools to complete the definition of the ψ_i, we let

$$\psi_{i+1}(\mathbf{N}) = \alpha_{i+1}(\mathbf{N}) \wedge \beta_{i+1}(\mathbf{N}) \wedge \gamma_{i+1}(\mathbf{N}) \wedge \delta_{i+1}(\mathbf{N}). \tag{5.15}$$

We still require some way of mapping between term labels and (network) labels. To achieve this we first define a different kind of network, unsurprisingly called a term network.

DEFINITION 5.3.1. A *term network* is a structure $\mathbf{N} = \langle N, F^{\mathbf{N}}, \tau\rangle$ such that $\langle N, F^{\mathbf{N}}\rangle$ is a finite partial algebra and τ is a term labelling.

Given a term network $\mathbf{N}$, we let $\mathrm{var}(\mathbf{N})$ denote the set of variables occurring in the term labels of $\mathbf{N}$, i.e. if $\tau : N \longrightarrow \mathrm{Term}_{\mathsf{BA}}(X)$ then $\mathrm{var}(\mathbf{N}) = X$, for the minimal possible X.

Now a (network) labelling should map from N to $\mathbf{B}$, each term $\tau(k)$ has an interpretation in $\mathbf{B}$, defined inductively using the construction of τ. Thus if we have some assignment of the variables of a term network $\mathbf{N}$ to elements of $\mathbf{B}$ we can easily associate a network with $\mathbf{N}$.

DEFINITION 5.3.2. Let $\mathbf{N} = \langle N, F^{\mathbf{N}}, \tau\rangle$ be a term network and $\mathbf{B}$ an F-algebra. Then for any assignment $h : \mathrm{var}(\mathbf{N}) \longrightarrow \mathbf{B}$ we define a network $\mathbf{N}^h = \langle N, F^{\mathbf{N}}, \tau^h\rangle$, where

$$\tau^h(k) = \tau^{\mathbf{B}}(k)(h(x_0), \ldots, h(x_{n-1}))$$

for $k \in N$, and with $x_0, \ldots, x_{n-1}$ the free variables occurring in $\tau(k)$.

In deference to our standard notation we will write $\mathbf{N}^{\mathbf{B}} = \langle N, F^{\mathbf{N}}, \tau^{\mathbf{B}}\rangle$ when the assignment is clear from the context, instead of explicitly indicating the assignment by writing $\mathbf{N}^h$

Previously we saw that $\exists$ having a winning strategy for the game $G_\omega(\mathbf{N}_\emptyset, \mathbf{B}, \mathcal{V})$ is equivalent to $\mathbf{B}$ being representable. From our current discussion we have seen that each round of the game produces a particular term related to $\mathbf{B}$. Put in another way, if $\mathbf{B} \models \psi_\eta(\mathbf{N})$ then we would expect $\exists$ to have a winning strategy for $G_\eta(\mathbf{N}^{\mathbf{B}}, F^{\mathbf{N}}, \lambda^{\mathbf{B}})$. Now if being able to win all games of finite length implied that $\exists$ could win the game of length ω we might very well be able to get an axiomatisation for $\mathbf{B}$ using the ψ_i. (The following theorem and proof were first presented in [**HMV99**] Theorem 5.1. Note that the result in [**HMV99**] did not explicitly mention the finiteness restriction on the network $\mathbf{N}$.)

THEOREM 5.3.3. *Let* $\mathbf{B}$ *be a countable* BAO *and* $\mathbf{N}$ *a finite network over* $\mathbf{B}$. *Then* $\exists$ *have a winning strategy for the game* $G_\omega(\mathbf{N}, \mathbf{B}, \mathcal{V})$ *if, and only if,* $\exists$ *have a winning strategy for every game* $G_\eta(\mathbf{N}, \mathbf{B}, \mathcal{V})$ *of length* $\eta < \omega$.

PROOF. The forward direction of the theorem follows directly from the definition of $G_\omega(\mathbf{N}, \mathbf{B}, \mathcal{V})$. For the converse assume that, for any $\eta \in \omega$, $\exists$ have a winning strategy for $G_\eta(\mathbf{N}, \mathbf{B}, \mathcal{V})$. We need to show that $\exists$ have a winning strategy for $G_\omega(\mathbf{N}, \mathbf{B}, \mathcal{V})$.

A network $\mathbf{M}$ is said to be *safe* for $\exists$ if for infinitely many j we have a winning strategy for $G_j(\mathbf{M}, \mathbf{B}, \mathcal{V})$.

Since we can win all finite games starting with $\mathbf{N}$, $\mathbf{N}$ is obviously safe for $\exists$. We will prove that $\exists$ can win the game $G_\omega(\mathbf{N}, \mathbf{B}, \mathcal{V})$ by maintaining the condition that at any round i the current network $\mathbf{N}_i$ is safe. We do this by induction on i. I.e. we show that, at round i, $\exists$ can survive one round while maintaining the condition that $\mathbf{N}_i$ is safe.

Suppose we are currently at round i of the game $G_\omega(\mathbf{N}, \mathbf{B}, \mathcal{V})$ and let $\mathbf{N}_i$ be the current board, where $\mathbf{N}_i$ is safe for $\exists$. From the discussion surrounding the definition of M_i^n (p. 10[illegible]) it should be clear that after $\forall$ make their ith move $\exists$ only have a finite number of possible networks to respond with. But, by assumption, there are infinitely many j for which $\exists$ have a winning strategy in the game $G_j(\mathbf{N}_i, \mathbf{B}, \mathcal{V})$. It follows that for at least one of the networks $\mathbf{N}'$ that $\exists$ can respond with $\exists$ have a winning strategy for the games $G_{j-1}(\mathbf{N}_i, \mathbf{B}, \mathcal{V})$, unless $j = 0$ (in which case we disregard this particular j). Since there were infinitely many j to start off with we will have infinitely many games of length $j - 1$ for which we have a winning strategy starting from the network $\mathbf{N}'$. Consequently such an $\mathbf{N}'$ will again be safe for $\exists$. It is not hard to see that if we choose $\mathbf{N}'$ to be our response in the ith round of the game $G_\omega(\mathbf{N}, \mathbf{B}, \mathcal{V})$ we will maintain our condition of safety. $\square$

We can now define the formulas that we expect will axiomatise a particular representable algebra $\mathbf{B}$. Recall that when we were characterising the representation games we always started with the network $\mathbf{N}_\emptyset$. Thus we let

$$\varphi_i = \psi_i(\mathbf{N}_\emptyset) \tag{5.16}$$

and

$$\Phi(\mathcal{V}) = \{\varphi_i : i \in \omega\}. \tag{5.17}$$

LEMMA 5.3.4. *Let* $\mathbf{B}$ *be a* BAO *and* $\mathbf{N} = \langle N, F^{\mathbf{N}}, \tau \rangle$ *be a finite term network . Then* $\mathbf{B} \models \psi_\eta(\mathbf{N})$ *if, and only if,* $\exists$ *have a winning strategy in the game* $G_\eta(\mathbf{N}^{\mathbf{B}}, \mathbf{B}, \mathcal{V})$.

PROOF. We prove this lemma by induction on η.

Claim 1: $\mathsf{B} \models \psi_0(\mathbf{N})$ if, and only if $\mathbf{N}^{\mathbf{B}}$ is a coherent $\mathcal{V}$-network.

Observe that the validity of $\pi(\mathbf{N})$ corresponds directly to $\langle N, F^{\mathbf{N}} \rangle$ being a partial $\mathcal{V}$-algebra, while $\chi(\mathbf{N})$ takes care of the coherence condition (5.1).

Claim 2: Assume the lemma holds for η. If $\mathbf{B} \models \psi_{\eta+1}(\mathbf{N})$ then $\exists$ have a winning strategy in the game $G_{\eta+1}(\mathbf{N}^{\mathbf{B}}, \mathbf{B}, \mathcal{V})$.

Assume that $\mathbf{B} \models \psi_{\eta+1}(\mathbf{N})$. We need to construct a winning strategy for $\exists$. Consider a particular match of length $\eta+1$ in the game $G_{\eta+1}(\mathbf{N}^{\mathbf{B}}, \mathbf{B}, \mathcal{V})$. Let

$$\mu_1, \ldots, \mu_{\eta+1} \text{ and } \mathbf{N}_0, \ldots, \mathbf{N}_\eta$$

respectively be enumerations of the moves $\forall$ makes during this match and the networks $\mathbf{N}_i$ arising from these moves, where $\mathbf{N}_0 = \mathbf{N}^{\mathbf{B}}$.

We here only contemplate the case where the move $\mu_{\eta+1}$ is of type (β) with $\forall$ choosing the node $k \in \mathbf{N}_\eta$, $f \in F$ and $b_0, \ldots, b_{n-1} \in \mathbf{B}$ $(ar(f) = n)$. The proofs for moves of types (α), (γ) and (δ) can be dealt with in a similar fashion using the explanations surrounding the definitions (5.10), (5.13) and (5.14).

By assumption $\mathbf{B} \models \beta_{\eta+1}(\mathbf{N})$ and hence, for some $\underline{b} = \langle b_0, \ldots, b_{n-1}\rangle$ either

$$(*) \qquad \mathbf{B} \models \psi_i\big(\langle N, F^{\mathbf{N}}, \tau_{[k \hookrightarrow f(\underline{b})]}\rangle\big)$$

or, for $H = F \setminus \{f\}$,

$$(**) \qquad \mathbf{B} \models \bigvee_{\underline{m} \in M_\eta^{ar(f)}} \psi_i\big(\langle N \cup \underline{m}, H^{\mathbf{N}} \cup \{f^{\mathbf{N}}_{[\underline{m} \mapsto k]}\}, \tau_{[\underline{m} \hookrightarrow \underline{b}]}\rangle\big).$$

Assuming that $(**)$ holds, there exist some $\underline{m} \in M_\eta^{ar(f)}$ such that

$$\mathbf{B} \models \psi_i\big(\langle N \cup \underline{m}, H^{\mathbf{N}} \cup \{f^{\mathbf{N}}_{[\underline{m} \mapsto k]}\}, \tau_{[\underline{m} \hookrightarrow \underline{b}]}\rangle\big).$$

By the inductive hypothesis it follows that $\exists$ can win any match of length η in the game $G_\eta(\mathbf{N}', \mathbf{B}, \mathcal{V})$, where $\mathbf{N}' = \langle N \cup \underline{m}, H^{\mathbf{N}'} \cup \{f^{\mathbf{N}'}_{[\underline{m} \mapsto k]}\}, \tau^{\mathbf{B}}_{[\underline{m} \hookrightarrow \underline{b}]}\rangle$. Let $\tau' = \tau_{[\underline{m} \hookrightarrow \underline{b}]}$. Since $N \subseteq N \cup \underline{m}$, $f^{\mathbf{N}}(\underline{x}) = f^{\mathbf{N}'}(\underline{x})$ for all $\underline{x} \neq \underline{m}$ and $\mathbf{B} \models \tau' \leq \tau$ it follows that $\exists$ can counter any move μ_i for $i < \eta + 1$ by using our winning strategy for the game $G_\eta(\mathbf{N}', \mathbf{B}, \mathcal{V})$. Clearly $\exists$ will accept and then survive round $\eta+1$ by choosing $\underline{m}$ and defining the new board $\mathbf{N}_{\eta+1}$ as required by acceptance.

The argument for the case where $(*)$ holds is a simplified version of the argument above and leads to $\exists$ rejecting $\forall$'s move.

Claim 3: Assume the lemma holds for η. If $\exists$ have a winning strategy in the game $G_{\eta+1}(\mathbf{N}^{\mathbf{B}}, \mathbf{B}, \mathcal{V})$, then $\mathbf{B} \models \psi_{\eta+1}(\mathbf{N})$.

Suppose $\exists$ have a winning strategy for $G_{\eta+1}(\mathbf{N}^{\mathbf{B}}, \mathbf{B}, \mathcal{V})$. We will demonstrate the proof for $\delta_{\eta+1}(\mathbf{N})$.

Let b be some non-zero element of $\mathbf{B}$ we will show that there exists a node m such that $\mathbf{B} \models \psi_\eta(\langle N \cup \{m\}, F^{\mathbf{N}}, \tau_{[m \hookrightarrow b]}\rangle)$. Since $\exists$ have a winning strategy we know that if $\forall$ plays a move of type (δ) at round $i < \eta$, specifying the element b, then $\exists$ can respond with a node m such that the new board is still coherent. Clearly $\exists$ then have a winning strategy for the game $G_\eta(\mathbf{N}', \mathbf{B}, \mathcal{V})$, where $\mathbf{N}' = \langle N \cup \{m\}, F^{\mathbf{N}}, \tau^{\mathbf{B}}_{[m \hookrightarrow b]}\rangle$. Using the inductive hypothesis it follows that $\mathbf{B} \models \psi_\eta(\langle N \cup \{m\}, F^{\mathbf{N}}, \tau_{[m \hookrightarrow b]}\rangle)$.

Similarly we use moves of type (α) to prove that $\mathbf{B} \models \alpha_{\eta+1}(\mathbf{N})$, type (β) for $\mathbf{B} \models \beta_{\eta+1}(\mathbf{N})$ and type (γ) for $\mathbf{B} \models \gamma_{\eta+1}$. Thus $\mathbf{B} \models \psi_{\eta+1}(\mathbf{N})$. □

COROLLARY 5.3.5. *Let* $\mathbf{B}$ *be a* BAO. *Then* $\mathbf{B} \models \Phi(\mathcal{V})$ *if, and only if,* $\exists$ *have a winning strategy in the game* $G_\eta(\mathbf{N}_\emptyset, \mathbf{B}, \mathcal{V})$, *for every* $\eta < \omega$.

PROOF. This follows directly from the definitions of φ_η, $\Phi(\mathcal{V})$, Theorem 5.3.3 and Lemma 5.3.4. □

Using the earlier characterisation theorem for representable algebras and the corollary above we can now show that $\Phi(\mathcal{V})$ gives our required characterisation. (This result was first present in [**HMV99**] Theorem 5.5.)

THEOREM 5.3 6. *Let* $\mathbf{B}$ *be a* BAO *with operators* F *and* $\mathcal{V}$ *a variety of* F*-algebras. Then the following are equivalent:*

(i) $\mathbf{B}$ *is representable over* $\mathcal{V}$,
(ii) $\exists$ *have a winning strategy for the game* $G_\omega(\mathbf{N}_\emptyset, \mathbf{B}, \mathcal{V})$,
(iii) $\exists$ *have a winning strategy for the games* $G_\eta(\mathbf{N}_\emptyset, \mathbf{B}, \mathcal{V})$, *where* $\eta < \omega$, *and*
(iv) $\mathbf{B} \models \Phi(\mathcal{V})$

PROOF. By Theorem 5.2.1 (i) implies (ii), by Theorem 5.3.3 (ii) implies (iii) and (iii) implies (iv) follows directly from Corollary 5.3.5. Clearly the backward implications would hold if $\mathbf{B}$ were a countable BAO.

Hence we consider the case where $|\mathbf{B}| > \omega$. We wish to prove that (iv) implies (i) for such a $\mathbf{B}$. Suppose $\mathbf{B} \models \Phi(\mathcal{V})$. Since the φ_η are universal formulas it follows that each subalgebra of $\mathbf{B}$ satisfies $\Phi(\mathcal{V})$. By our previous observation this implies that every countable subalgebra of $\mathbf{B}$ is representable over $\mathcal{V}$, i.e. for every $\mathbf{A} \leq \mathbf{B}$ with $|\mathbf{A}| \leq \omega$ we have $\mathbf{A} \in \mathbf{SCm}\mathcal{V}$. Since $\mathcal{V}$ is an elementary class it follows, by Theorem 4.3.5 (p. 82), that $\mathbf{SCm}\mathcal{V}$ is an elementary class. Consequently using the Downward Löwenheim-Skolem Theorem (p. 20) there exists a countable algebra $\mathbf{A}$ such that $\mathbf{A} \preceq \mathbf{B}$. But by our previous observation $\mathbf{A} \models \Phi(\mathcal{V})$ and hence $\mathbf{A} \in \mathrm{SCm}\mathcal{V}$ and since $\mathbf{A} \preceq \mathbf{B}$ we get $\mathbf{B} \in \mathrm{SCm}\mathcal{V}$ as required by (i). □

There is one underlying assumption to constructing a recursive axiomatisation that needs some elaboration. In defining $\pi(\mathbf{N})$ above we are assuming that there is some effective way of deciding whether $\mathbf{N}$ is in fact a partial $\mathcal{V}$- algebra. Clearly if $\mathcal{V}$ itself only had a finite set of axioms we could easily define an effective decision procedure. For most classes of algebras that will play the role of $\mathcal{V}$ they in fact have finite axiomatisations. However there do exist varieties of BAOs that only have countable axiomatisations.

To deal with such axiomatisations we need a slightly modified game. We firstly enumerate all the axioms of the variety $\mathcal{V}$. At each round i of this modified game we only consider whether our current board is a partial $\mathcal{V}$- algebra to the extent that it satisfies the first i axioms of $\mathcal{V}$. Since our game has a countable number of rounds this would ensure that the final playing board $\mathbf{N}$ were in fact in $\mathcal{V}$. For a formal definition of this game we refer the reader to [**HMV99**].

5.4. Extending the game

In this the penultimate section of this chapter we informally discuss two modifications to the game already presented.

Games for $\mathbf{SPCm}\mathcal{K}$. As we have seen in the previous chapter any complex variety $\mathcal{V}$ can in fact be characterised by a class $\mathcal{K}$ in two ways, either $\mathcal{V} = \mathbf{SPCm}\mathcal{K}$ or otherwise $\mathcal{V} = \mathbf{SCm}\mathcal{K}$. Let $\mathcal{K}$ be a class of algebras, to show that an algebra $\mathbf{B}$ is a member of $\mathbf{SPCm}\mathcal{K}$ we need to show that there is an embedding of $\mathbf{B}$ into a product of members of $\mathbf{Cm}\mathcal{K}$. The smallest element of a product is always the empty product and hence it suffices to show that at least one element of $\mathbf{B}$ gets mapped to a member of $\mathbf{Cm}\mathcal{K}$.

If we take a look back at the proof of Theorem 5.2.1 we will note that moves of type (δ) were required throughout the game to ensure injectivity. However in the case at hand it will suffice for this kind of move to happen only once. To play the game over $\mathbf{SPCm}\mathcal{K}$ we thus modify the rules of the game such that $\forall$ is only allowed to make a move of type (δ) in the first round of the game. (This modification of the game is by no means new and receives a cursory mention in [**HMV99**].)

Games for partial algebras. Another modification to the game arises from representing BAOs over partial algebras. As we have already noted in an earlier discussion, moves of type (γ) ensure that each of the functions of our representation is total. As the following result shows the complex algebra of an partial algebra is in fact a BAO.

PROPOSITION 5.4.1. *Assume* $\mathbf{A}$ *is a partial* F*-algebra. Then* $\mathrm{Cm}\mathbf{A}$ *is a boolean algebras with operators* F.

PROOF. Let $f \in F$, with $ar(f) = n$ and let $X_0, \ldots, X_{n-1} \subseteq \mathbf{A}$. If for some X_j $f(x_0, \ldots, x_{n-1})$ is undefined for all $x_j \in X_j$, then $f^{\mathrm{Cm}\mathbf{A}}(X_0, \ldots, X_{n-1}) = \emptyset \in \mathrm{Cm}\mathbf{A}$. Consequently $f^{\mathrm{Cm}\mathbf{A}}$ is a total function. Clearly $f^{\mathrm{Cm}\mathbf{A}}$ will be both normal and additive. □

Thus it would not be too surprising it we can modify the games presented in this chapter to include representations over partial algebras. All we need is to find the appropriate partial algebraic version of equational classes of (total) algebras.

DEFINITION 5.4.2. Let $\mathbf{A}$ be a partial F-algebra. We write $\mathbf{A} \models \sigma = \tau$ if $\mathbf{A}$ satisfies the equation $\tau = \sigma$ (in the sense of Definition 5.1.3). For a set of equations Σ we write $\mathbf{A} \models \Sigma$ if, and only if, $\mathbf{A} \models \sigma$ for every $\sigma \in \Sigma$.

Now we are ready to define the partial algebraic equivalent of an equational class.

DEFINITION 5.4.3. Let $\mathcal{S}$ be a class of partial algebras. We call $\mathcal{S}$ an *equational class of partial algebras* if there exists a set of equations Σ such that

$$\mathbf{A} \in \mathcal{S} \text{ iff } \mathbf{A} \models \Sigma.$$

$\mathbf{A}$ is called a partial $\mathcal{S}$-algebra if $\mathbf{A} \in \mathcal{S}$.

We then give a modified definition of the networks presented at the beginning of this chapter.

DEFINITION 5.4.4. Assume $\mathcal{S}$ is an equational class of partial algebras. A network $\mathbf{N} = \langle N, F^{\mathbf{N}}, \lambda \rangle$ is an $\mathcal{S}$-network if $\langle N, F^{\mathbf{N}} \rangle$ is a partial $\mathcal{S}$-algebra.

To represent a boolean algebra with operators over an equational class of partial algebras we play the game over these $\mathcal{S}$-networks and do not allow $\forall$ to make any moves of type γ.

5.5. Further research

To conclude we present two avenues along which further research in this field can be done.

(1) The two modifications to the original game above present tools to study several new classes of representations. However these modifications are not applicable

to classes of partial algebras such as Brandt groupoids and full pair arrow frames since the definition of equality is too weak. However these two classes are the canonical examples used in representing relation algebras. The question thus is how do we modify this game to deal with stronger versions of equality for partial algebras.

(2) In [**HiH99**] R. Hirsh and I. Hodkinson present a game to cope with axiomatisations of representable classes over pseudo-elementary classes. This is a much broader setting than ours (which only considers representing classes over varieties and equational classes of partial algebras). Thus it would be interesting to see how the games described in this chapter can be modified to include a wider range of underlying classes.

CHAPTER 6

Case Studies

We conclude this dissertation by presenting some case studies and avenues for further research that relate to the theory introduced in the previous chapters. We limit the examples we look at to certain subvarieties and expansions of the variety of groups One well-known example of such a subvariety is the class of Brandt groupoids used in [**JoT52**] to give a concrete representation of the class of representable relation algebras. Another early example is that of the complex algebras of groups which generate the variety GRA of group relation algebras (c.f. [**Mon64**]).

6.1. Rectangular bands

The class of semigroups has also provided fertile ground in the study of complex varieties. Two such examples are the varieties Lz and Rz of left-zero and right-zero semigroups, defined by $x \cdot y = x$ and $x \cdot y = y$ respectively. C. Bergman (c.f. [**Ber**]) showed that the complex varieties generated by Lz and Rz are finitely based (c.f. Definition 2.5.10). An equational basis for $\mathrm{V}(\mathbf{Cm}\mathsf{Lz})$ is given by the standard equations for the boolean part, along with

$$x \cdot (y \vee z) = (x \cdot y) \vee (x \cdot z), \tag{6.1}$$

$$(x \vee y) \cdot z = (x \cdot z) \vee (y \cdot z), \tag{6.2}$$

$$\mathbb{C} \cdot x = \mathbb{0} = x \cdot \mathbb{0}, \tag{6.3}$$

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z, \text{ and} \tag{6.4}$$

$$x \cdot \mathbb{1} = x = x \cdot x. \tag{6.5}$$

Equations (6 1) through (6.4) along with the boolean part define the variety BSg of Boolean semigroups. Thus $\mathrm{V}(\mathbf{Cm}\mathsf{Lz})$ and $\mathrm{V}(\mathbf{Cm}\mathsf{Rz})$ are subvarieties of BSg.

6.1.1. Hyper-rectangular bands.

based (this follows from a result of Andreka and Nemeti (c.f. [**AnN96**]) that the composition of reducts of representable relation algebras generates a non-finitely based variety).

In this case study we will consider the variety of rectangular bands. This variety covers Lz and Rz in the lattice of semigroup varieties.

DEFINITION 6.1.1. RB, the variety of *rectangular bands* is the class of semigroups defined by the identity $x \cdot y \cdot x = x$. By the variety of *hyper-rectangular bands*, denoted HRB, we mean the class $\mathrm{V}(\mathbf{Cm}\mathsf{RB})$.

As we have mentioned the subvarieties $\mathsf{V}(\mathbf{Cm}\mathsf{Lz})$ and $\mathsf{V}(\mathbf{Cm}\mathsf{Rz})$ of BSg are finitely based. Which motivates us to ask whether HRB is finitely based.

For the rest of this section we will leave out the operation symbol '$\cdot$' where confusion is unlikely, thus abbreviating $x \cdot y$ to xy. We also assume that '$\cdot$' has precedence thus reducing the number of brackets we use.

6.1.2. Axiomatising rectangular bands.

LEMMA 6.1.2. *Let* $\mathbf{B}$ *be a rectangular band. For all* $X, Y, Z \subseteq \mathbf{B}$ *we have*

(i) $X \cap YZ \neq \emptyset$ *implies* $XY \cap Y \neq \emptyset$ *and* $ZX \cap Z \neq \emptyset$ *and*
(ii) $X \cap XY \cap YX \neq \emptyset$ *implies* $X \cap YY \neq \emptyset$.

PROOF. To see why (i) holds note that in a rectangular band if $x = yz$ then $xy = yzy = y$ and $zx = zyz = z$.

For (ii) suppose $x_0, x_1, x_2 \in X$ and $y_1, y_2 \in Y$ such that $x_0 = x_1y_1 = y_2x_2$. Then $y_1x_0 = y_1x_1y_1 = y_1$ and $x_0y_2 = y_2x_2y_2 = y_2$, so $y_2y_1 = x_0y_2y_1x_0 = x_0$. □

Note that the term $d(x) = \mathbb{1}x\mathbb{1}$ is a unary discriminator for complex algebras of rectangular bands (i.e. $d(\mathbb{0}) = \mathbb{C}$ and $d(x) = \mathbb{1}$ if $x \neq \mathbb{C}$). Thus $\mathbf{Cm}\mathsf{RB}$ forms a class of discriminator algebras. It follows by (4.17) p. 87 that any inequality that holds in HRB can be translated into an equation that holds in HRB.

THEOREM 6.1.3. *An equational basis for* HRB *is given by all the identities for* BSg *together with the following identities:*

(i) $x \leq xx$,
(ii) $x\mathbb{1}y = xy$,
(iii) $\mathbb{1}x\mathbb{1} \wedge y \leq \mathbb{1}(x \wedge \mathbb{1}y\mathbb{1})\mathbb{1}$,
(iv) $\mathbb{1}(x \wedge yz)\mathbb{1} \leq \mathbb{1}(xy \wedge y)\mathbb{1} \wedge \mathbb{1}(zx \wedge z)\mathbb{1}$, *and*
(v) $\mathbb{1}(x \wedge xy \wedge yx)\mathbb{1} \leq \mathbb{1}(x \wedge yy)\mathbb{1}$.

PROOF. Considering that in all rectangular bands we have that $x = xxxx = xx$ and $xzy = xzxyzy = xy$, it easily follows from Lemma 4.3.27 that (i) and (ii) hold in $\mathbf{Cm}\mathsf{RB}$. Since $d(x)$, defined above, is a discriminator term it follows that (iii) must hold. Finally, (iv) and (v) are just (in)equational translations of Lemma 6.1.2.

Let $\mathbf{B}$ be any boolean semigroup that satisfies (i) to (v). Then for any $x, y \in \mathbf{B}$, using (6.1) and (6.2), it is easy to show that $xy \leq x\mathbb{1}$ and $xy \leq \mathbb{1}y$. Hence the unary term $d(x)$, defined by $d(x) = \mathbb{1}x\mathbb{1}$, satisfies the following properties:

$$\mathbb{1}d(x) \leq d(x) \tag{1}$$

$$d(x)\mathbb{1} \leq d(x) \tag{2}$$

$$x \leq d(x) \text{ from (i)} \tag{3}$$

$$d(\mathbb{C}) = \mathbb{0} \text{ from (6.3)} \tag{4}$$

From property (iii) it follows that $x \wedge d(y) = \mathbb{C}$ if, and only if, $d(x) \wedge y = \mathbb{C}$. Consequently $d({\sim}d(x)) \wedge d(x) = \mathbb{C}$ and so

$$\mathbb{1}({\sim}d(x)) \leq {\sim}d(x). \tag{5}$$

Let $\mathbf{A}$ be a subdirectly irreducible Boolean semigroup that satisfies (i) to (v). Then from properties (1) through (5) it follows, by Theorem 4.3.22 (p. 87), that d is a unary discriminator for $\mathbf{A}$. This property and the positive identities (i) to (v) are preserved by canonical extension (c.f. Theorem 4.3.33 and 4.3.34 p. 89), so d is a unary discriminator for Em**A**.

We will now show that Em**A** $\in$ **SCm**RB by constructing a rectangular band **B** such that Em**A** is embedded in Cm**B**. Let $\mathbf{D} = \langle D, \cdot^{\mathbf{D}} \rangle$ be the structure At(Em**A**) and for $a, b \in \mathbf{D}$, define $a \cong b$ if $a \leq bb$. Observe that, since '$\cdot$' is an operator, if $a, b \in D$ then $ab > \mathbb{0}$.

Claim 1: $\cong$ is an equivalence relation.

From (i) we get $a \cong a$. Let $a \cong b$, then $a \wedge bb \neq \mathbb{0}$, hence using (iv) $ab \wedge b \neq \mathbb{0}$. Since $a, b \in D$ it follows that $b \wedge ba \wedge ab \neq \mathbb{0}$. Thus, by (v), $b \wedge aa \neq \mathbb{0}$, and so $b \cong a$. Lastly, let $a \leq bb$ and $b \leq cc$ then $a \leq cccc$. Hence, by (ii) and since c is an atom, it follows that $a \leq c\mathbb{1}cc\mathbb{1}c = c\mathbb{1}c = cc$.

Note that $aa = \bigvee\{b \in D : b \leq aa\} = \bigvee(a/\cong)$, for each atom a.

Claim 2: The set $\{aa : a \in D\}$ is the universe of a rectangular band $\mathbf{C} = \langle C, \cdot \rangle$.

First we need to show that C is closed under multiplication. For $a, b \in D$, we have $ab \leq aabb \leq a\mathbb{1}b = ab$. Hence it suffices to show that $ab \in C$, i.e. $ab = cc$ for some atom c.

In fact, for any atom $d \leq ab$ we have $dd \leq abab \leq a\mathbb{1}b = ab$. For the reverse inequality we use (iv) and the fact that both a and b are atoms. Since $d \wedge ab \neq \mathbb{0}$, (iv) implies $a \wedge da \neq \mathbb{0}$, whence $a \leq da$. Similarly, $d \leq ab$ implies $b \leq bd$ by (iv). Therefore $ab \leq dabd \leq d\mathbb{1}d = dd$. (Observe that if $ab = c \in C$ and $c \geq d$ this argument shows that $c = dd$.)

To see that **C** is a rectangular band observe that $aabbaa = a\mathbb{1}abba\mathbb{1}a = a\mathbb{1}a = aa$, since $abba \neq \mathbb{0}$ for $a, b \in D$.

We define two sets L and R by $L = R = C$. It is not hard to see that the partial function $g : D \longrightarrow L \times R$, defined by $g(aa) = \langle aa, aa \rangle$ for $a \in D$, can be extended to an isomorphism $g' : \mathbf{C} \longrightarrow \langle L \times R, \circ \rangle$, where the operation '$\circ$' is given by $\langle u, v \rangle \circ \langle w, x \rangle = \langle u, x \rangle$.

Let $\mathbf{B} = \langle (L \times D) \times (R \times D), \circ \rangle$. Since $|(c] \cap D|^2 \leq |D \times D|$, we can choose, for each $c \in C$, a map $f_c : (c] \cap D \longrightarrow \mathcal{P}(D \times D)$ such that

(6) $a \neq b$ implies $f_c(a) \cap f_c(b) = \emptyset$ (for all $a, b \in D$),
(7) $\{u : \langle u, v \rangle \in f_c(a)\} = D = \{v : \langle u, v \rangle \in f_c(a)\}$ (for all $a \in D$), and
(8) $\bigcup\{f_c(a) : c \geq a \in D\} = D \times D$.

Define the map $h : \mathbf{D} \longrightarrow$ Cm**B** by

$$h(a) = \{ ((\pi_L g'(aa), u), (\pi_R g'(aa), v)) : (u, v) \in f_{aa}(a) \}$$

and extend it to a map $\hat{h} :$ Em**A** $\longrightarrow$ Cm**B** by distributivity over $\vee$. Then, by (6), $\hat{h}$ is a Boolean embedding. It remains to show that $\hat{h}(ab) = h(a) \circ h(b)$.

Consider any $a, b \in D$ then $ab \leq aabb \leq a\mathbb{1}b \leq ab$, whence $ab \in C$. Hence we let $ab = c \in C$. Now

$$\begin{aligned}
\hat{h}(c) &= \bigcup\{h(d) : c \geq d \in D\} \\
&= \bigcup\{\{((\pi_L g'(c), u), (\pi_R g'(c), v)) : (u, v) \in f_c(d)\} : c \geq d \in D\} \\
&= \{((\pi_L g'(c), u), (\pi_R g'(c), v)) : (u, v) \in D \times D\}
\end{aligned}$$

where we used the observation that $c = ab = dd$ in the second step, and property (8) in the third step. On the other hand,

$$\begin{aligned}h(a)\circ h(b) &= \big\{\big((\pi_L\, g'(aa), u), (\pi_R\, g'(aa), v)\big) : (u, v) \in f_{aa}(a)\big\}\\ &\quad\circ \big\{\big((\pi_L\, g'(bb), u), (\pi_R\, g'(bb), v)\big) : (u, v) \in f_{bb}(b)\big\}\\ &= \big\{\big((\pi_L\, g'(aa), u), (\pi_R\, g'(bb), v)\big) : (u, v) \in D\times D\big\}.\end{aligned}$$

by property (7). We observe that $c = ab = aaab = aac$, from which it follows that $\pi_L\, g'(c) = \pi_L\, g'(aac) = \pi_L\,(g'(aa)\circ g'(c)) = \pi_L\, g'(aa)$. Similarly $\pi_R\, g'(c) = \pi_R\, g'(bb)$. Hence $\hat{h}(ab) = h(a)\circ h(b)$. (iv). Using 6.1 and (i) it is easy to show that $x \leq x\mathbb{1}$ and $x \leq \mathbb{1}x$ hence the unary term $d(x)$, defined by $d(x) = \mathbb{1}x\mathbb{1}$, is a congruence element for each $x \in \mathbf{A}$. □

The above result and its proof was first presented in [**Jip01**]. In this paper it is also observed that HRB is term-equivalent to the variety Df_2 of diagonal free 2-dimensiona cylindrical algebras (c.f. [**HMT85**] for a definition). Hence the above proof is an alternative demonstration to the one in [**HMT85**] that all members of Df_2 are representable. Since the equational theory of Df_2 is known to be decidable [**HMT85**], the same holds for HRB.

6.2. Semigroups

In her dissertation [**Rei96**] P. J. Reich studied the class of complex algebras of semigroups. (We denote the variety of semigroups by Sg.) In this informal case study we present some comments and results relating to her original monograph. First we define the partial algebraic equivalent of semigroups.

DEFINITION 6.2.1. A partial binary operation on a set S is said to be *strongly associative* if, for any $a, b, c \in S$, $a * (b * c)$ is defined if, and only if, $(a * b) * c$ is defined, and $a * (b * c) = (a * b) * c$.

A *partial semigroup* is a partial algebra $\mathbf{S} = \langle S, *\rangle$ where '$*$' is a binary operation that is strongly associative. We denote the class of all partial semigroups by PSg.

Note that the concept of equality used in the definition above is stronger than that required by Definition 5.1.3 (p. 92).

Let $\mathbf{S} = \langle S, *\rangle$ be a partial semigroup. For some $0 \notin S$ we define an algebra $\mathbf{S}_0 = \langle S_0, \cdot\rangle$, where $S_0 = S \cup \{0\}$ and '$\cdot$' is a binary operation defined by

$$a\cdot b = \begin{cases} a * b & \text{if } a * b \text{ is defined}\\ 0 & \text{otherwise}\end{cases} \tag{6.6}$$

with $a, b \in S_0$ (in particular $0 * b = 0 = a * 0$). It follows from the definition of strong associativity that $\mathbf{S}_0$ is a semigroup.

PROPOSITION 6.2.2. $\mathbf{CmPSg} \subseteq \mathbf{HCmSg}$.

PROOF. Let $\mathbf{S} = \langle S, *\rangle \in \mathsf{PSg}$ and define $\mathbf{S}_0$ as above. Observe that if $a{*}b = c \in S$ then $a\cdot b = c \in S_0$. Hence $\mathbf{S}$ is an inner substructure of $\mathbf{S}_0$. Thus $\mathrm{Cm}\mathbf{S}$ is a homomorphic image of $\mathrm{Cm}\mathbf{S}_0$ (c.f. by Lemma 3.1.15 p. 37). □

Consequently $\mathrm{V}(\mathbf{CmPSg}) \subseteq \mathrm{V}(\mathbf{CmSg})$. It is easily seen that any semigroup is in fact a partial semigroup. Thus giving us the following corollary.

COROLLARY 6.2.3. $\mathrm{V}(\mathbf{Cm}\mathsf{PSg}) = \mathrm{V}(\mathbf{Cm}\mathsf{Sg})$.

It follows from **zag** that any inner substructure of a partial semigroup is again a partial semigroup. Hence PSg is closed under one-generated inner substructures and it is not hard to see that PSg is also closed under ultraproducts. Hence, by Theorem 4.3.17 (p. 85), $\mathrm{V}(\mathbf{Cm}\mathsf{PSg}) = \mathbf{SPCm}\mathsf{PSg}$. However, the 2-element constant semigroup has an inner substructure that is a partial semigroup but not a semigroup. Thus $\mathrm{V}(\mathbf{Cm}\mathsf{Sg}) \neq \mathbf{SPCm}\mathsf{Sg}$ by Theorem 4.3.18.

Consider any transitive relation R, we define a binary operation on R by taking $\langle a, b\rangle * \langle c, d\rangle = \langle a, d\rangle$ if $b = d$ and undefined otherwise. The transitivity of R then implies that $\langle R, *\rangle$ is a partial semigroup. Let $\mathcal{K}$ be the class of all pair semigroups. It can be shown that for any semigroup $\mathbf{S}$, $\mathrm{Cm}\mathbf{S} \in \mathcal{K}$. Hence

$$\mathrm{V}(\mathbf{Cm}\mathsf{Sg}) \subseteq \mathrm{V}(\mathbf{Cm}\mathcal{K}) \subseteq \mathrm{V}(\mathbf{Cm}\mathsf{PSg}).$$

By the corollary above it follows that these varieties coincide. Consequently to decide if a BAO is in $\mathrm{V}(\mathbf{Cm}\mathsf{Sg})$, one may instead check if it is in $\mathrm{V}(\mathbf{Cm}\mathcal{K})$. It turns out that $\mathcal{K}$ is closed under one-generated inner substructures and ultraproduct. Therefore $\mathrm{V}(\mathbf{Cm}\mathcal{K}) = \mathbf{SPCm}\mathcal{K}$. Recently Jipsen (c.f. [**Jip01**]) has shown that each of the 4-element BAOs listed in Appendix A of [**Rei96**] are members of $\mathrm{V}(\mathbf{Cm}\mathsf{Sg})$. However there are many 8-element BAOs with an associative operator that are not in $\mathrm{V}(\mathbf{Cm}\mathsf{Sg})$. Moreover there exist sequences of such BAOs (with > 8 elements) whose ultraproduct is in $\mathrm{V}(\mathbf{Cm}\mathsf{Sg})$, which shows that this variety is not finitely based.

6.3. Boolean algebras and rings

In this case study we turn to the varieties generated by complex algebras of Boolean algebras (HBA) and Boolean rings (HBR). Previously (c.f. [**GoV99**]) the variety HBA was studied as the algebraic semantics for Hyperboolean Modal Logic, described below. (Note that we diverge from [**GoV99**] in that they denote the class of all full complex Boolean algebras $\mathbf{Cm}\mathsf{BA}$ by HBA.)

Notationally we shall use lowercase letters for variables when writing formulas in the language of HBA and HBR whereas uppercase is used for $\mathbf{Cm}\mathsf{BA}$ and $\mathbf{Cm}\mathsf{BR}$. For the boolean operations in the language of BA and BR we use $+$, $\cdot$, $\oplus$, $-$, 1 and 0. In the case of the complex algebras we use $\vee$, $\wedge$, $\sim$, $\mathbb{1}$ and $\mathbb{0}$, where $x \wedge y = \sim(\sim x \vee \sim y)$ and $\mathbb{1} = \sim\mathbb{0}$.

The language of Hyperboolean Modal Logic HBML contains:

- $\Phi_\omega = \{p_1, p_2, \ldots\}$ — a countable set of propositional variables,
- $\sim$, $\vee$, $\wedge$ $\mathbb{0}$, $\mathbb{1}$ — the classical boolean connectives,
- $\langle\rightarrow\rangle$ — binary diamond modality (internal implication) and
- $\langle 0\rangle$ — propositional constant (internal zero).

Observe that the modalities $\langle\rightarrow\rangle$ and $\langle 0\rangle$ make HBML a modal logic. Note that we take the notion of formulas in HBML to be the usual one for modal logics.

The algebraic semantics of HBML is over $\mathbf{Cm}\mathsf{BA}$. Let $\mathbf{B} = \langle B, +, -, 0\rangle$ be a Boolean algebra. By a valuation we mean any function $V : \Phi_\omega \longrightarrow \mathcal{P}(B)$, i.e. for each $p \in \Phi_\omega$, $V(p)$ is a subset of B. Each valuation V is then extended to arbitrary formulas by induction:

- $V(\mathbb{0}) = \emptyset$, $V(X \vee Y) = V(X) \cup V(Y)$ and $V({\sim}X) = {\sim}V(X) = B \setminus V(X)$,
- $V(X \langle\rightarrow\rangle Y) = -V(X) + V(Y)$ and
- $V(\langle 0 \rangle) = \{0\}$.

A formula A is said to be *valid* in Cm**B** if for any valuation V we have $V(A) = B$.

6.3.1. Hyperboolean algebras and rings.

A *Boolean ring* is a ring R with unit 1 (i.e. R $= \langle R, \oplus, \cdot, 0, 1\rangle$ where '$\oplus$' denotes ring addition and '$\cdot$' denotes ring multiplication), such that for every $x \in R$ we have $x \cdot x = x$. Given a Boolean ring R and by letting $B = R$ we can define a Boolean algebra $\mathbf{B} = \langle B, +, -, 0\rangle$, in the following way

$$-x = x \oplus 1 \quad \text{and} \quad x + y = -(-x \cdot -y) \tag{6.7}$$

Similarly given a Boolean algebra $\mathbf{B} = \langle B, +, -, 0\rangle$ and by letting $R = B$ we can define a Boolean ring R $= \langle R, \oplus, \cdot, 0, 1\rangle$, in the following way

$$x \cdot y = -(-x + -y)\ .\ 1 = -0 \quad \text{and} \quad x \oplus y = (x \cdot -y) + (-x \cdot y) \tag{6.8}$$

This gives us the natural term definitional equivalence between Boolean algebras and Boolean rings first noted in [**Sto36**].

PROPOSITION 6.3.1. V(**Cm**BA) = **SPCm**BA *and* V(**Cm**BR) = **SPCm**BA.

PROOF. We will show that no Boolean algebra has a proper inner substructure from which it follows that BA is closed under one-generated inner substructures. Then by Theorem 4.3.18 (p. 85) the first result will follow.

Take any $\mathbf{A} \in$ BA such that $\mathbf{B}$ is a proper inner substructure of $\mathbf{A}$. Then the embedding function $\gamma : \mathbf{B} \longrightarrow \mathbf{A}$ is a bounded morphism. Note that $1 \in \mathbf{B}$ since $\mathbf{B}$ is a subalgebra of $\mathbf{A}$. Clearly γ is a BA-homomorphism and hence $\gamma(1) = 1$. Since $\mathbf{B}$ is a proper substructure there exists an $x \in \mathbf{A} \setminus \mathbf{B}$. Then $x + \gamma(1) = \gamma(1) = 1 \in \mathbf{B}$, contradicting **zag**, since for all $y \in \mathbf{B}$ $\gamma(y) \neq x$.

The result for BR follows a similar argument using the observation that $x \oplus x = 0$ and $\gamma(0) = 0$ □

We now formally introduce the complex varieties associated with BA and BR.

DEFINITION 6.3.2. By *hyperboolean algebras*, denoted HBA, we mean the variety V(**Cm**BA). We call the elements of the variety V(**Cm**BR) *hyperboolean rings* and denote this class by HBR.

Clearly by Proposition 6.3.1 and Proposition 4.3.6 (p. 82) these are complex varieties.

DEFINITION 6.3.3. Let $\mathcal{V}$ and $\mathcal{W}$ be varieties, not necessarily of the same type, and let D be a map relating each operation f of the language of $\mathcal{V}$ to a term $D(f)$ in the language of $\mathcal{W}$. Then D can naturally be extended to terms. $\mathcal{V}$ is said to be *interpretable* in $\mathcal{W}$, written $\mathcal{V} \leq_D \mathcal{W}$, if for all terms σ and τ, $\mathcal{V} \models \sigma = \tau$ implies $\mathcal{W} \models D(\sigma) = D(\tau)$.

Note that $\mathcal{V}$ is interpretable in $\mathcal{W}$ if each $\mathcal{W}$-algebra $\mathbf{A} = \langle A, \langle g^{\mathbf{A}} : g \in F_{\mathcal{W}} \rangle \rangle$ can be considered as a $\mathcal{V}$-algebra $D(\mathbf{A}) = \langle A, \langle D(f)^{\mathbf{A}} : f \in F_{\mathcal{V}} \rangle \rangle$. A more detailed discussion can be found in [**GaT84**].

Suppose now that $D(f)$ is linear for each fundamental operation f of the language of $\mathcal{V}$. The following calculation shows that for any $\mathbf{A} \in \mathcal{W}$, the algebra $\mathrm{Cm}D(\mathbf{A})$ is equal to the algebra $D(\mathrm{Cm}\mathbf{A})$

$$\begin{aligned} f^{\mathrm{Cm}D(\mathbf{A})}(X_1, \dots, X_n) &= \{f^{D(\mathbf{A})}(x_1, \dots, x_n) : x_i \in X_i\} \\ &= \{D(f)^{\mathbf{A}}(x_1, \dots, x_n) : x_i \in X_i\} \\ &= D(f)^{\mathrm{Cm}\mathbf{A}}(X_1, \dots, X_n) \end{aligned}$$

where the first and last equality follows from Lemma 4.3.25.

In our setting, $D(-x) = x \oplus 1$ and $D(x+y) = ((x \oplus 1)(y \oplus 1)) \oplus 1$ gives the interpretation from BA to BR (c.f. equation (6.7)). Note that these terms are linear.

COROLLARY 6.3.4. HBA *is interpretable in* HBR.

PROOF. Assume $\mathsf{HBA} \models \sigma{=}\tau$, then $\mathbf{Cm}\mathsf{BA} \models \sigma{=}\tau$. We wish to show that $\mathbf{Cm}\mathsf{BR} \models D(\sigma){=}D(\tau)$. Let $\mathrm{R} \in \mathsf{BR}$. Then $\mathrm{Cm}D(\mathrm{R}) \in \mathbf{Cm}\mathsf{BA}$, so $\mathrm{Cm}D(\mathrm{R}) \models \sigma{=}\tau$. By the preceding observation $\mathrm{Cm}D(\mathrm{R}){=}D(\mathrm{CmR})$, so $D(\mathrm{CmR}) \models \sigma{=}\tau$. Hence $\mathrm{CmR} \models D(\sigma){=}D(\tau)$. □

However HBA and HBR are not term definitionally equivalent. To demonstrate this we turn to the Boolean algebra 2^4. The elements of 2^4 will be represented in binary 4-tuples, so that the top element of 2^4 will be 1111 and the bottom element 0000.

Let $X = \{0001, 0010, 0100, 1000, 1100, 1010, 0110\}$ and consider the Boolean subalgebra B of $\mathcal{P}(2^4)$ with $\{0000\}$, $\{1111\}$, X and $-X$ as atoms. To see that B is the universe of an HBA $\mathbf{B}$ observe that $1^{\mathbf{B}} = \{1111\}$, $0^{\mathbf{B}} = \{0000\}$,

$+$	$0^{\mathbf{B}}$	$1^{\mathbf{B}}$	X	$-X$
$0^{\mathbf{B}}$	$0^{\mathbf{B}}$	$1^{\mathbf{B}}$	X	$-X$
$1^{\mathbf{B}}$	$1^{\mathbf{B}}$	$1^{\mathbf{B}}$	$1^{\mathbf{B}}$	$1^{\mathbf{B}}$
X	X	$1^{\mathbf{B}}$	$X \vee -X$	$-X \vee 1^{\mathbf{B}}$
$-X$	$-X$	$1^{\mathbf{B}}$	$-X \vee 1^{\mathbf{B}}$	$-X \vee 1^{\mathbf{B}}$

and $\{0^{\mathbf{B}}, 1^{\mathbf{B}}, X, -X\}$ is clearly closed under internal complementation.

If the symmetric difference '$\oplus$' were term definable in HBA then B would be closed under this operation. However this is not the case since $1000 = 0010 \oplus 1010 \in X \oplus X$, but $0001 \notin X \oplus X$ since to find $x_0, x_1 \in X$ with $x_0 \oplus x_1 = 0001$ we need x_0 and x_1 to agree on their first three coordinates but disagree on the last. It is easily checked that no two elements of X satisfy this criteria. It follows that HBR is not interpretable in HBA.

6.3.2. Equations valid for hyperboolean algebras.

[**GoV99**] use so-called Gabbay style irreflexivity rules (c.f. [**Gab81**]) to show that all valid equations of HBA can be deduced from a finite list of equations.

Because of the nonstandard deduction rules, this result does not answer the question whether HBA is finitely based. Here we give a list of equations that are valid on all hyperboolean algebras, and for an algebra of type $\langle 2, 1, 0, 2, 1, 0\rangle$ with corresponding abstract operations $\{\vee, \sim, \mathbb{0}, +, -, 0\}$ and size $\leq 2^4$, these equations hold if, and only if, the algebra in question is a member of HBA.

[**GoV99**]) and $\mathcal{K}$ is a class of simple algebras that generates HBA, it is well known that for any universal formula ϕ there is a corresponding equation ϕ^* such that $\mathcal{K} \models \phi$ iff HBA $\models \phi^*$ (c.f. [**McK75**] and [**Jip93**] for more detailed discussions of this result). Since ϕ is often much shorter than ϕ^*, we will usually only give the universal formula and check it holds in all Cm**B**.

We begin with some formulas that are derived from equations which hold in the underlying Boolean algebras. Since the commutativity, associativity, involution and De Morgan's laws are linear and each variable appears on both sides it is easily seen (c.f. Lemma 4.3.27 p. 88) that they lift to HBA.

comm: $x + y = y + x$
assoc: $(x + y) + z = x + (y + z)$
ident: $x + 0 = x$
invol: $- - x = x$
deMorgan: $x \cdot y = -(-x + -y)$

The following are typical BAO equations (c.f. Definition 3.1.1 p. 30).

operator: $x + (y \vee z) = (x + y) \vee (x + z)$ $\quad -(y \vee z) = -y \vee -z$
normal: $x + \mathbb{0} = \mathbb{0}$ $\quad -\mathbb{0} = \mathbb{0}$

The next few formulas are derived from BA equations that are not linear or do not have the same variables on each side of the equation. In this case they still lift partially, as can be checked easily using Lemma 4.3.27(i)–(iii) (p. 88). In the formulas below $\mathbb{1} = \sim\mathbb{0}$ and $x \wedge y = \sim(\sim x \vee \sim y)$.

idem: $x \leq x + x$
distr: $x + (y \cdot z) \leq (x + y) \cdot (x + z)$
zero: $x \neq \mathbb{0}$ implies $x + 1 = 1$
least: $x \wedge 0 = \mathbb{0}$ implies $(x + \mathbb{1}) \wedge 0 = \mathbb{0}$

To obtain further equations, and to check the independence of some of the above equations, it is useful to consider small simple algebras of the HBA similarity type. Since HBA $= \mathbb{V}(\mathbf{Cm}$BA) it follows from Jónsson's Lemma[‡] that 0 and 1 are atoms of any simple HBA. If $0 = 1$, then it follows from the **ident** and **zero** formulas that $x \neq \mathbb{0}$ implies $x = 0$, so this characterises the algebra Cm$\mathbf{B}_0$, where $\mathbf{B}_0$ is the 1-element Boolean algebra. (The finite Boolean algebra $\langle B_n, +, -, 0\rangle$ with 2^n elements is denoted by $\mathbf{B}_n$.)

‡Let $\mathcal{V}$ be a congruence variety and $\mathbf{A}_\lambda \in \mathcal{V}$ for $\lambda \in \Lambda$. If $\mathbf{B} \leq \mathbf{A} = \prod_{\lambda\in\Lambda} \mathbf{A}_\lambda$ and $\theta \in \mathrm{Con}\mathbf{B}$ is such that $\mathbf{B}/\theta$ is a non-trivial subdirectly irreducible algebra, then there is an ultrafilter $\mathcal{F}$ over Λ such that $\theta_{\mathcal{F}}|_{\mathbf{B}} \subseteq \theta$, where $\theta_{\mathcal{F}}$ is the congruence on $\mathbf{A}$ defined by

$$\langle a, b\rangle \in \theta_{\mathcal{F}} \text{ iff } \{\lambda \in \Lambda : a(\lambda) = b(\lambda)\} \in \mathcal{F}.$$

(C.f.[**BuS81**] Corollary 6.9 for a proof.)

We now list the possible HBA-type algebras with 2^2 and 2^3 elements. For a finite BAO it suffices to define the operators on the atoms.

$+$	0	1	$-$
0	0	1	1
1	1	1	0

$+$	0	1	R	$-$
0	0	1	R	1
1	1	1	1	0
R	R	1	$R \vee 1$	R

(The last column lists the result of applying '$-$' to the corresponding atom in the first column.)

The first of these two algebras is $\mathrm{Cm}\mathbf{B}_1$. Its structure is determined by the **ident** and **zero** formulas. The second algebra is a subalgebra of $\mathrm{Cm}\mathbf{B}_2$, where $R = B_2 \setminus \{0, 1\}$. To see that it is the only possible model of the above formulas, we note that all entries in the table except the $R + R$ entry are determined by the **invol**, **ident** and **zero** formulas. On the other hand, the **idemp** and **least** formulas imply that $R + R = R \vee 1$ or $R + R = R$. The latter possibility is excluded by part (i) of the lemma below.

LEMMA 6.3.5. *The class of all full complex Boolean algebras* $\mathbf{Cm}\mathsf{BA}$ *satisfies the following formulas (for all x, y).*

(i) $1 \wedge (x + -y) \neq \mathbb{0}$ *if, and only if,* $x \wedge (x + y) \neq \mathbb{C}$,

(ii) *(*$x \vee y \leq x + y$ *and* $x = -y$ *and* $x \vee y = \sim(0 \vee 1)$ *and* $y \wedge (x + x) = 0$*) imply* $x = \mathbb{0}$, *and*

(iii) *(*$x \leq x + y$ *and* $x = -x$ *and* $y = -y$ *and* $x \vee y = \sim(0 \vee 1)$ *and* $x \wedge (y + y) = \mathbb{0}$ *and* $y \wedge (x + x) = \mathbb{C}$*) imply* $x = \mathbb{0}$.

PROOF. Consider any Boolean algebra, $\mathbf{B} = (B, +, -, 0)$, and X, Y such that:

(i): X, Y are any subsets of B. Consider the following calculation

$$\begin{aligned} \{1\} \cap (X + -Y) \neq \emptyset \quad &\text{iff} \quad 1 \in X + -Y \\ &\text{iff} \quad 1 = x + -y \text{ for some } x \in X, y \in Y \\ &\text{iff} \quad y \leq x \text{ for some } x \in X, y \in Y. \end{aligned}$$

From the final statement it follows that $x = x + y$ for some $x \in X$ and $y \in Y$ and so $X \cap (X + Y) \neq \emptyset$. To prove the right to left direction we assume $X \cap (X + Y) \neq \emptyset$ then there exist $x, x_0 \in X$ and $y \in Y$ such that $x = x_0 + y$ from which it follows that $y \leq x$.

(ii): X, Y are subsets of B, all the premises hold and $X \neq \emptyset$. Then there exists an $x \in X$ and since $X \subseteq X + Y$ we see that for some $x_0 \in X$ and $y \in Y$ we have that $x = x_0 + y$ and thus $y \leq x$. Similarly since $Y \subseteq X + Y$ we can find a $w \in X$ such that $w \leq y$. Since $x \in X = -Y$ we have $-x \in Y \subseteq X + Y$ and so there exists $z \in X$ with $x \leq -z$.

The following calculation shows that $(-y \cdot -z + w) + x = -z \in Y$.

$$\begin{aligned} -y \cdot -z + w + x &= -y \cdot -z + x \quad (\text{since } w \leq x) \\ &= -y \cdot -z + x \cdot -z \quad (\text{since } x \leq -z) \\ &= -z \cdot (-y + x). \end{aligned}$$

But $-x \leq -y$, from which $-y + x = 1$ and so the equality follows. Since $(X + X) \cap Y = \emptyset$ and $X \cup Y = \sim\{0, 1\} = B \setminus \{0, 1\}$ we have $(-y \cdot -z + w) \in Y$, thus $-(-y \cdot -z + w) = (y + z) \cdot -w \in X$. From the following calculation we see that

$(y+z)\cdot -w + -y = -w$.

$$\begin{aligned}(y+z)\cdot -w + -y &= y\cdot -w + z\cdot -w + -y \quad (\text{since } z \le -w)\\ &= (y+z+-y)\cdot -w + -y\\ &= -w + -y.\end{aligned}$$

Since $w \le y$ the result follows. But $(y+z)\cdot -w \in X$ and $-y \in X$ contradicting the assumption that $(X+X)\cap Y = \emptyset$.

(iii): X, Y are subsets of B, the premises are satisfied and $X \neq \emptyset$. Then there exists an $x \in X$ and, as in (ii), since $X \subseteq X+Y$ we can deduce that there is some $y \in Y$ such that $y \le x$. Then

$$-y\cdot x + y = (-y+y)\cdot(x+y) = 1\cdot x = x.$$

So $-y\cdot x \in X$ since $(Y+Y)\cap X = \emptyset$ and $X \cup Y = {\sim}\{0,1\}$. But

$$-y\cdot -x + -x = (-y+x)\cdot(x+-x) = -y\cdot 1 = -y \in Y$$

and $-x \in X$ which contradicts $(X+X)\cap Y = \emptyset$. $\square$

We note that since **CmBA** turns out to be a class of discriminator algebras ([**GoV99**]), using (4.17) p. 87 the above formulas can be translated to equations that hold in $\mathsf{V}(\mathbf{CmBA}) = \mathsf{HBA}$.

Because of the **ident** and **zero** formulas, the rows and columns for 0 and 1 are always the same, and will be omitted. The two remaining atoms are denoted by R and S. Because of commutativity we further omit one of the entries. The following algebras labeled (i)–(iii) show that each of the formulas (i)–(iii) is necessary and is not implied by the others.

$$\text{(i)}\ \begin{array}{c|c||c} + & R & - \\ \hline R & R & R \end{array} \qquad \text{(ii)}\ \begin{array}{c|cc||c} + & R & S & - \\ \hline R & R\vee 1 & \sim 0 & S \\ S & & \sim 0 & R \end{array} \qquad \text{(iii)}\ \begin{array}{c|cc||c} + & R & S & - \\ \hline R & R\vee 1 & \sim 0 & S \\ S & & S\vee 1 & R \end{array}$$

The list of HBA-type algebras with 2^4 elements (thus 4 atoms) which satisfy the above formulas is somewhat longer. We first consider the case where $-R = S$. In this case $R + S \ge 1$ by the **idemp** formula and Lemma 6.3.5(i), and $R + R \ge R$, $S+S \ge S$ by the **idemp** formula. It follows from the **least** formula that none of these entries include 0. Hence there remain four cases for each entry depending on whether they include none, one or two of the remaining two atoms. These 64 possibilities are further reduced by eliminating isomorphic copies and applying Lemma 6.3.5(ii) until we are left with the algebras $\mathbf{C}_1$ to $\mathbf{C}_4$. A similar analysis, using Lemma 6.3.5(iii), of the algebras with $R = -R$ and $S = -S$ yields the remaining two algebras $\mathbf{C}_5$ and $\mathbf{C}_6$.

$$(\mathbf{C}_1)\ \begin{array}{c|cc||c} + & R & S & - \\ \hline R & R & 1 & S \\ S & & S & R \end{array} \qquad (\mathbf{C}_2)\ \begin{array}{c|cc||c} + & R & S & - \\ \hline R & R\vee 1 & R\vee 1 & S \\ S & & S & R \end{array}$$

$$(\mathbf{C}_3)\ \begin{array}{c|cc||c} + & R & S & - \\ \hline R & R\vee 1 & R\vee 1 & S \\ S & & R\vee S & R \end{array} \qquad (\mathbf{C}_4)\ \begin{array}{c|cc||c} + & R & S & - \\ \hline R & \sim 0 & \sim 0 & S \\ S & & \sim 0 & R \end{array}$$

$$(\mathbf{C}_5)\ \begin{array}{c|cc||c} + & R & S & - \\ \hline R & R\vee 1 & \sim 0 & R \\ S & & \sim 0 & S \end{array} \qquad (\mathbf{C}_6)\ \begin{array}{c|cc||c} + & R & S & - \\ \hline R & \sim 0 & \sim 0 & R \\ S & & \sim 0 & S \end{array}$$

These algebras were found by listing all (non-isomorphic) 4-atom BAOs with one binary and one unary operator and then checking which of them satisfy the axioms for HBA.

LEMMA 6.3.6. (i) $\mathrm{Cm}\mathbf{B}_2$ *is a representation of* $\mathbf{C}_1$.
(ii) *By taking R as a non-principal ultrafilter of any infinite Boolean algebra we get a representation of* $\mathbf{C}_2$.
(iii) *By taking R as the set of co-atoms of* $\mathbf{B}_3$ *we get a a representation of* C_3.
(iv) $\mathbf{C}_4$, $\mathbf{C}_5$ *and* $\mathbf{C}_6$ *are representable in the free Boolean algebra on ω generators,* $\mathbf{F}_{\mathsf{BA}}(\omega)$.

PROOF. The proofs of (i), (ii) and (iii) are fairly simple and are left to the reader. The more interesting constructions are found when looking at (iv).

First we introduce a bit of notation. For $i \in \omega$ we write $\mathbf{F}_{\mathsf{BA}}(x_0, \dots, x_{i-1})$ for the free Boolean algebra on i generators (i.e. $\mathbf{F}_{\mathsf{BA}}(x_0, \dots, x_{i-1}) = \mathbf{F}_{\mathsf{BA}}(\{x_0, \dots, x_{i-1}\})$ and $\mathbf{F}_{\mathsf{BA}}(\omega)$ for the free algebra with free generating set $\{x_i : i \in \omega\}$.

$\mathbf{C}_4$: For $a, b \in \mathbb{R}$, let $[a, b)$ be the right-open interval, and define $\mathcal{I}_k = \{[\frac{i}{3^k}, \frac{i+1}{3^k}) : i < 3^k\}$ for $k < \omega$. The set F of all finite unions of intervals in $\bigcup_{k<\omega} \mathcal{I}_k$ is a countable atomless Boolean algebra $\mathbf{F} = \langle F, +, -, 0\rangle$, where $+$ is union, $-$ is complementation with respect to $[0, 1)$, and $0 = \emptyset$. Note that since all countable atomless Boolean algebras are isomorphic (c.f. [**ChK77**] Proposition 1.4.5), $\mathbf{F}$ is isomorphic to $\mathbf{F}_{\mathsf{BA}}(\omega)$. We show that $\mathbf{C}_4$ is isomorphic to a subalgebra of $\mathrm{Cm}\mathbf{F}$.

For any $X \in F$, there exists $k \in \omega$ such that X is a union of intervals in $\mathcal{I}_k$. The subsets R and S of F are defined as follows: For $X \in F \setminus \{[0, 1), \emptyset\}$, let $X \in R$ if X is the union of an odd number of distinct intervals in $\mathcal{I}_k$, and let $X \in S$ otherwise. This is well-defined since if $j > k$ then each interval in X splits into 3^{j-k} intervals, so X is also the union of an odd, respectively even, number of intervals in $\mathcal{I}_j$. Since $\mathcal{I}_k$ has an odd number of elements, $S = -R = \{-X : X \in R\}$. We have $1^{\mathbf{F}}$ in $R + S$, $S + S$ and $R + R$ since

$$1^{\mathbf{F}} = [0, 1) = [0, \tfrac{1}{3}) \cup [\tfrac{1}{3}, 1) = [0, \tfrac{2}{3}) \cup [\tfrac{1}{3}, 1) = [0, \tfrac{7}{9}) \cup [\tfrac{2}{3}, 1).$$

To see that $R + S \supseteq R$ and $S + S \supseteq R$, consider $X \in R$. Then $I = [\frac{i}{3^k}, \frac{i+1}{3^k}) \subseteq X$ for some $i < 3^k$ and $k \in \omega$. Since $I \in R$ and $X \setminus I \in S$ we have $X \in R + S$. Moreover, we have

$$X = [\tfrac{i}{3^k}, \tfrac{3i+2}{3^{k+1}}) \cup (X \setminus [\tfrac{i}{3^k}, \tfrac{3i+1}{3^{k+1}}) \in S + S.$$

The argument for $R + S \supseteq S$ and $R + R \supseteq S$ is similar. Since one always has $R + R \supseteq R$ and $S + S \supseteq S$, it follows that R and S are the atoms of a Boolean subalgebra of $\mathrm{Cm}\mathbf{F}$ isomorphic to $\mathbf{C}_4$.

$\mathbf{C}_5$: The representation of $\mathbf{C}_5$ is achieved by starting in $\mathbf{F}_{\mathsf{BA}}(x_0)$ and colouring x_0 and $-x_0$ red (i.e. $x_0, -x_0 \in R$). We then generate the Boolean subalgebra of $\mathbf{F}_{\mathsf{BA}}(x_0)$ containing all red elements and ensure that each of these red elements will have two silver elements (i.e. elements of S) below it. We then produce new red elements such that each silver element has one red and one silver below it. We iterate this process until we've coloured the whole of $\mathbf{F}_{\mathsf{BA}}(\omega)$.

An explicit description of the atoms of R_i, i.e. all the red elements after i iterations, is given by the following definition.

- $A_0 = \emptyset \qquad D_0 = \{\{x_0, -x_0\}\}$,
- $A_i = \{\{t \cdot x_i\} : t \in X \text{ for some } X \in D_{i-1}\}$, for $i > 0$ and
- $D_i = \{X \cdot \{-x_i\} : X \in D_{i-1}\} \cup \{Y \cdot \{x_i, -x_i\} : Y \in A_{i-1}\}$, for $i > 0$.

Let $\hat{A}_i = \{t : \{t\} \in A_i\}$ and let $\hat{D}_i = \{s \vee t : \{s,t\} \in D_i\}$. Finally we let $R_i = \{\bigvee X : X \subseteq \hat{A}_i \cup \hat{D}_i\}$ and $S_i = \mathbf{F}_{\mathsf{BA}}(x_0, \ldots, x_i) \setminus (R_i \cup \{0,1\})$. We now define $R = \bigcup_{i\in\omega} R_i \setminus \{0,1\}$ and $S = \bigcup_{i\in\omega} S_i$.

Note that, for each $i \in \omega$, $\hat{A}_i \cup \hat{D}_i$ is the set of atoms of $\mathbf{F}_{\mathsf{BA}}(x_0, \ldots, x_i)$. Thus it follows that R_i is a Boolean subalgebra of $\mathbf{F}_{\mathsf{BA}}(x_0, \ldots, x_i)$ and so $\bigcup D_i \subseteq S_i$. Since each R_i is a Boolean subalgebra of $\mathbf{F}_{\mathsf{BA}}(x_0, \ldots, x_i)$ it follows that $R = -R$, $S = -S$ and $R + R \supseteq R \cup \{1\}$. To see that $R + R \subseteq R \cup \{1\}$, observe that $t \in S_i$ if, and only if, there exists a $\{t_1, t_2\} \in D_{i-1}$ such that $t_1 \leq t$ and $t_2 \leq -t$, thus $t \in S_{i-1}$ implies $t \in S_i$. Hence $R_i \cap S_{i-1} = \emptyset$ and so $R + R \subseteq R \cup \{1\}$. From $D_i \supseteq \{Y \cdot \{x_i, -x_i\} : Y \in A_{i-1}\}$ it follows that $S_i + S_i \supseteq R_{i-1}$ and hence $S + S \supseteq R$, and so $S + S = {\sim}0$ and $R + S \supseteq R$. Recall that $\bigcup D_{i-1}$ is the set of atoms of $\mathbf{F}_{\mathsf{BA}}(x_0, \ldots, x_{i-1})$ in S_{i-1} and that $R_{i-1} \cap S_{i-1} = \emptyset$. Thus by construction A_i ensures that there are elements of R_i below each element of S_{i-1} and thus $R + S \subseteq S$.

element of S_{i-1}.

$\mathbf{C}_6$: Take X_i to be the universe of $\mathbf{F}_{\mathsf{BA}}(x_0, \ldots, x_{i-1})$. To demonstrate the representation of $\mathbf{C}_6$, let $S_0 = S_1 = R_0 = \emptyset$ and $R_1 = X_1 \setminus \{0,1\}$ and then recursively construct R_k and S_k, for $k > 1$, in the following way.

- $R_{2i} = R_{2i-1}$ for $i \geq 1$.
- $S_{2i} = X_{2i} \setminus (R_{2i-1} \cup X_{2i-2} \cup \{0,1\})$ for $i \geq 1$.
- $R_{2i+1} = X_{2i+1} \setminus (S_{2i} \cup X_{2i-1} \cup \{0,1\})$ for $i \geq 1$.
- $S_{2i+1} = S_{2i}$ for $i \geq 1$.

We let $R = \bigcup_{k\in\omega} R_k$ and $S = \bigcup_{k\in\omega} S_k$.

It follows that $R = -R$, $S = -S$, $R \cap S = \emptyset$ and that $R \cup S \cup \{1\} \cup \{0\}$ is the universe of $\mathbf{F}_{\mathsf{BA}}(\omega)$. To see that $R \subseteq S + S$ observe that if $r \in R$ then there exists $i \in \omega$ such that $r \in R_i$, but $r = r \cdot x_{i+1} + r \cdot -x_{i+1}$ and from our construction $r \cdot x_{i+1}$ and $r \cdot -x_{i+1}$ are elements of S_{i+1} and hence of S. Note that if $r \in R_i$ then $r = r + r \cdot x_{i+1}$ with $r \cdot x_{i+1} \in S_{i+1}$ and so $R \subseteq R + S$ follows from a similar argument as before. □

From the comment earlier we can see that all the universal formulas described so far give rise to equations that fully characterise any algebra $\mathbf{B}$ of the HBA similarity type up to size $\leq 2^4$, i.e. $\mathbf{B}$ satisfies the equations if, and only if, $\mathbf{B} \in$ HBA.

6.3.3. The undecidability of HBR.

First we note a result from [**AGN97**] (c.f. Corollary 2.10(i)).

THEOREM 6.3.7. *Let $\mathcal{K}$ be a class of relation algebras that contains the complex algebra of an infinite group. Then the equational theory of $\mathcal{K}$ is undecidable.*

So we restrict ourselves to reducts of HBR consisting of the external boolean operations '$\vee$', '$\sim$' and '$\complement$', the internal symmetric difference '$\oplus$' and 0. These algebras are symmetric relation algebras (symmetric means that the converse operation, a fundamental operation of relation algebras, is the identity map), with relation composition given by $\oplus$ and identity element 0. Let $\langle 2^\omega, \oplus, \cdot, 1, 0\rangle$ be the countable product of 2-element Boolean rings. Its complex algebra is in HBR, and its reduct is the complex algebra of $\langle 2^\omega, \oplus, 0\rangle$ which is an infinite group. Hence the result follows.

COROLLARY 6.3.8. *The equational theory of* HBR *is undecidable*

6.4. Open problems

We conclude this chapter with a few open problems related to the work presented here.

(1) In the previous section we presented a finite set of equations which determine BAOs of the HBA similarity type of up to size 2^4. It is however still unknown whether either of the varieties HBA and HBR are finitely axiomatisable.
(2) As we have seen the equational theory of HBR is undecidable. In [**GoV99**] Goranko and Vakarelov proved that HBML does not have the finite model property, which still begs the question as to whether the equational theory of HBA is decidable.

Bibliography

[AGN97] H. Andreka, S. R. Givant and I. Nemeti. Decision problems for equational theories of relation algebras. *Memoirs of the American Mathematical Society*, **126**, no. 604, [1997].

[AnH88] I. H. Anellis and N. R. Houser. Nineteenth Century Roots of Algebraic Logic and Universal Algebra. *Algebraic Logic*. [1988]. p. 1–36.

[AnN96] H. Andréka and I. Németi. Axiomatization of identity-free equations valid in relation algebra. *Algebra Universalis*, **35**, [1996]. p. 256–264.

[BDV01] P. Blackburn, M. de Rijke and Y. Venema. Modal Logic. *Cambridge Tracts in Theoretical Computer Science*, no. 53. [2001]. 576p.

[vBe79] van Benthem, J. A. F. K. Canonical modal logics and ultrafilter extensions. *Journal of Symbolic Logic*, **41**, [1979]. p. 436–438.

[vBe80] van Benthem, J. A. F. K. Some kinds of modal completeness. *Studia Logica*, **39**, [1980]. p. 125–141.

[vBe84] van Benthem, J. A. F. K. Correspondence theory. *Handbook of Philosophical Logic*, **2**, [1984]. p. 167–247.

[Ber] Bergman, C. Complex algebras of left-zero semigroups. *Private communication.*

[Bir33] Birkhoff, G. On the combination of subalgebras. *Proc. Cambridge Philosophical Society*, **29**, [1933]. p. 441–464.

[Bir35] Birkhoff, G. On the structure of abstract algebras. *Proc. Cambridge Philosophical Society*, **31**, [1935]. p. 433–454.

[Bir44] Birkhoff, G. Subdirect unions of universal algebra. *Bulletin of the American Mathematical Society*, **50**, [1944]. p. 764–768.

[Bir48] Birkhoff, G. *Lattice Theory*. Rhode Island: AMS Colloquium Publications Vol. 25. 1948. 283p.

[BKP84] W. J. Blok, P. Köhler and D. Pigozzi. On the structure on varieties with equationally definable principal congruences II. *Algebra Universalis*, **18**, [1984]. p. 334–379.

[Bla90] Blackburn, P. *Nominal tense logic and other sorted intensional frameworks*. Dissertation, University of Edinburgh. [1990]

[Blo78] Blok, W. J. On the degree of incompleteness of modal logics and the covering relation in the lattice of modal logics. [1978]. Amsterdam: Mathematisch Instituut Report 78-07.

[BlP82] W. J. Blok and D. Pigozzi. On the structure on varieties with equationally definable principal congruences I. *Algebra Universalis*, **15**, [1982]. p. 195–227.

[BlP89] W. J. Blok and D. Pigozzi. Algebraizable logics. *Memoirs of the American Mathematical Society*, **77** no. 396, [1989].

[Boo54] Boole, G. *An investigation of the laws of thought*. London: Walton and Maberly. [1854].

[Bri84] Brink, C. Second-order Boolean algebras. *Quaestiones Mathematicae*, **7**, no. 2, [1984]. p. 93–100.

[Bri92] Brink, C. *Power Structures and Their Applications*. D.Phil Thesis, Randse Afrikaanse Universiteit. [1992]

[Bri93] Brink, C. Power structures. *Algebra Universalis*, **30**, [1993]. p. 177–216.

[BGO95] C. Brink, D. M. Gabbay and H. J. Ohlbach. Towards automating duality. *Computers and Mathematics with Applications*, **29** no. 2, [1995]. p. 73–90.

[BuS81] S. Burris and H. Sankappanavar. *A course in universal algebra*. New York: Springer-Verlag [1981]. 276p.

[ChK77] C. C. Chang and H. J. Keisler. *Model Theory*. 2. Amsterdam: North-Holland Publishing Company. [1977]. 554p. (Studies in Logic).

[Coh65] Cohn, P. M. Universal Algebra. New York: Harper and Row. [1965].

[Com86] Comer, S. D. Multi-Valued Algebras and their Graphical Representations. [1986]. Mathematics and Computer Science Department, Charleston, South Carolina. (Preliminary Draft)

[Dev93] Devlin, K. *The Joy of Sets, Fundamentals of Contemporary Set Theory*. 2. New York: Springer-Verlag. [1993]. 192p. (Undergraduate Texts in Mathematics).

[Ehr61] Ehrenfeucht, A. An application of games to the completeness problem for formalised theories. *Fundamenta Mathematicae*, **49**, [1961]. p. 129–141.

[Fin74] Fine, K. An incomplete logic containing S4. *Theoria*, **40**, [1974]. p.23–29.

[Fin75] Fine, K. Some connections between modal and elementary logic. *Proc. First Scandinavian Logic Symposium*, North-Holland Publishing Company, [1975]. p.15–31. (Studies in Logic **82**.)

[Fra52] Fraïssé R. Sur les rapports entre la theorie des ralations et la semantique au sens de A. Tarski. *Colloque de logique mathematique*, [1952].

[Gab81] Gabbay, D. An irreflexivity lemma. *Aspects of Philosophical Logic*, [1981]. p. 67–89.

[GaT84] O. C. Garcia and W. Taylor. The lattice of interpretability types of varieties. *Memoirs of the American Mathematical Society*, **305**, [1984]. 125p.

[Gau57] Gautam, N. D. The validity of equations of complex algebras. *Arch. Math. Logik Grundlagen*, **3**, [1957]. p. 117–124.

[Gol89] Goldblatt, R. Varieties of complex algebras. *Annals of Pure and Applied Logic*, **44**, [1989]. p 173–242.

[Gol99] Goldblatt, R. Algebraic Polymodal Logic: A Survey. Preprint. [1999].

[GoV99] V. Goranko and D. Vakarelov. Hyperboolean algebras and hyperboolean modal logic. *Journal of Applied Non-classical Logics*, **9** no 2–3, [1999]. p. 345–368.

[Grä68] Grätzer, G. *Universal Algebra*. New York: D. Van Nostrand, Princeton. [1968].

[Grä79] Grätzer, G. *Universal Algebra*. 2. Springer-Verlag. [1979].

[GrL88] G. Grätzer and H. Lakser. Identities for global (complex algebras) of algebras. *Colloq. Math.*, **56**, [1988]. p. 19–29.

[GrS63] G. Grätzer and E. T. Schmidt. Characterisation of congruence lattices of abstract algebras. *Acta Sci. Math.*, **24**, [1963]. p.34–59.

[GrW84] G. Grätzer and S. Whitney. Infinitary varieties of structures closed under the formation of complex structures. *Colloq. Math*, **48**, [1984]. p. 1–5.

[HaH79] J. Hagemann and C. Hermann. A concrete ideal multiplication for algebraic systems and its relation to congruence distributivity. *Arch. Math. (Basel)*, **32**, [1979]. p234–245.

[HiH99] R. Hirsh and I. Hodkinson. Relational Algebras by Games. *to appear*. [1999]. 363p.

[Hod94] Hodges, W. *Model theory*. Wiltshire: Antony Rowe Ltd. [1994]. 772p.

[HMT71] L. Henkin, D. Monk and A. Tarski. *Cylindric Algebras I*. North-Holland. [1971]. 508p.

[HMT85] L. Henkin, D. Monk and A. Tarski. *Cylindric Algebras II*. North-Holland. [1985]. 302p.

[HMV99] I. Hodkinson, S. Mikulas and Y. Venema. Axiomatizing complex algebras by games. *Preprint*. currently available at http://www.doc.ic.ac.uk/ imh/. [1999]. 19p.

[Jec78] Jech, T. *Set Theory*. New York: Academic Press. [1978]. 621p. (Pure and Applied Mathematics).

[Jip93] Jipsen, P. Discriminator varieties of Boolean algebras with residuated operators. *Alg. Methods in Logic and Computer Science*, **28**, [1993]. p. 239–252

[Jip01] Jipsen, P. Some results about complex algebras of algebras. *Preprint*, [2001]. 8p.

[Jón67] Jónsson, B. Algebras whose congruence lattices are distributive. *Lecture Notes in Mathematics*, **250**, [1967]. New York: Springer-Verlag.

[Jón89] Jónsson, B. Introduction to Universal Algebra. [1989]. (Unpublished notes.)

[Jón93] Jónsson, B. A Survey of Boolean Algebras with Operators. *Algebras and Orders*, [1993]. p. 239–286

[Jón95] Jónsson, B. The Preservation Theorem for Canonical Extensions of Boolean Algebras with Operators. *Lattice Theory and its Applications*, [1995]. p. 121–130.

[JoT48] B. Jónsson and A. Tarski, A. Representation problems for relation algebras. *Bulletin of the American Mathematical Society*, **54**. [1948].

[JoT51] B. Jónsson and A. Tarski, A. Boolean algebras with operators I. *American Journal of Mathematics*, **73**, [1951]. p. 891–939.

[JoT52] B. Jónsson and A. Tarski. Boolean algebras with operators II. *American Journal of Mathematics*, **74**, [1952]. p. 127–162.

[Kei61] Keisler H. J. Ultraproducts and elementary classes. *Koninkl. Ned. Akad. Wetenschap Proc. Ser.*, **A 64**, [1961]. p. 477–495.

[Kei64] Keisler H. J. Good ideals in fields of sets. *Annals of Math.*, **79**, [1964]. p. 338–359.

[Kri59] Kripke S. A completeness theorem in modal logic. *The Journal of Symbolic Logic*, **24**, [1959]. p. 1–14.

[Kun72] Kunen, K. Ultrafilters and independent sets. *Trans. Amer. Math. Soc.*, **172**, [1972]. p. 199–206.

[LeS77] E. J. Lemmon and D. Scott. *An introduction to Modal Logic*, Oxford: Basil Blackwell. [1977]. 87p. (American Philosophical Quarterly).

[Łoś55] Łos, J. Quelques remarques théorème et problèmes sur les classes dèfinissables d'algèbres. *Mathematical interpretation of formal systems.* Amsterdam: North-Holland. [1955]. p.98–113

[Lyn50] Lyndon, R. The representation of relation algebras. *Annals of Mathematics*, **53**, no. 3, [1950]. p. 707–729.

[Lyn56] Lyndon, R. The representation of relation algebras, II. *Annals of Mathematics*, **63**, no. 2, [1956]. p. 294–307.

[Lyn61] Lyndon, R. Relation algebras and projective geometries. *Michigan Math J.*, **8**, [1961]. p. 21–28.

[Mal73] Maltsev, A. I. Algebraic systems. New York: Springer-Verlag. [1973]. (Grundlehren der mathematischen Wissenschaften, **192**).

[McK70] McKenzie, R. N. Representations of Integral Relation Algebras. *Michigan Mathematical Journal*, **17**, [1970]. p.279–287.

[McK75] McKenzie, R. N. On spectra, and the negative solution of the decision problem for identities having a finite nontrivial model. *Journal of Symbolic Logic*, **40**, no 2, [1975]. p.186–196.

[McL95] McLarty, C. Elementary Categories, Elementary Toposes. Oxford: Clarendon Press. [1995]. 265p. (Oxford Logic Guides).

[McT41] J. C. C. McKinsey and A. Tarski. A solution of the decision problem for the Lewis system S2 and S4. *Journal of Symbolic Logic*, **6**, [1941]. p. 117–124.

[McT48] J. C. C. McKinsey and A. Tarski. Some theorems about the sentential calculi of Lewis and Heyting. *Journal of Symbolic Logic*, **13**, [1948]. p. 1–15.

[MMT87] R. N. McKenzie, G. F. McNulty and W. F. Taylor. *Algebras, lattices, varieties.* California: Wadsworth, Inc. [1987]. 361p.

[Mon64] Monk, J. D. On representable relation algebras. *Michigan Mathematical Journal*, **11**, [1964]. p.207–210.

[DeM47] De Morgan, A. *Formal logic.* London: Taylor and Walton. [1847].

[DeM64] De Morgan, A. On the syllogism: IV, and on the logic of relations. *Transactions of the Cambridge Philosophical Society*, **10**, [1847]. p. 331–358.

[Ore35] Ore, O. On the foundations of abstract algebra. I. *Annals of Math.*, **36**, [1935]. p. 406–437.

[Ore36] Ore, O. On the foundations of abstract algebra. II. *Annals of Math.*, **37**, [1936]. p. 265–292.

[Pei80] Peirce, C. S. On the algebra of logic. *American Journal of Mathematics*, **3**, [1880]. p. 15–57.

[Pei83] Peirce, C. S. Note B: the logic of relatives. *Studies of Logic by Members of the Johns Hopkins University*, [1883]. p. 187–203.

[Pie68] Pierce, R. S. *Introduction to the theory of abstract algebra.* New York: Holt, Rinehart and Winston. [1968].

[Pos41] Post, E. L. The two-valued iterative systems of mathematical logic. *Annals of Math. Studies*, **5**, [1941].

[Rei96] Reich, P. J. Complex algebras of semigroups. Ph.D. Thesis. Iowa State University. [1996].

[DeR92] de Rijke, M. The modal logic of inequality. *The Journal of Symbolic Logic*, **57**, no 2, [1992]. p 566–584.

[Sch95] Schröder, F. W. K. E. *Vorlesungen üder die Algebra der Logik (exakte Logik).* Leipzig: E. Müller and B. G. Teubner. [1895]. 4vol.

[Sch66] Schröder, F. W. K. E. *Vorlesungen üder die Algebra der Logik (exakte Logik).* 2. New York: Chelsea Publ. Co. [1966]. 3vol.

[Sch00] Schalekamp, H. J. Some results on hyperboolean algebras and rings. *Proceedings of the ESSLLI-2000 Student Session* , [2000]. p. 244–253.

[She72] Shelah, S.. Every two elementarily equivalent models have isomorphic powers. *Israel Journal of Math.*, **10**, [1972]. p. 224–233.

[Sik64] Sikorski, R. *Boolean Algebras.* Berlin: Springer-Verlag. [1964]. (Ergebnisse der Mathematik und ihrer Grenzgebiete, Band **25**).

[Smi76] Smith, J. D. H. Mal'cev Varieties. *Lecture Notes in Mathematics*, **554**, [1976]. Berlin: Springer-Verlag.

[Sto36] Stone, M. H. The theory of representation for Boolean algebras. *Transcripts of the American Mathematical Society*, **40**, [1936]. p37–111.

[Tar31] Tarski, A. Sur les ensembles définissables de nombres réels I. *Fund. Math.*, **17**, [1931]. p. 210–239.

[Tar35] Tarski, A. Grundzüge des Systemenkalkul, Erster Teil. *Fundamenta Mathematicae*, **25**, [1935]. p503–526.

[Tar36] Tarski, A. Grundzüge des Systemenkalkul, Zweiter Teil. *Fundamenta Mathematicae* **26**, [1936]. p 283–301.

[Tar41] Tarski, A. On the calculus of relations. *Journal of Symbolic Logic*, **6**, [1941]. p. 73–89.

[Tar46] Tarski, A. A remark on functionally free algebras. *Annals of Mathematics*, **47**, [1946]. p. 163–165.

[Tar56] Tarski, A. *Logic, Semantics, Metamathematics*. Oxford: Oxford University Press. [1956]. p. 320–341.

[TaS67] T. Tamura and J. Shafer. Power semigroups. *Math. Japon*, **12**, [1967]. p. 25–32.

[Tho72] Thomason, S. K. Semantic Analysis of Tense Logic. *Journal of Symbolic Logic*, **37**, [1972]. p. 150–158.

[Tho74] Thomason, S. K. An incompleteness theorem in modal logic. *Theoria*, **40**, [1974]. p. 30–34.

[vdW31] van der Waerden, B. L. *Moderne Algebra*. Berlin: Julius Springer. [1931].

[Wer78] Werner, H. *Discriminator-Algebras*. Berlin: Akademie-Verlag. [1978]. 92p. (Studien zur Algebra und Ihre Anwendungen).

[Whi98] Whitehead, A. N. *A Treatise on Universal Algebra*. United Kingdom: Cambridge University Press. [1898].