A quasi-pseudometrizability problem for ordered metric spaces

by

Zechariah Mushaandja

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Department of Mathematics and Applied Mathematics
Faculty of Science, University of Cape Town

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Dedication

I dedicate this dissertation to all Namibians, young or old who have interest in intellectual matters of any kind, hungry for success in good activities that contribute to others and the world around, and particularly those who are willing to learn. Just like physical exercises are good for a healthy body, mental work-out is equally essential for a productive person. It gives us tools and power to deal with our daily activities successfully in any area of life. Moreover, as a fundamental human skill, thinking determines our achievements and our happiness as people.

"Intelligence, imagination and knowledge are essential resources but only effectiveness converts them into results." John C. Maxwell

Contents

Abstract

In this dissertation we obtain several results in the setting of ordered topological spaces related to the Hanai-Morita-Stone Theorem. The latter says that if f is a closed continuous map of a metric space X onto a topological space Y then the following statements are equivalent:

(i) Y satisfies the first countability axiom;

(ii) For each $y \in Y$, $f^{-1}\{y\}$ has a compact boundary in X;

(iii) Y is metrizable.

A partial analogue of the above theorem for ordered topological spaces is herein obtained. We particularly investigate the upper and lower topologies of metrizable ordered spaces which are both C- and I-spaces in the sense of Priestley. Among other results, we show that the bitopological spaces consisting of the upper topology and the lower topology associated with metrizable ordered spaces which are C- and I- satisfying properties like separability and local connectedness are quasi-pseudometrizable. Also, a partial order called friendly partial order is introduced and characterized. Furthermore, we show that a specified bispace associated with any uniform space endowed with this kind of partial order is quasi-uniformizable. Some interesting examples are also discussed.

Historical Background

Order is undoubtedly an important concept in mathematics but also in life in general. It is intrinsic to all our daily activities and it particularly injects conciseness in our communications. Its origin is in both logic and mathematics, and it is as old as the idea of number. Clearly, order was central to the work of G. Cantor in the development of set theory in the nineteenth century. In his remarkable work, Cantor introduced rich classes of the ordered sets, namely cardinals and ordinals (see for instance [11]). Furthermore, among other results, Cantor established that a countable linearly ordered set which is densely ordered with no end points is order-isomorphic to the rationals with the usual order. This was later generalized by Hausdorff.

It seems that the earliest documented work that marries order and topology was presented at the St Louis meeting in 1904 under the title *The fundamental Theorem of Analysis situs* and later published [56] in 1905. In this paper, Veblen, a student of E. H Moore, uses the idea of an ordered set and a topological space to define a simple arc and show that every metric continuum with exactly two non-cut points is homeomorphic to the unit interval. In his work, presented in the 1905, N. J. Lennes also a student of Moore, proved Veblen's result using more modern terminology and a purely topological definition of an arc [39]. More of such classical results appear in Fundamenta Mathematica volume 1-2, 1920-21. As from 1941, characterizations of orderability for other classes were found, such as for connected spaces, metric spaces and scattered spaces. For instance, Eilenberg proved that a connected space X is *weakly orderable* if and only if the subset of $X \times X$ obtained by deleting the diagonal is not connected [15, Theorem 1], where a space is weakly orderable if there exists a linear order on it such that the given topology on the space is finer than its open interval topology. This result came to be generalized by Banaschewski two decades later in [5]. The interested reader may consult [51] for more historical details on this.

In 1965, Nachbin's book *Topology and Order* [46] appeared. It is one of the general references on the subject available today, and it covers results obtained by the author in his research on spaces with structures of order and topology. In 1966, Borges introduced the property of monotone normality in [9] without calling it as such, and later Zenor [59] named the property and related it to metrizability and stratifiability. Then Borges [9] and Heath and Lutzer ([24], [25]) made a follow up on Zenor's original work. Borges gave characterizations of monotone normality and observed that this property holds in linearly ordered spaces. Further development of separation axioms for ordered spaces by McCartan in 1968 was another remarkable contribution to the body of knowledge on ordered spaces [43]. Later, McCartan [44] studied bicontinuous (herein called C-space and I-space) preordered topological spaces and investigated the relationship between the topology of such a space and two associated convex topologies. This partly motivated the joint work of Burgess and Fitzpatrick in [10], whose methods allow shorter proofs of earlier results of McCartan [43] and Nachbin [46] about the equivalence of T_2 and T_4-(T_3-)order axioms in compact (locally compact) spaces. The body of this thesis makes it evident that works of authors cited in this paragraph richly feed our current investigation of the main problem stated in the introduction.

In 1973, the independent but related works of Heath, Lutzer and Zenor were combined in [26] to study monotone normality in ordered spaces and, among others, characterize metrizability in terms of monotone normality. The Amsterdam contingent under the leadership of Maarten Maurice also comes to the show. Their contribution to the study of ordered spaces began with Maurice's PhD thesis [42] in 1964 at the Universiteit van Amsterdam and later through the Doctoral theses of his students like J. van Dalen, H. Hok, M. J. Faber, A. E. Brouwer, J. M. van Wouwe and K. P. Hart at the Vrije Universiteit. Among other topics, Faber studied metrizability in generalized ordered spaces which is in line with our current work. For the interested reader, the contents of most of these monographs are outlined in [23].

Completely regular ordered spaces are of particular interest in this subject. These are precisely the quasi-uniformizable ordered spaces. In his preprint *Order and strongly sober compactification* of 25 August 1989, Lawson introduced a stronger version of complete regularity namely, strictly complete regularity. He then asked for an example of a completely regular ordered space that is not strictly completely regular ordered. This was provided by Künzi in [32] in the following year, and it clearly demonstrates that Lawson's condition is much stronger. In the same paper, Künzi also shows that a completely regular ordered I-space is strictly completely regular ordered if the space satisfies at least one of the following conditions: locally compact, C-space or topological lattice. In the sequel [36] of 1994, Künzi and Watson studied metrizable completely regular ordered spaces, affirmatively settling the problem posed by Künzi in 1990. They achieved this by constructing a completely regular ordered space $(X, \mathcal{T}, \leqslant)$ where X is an I-space and $\mathcal{T}$ is metrizable such that the associated bitopological space $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is pairwise regular but not pairwise completely regular. In [47], Nailana pursues quasi-pseudometrizability of ordered spaces in the subclasses of point-open ordered spaces and the compact-open ordered spaces. He generalizes conditions for metrizability of the point-open topology and the compact-open topology for function spaces.

Furthermore, following the observation of Schwarz and Weck-Schwarz that a T_2-orderded space $(X, \mathcal{T}, \leqslant)$ with a completely regular underlying topological space $(X, \mathcal{T})$ need not be completely regularly ordered, Künzi and Richmond [34] show that this fact holds even when the topology is convex. In a more recent paper [2], though not on ordered spaces, Andrikopoulos considers the quasi-pseudometrization problem in (bi)topological spaces and generalizes well-known results on the subject in a "Bing-Nagata-Smirnov style", citing earlier authors like Kelly, Lane, Kopperman, Hung, Patty, Williams, Künzi, Salbany and many more.

Introduction

It is a well-known fact that metrics generate topologies. One may then wonder which topologies are obtained this way, and whether they can be characterized in terms of open sets. This question attracted attention of many researchers. In the early 1950s, Bing with Nagata and Smirnov independently provided similar characterizations, which essentially say that a topological space is metrizable if and only if it is T_3 and has a σ-locally finite base. Then in 1956 Hanai, Morita, and Stone established the following well-known Theorem (see [16, Theorem 4.4.17]):

Let f be a closed continuous mapping of a metric space X onto a topological space Y. Then the following statements are equivalent:

(i) Y satisfies the first countability axiom;

(ii) For each $y \in Y$, $f^{-1}\{y\}$ has a compact boundary in X;

(iii) Y is metrizable.

In this thesis we present some related results in the setting of ordered topological spaces $(X, \mathcal{T}, \leq)$. We particularly investigate the upper topology $\mathcal{T}^\natural$ and the lower topology $\mathcal{T}^\flat$ of those metrizable ordered spaces $(X, \mathcal{T}, \leq)$ which are both C- and I-spaces in the sense of Priestley [50].

Recall that, in some sense, the spaces $(X, \mathcal{T}^\natural)$ and $(X, \mathcal{T}^\flat)$ of an ordered C-space (resp. ordered I-space) $(X, \mathcal{T}, \leq)$ behave like closed (resp. open) images of $(X, \mathcal{T})$, where for each $x \in X$, the sets $i(x)$ and $d(x)$ correspond to the fibers of the map.[1]

In the light of this analogy, the following result due to Balachandran [4] involves fibers and is of some importance in this work.

If $f : X \to Y$ is a closed and open continuous map from a bounded metric space (X, e) onto a topological space Y, then Y is metrizable by the Hausdorff metric r, that is, r defined by

$$r(y_1, y_2) = \max\{ \sup_{y_1' \in f^{-1}\{y_1\}} e(y_1', f^{-1}\{y_2\}), \sup_{y_2' \in f^{-1}\{y_2\}} e(f^{-1}\{y_1\}, y_2')\}$$

[2] whenever $y_1, y_2 \in Y$.

In view of the above, one may then hope that if $(X, \mathcal{T}, \leq)$ is an ordered C- and I-space and e is a (bounded) metric inducing the topology $\mathcal{T}$, then

$$s(x, y) = \max\{ \sup_{x' \in d(x)} e(x', d(y)), \sup_{y' \in i(y)} e(i(x), y')\}$$

whenever $x, y \in X$ yields a quasi-pseudometric inducing the topologies of the bitopological space $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ associated with $(X, \mathcal{T}, \leq)$. Unfortunately, we have not been able to produce a counterexample to rebut this doubtful conjecture. (Indeed, note that in any case s, as defined above, is not the standard Hausdorff quasi-pseudometric (see e.g. [35]), since, of course, given $x \in X$, in general $d(x) \neq i(x)$). However, we shall demonstrate that under the assumption that both $d(x)$ and $i(x)$ are totally bounded whenever $x \in X$, we get an affirmative

[1]For example, compare [16, Problem 1.7.16] with the definition of the maps f_* and f^* discussed later in the thesis. However, this analogy cannot be pushed too far. While each fiber of a map is open or compact under the hypotheses mentioned in the result of Balachandran [4], the sets $i(x)$ and $d(x)$ ($x \in X$) of a metrizable ordered C- and I-space X need neither be open nor compact, as Example 5.4.8 below shows.

[2]Here $e(A, B) = \inf\{e(a, b) : a \in A, b \in B\}$ for given nonempty subsets $A, B \subseteq X$.

solution. Here we focus on the topological version underlying our problem, which does not demand for an explicit formula of the quasi-pseudometric in terms of the starting metric and hence can be formulated as follows:

Given an ordered topological space $(X, \mathcal{T}, \leq)$ which is a C-space and I-space such that $\mathcal{T}$ is metrizable, is the bitopological space $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ quasi-pseudometrizable?

The purpose of this thesis is to investigate this problem, provide some solutions under certain assumptions, and in addition present related results which may be of interest in their own rights. And of course, we achieve this by tackling several questions, and as expected this leads us to, and leaves us with some new open problems. The thesis is organized as described below.

A Brief Outline Of The Thesis

Chapter 0. In this first chapter many basic concepts from the theory of ordered topological spaces and some elementary results to be used throughout the thesis are presented. This chapter serves as a quick memory refresher to the reader with some background in the subject, and perhaps as a skeleton of the thesis language to all readers.

Chapter 1. Some known results are echoed as a motivation to the problem of study. Then the main problem investigated in this project is introduced in Subsection 1.2.1. As a smooth and encouraging start, a positive partial solution to the problem is given under the assumption that the topology we begin with is separable. In addition, we show that metrizability of an ordered space $(X, \mathcal{T}, \leq)$ which is a C-space and I-space imply that $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is pairwise stratifiable and doubly first countable. Furthermore, over intermediate steps, we prove that the bitopological space $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is quasi-pseudometrizable if and only if both the upper topology $\mathcal{T}^\natural$ and the lower topology $\mathcal{T}^\flat$ are quasi-pseudometrizable provided that $(X, \mathcal{T}, \leq)$ is an ordered metrizable C-space.

Chapter 2. Here we relate compactness to quasi-pseudometrizability in this setting. The notion of locally finite collections plays a central role in this part of the investigation. We show that if $(X, \mathcal{T}, \leq)$ is a metrizable ordered C-space and I-space such that $i(x)$ and $d(x)$ are compact for any $x \in X$, then the bitopological space $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is quasi-pseudometrizable. We then present a partial analogue of the Hanai-Morita-Stone Theorem in the ordered topological space setting. Also, a quasi-pseudometric is directly constructed on an ordered metric space such that it generates the upper and lower topology.

Chapter 3. We devote this chapter to relating uniform local connectedness to quasi-pseudometrizability of the bitopological space in Problem 1.2.1. We show that under the somewhat unexpected assumption of uniform local connectedness of an ordered metric space $(X, d, \leq)$ which is a C-space and I-space, the associated bitopological space $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is quasi-pseudometrizable.

Chapter 4. We provide a uniform version of the result obtained in Chapter 2 and in the process prove other related results. We do the earlier by carrying out a construction of a quasi-uniformity in two steps, and under the assumption of total boundedness, give an affirmative answer to our main problem. More precisely, given an ordered uniform space $(X, \mathcal{U}, \leq)$, we define quasi-uniformities $\mathcal{U}^i$ and $\mathcal{U}^d$ in terms of entourages of $\mathcal{U}$ whose corresponding topological space $(X, \mathcal{T}(\mathcal{U}), \leq)$ is assumed to be a metrizable C- and I-space such that for any $x \in X$, $i(x)$ and $d(x)$ are totally bounded. Then using these two quasi-uniformities we construct another one which we denote by $\mathcal{U}_\Uparrow$ so that its induced topology and that of its conjugate are quasi-pseudometrizabe.

Chapter 5. In the first section of this chapter we introduce a type of partial order on a metric space (X, m) and call it an m-friendly partial order (Definition 5.2.2) and continue to investigate Problem 1.2.1 in the setting where the metric topology and the m-friendly partial order are compatible in the sense described in the section. In the second section we study another version of the m-friendly partial order, this time defined on a uniform space $(X, \mathcal{U})$. We call it a $\mathcal{U}$-friendly partial order (see Definition 5.3.1), and prove results similar to those in the earlier section and more.

Chapter 6. In this last chapter we conclude this work by reflecting on the main results of the thesis and highlight some connections of this current work with old work in literature which we believe also provides a rich mine for future exploration. Furthermore, we pose a generalized version of the main problem of study here in a way which may also stimulate different approaches and probably more insight.

Chapter 0

Preliminaries

0.1 Introduction

In this chapter we mainly give an overview of notation, terminology and some elementary results, most of which already appear in the literature, to be used throughout the thesis. For more on these basics, the reader is referred to, among others, [29], [32], [33] and [46].

0.2 Ordered Topological Spaces

An *ordered topological space* is a triple $(X, \mathcal{T}, \leq)$ where X is a set equipped with a topology $\mathcal{T}$ and a partial order $\leq$. Sometimes we shall write X instead of $(X, \mathcal{T}, \leq)$ when no confusion is likely to arise. A mapping $f : X \to Y$ between ordered topological spaces X and Y is said to be *increasing* (*decreasing*) if and only if $f(x) \leq f(y)$ $(f(x) \geq f(y))$ whenever $x, y \in X$ and $x \leq y$.

Let X be an ordered topological space and $A \subseteq X$, its subset. Then we call A an *upper set* of X if and only if $x \in A$, $y \in X$ and $x \leq y$ together imply $y \in A$. Dually, we say that A is a *lower set* of X if and only if $x \in A$, $y \in X$ and $y \leq x$ together imply $y \in A$. For any subset A of X, $i(A)$ $(d(A))$ will denote the intersection of all upper (lower) sets of X containing A. Note that $i(A)$ $(d(A))$

is the smallest upper (lower) set containing A. It is easy to see that $A = i(A)$ if and only if A is an upper set. Similarly, $A = d(A)$ if and only if A is a lower set. We recall that an ordered topological space $(X, \mathcal{T}, \leq)$ is called a *C-space* if and only if $d(F)$ and $i(F)$ are closed whenever F is a closed subset of X. Similarly, an ordered topological space X is called an *I-space* if and only if $d(G)$ and $i(G)$ are open whenever G is an open subset of X. We shall say that $(X, \mathcal{T}, \leq)$ is a *C- and I-space* if it is both a C-space and an I-space. Furthermore, a topological space X is *completely regular* if and only if whenever F is a closed subset of X and $x \in X \setminus F$, there is a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ and $f(F) = \{1\}$. Here $[0,1]$ is equipped with the Euclidean topology herein denoted by $\mathcal{J}$.

We now define the maps f_* and f^* mentioned in the introduction above.
Let $f : X \to [0,1]$ be a continuous function, where $[0,1]$ is equipped with the Euclidean topology and the usual order. Then for any $x \in X$, put

$$f^*(x) := \sup\{f(y) \mid y \in d(x)\} \text{ and } f_*(x) := \inf\{f(y) \mid y \in i(x)\}.$$

In [32, Proof of Proposition 2], Künzi shows that, on an ordered topological space $(X, \mathcal{T}, \leq)$ which is a C- and I-space with $\mathcal{T}$ completely regular, f^* and f_* are continuous whenever f is continuous. It is immediate from the definitions of f^* and f_* that $f_* \leq f \leq f^*$. Furthermore, it is easy to see that both f^* and f_* are increasing.
The following may help illustrate the analogy announced in the introduction.
Let $f : X \to Y$ be an open continuous mapping of X onto Y, and $g : X \to \mathbb{R}$ be a continuous function that is bounded on $f^{-1}(y)$ for all $y \in Y$. Then it is known that $g^*(y) := \sup\{g(x) \mid x \in f^{-1}(y)\}$ is a lower semicontinuous function and $g_*(y) := \inf\{g(x) \mid x \in f^{-1}(y)\}$ is an upper semicontinuous function on Y. In the case f is closed then a similar result with lower and upper interchanged holds (see for instance [27, Lemma 4.1]).

0.3 Bitopological Spaces and Quasi-pseudometrics

A *bitopological space* (*bispace*) is a triple $(X, \mathcal{T}_1, \mathcal{T}_2)$ where X is a set equipped with two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$. Hereafter, the terms bitopological space and bispace shall be used interchangeably. In a bispace $(X, \mathcal{T}_1, \mathcal{T}_2)$, the topology $\mathcal{T}_1$ is said to be *regular with respect to* $\mathcal{T}_2$ if and only if for each point $x \in X$ and each $\mathcal{T}_1$-closed set F with $x \notin F$, there are a $\mathcal{T}_1$-open set U and a $\mathcal{T}_2$-open set V such that $x \in U$, $F \subseteq V$ and $U \cap V = \emptyset$. Then we say that the two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ are, or simply the bispace $(X, \mathcal{T}_1, \mathcal{T}_2)$ is *pairwise regular* if and only if $\mathcal{T}_1$ is regular with respect to $\mathcal{T}_2$ and $\mathcal{T}_2$ is regular with respect to $\mathcal{T}_1$.
In the following, we shall consider the bispace $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ associated with a given ordered topological space $(X, \mathcal{T}, \leq)$ where the upper topology $\mathcal{T}^\natural$ and the lower topology $\mathcal{T}^\flat$ are respectively given by

$$\mathcal{T}^\natural := \{U \in \mathcal{T} \mid U \text{ is an upper set}\} \text{ and } \mathcal{T}^\flat := \{D \in \mathcal{T} \mid D \text{ is a lower set}\}.$$

Recall that a *bicontinuous function* $f : (X, \mathcal{T}_1, \mathcal{T}_2) \to (Y, \mathcal{V}_1, \mathcal{V}_2)$ is a function $f : X \to Y$ such that $f : (X, \mathcal{T}_i) \to (Y, \mathcal{V}_i)$ is continuous for each $i = 1, 2$. A bitopological space $(X, \mathcal{T}_1, \mathcal{T}_2)$ is *pairwise completely regular* if and only if for each $x \in X$ and each $\mathcal{T}_1$-closed set F with $x \notin F$, there exists a bicontinuous function $f : (X, \mathcal{T}_1, \mathcal{T}_2) \to ([0,1], \mathcal{J}^\natural, \mathcal{J}^\flat)$ such that $f(x) = 1$ and $f(F) = \{0\}$; and for each $\mathcal{T}_2$-closed set Q with $x \notin Q$, there exists a bicontinuous function $g : (X, \mathcal{T}_1, \mathcal{T}_2) \to ([0,1], \mathcal{J}^\natural, \mathcal{J}^\flat)$ such that $g(x) = 0$ and $g(Q) = \{1\}$. A function $q : X \times X \to [0, \infty)$ is called a *quasi-pseudometric* on X if and only if $q(x,x) = 0$ and $q(x,z) \leq q(x,y) + q(y,z)$ where $x, y, z \in X$. If, in addition, $q(x,y) = 0$ if and only if $x = y$ then q is called a *quasi-metric*. If also q is symmetric, that is, $q(x,y) = q(y,x)$ for all $x, y \in X$ then q is called a *metric*. Given a quasi-pseudometric q on X, then $q^{-1} : X \times X \to [0, \infty)$ defined by $q^{-1}(x,y) = q(y,x)$ is also a quasi-pseudometric on X. The two quasi-pseudometrics q and q^{-1} are said to be conjugate. For any quasi-pseudometric q, and $x \in X$, $\epsilon > 0$, the collection of open sets $B_q(x, \epsilon) := \{y \in X \mid q(x,y) < \epsilon\}$ form a base for the

topology $\mathcal{T}(q)$, which is called the *quasi-pseudometric topology on X*. Similarly, q^{-1} determines a topology $\mathcal{T}(q^{-1})$ on X.
We say that a bitopological space $(X, \mathcal{T}_1, \mathcal{T}_2)$ is *quasi-pseudometrizable* if and only if there is a quasi-pseudometric q on X such that $\mathcal{T}_1 = \mathcal{T}(q)$ and $\mathcal{T}_2 = \mathcal{T}(q^{-1})$.

0.3.1 Examples

The examples below are well-known. They appear for instance in [29].

a) Let $X = \mathbb{R}$, and define $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ by $d(x, y) = min\{1, |x - y|\}$ for any $x, y \in \mathbb{R}$. Then d is a metric on $\mathbb{R} \times \mathbb{R}$.

For $x, y \in \mathbb{R}$, let $p(x, y) = \begin{cases} d(x, y), & \text{if } x \leq y\,; \\ 1, & \text{if } x > y\,; \end{cases}$

and

$$q(x, y) = \begin{cases} d(x, y), & \text{if } x \geq y\,; \\ 1, & \text{if } x < y. \end{cases}$$

Then note that $q(x, y) = p(y, x)$. Also, one can easily check that p is a quasi-pseudometric on $\mathbb{R}$ and hence so is q. The quasi-pseudometric p generates the topology $T(p)$ on $\mathbb{R}$ with base $\{[a, b) \,|\, a, b \in \mathbb{R}\}$ and q generates the topology $T(q)$ on $\mathbb{R}$ with $\{(a, b] \,|\, a, b \in \mathbb{R}\}$ as its base. Thus $(\mathbb{R}, T(p), T(q))$ is a quasi-pseudometrizable bispace.

b) Similarly, with the metric d as in a) above and $x, y \in \mathbb{R}$, define u on $\mathbb{R}^2$ by

$$u(x, y) = \begin{cases} 0, & \text{if } x \leq y\,; \\ d(x, y), & \text{if } x > y\,; \end{cases}$$

and put $v(x, y) = u(y, x)$. Then u and v are conjugate quasi-pseudometrics on $\mathbb{R}$ generating the topologies $T(u)$ and $T(v)$ with the sets $\{(-\infty, a) \,|\, a \in \mathbb{R}\}$ and $\{(a, \infty) \,|\, a \in \mathbb{R}\}$ as their bases respectively. Hence the bispace $(\mathbb{R}, T(u), T(v))$ is quasi-pseudometrizable.

Chapter 1

On The Main Problem

1.1 Introduction

We begin this chapter by recalling some known results which build up the motivation for the main problem (Subsection 1.2.1) discussed in this thesis. Then we present an affirmative solution to the problem for a special case of separable spaces. Using the concept of monotone normality, we show that stratifiability of an ordered topological C-space implies pairwise stratifiability of the associated bispace given by the lower topology and the upper topology. We shall close the chapter by showing that metrizability of an ordered topological C- and I-space is sufficient for pairwise stratifiability and double first countability of the bispace given by the upper topology and the lower topology.

1.2 Towards the Problem

According to Burgess and Fitzpatrick [10, Corollary 6.2], if $(X, \mathcal{T}, \leq)$ is an ordered topological C-space and I-space such that $\mathcal{T}$ is completely regular, then the associated bitopological space $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is pairwise completely regular. Künzi gives an elementary proof of this result in [32, Proposition 2]. Recall that a topological space $(X, \mathcal{T})$ is *uniformizable* if and only if there is a uniformity $\mathcal{U}$

on X such that $\mathcal{T} = \mathcal{T}(\mathcal{U})$. Concepts related to uniformities will be defined in Section 3.2. It is known that a topology is uniformizable if and only if it is completely regular. Similarly, a bitopological space $(X, \mathcal{T}_1, \mathcal{T}_2)$ is quasi-uniformizable if and only if there exists a quasi-uniformity $\mathcal{Q}$ on X such that $\mathcal{T}_1 = \mathcal{T}(\mathcal{Q})$ and $\mathcal{T}_2 = \mathcal{T}(\mathcal{Q}^{-1})$. In [37, Theorem 4.2], Lane establishes that a bitopological space is pairwise completely regular if and only if it is quasi-uniformizable.

Therefore we can reformulate the result of Burgess and Fitzpatrick as follows: If $(X, \mathcal{T}, \leq)$ is an ordered topological C-space and I-space such that $\mathcal{T}$ is uniformizable then the bispace $(X, \mathcal{T}^{\natural}, \mathcal{T}^{\flat})$ is quasi-uniformizable. Of course, then the question can be asked whether there exists a uniformized version of the aforementioned result, in the sense that a simple construction could be found that transforms a given uniformity $\mathcal{U}$ compatible with $\mathcal{T}$ under our assumptions into a quasi-uniformity $\mathcal{Q}$ inducing both topologies $\mathcal{T}^{\natural}$ and $\mathcal{T}^{\flat}$.

Recall that the *weight of a quasi-uniformity* $\mathcal{Q}$ denoted by $w(\mathcal{Q})$ is defined as the minimal cardinality of a base of $\mathcal{Q}$. Then, in particular, it is natural to wonder whether the desired construction could satisfy even the inequality $w(\mathcal{Q}) \leq w(\mathcal{U})$. In the case of countable weight, the above question leads to the following intriguing question which is our main problem.

1.2.1 The Problem

If $(X, \mathcal{T}, \leq)$ is an ordered topological C- and I-space which is pseudometrizable, is the associated bispace $(X, \mathcal{T}^{\natural}, \mathcal{T}^{\flat})$ quasi-pseudometrizable?

We begin by recalling the following result of 1963 due to J. C. Kelly.

1.2.2 Lemma

([29, Theorem 2.8]) *Any pairwise regular bitopological space $(X, \mathcal{T}_1, \mathcal{T}_2)$ satisfying the second axiom of countability is quasi-pseudometrizable.* $\square$

As announced earlier, for a separable pseudometrizable space, we provide a positive solution to this problem.

1.2.3 Proposition

If $(X, \mathcal{T}, \leq)$ is an ordered topological C- and I-space such that $\mathcal{T}$ is separable and pseudometrizable, then $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is quasi-pseudometrizable.

Proof. Suppose $(X, \mathcal{T}, \leq)$ is an ordered topological C- and I-space which is separable and pseudometrizable. Since any pseudometrizable space is completely regular, $(X, \mathcal{T}, \leq)$ is completely regular. Then it follows that the bitopological space $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is pairwise completely regular [10] or [32, Proposition 2]. Since $\mathcal{T}$ is separable, it has a countable base, say $\mathcal{B}$. Let $\mathcal{B}_1 = \{d(B) \mid B \in \mathcal{B}\}$ and $\mathcal{B}_2 = \{i(B) \mid B \in \mathcal{B}\}$. We show that $\mathcal{B}_1$ is a countable base for the lower topology $\mathcal{T}^\flat$ and $\mathcal{B}_2$ is a countable base for the upper topology $\mathcal{T}^\natural$. Let $x \in G$ where G is an open lower set, that is, $G \in \mathcal{T}^\flat$. Since $G \in \mathcal{T}$ and $\mathcal{B}$ is a base for $\mathcal{T}$, there exists some $B \in \mathcal{B}$ such that $x \in B \subseteq G$. Since X is an I-space, $d(G)$ is open. But $d(G)$ is a lower set, so it lies in $\mathcal{T}^\flat$. Also, $G = d(G)$, and since d preserves inclusion, we have $d(B) \subseteq d(G)$ and thus $x \in d(B) \subseteq d(G)$. Since $\mathcal{B}$ is countable so is $\mathcal{B}_1$. Hence $\mathcal{B}_1$ is a countable base for $\mathcal{T}^\flat$.
Similarly, let $U \in \mathcal{T}^\natural$ and $x \in U$. Since $U \in \mathcal{T}$, there exists some $B \in \mathcal{B}$ such that $x \in B \subseteq U$. Also, $i(B)$ is open because X is an I-space. Thus $x \in i(B) \subseteq i(U) = U$. Since $\mathcal{B}$ is countable so is $\mathcal{B}_2$. Hence $\mathcal{B}_2$ is a countable base of $\mathcal{T}^\natural$. Now we have a pairwise completely regular bispace $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ where each of the two topologies $\mathcal{T}^\natural$ and $\mathcal{T}^\flat$ has a countable base. By Kelly's result above, it follows that $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is quasi-pseudometrizable. $\square$

1.3 Stratifiable Spaces and Quasi-pseudometrization

For the sake of completeness and a smooth flow of things, we briefly recall some definitions and relevant basic facts regarding stratifiable spaces.

1.3.1 Definition

([26]) A topological space $(X, \mathcal{T})$ is said to be *monotonically normal* if and only if it is T_1 and there is a function G which assigns to each ordered pair (A, B) of disjoint closed subsets of X an open set $G(A, B)$ such that

(i) $A \subseteq G(A, B) \subseteq cl_{\mathcal{T}} G(A, B) \subseteq X \setminus B$;

(ii) If (A', B') is a pair of disjoint closed sets such that $A \subseteq A'$ and $B' \subseteq B$ then $G(A, B) \subseteq G(A', B')$.

The function G is called a *monotone normality operator* for X.

Remark. There are several equivalent characterizations of monotonically normal space (see for instance [8] and [21]). For any monotone normality operator G for X, one can define another monotone normality operator $\tilde{G}$ for X by
$\tilde{G}(A, B) = G(A, B) \setminus \overline{G(B, A)}$ for any pair (A, B) of disjoint closed subsets of X. It is not hard to see that $\tilde{G}$ satisfies $\tilde{G}(A, B) \cap \tilde{G}(B, A) = \emptyset$. Furthermore, we point out that every monotonically normal space is normal.

1.3.2 Definition

([26]) A topological space $(X, \mathcal{T})$ is said to be *semi-stratifiable* if and only if there is a map G which assigns to each $n \in \mathbb{N}$ and a closed subset $F \subseteq X$, an open set $G(n, F)$ containing F such that

(i) $F = \bigcap_{n=1}^{\infty} G(n, F)$;

(ii) $F \subseteq K \Longrightarrow G(n, F) \subseteq G(n, K)$.

If also

(iii) $F = \bigcap_{n=1}^{\infty} \overline{G(n, F)}$

then we say that $(X, \mathcal{T})$ is *stratifiable.*

Here the map $G : \mathbb{N} \times \{F \subseteq X \mid F \,\text{closed}\,\} \to \mathcal{T}$ is called a *(semi-)stratification* for X.

We now turn to some examples. These have been studied by Gruenhage in [21]. However, to help the concept sink in well, we give detailed proofs.

1.3.3 Examples

a) ([21]) *Every metric space (X, d) is stratifiable.*

Proof. Set $G(n, F) = \{x \in X \mid d(x, F) < 2^{-n}\}$ for any closed set $F \subseteq X$ and $n \in \mathbb{N}$. Let $a \in \bigcap_{n=1}^{\infty} G(n, F)$. Then $a \in G(n, F)$ for each $n \in \mathbb{N}$ and so $d(a, F) < 2^{-n}$ for all $n \in \mathbb{N}$. This means $d(a, f) < 2^{-n}$ for all $n \in \mathbb{N}$ and $f \in F$. Thus $a \in \overline{F} = F$. Hence $\bigcap_{n=1}^{\infty} G(n, F) \subseteq F$. Conversely, note that $F \subseteq G(n, F)$ for each $n \in \mathbb{N}$ because $d(f, F) = 0$ for all $f \in F$. Thus $F \subseteq \bigcap_{n=1}^{\infty} G(n, F)$, and together we have $F = \bigcap_{n=1}^{\infty} G(n, F)$. Also, given two closed subsets of X, say F and K with $F \subseteq K$, then every lower bound of $\{d(x, b) \mid b \in K\}$ is a lower bound of $\{d(x, a) \mid a \in F\}$ and hence $\inf\{d(x, b) \mid b \in K\} \leq \inf\{d(x, a) \mid a \in F\}$, that is, $d(x, K) \leq d(x, F)$. Then clearly $d(x, K) < 2^{-n}$ whenever $d(x, F) < 2^{-n}$ for each $n \in \mathbb{N}$. Hence $G(n, F) \subseteq G(n, K)$ for each $n \in \mathbb{N}$.

For the third condition, we already have $F \subseteq G(n, F)$. Since $G(n, F) \subseteq \overline{G(n, F)}$ for each $n \in \mathbb{N}$ and closed $F \subseteq X$, it follows that $\bigcap_{n=1}^{\infty} G(n, F) \subseteq \bigcap_{n=1}^{\infty} \overline{G(n, F)}$. Thus $F \subseteq \bigcap_{n=1}^{\infty} \overline{G(n, F)}$. Conversely, pick any $s \in \bigcap_{n=1}^{\infty} \overline{G(n, F)}$. Then $s \in \overline{G(n, F)}$ for each $n \in \mathbb{N}$. This implies that there is a sequence $(s_n)_{n\in\mathbb{N}}$ in $G(n, F)$ which converges to s. So $d(s_n, F) < 2^{-n}$ and $d(s_n, s) \to 0$. But $d(s, F) = \inf\{d(s, f) \mid f \in F\} \leq \inf\{d(s_n, s) \mid n \in \mathbb{N}\} = 0$, that is, $d(s, F) = 0$ so that $s \in F$. Thus $\bigcap_{n=1}^{\infty} \overline{G(n, F)} \subseteq F$ and hence $F = \bigcap_{n=1}^{\infty} \overline{G(n, F)}$ for each $n \in \mathbb{N}$ and any closed $F \subseteq X$. Therefore G is a stratification for X. $\square$

b) ([21]) *Every developable space is semi-stratifiable.*

Proof. We first recall that a space X is said to be *developable* if it has a development, where a *development* for X is a sequence $(\mathcal{U}_n)_{n\in\mathbb{N}}$ of open covers of X such that $\mathcal{U}_n < \mathcal{U}_{n-1}$ and the collection $\{St(x,\mathcal{U}_n) \,|\, n = 1, 2, ...\}$ is a neighbourhood base at each $x \in X$. Here, $\mathcal{U}_n < \mathcal{U}_{n-1}$ means each $U \in \mathcal{U}_n$ is contained in some $V \in \mathcal{U}_{n-1}$ (see [57, Definition 20.1]). For any closed subset $F \subseteq X$, define $G(n, F) := St(F, \mathcal{G}_n)$ where, as usual $St(F, \mathcal{G}_n) = \bigcup\{D \in \mathcal{G}_n \,|\, D \cap F \neq \emptyset\}$ and $\mathcal{G}_n$ is an open cover of X where $n \in \mathbb{N}$. Note that $x \in \bigcap_{n=1}^{\infty} G(n, F) \iff x \in St(F, \mathcal{G}_n) \iff x \in D$ for some $D \in \mathcal{G}_n$ with $D \cap F \neq \emptyset \iff St(x, \mathcal{G}_n) \cap F \neq \emptyset \iff x \in F$. Thus $F = \bigcap_{n=1}^{\infty} G(n, F)$. To see that G is monotone, suppose F and K are closed subsets of X such that $F \subseteq K$. If $D \in \mathcal{G}_n$ with $D \cap F \neq \emptyset$ then obviously $D \cap K \neq \emptyset$. Hence, without further effort, $St(F, \mathcal{G}_n) \subseteq St(K, \mathcal{G}_n)$. $\square$

Recall that a topological space X is called an *M_1-space* if and only if X is regular and has a σ-closure preserving base [13].

c) ([13], [21]) *Every M_1-space is stratifiable.*

Proof. Suppose X is an M_1-space. Then X is regular and has a σ-closure preserving base for X, say $\mathcal{B} = \bigcup\limits_{n\in\mathbb{N}} \mathcal{B}_n$. Put $G(n, H) = X \setminus \bigcup\{\overline{B} \,|\, B \in \mathcal{B}_n,\ \overline{B} \cap H = \emptyset\}$ for any closed subset $H \subseteq X$ and $n \in \mathbb{N}$. Note that $G(n, H) = \bigcap\{X \setminus \overline{B} \,|\, B \in \mathcal{B}_n,\ \overline{B} \cap H = \emptyset\}$. Since $\overline{B} \cap H = \emptyset$ implies $H \subseteq X \setminus \overline{B}$ for each $B \in \mathcal{B}_n$, it follows that $H \subseteq \bigcap(X \setminus \overline{B})$ for each such B. Thus for each $n \in \mathbb{N}$ and a closed $H \subseteq X$ we get $H \subseteq G(n, H)$ so that $H \subseteq \bigcap\limits_{n\in\mathbb{N}} G(n, H)$. For the other inclusion, let $x \in X \setminus H$. Since $X \setminus H$ is open, it contains a neighbourhood of each of its points. So there is some $B \in \mathcal{B}_n$ such that $x \in B \subseteq X \setminus H$. This says that there exists $B \in \mathcal{B}_n$ such that $x \notin X \setminus B$. Since $B \subseteq \overline{B}$ we deduce that for such $B \in \mathcal{B}_n$ we have $x \notin X \setminus \overline{B}$. Equivalently, for each $B \in \mathcal{B}_n$, with $\overline{B} \cap H = \emptyset$ we have $X \setminus \overline{B} \subseteq H$. Thus $\bigcap(X \setminus \overline{B}) \subseteq H$, and hence $\bigcap\limits_{n\in\mathbb{N}} G(n, H) \subseteq H$. So $\bigcap\limits_{n\in\mathbb{N}} G(n, H) = H$. Suppose H_1 and H_2 are closed subsets of X such that $H_1 \subseteq H_2$. Then $\{\overline{B} \,|\, B \in \mathcal{B}_n,\ \overline{B} \cap H_2 = \emptyset\} \subseteq \{\overline{B} \,|\, B \in \mathcal{B}_n,\ \overline{B} \cap H_1 = \emptyset\}$ which implies that $\bigcup\{\overline{B} \mid B \in \mathcal{B}_n, \overline{B} \cap H_2 = \emptyset\} \subseteq \bigcup\{\overline{B} \mid B \in \mathcal{B}_n, \overline{B} \cap H_1 = \emptyset\}$ and hence $G(n, H_1) \subseteq G(n, H_2)$. The third condition holds essentially because H is closed by assumption and $G(n, H) \subseteq \overline{G(n, H)}$. Therefore X is stratifiable. $\square$

Next, we look at the corresponding notion for bispaces.

1.4 Pairwise Stratifiable Spaces and Quasi-pseudometrization

1.4.1 Definition

([41]) A bitopological space $(X, \mathcal{T}_1, \mathcal{T}_2)$ is said to be *pairwise semi-stratifiable* if and only if for all $\mathcal{T}_i$-closed set $F, K \subseteq X$ there exist sequences of $\mathcal{T}_j$-open sets $(F_n)_{n\in\mathbb{N}}$ and $(K_n)_{n\in\mathbb{N}}$ satisfying the following two conditions ($i, j \in \{1, 2\}$ and $i \neq j$):

(i) If $F \subseteq K$ then $F_n \subseteq K_n$ for all $n \in \mathbb{N}$;

(ii) $F = \bigcap_{n=1}^{\infty} F_n$.

If, in addition, we also have

(iii) $F = \bigcap_{n=1}^{\infty} cl_{\mathcal{T}_i} F_n$,

then $(X, \mathcal{T}_1, \mathcal{T}_2)$ is said to be *pairwise stratifiable*.

In what follows, we apply the above concept to the bispace $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ associated with an ordered topological space $(X, \mathcal{T}, \leq)$. But we first recall the following characterization of stratifiable spaces. As Gruenhage puts it, it tells us that the difference between stratifiable and semi-stratifiable spaces is monotone normality.

1.4.2 Theorem

[20, Theorem 5.16] *A topological space X is stratifiable if and only if it is semi-stratifiable and monotonically normal.* $\square$

We now establish the following new fact.

1.4.3 Proposition

If $(X, \mathcal{T}, \leq)$ is a stratifiable ordered topological C-space then the bispace $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is pairwise stratifiable.

Proof. Suppose $(X, \mathcal{T}, \leq)$ is stratifiable. Then there exists a stratification S_n for $(X, \mathcal{T}, \leq)$. By Theorem 1.4.2, $(X, \mathcal{T}, \leq)$ is semi-stratifiable and monotonically normal. This guarantees the existence of a monotone normality operator D for $\mathcal{T}$. Put $\psi_n(F) = X \setminus d(X \setminus D(F, d(X \setminus S_n(F))))$ for any closed upper set $F \subseteq X$. We claim that ψ_n gives us a $\mathcal{T}^\natural$-stratification of $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ with respect to $\mathcal{T}^\flat$. Since S_n is a stratification, $S_n(F)$ is an open subset of X, and so $X \setminus S_n(F)$ is closed. Given that $(X, \mathcal{T}, \leq)$ is a C-space, it follows that $d(X \setminus S_n(F))$ is a closed lower set. Thus the operator D in $\psi_n(F)$ acts on a pair of closed sets. Also, $F \cap d(X \setminus S_n(F)) = \emptyset$. Otherwise, there exists some $a \in F \cap d(X \setminus S_n(F))$ so that $a \in F$ and $a \in d(X \setminus S_n(F))$. The latter implies that $a \in F$ and $a \leq t$ for some $t \in X \setminus S_n(F)$. Since F is an upper set, we then deduce that $t \in F$. Now we have $t \in F$ and $t \notin S_n(F)$. But $F \subseteq S_n(F)$, so we have reached a contradiction. Hence $F \cap d(X \setminus S_n(F)) = \emptyset$ as asserted. Since D is a monotone normality operator, it follows that $D(F, d(X \setminus S_n(F)))$ is open and so $X \setminus D(F, d(X \setminus S_n(F)))$ is closed. Since X is a C-space, $d(X \setminus D(F, d(X \setminus S_n(F))))$ is a closed lower set and then $X \setminus d(X \setminus D(F, d(X \setminus S_n(F)))) = \psi_n(F)$ is an open upper set, that is, $\psi_n(F) \in \mathcal{T}^\natural$.

Next we show that $F_1 \subseteq F_2$ implies $\psi_n(F_1) \subseteq \psi_n(F_2)$ for any $n \in \mathbb{N}$. Suppose $F_1 \subseteq F_2$. Then $S_n(F_1) \subseteq S_n(F_2)$ because S_n being a stratification preserves inclusion, and then $X \setminus S_n(F_2) \subseteq X \setminus S_n(F_1)$. This implies that $d(X \setminus S_n(F_2)) \subseteq d(X \setminus S_n(F_1))$. Now we have $F_1 \subseteq F_2$ and $d(X \setminus S_n(F_2)) \subseteq d(X \setminus S_n(F_1))$, hence $D(F_1, d(X \setminus S_n(F_1))) \subseteq D(F_2, d(X \setminus S_n(F_2)))$. Then $X \setminus D(F_2, d(X \setminus S_n(F_2))) \subseteq X \setminus D(F_1, d(X \setminus S_n(F_1)))$ so that $d(X \setminus D(F_2, d(X \setminus S_n(F_2)))) \subseteq d(X \setminus D(F_1, d(X \setminus S_n(F_1))))$. Thus $X \setminus d(X \setminus D(F_1, d(X \setminus S_n(F_1)))) \subseteq X \setminus d(X \setminus D(F_2, d(X \setminus S_n(F_2))))$ which says $\psi_n(F_1) \subseteq \psi_n(F_2)$.

Now we show that $F = \bigcap\{\psi_n(F) \mid n \in \mathbb{N}\}$ for any closed upper set $F \subseteq X$.

Always $F \subseteq D(F, d(X \setminus S_n(F)))$ by definition of a monotone normality operator so, $X \setminus D(F, d(X \setminus S_n(F))) \subseteq X \setminus F$ and hence $d(X \setminus D(F, d(X \setminus S_n(F)))) \subseteq d(X \setminus F)$. Since $X \setminus F$ is a lower set,

$$d(X \setminus F) = X \setminus F \text{ hence } d(X \setminus D(F, d(X \setminus S_n(F)))) \subseteq X \setminus F$$

and this implies that

$F \subseteq X \setminus d(X \setminus D(F, d(X \setminus S_n(F)))) = \psi_n(F)$ for each $n \in \mathbb{N}$. Hence $F \subseteq \bigcap \psi_n(F)$.

For the other inclusion, note that $D(F, d(X \setminus S_n(F))) \subseteq X \setminus d(X \setminus S_n(F))$ since D is a monotone normality operator, and hence $d(X \setminus S_n(F)) \subseteq X \setminus D(F, d(X \setminus S_n(F)))$. Since $d^2 = d$, it follows that $d(X \setminus S_n(F)) \subseteq d(X \setminus D(F, d(X \setminus S_n(F))))$. Always $X \setminus S_n(F) \subseteq d(X \setminus S_n(F))$ and this implies that $X \setminus d(X \setminus S_n(F)) \subseteq S_n(F)$. Hence $X \setminus d(X \setminus D(F, d(X \setminus S_n(F)))) \subseteq S_n(F)$ for each $n \in \mathbb{N}$.

Therefore $\bigcap\{X \setminus d(X \setminus D(F, d(X \setminus S_n(F))))\} \subseteq \bigcap\{S_n(F) \,|\, n \in \mathbb{N}\} = F$, that is, $\bigcap\{\psi_n(F) \,|\, n \in \mathbb{N}\} \subseteq F$. Now the two inclusions $\bigcap\{\psi_n(F) \,|\, n \in \mathbb{N}\} \subseteq F$ and $F \subseteq \bigcap\{\psi_n(F) \,|\, n \in \mathbb{N}\}$ together give us $F = \bigcap\{\psi_n(F) \,|\, n \in \mathbb{N}\}$ as desired.

Furthermore, we check that $F = \bigcap\{cl_{\mathcal{T}^\flat}\psi_n(F) \,|\, n \in \mathbb{N}\}$. By construction of $\psi_n(F)$ and the fact that $X \setminus d(X \setminus A) \subseteq A$ we have $\psi_n(F) \subseteq D(F, d(X \setminus S_n(F)))$ where $D(F, d(X \setminus S_n(F))) \subseteq cl_{\mathcal{T}} D(F, d(X \setminus S_n(F))) \subseteq X \setminus d(X \setminus S_n(F))$.

Since i preserves inclusion and $X \setminus d(X \setminus S_n(F))$ is an upper set, it follows that

$$i(cl_{\mathcal{T}} D(F, d(X \setminus S_n(F)))) \subseteq X \setminus d(X \setminus S_n(F)) \subseteq S_n(F).$$

Thus by definition of closure in $\mathcal{T}^\flat$ we obtain

$cl_{\mathcal{T}^\flat} D(F, d(X \setminus S_n(F))) \subseteq i(cl_{\mathcal{T}} D(F, d(X \setminus S_n(F)))) \subseteq X \setminus d(X \setminus S_n(F)) \subseteq S_n(F)$.

Recall that $\psi_n(F) \subseteq D(F, d(X \setminus S_n(F)))$. Then $\bigcap cl_{\mathcal{T}^\flat}\psi_n(F) \subseteq F$. The other inclusion holds simply because $F \subseteq \psi_n(F)$. Hence $\bigcap cl_{\mathcal{T}^\flat}\psi_n(F) = F$ for any $n \in \mathbb{N}$ and closed upper set $F \subseteq X$. The above says that $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is $\mathcal{T}^\natural$-stratifiable with respect to $\mathcal{T}^\flat$.

Similarly, let $\sigma_n(G) = X \setminus i(X \setminus D(G, i(X \setminus S_n(G))))$ for each closed lower set G.

Then by mimicking the above argument, one can easily show that $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is $\mathcal{T}^\flat$-stratifiable with respect to $\mathcal{T}^\natural$. In all, $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is pairwise stratifiable. $\square$

Without further effort, the following fact emerges.

1.4.4 Corollary

If $(X, \mathcal{T}, \leq)$ is an ordered M_1 topological C-space then the bispace $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is pairwise stratifiable. $\square$

Remark. In this corollary, the condition M_1 can be replaced with metrizable.

1.4.5 Lemma

If $(X, \mathcal{T}, \leq)$ is a first countable ordered topological I-space then $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is doubly first countable.

Proof. Suppose $(X, \mathcal{T}, \leq)$ is a first countable ordered I-space. Then every point $x \in X$ has a countable neighbourhood base $\mathcal{B} = \{U_n \,|\, n \in \mathbb{N}\}$. Let $\mathcal{B}_1 = \{i(U_n) \,|\, U_n \in \mathcal{B}\}$. We claim that $\mathcal{B}_1$ is an open neighbourhood base for $\mathcal{T}^\natural$. Let $V \in \mathcal{T}^\natural$ and $x \in V$. Then V is an upper set and $V \in \mathcal{T}$. So there exists some $U_n \in \mathcal{B}$ such that $x \in U_n \subseteq V$. Clearly $i(U_n) \subseteq i(V)$ where $i(U_n)$ is open because X is an I-space. Also, $i(V) = V$ because V is an upper set. Hence $x \in i(U_n) \subseteq V$. Since each member of $\mathcal{B}_1$ is determined by a member of $\mathcal{B}$, it follows that $\mathcal{B}_1$ is also countable. Thus $\mathcal{B}_1$ is a countable open neighbourhood base for $\mathcal{T}^\natural$. Similarly, one can easily check that $\mathcal{B}_2 = \{d(U_n) \,|\, U_n \in \mathcal{B}\}$ is a countable open neighbourhood base for $\mathcal{T}^\flat$. Hence both $\mathcal{T}^\natural$ and $\mathcal{T}^\flat$ are first countable. Therefore $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is doubly first countable. $\square$

The following new observation is now immediate.

1.4.6 Theorem

If $(X, \mathcal{T}, \leq)$ is an ordered metrizable topological C- and I-space then $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is pairwise stratifiable and doubly first countable.

Proof. Let $(X, \mathcal{T}, \leq)$ be an ordered metrizable C- and I-space. Recall that every metric space is first countable. Then $(X, \mathcal{T}, \leq)$ is a first countable I-space and hence by Lemma 1.4.5, $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is doubly first countable. Also, recall that any metrizable space is stratifiable. So $(X, \mathcal{T}, \leq)$ is an ordered stratifiable C-space. Hence by Proposition 1.4.3, we deduce that $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is pairwise stratifiable. $\square$

We recall the following important result which is due to Fox.

1.4.7 Proposition

([18, Corollary 8.2], [31])*A bitopological space $(X, \mathcal{T}_1, \mathcal{T}_2)$ is quasi-pseudometrizable provided that it is pairwise stratifiable and each of the topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ admits a local quasi-uniformity with a countable base.* $\square$

It may be necessary to refresh our minds on some terminology involved here.

1.4.8 Definition

([18], [31]) A *local quasi-uniformity* $\mathcal{L}$ on a set X is a filter on $X \times X$ such that $\triangle \subseteq U$ for every $U \in \mathcal{L}$ and that for any $x \in X$ and $U \in \mathcal{L}$ there exists $V \in \mathcal{L}$ such that $(V \circ V)(x) \subseteq U(x)$. If $\mathcal{L} = \mathcal{L}^{-1}$ then $\mathcal{L}$ is called a *local uniformity.*

Contrary to the situation with quasi-uniformities, a conjugate of a local quasi-uniformity need not be a local quasi-uniformity. For instance [1, Example 3.7] due to T. Abreu and E. Corbacho demonstrates this fact.

1.4.9 Proposition

([31]) *A topology is induced by a quasi-uniformity with a countable base if and only if it is induced by a local quasi-uniformity with a countable base whose inverse is a local quasi-uniformity.* $\square$

Variations of the following result (recalling that if a uniformity is metrizable then so is the topology it induces) appear in literature, for instance [17, p. 162].

1.4.10 Proposition

([31, Theorem 3.3]) *A topology induced by a local quasi-uniformity with a countable base whose inverse is a local quasi-uniformity is quasi-pseudometrizable.* $\square$

In view of Proposition 1.4.9, we make the following observation.

1.4.11 Corollary

A topology which is induced by a quasi-uniformity with a countable base is quasi-pseudometrizable. $\square$

In the light of Fox's result and Proposition 1.4.3 together with the fact that any metrizable space is stratifiable (Example 1.3.3 above) we obtain the following:

1.4.12 Corollary

Let $(X, \mathcal{T}, \leq)$ be an ordered topological C-space such that $\mathcal{T}$ is metrizable. Then the bitopological space $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is quasi-pseudometrizable if and only if each of the topologies $\mathcal{T}^\natural$ and $\mathcal{T}^\flat$ is quasi-pseudometrizable. $\square$

Chapter 2

Compactness of Boundaries and Quasi-pseudometrics

2.1 Introduction

In this chapter we employ the idea of a locally finite collection to prove that given an ordered metrizable topological C- and I-space $(X, \mathcal{T}, \leq)$ such that $i(x)$ and $d(x)$ are compact for any $x \in X$ then the bispace $(X, \mathcal{T}^{\natural}, \mathcal{T}^{\flat})$ is quasi-pseudometrizable. We then establish a partial analogue of the Hanai-Morita-Stone Theorem (see Theorem 2.3.4). Furthermore, in the last part of this chapter, we construct a quasi-pseudometric on an ordered metric space which captures the upper topology and the lower topology in the desired manner (see Theorem 2.4.2).

2.2 Locally Finite Collections

The concept of local finiteness is fundamental in the study of paracompact spaces, which were first studied by Dieudonné in 1944 as a generalization of compact spaces. Paracompact spaces have been characterized in terms of locally finite collections (see [57, Theorem 20.7] or [21, Proposition 1.1, p. 350]). The celebrated theorem of A. H. Stone which says that *every metric space is paracompact* [57, Theorem 20.9] brings to the surface the connection between locally finite collections and metrizable spaces. In fact, paracompact spaces include both compact spaces and metrizable spaces.

2.2.1 Definition

([21]) Let $(G_i)_{i\in I}$ be a collection of subsets of a topological space X. Then $(G_i)_{i\in I}$ is said to be *locally finite* if and only if every $x \in X$ has a neighbourhood $\mathcal{N}_x$ such that the set $\{i \in I \,|\, \mathcal{N}_x \cap G_i \neq \emptyset\}$ is finite. Furthermore, we say that $(G_i)_{i\in I}$ is *point-finite* if and only if $\{i \in I \,|\, x \in G_i\}$ is finite for any $x \in X$.

2.2.2 Example

([57]) Let $n \in \mathbb{N}$. The collection of all subsets of $\mathbb{R}$ of the form $(n, n+2)$ is locally finite.

The first part of the next lemma gives more examples of locally finite collections and the second part says that locally finite collections are closure preserving. We omit the proofs, which are easy and available in the literature (see for instance [57, Lemmas 20.4 and 20.5]).

2.2.3 Lemma

([20]) Let $\{G_i \,|\, i \in I\}$ be a collection of subsets of a topological space $(X, \mathcal{T})$. Then

a) *If* $\{G_i \,|\, i \in I\}$ *is locally finite then so is* $\{cl_{\mathcal{T}}(G_i) \,|\, i \in I\}$.

b) *If* $\{G_i \,|\, i \in I\}$ *is locally finite then* $\bigcup cl_{\mathcal{T}}(G_i) = cl_{\mathcal{T}}(\bigcup G_i)$. □

The following known fact relates local finiteness to point-finiteness. We give its proof below.

2.2.4 Proposition

([21, Proposition 1.1, p. 350]) *Any locally finite collection of subsets of a topological space is point-finite.*

Proof. Let $(G_i)_{i\in I}$ be a locally finite collection of subsets of a topological space $(X, \mathcal{T})$ and $x \in X$. Then by definition of local finiteness, $S := \{i \in I \,|\, \mathcal{N}_x \cap G_i \neq \emptyset\,\}$ is finite. Pick any $j \in T := \{i \in I \,|\, x \in G_i\,\}$. Then $x \in G_j$, and since $\mathcal{N}_x$ is a neighbourhood of x, we have $x \in G_j \cap \mathcal{N}_x$ so that $G_j \cap \mathcal{N}_x \neq \emptyset$. Thus $j \in S$ and hence $T \subseteq S$. Since S is finite, T has no choice but to be finite. Therefore $(G_i)_{i\in I}$ is point-finite. $\square$

Remark. The converse of the above proposition does not hold since for any space X, the collection $\{\{x\} \,|\, x \in X\}$ is a point-finite cover of X which is, in general, not locally finite.

The following three lemmas will be used in the proof of the theorem to follow.

2.2.5 Lemma

([20]) *Any locally finite collection of subsets of a topological space is compact-finite, that is, if K is a compact subset of a topological space X and $(A_i)_{i\in I}$ is a locally finite collection in X then $\{i \in I \,|\, A_i \cap K \neq \emptyset\}$ is finite.*

Proof. Suppose $(A_i)_{i\in I}$ is locally finite and K is a compact subset of X. Then for any $x \in K$ there exists an open neighbourhood $\mathcal{N}_x$ such that $\mathcal{N}_x \cap A_i \neq \emptyset$ for only finitely many $i \in I$. Let $I_x = \{i_1, ..., i_n\}$ be the collection of all such i, and suppose $\{\mathcal{N}_x \,|\, x \in K\}$ is an open cover for K. Then $K \subseteq \bigcup\{\mathcal{N}_x \,|\, x \in K\,\}$. Since K is compact, there exists $x_1, ..., x_m$ such that $K \subseteq \bigcup\{\mathcal{N}_{x_i} \,|\, i = 1, ..., m\,\}$. For any $i \in I$ with $A_i \cap K \neq \emptyset$ we have $i \in I_{x_i}$ where I_{x_i} is finite and hence $\{i \,|\, A_i \cap K \neq \emptyset\,\} \subseteq \bigcup\{I_{x_i} \,|\, i = 1, ..., m\,\}$. Since $\bigcup\{I_{x_i} \,|\, i = 1, ..., m\,\}$ is finite then so is $\{i \,|\, A_i \cap K \neq \emptyset\,\}$. Therefore $(A_i)_{i\in I}$ is compact-finite. $\square$

2.2.6 Lemma

Let $(X, \leq)$ be an ordered set and $x \in X$. Then $d(x) \cap M \neq \emptyset$ if and only if $x \in i(M)$ for any subset $M \subseteq X$.

Proof. Suppose $d(x) \cap M \neq \emptyset$. Then there exists some $a \in d(x) \cap M$ which means that $a \leq x$ and $a \in M$. Hence $x \in i(M)$. Conversely, suppose $x \in i(M)$. Then by definition of $i(M)$, there exists some $a \in M$ such that $a \leq x$. This implies that $a \in d(x) \cap M$. Therefore $d(x) \cap M \neq \emptyset$. $\square$

As it might be expected, we also obtain

2.2.7 Lemma

Let $(X, \leq)$ be an ordered set and $x \in X$. Then $i(x) \cap M \neq \emptyset$ if and only if $x \in d(M)$ for any subset $M \subseteq X$.

Proof. Suppose $i(x) \cap M \neq \emptyset$. Then we can find some $a \in i(x) \cap M$ which means that $x \leq a$ and $a \in M$. Hence $x \in d(M)$. Conversely, let $x \in d(M)$. Then by definition of $d(M)$, there exists some $a \in M$ such that $x \leq a$. Thus $a \in i(x) \cap M$. Therefore $i(x) \cap M \neq \emptyset$. $\square$

Now recall that a function $q : X \times X \to [0, \infty)$ is called a *non-Archimedean quasi-pseudometric* on X if and only if $q(x, x) = 0$ for all $x \in X$ and $q(x, z) \leq max\{q(x, y),\ q(y, z)\}$ where $x, y, z \in X$. Evidently, every non-Archimedean quasi-pseudometric is a quasi-pseudometric. We employ the above lemmas to establish the next theorem. But first, we recall the following:

2.2.8 Definition

([57]) A collection $\mathcal{G}$ of subsets of a topological space X is said to be *σ-locally finite* if and only if $\mathcal{G}$ can be expressed as $\mathcal{G} = \bigcup_{n \in \mathbb{N}} \mathcal{G}_n$ where each $\mathcal{G}_n$ is locally finite. Similarly, $\mathcal{G}$ is said to be *σ-point finite* if and only if $\mathcal{G} = \bigcup_{n \in \mathbb{N}} \mathcal{G}_n$ where each $\mathcal{G}_n$ is point finite.

2.2.9 Theorem

If $(X, \mathcal{T}, \leq)$ is an ordered metrizable C- and I-space with $i(x)$ and $d(x)$ compact whenever $x \in X$, then $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is quasi-pseudometrizable.

Proof. Suppose $(X, \mathcal{T}, \leq)$ is a metrizable C- and I-space with $i(x)$ and $d(x)$ compact whenever $x \in X$. By the Bing-Nagata-Smirnov Metrization Theorem [57, Theorem 23.9], $\mathcal{T}$ has a σ-locally finite base $\mathcal{B}$. Then $\mathcal{B} = \bigcup\limits_{n\in\mathbb{N}} \mathcal{B}_n$ where each $\mathcal{B}_n$ is a locally finite open collection of subsets of X. Put $\mathcal{B}_n = \{B_{n_j} \,|\, j \in I_n\}$. Since $\mathcal{B}_n$ is locally finite it follows that $d(x) \cap B_{n_j} \neq \emptyset$ for finitely many $j \in I_n$. Equivalently, $x \in i(B_{n_j})$ for finitely many $j \in I_n$. Thus using this equivalence, $\{j \in I_n \,|\, d(x) \cap B_{n_j} \neq \emptyset\} = \{j \in I_n \,|\, x \in i(B_{n_j})\}$. Note that $i(\mathcal{B}_n) = \{i(B_{n_j}) \,|\, j \in I_n\}$. Since $\mathcal{B}_n$ is an open collection in an I-space X, it follows that $i(\mathcal{B}_n)$ is also an open collection. We show that $i(\mathcal{B}_n)$ is point-finite. Given that $d(x)$ is compact and $\mathcal{B}_n$ is locally finite we deduce that the set $\{j \in I_n \,|\, d(x) \cap B_{n_j} \neq \emptyset\}$ is finite and hence $\{j \in I_n \,|\, x \in i(B_{n_j})\}$ is finite which means $i(\mathcal{B}_n)$ is point-finite for any $n \in \mathbb{N}$.

We claim that $\bigcup\limits_{n\in\mathbb{N}} i(\mathcal{B}_n) = i(\mathcal{B})$. Let $A \in \bigcup\limits_{n\in\mathbb{N}} i(\mathcal{B}_n)$. Then $A \in i(\mathcal{B}_n)$ for some $n \in \mathbb{N}$. By definition of $i(\mathcal{B}_n)$ there exists some $A_0 \in \mathcal{B}_n$ such that $A_0 \subseteq A$. Since $\bigcup\limits_{n\in\mathbb{N}} \mathcal{B}_n = \mathcal{B}$, we have $A_0 \in \mathcal{B}$ such that $A_0 \subseteq A$. Thus $A \in i(\mathcal{B})$. This establishes the inclusion $\bigcup\limits_{n\in\mathbb{N}} i(\mathcal{B}_n) \subseteq i(\mathcal{B})$. Conversely, if $A \in i(\mathcal{B})$ then $A \in i(\bigcup\limits_{n\in\mathbb{N}} \mathcal{B}_n)$. So there exists some $C \in \bigcup\limits_{n\in\mathbb{N}} \mathcal{B}_n$ such that $C \subseteq A$. This implies that $C \in \mathcal{B}_n$ for some $n \in \mathbb{N}$ and $C \subseteq A$. Thus $A \in i(\mathcal{B}_n)$ for some $n \in \mathbb{N}$. Hence $A \in \bigcup\limits_{n\in\mathbb{N}} i(\mathcal{B}_n)$, verifying that $i(\mathcal{B}) \subseteq \bigcup\limits_{n\in\mathbb{N}} i(\mathcal{B}_n)$. Thus $\bigcup\limits_{n\in\mathbb{N}} i(\mathcal{B}_n) = i(\mathcal{B})$, and by the proof of Proposition 1.2.3, it is a base for $\mathcal{T}^\natural$. Therefore $i(\mathcal{B})$ is a σ-point-finite base for $\mathcal{T}^\natural$. So, $\mathcal{T}^\natural$ is non-Archimedeanly quasi-pseudometrizable and thus quasi-pseudometrizable.

We show that the same is true for $\mathcal{T}^\flat$.
Since $\mathcal{B} = \bigcup\limits_{n\in\mathbb{N}} \mathcal{B}_n$ is a σ-locally finite base for $\mathcal{T}$, $\mathcal{B}_n$ is a locally finite open collection for each $n \in \mathbb{N}$, and hence for each $x \in X$ we have $i(x) \cap B_{n_j} \neq \emptyset$

for finitely many $j \in I_n$ if and only if $x \in d(B_{n_j})$ for finitely many $j \in I_n$. So, $\{j \in I_n \,|\, i(x) \cap B_{n_j} \neq \emptyset\} = \{j \in I_n \,|\, x \in d(B_{n_j})\}$. Note that $d(\mathcal{B}_n) = \{d(B_{n_j}) \,|\, j \in I_n\}$ and it is an open collection because $\mathcal{B}_n$ is open and X is an I-space. Now we show that $d(\mathcal{B}_n)$ is point-finite. $\mathcal{B}_n$ being locally finite and $i(x)$ being compact together imply that $\{j \in I_n \,|\, i(x) \cap B_{n_j} \neq \emptyset\}$ is finite and hence so is $\{j \in I_n \,|\, x \in d(B_{n_j})\}$. This means $d(\mathcal{B}_n)$ is point finite. Note that $\bigcup\limits_{n\in\mathbb{N}} d(\mathcal{B}_n) = d(\mathcal{B})$. Now we have $d(\mathcal{B})$ as a σ-point-finite base for $\mathcal{T}^\flat$. Hence $\mathcal{T}^\flat$ is non-Archimedeanly quasi-pseudometrizable and hence quasi-pseudometrizable. In all, $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is (doubly) quasi-pseudometrizable. $\square$

Remark. We do not know whether, without compactness, our standard assumptions in the above theorem are sufficient for the two topologies $\mathcal{T}^\natural$ and $\mathcal{T}^\flat$ to be (separately) even non-Archimedeanly quasi-pseudometrizable.

We now turn to compactness of boundaries.

2.3 Compact Boundaries

We begin by recalling some well-known definitions from general topology and work towards the results advertised in the introduction of this chapter.

2.3.1 Definition

([57]) Let X be a topological space and $A \subseteq X$ its subset.

(i) The *boundary of* A, denoted by $bd(A)$, is defined by $bd(A) := cl(A) \cap cl(X \setminus A)$. By duality of closure and interior, it is immediate that $bd(A) = cl(A) \setminus int(A)$.

(ii) An *open cover* of a topological space X is a family $(O_i)_I$ of open subsets of X such that $X = \bigcup\{O_i \,|\, i \in I\}$.

(iii) X is said to be *compact* if and only if for every open cover $(O_i)_I$ of X there exists some finite $I_0 \subseteq I$ such that $X = \bigcup\{O_i \,|\, i \in I_0\}$, that is, every open cover of X has a finite subcover. An open cover and compactness of a subset of X are defined similarly.

(iv) A metric space (X, ρ) is said to be *countably compact* if and only if every sequence in X has a cluster point. A point $a \in X$ is a *cluster point* of a sequence $(x_n)_{n\in\mathbb{N}}$ if and only if $a \in \bigcap_{k\in\mathbb{N}} cl\{x_n \,|\, n \in \mathbb{N},\, n \geqslant k\}$. Also recall that a metric space (X, ρ) is *sequentially compact* iff every sequence in X has a subsequence that converges to a point in X. It is well known that, in metric spaces, countable compactness, compactness, and sequential compactness are all equivalent (see [16, Theorem 4.1.17]).

Among other facts in [44, Theorem 1], McCartan proved that if $(X, \mathcal{T}, \leq)$ is an I-space then $i(\overline{A}) \subseteq \overline{i(A)}$ for any $A \subseteq X$. This seems to be related to the following:

2.3.2 Lemma

If $(X, \mathcal{T}, \leq)$ is an ordered I-space then $int_{\mathcal{T}} i(x) \in \mathcal{T}^\natural$ whenever $x \in X$.

Proof. Suppose $(X, \mathcal{T}, \leq)$, is an ordered I-space and let $x \in X$. Always $\mathrm{int}_{\mathcal{T}} i(x) \subseteq i(x)$. Thus $i(\mathrm{int}_{\mathcal{T}} i(x)) \subseteq i(x) = i(i(x))$. Since $\mathrm{int}_{\mathcal{T}} i(x)$ is $\mathcal{T}$-open and X is an I-space, it follows that $i(\mathrm{int}_{\mathcal{T}} i(x))$ is $\mathcal{T}$-open. Given that $\mathrm{int}_{\mathcal{T}} i(x)$ is the largest $\mathcal{T}$-open subset of $i(x)$ and $\mathrm{int}_{\mathcal{T}} i(x) \subseteq i(\mathrm{int}_{\mathcal{T}} i(x))$ we deduce that $\mathrm{int}_{\mathcal{T}} i(x) = i(\mathrm{int}_{\mathcal{T}} i(x))$. Hence $\mathrm{int}_{\mathcal{T}} i(x) \in \mathcal{T}^\natural$ whenever $x \in X$. $\square$

A similar proof establishes the following:

2.3.3 Lemma

If $(X, \mathcal{T}, \leq)$, is an ordered I-space then $int_{\mathcal{T}} d(x) \in \mathcal{T}^\flat$ whenever $x \in X$. $\square$

In the light of the Hanai-Morita-Stone Theorem cited in the introduction one might wonder whether the hypotheses of Problem 1.2.1 imply $\mathcal{T}$-compactness of the boundaries of the sets $d(y)$ and $i(y)$ whenever $y \in X$. Indeed the answer to this question is positive, as our next theorem implies (compare Lemma 1.4.5). It also provides the promised partial analogue of the Hanai-Morita-Stone Theorem.

Unfortunately we do not know whether, in this result,‘first-countable’ can be replaced by ‘quasi-pseudometrizable’ (compare Corollary 1.4.12).

2.3.4 Theorem

Let $(X, \mathcal{T}, \leq)$ be an ordered C-space such that $\mathcal{T}$ is a metrizable topology. Then both the upper topology $\mathcal{T}^{\natural}$ and the lower topology $\mathcal{T}^{\flat}$ are first countable if and only if for each $y \in X$, $bd_{\mathcal{T}}d(y)$ and $bd_{\mathcal{T}}i(y)$ are compact in $(X, \mathcal{T})$.

Proof. Let $(X, \mathcal{T}, \leq)$ be an ordered C-space such that $\mathcal{T}$ is a metrizable topology, and suppose r is a compatible metric on X and let

$$B_{2^{-n}} = \{(x, y) \in X \times X \mid r(x, y) < 2^{-n}\} \text{ for each } n \in \mathbb{N}.$$

Suppose that both the lower topology $\mathcal{T}^{\flat}$ and the upper topology $\mathcal{T}^{\natural}$ are first countable. Furthermore, let $y \in X$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathrm{bd}_{\mathcal{T}}i(y)$. Observe that $i(y)$ is $\mathcal{T}$-closed, since X is a metrizable C-space. Moreover, let $\{I_n \mid n \in \mathbb{N}\}$ be a $\mathcal{T}^{\natural}$-neighbourhood base at y consisting of $\mathcal{T}^{\natural}$-open sets. Fix $n \in \mathbb{N}$. Note that $I_n \cap B_{2^{-n}}(x_n)$ is a $\mathcal{T}$-neighbourhood of x_n, hence $[I_n \cap B_{2^{-n}}(x_n)] \setminus i(y) \neq \emptyset$. Choose $x'_n \in [I_n \cap B_{2^{-n}}(x_n)] \setminus i(y)$. Assume first that $\mathrm{cl}_{\mathcal{T}}\{x'_n \mid n \in \mathbb{N}\} \cap i(y) = \emptyset$. Since X is a C-space, $d(\mathrm{cl}_{\mathcal{T}}\{x'_n \mid n \in \mathbb{N}\})$ is $\mathcal{T}$-closed and obviously disjoint from $i(y)$. Therefore there is $m \in \mathbb{N}$ such that $I_m \cap d(\mathrm{cl}_{\mathcal{T}}\{x'_n \mid n \in \mathbb{N}\}) = \emptyset$ — a contradiction. Hence we conclude that there is $a \in \mathrm{cl}_{\mathcal{T}}\{x'_n \mid n \in \mathbb{N}\} \cap i(y)$. Then a is a $\mathcal{T}$-cluster point of the sequence $(x_n)_{n\in\mathbb{N}}$ and therefore belongs to the $\mathcal{T}$-closed set $\mathrm{bd}_{\mathcal{T}}i(y)$. We have shown that each sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathrm{bd}_{\mathcal{T}}i(y)$ has a $\mathcal{T}$-cluster point. Hence $\mathrm{bd}_{\mathcal{T}}i(y)$ is countably compact and thus $\mathcal{T}$-compact, since countably compact metrizable spaces are compact. Similarly, it can be shown that $\mathrm{bd}_{\mathcal{T}}d(y)$ is $\mathcal{T}$-compact in X.

For the converse, suppose that for each $y \in X$, $\mathrm{bd}_{\mathcal{T}}d(y)$ and $\mathrm{bd}_{\mathcal{T}}i(y)$ are compact in $(X, \mathcal{T})$. We show that $\mathcal{T}^{\natural}$ and $\mathcal{T}^{\flat}$ are first countable. Fix $x \in X$. Then for each $m \in \mathbb{N}$ set $H_m = X \setminus d(X \setminus [\mathrm{int}_{\mathcal{T}}(i(x)) \cup B_{2^{-m}}(\mathrm{bd}_{\mathcal{T}}i(x))])$. We claim that $\{H_m \mid m \in \mathbb{N}\}$ is a neighbourhood base at x for the upper topology $\mathcal{T}^{\natural}$. Fix $n \in \mathbb{N}$. Clearly $i(x) \subseteq \mathrm{int}_{\mathcal{T}}(i(x)) \cup B_{2^{-n}}(\mathrm{bd}_{\mathcal{T}}i(x))$. Hence H_n is an open upper set containing x, since X is a C-space. Let G be any open upper set such that $x \in G$. Because G is an open upper set containing the $\mathcal{T}$-closed set $i(x)$, then by compactness of $\mathrm{bd}_{\mathcal{T}} i(x)$, there is $p \in \mathbb{N}$ such that $B_{2^{-p}}(\mathrm{bd}_{\mathcal{T}} i(x)) \subseteq G$ (see [33, Remark 2.5.4]). Thus $[\mathrm{int}_{\mathcal{T}}(i(x)) \cup B_{2^{-p}}(\mathrm{bd}_{\mathcal{T}} i(x))] \subseteq G$. We conclude that $H_p \subseteq G$. Thus the upper topology $\mathcal{T}^\natural$ is proven to be first countable. A similar argument shows that $\mathcal{T}^\flat$ is also first countable, and this completes the proof. $\square$

2.3.5 Corollary

If $(X, \mathcal{T}, \leqslant)$ is a metric C-space and I-space and $y \in X$ then the boundaries of $i(y)$ and $d(y)$, ($bd_{\mathcal{T}} i(y)$ and $bd_{\mathcal{T}} d(y)$) are compact.

Proof. Since X is an I-space then both $\mathcal{T}^\natural$ and $\mathcal{T}^\flat$ are first countable and therefore by the above theorem, the boundaries are compact. $\square$

2.4 Construction of a Quasi-pseudometric on an Ordered Metric Space

Let $(X, \rho, \leq)$ be an ordered metric space which is a C-space and I-space such that ρ is bounded. We aim to construct a quasi-pseudometric q on $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ such that $\mathcal{T}(q) = \mathcal{T}^\natural$ and $\mathcal{T}(q^{-1}) = \mathcal{T}^\flat$. We borrow the idea from the Hausdorff quasi-pseudometric in the proof of [35, Proposition 4].

For any $x, y \in X$, define q on X in terms of the given metric ρ as follows:

$$q(x,y) = \sup_{y' \in i(y)} \rho(i(x), y') \vee \sup_{x' \in d(x)} \rho(x', d(y)) \text{ where}$$

$$\rho(i(x), y') = \inf_{a \in i(x)} \rho(a, y').$$

The next result gives us some properties of the above defined function q.

2.4.1 Proposition

The above-defined function q is a quasi-pseudometric on X.

Proof. We must show that $q(x,x) = 0$ for every $x \in X$ and $q(x,z) \leq q(x,y) + q(y,z)$ for all $x, y, z \in X$.

$$
\begin{aligned}
q(x,x) &= \sup_{x' \in i(x)} \rho(i(x), x') \vee \sup_{x' \in d(x)} \rho(x', d(x)) \\
&= \sup_{x' \in i(x)} \inf_{a \in i(x)} \rho(a, x') \vee \sup_{x' \in d(x)} \inf_{b \in d(x)} \rho(x', b) \\
&= \sup_{x' \in i(x)} \rho(x', x') \vee \sup_{x' \in d(x)} \rho(x', x') \qquad \text{(because } x' \in i(x) \cap d(x) \text{)} \\
&= 0 \vee 0 = 0.
\end{aligned}
$$

For the triangle inequality, note that $q(x,z) = \sup_{z' \in i(z)} \rho(i(x), z') \vee \sup_{x' \in d(x)} \rho(x', d(z))$ by definition of q. Considering each of these disjuncts separately, we get

$$
\begin{aligned}
\sup_{z' \in i(z)} \rho(i(x), z') &= \sup_{z' \in i(z)} \{\inf \rho(a, z') \mid a \in i(x)\} \\
&\leq \sup_{z' \in i(z)} \{ \inf_{a \in i(x)} [\rho(a, y') + \rho(y', z')] \mid y' \in i(y)\} \qquad (\rho \text{ is a metric}) \\
&= \sup_{z' \in i(z)} \{ \inf_{a \in i(x)} \rho(a, y') \mid y' \in i(y)\} + \sup_{z' \in i(z)} \{ \inf_{a \in i(x)} \rho(y', z') \mid y' \in i(y)\} \\
&= \{ \inf_{a \in i(x)} \rho(a, y') \mid y' \in i(y)\} + \sup_{z' \in i(z)} \{\rho(y', z') \mid y' \in i(y)\} \\
&\leq \sup_{y' \in i(y)} \{ \inf_{a \in i(x)} \rho(a, y') \} + \sup_{z' \in i(z)} \{ \inf_{y' \in i(y)} \rho(y', z') \} \\
&= \sup_{y' \in i(y)} \rho(i(x), y') + \sup_{z' \in i(z)} \rho(i(y), z').
\end{aligned}
$$

For the other disjunct,

$$
\begin{aligned}
\sup_{x' \in d(x)} \rho(x', d(z)) &= \sup_{x' \in d(x)} \{\inf \rho(x', z') \mid z' \in d(z)\} \\
&\leq \sup_{x' \in d(x)} \{ \inf_{z' \in d(z)} [\rho(x', y') + \rho(y', z')] \mid y' \in d(y)\} \qquad (\rho \text{ is a metric }) \\
&= \sup_{x' \in d(x)} \{ \inf_{z' \in d(z)} \rho(x', y') \mid y' \leq y\} + \sup_{x' \in d(x)} \{ \inf_{z' \in d(z)} \rho(y', z') \mid y' \leq y\}
\end{aligned}
$$

$$
\begin{aligned}
&= \sup_{x' \in d(x)} \{\rho(x', y') \,|\, y' \in d(y)\} + \{ \inf_{z' \in d(z)} \rho(y', z') \,|\, y' \in d(y)\} \\
&\leq \sup_{x' \in d(x)} \{ \inf_{y' \in d(y)} \rho(x', y')\} + \sup_{y' \in d(y)} \{ \inf_{z' \in d(z)} \rho(y', z')\} \\
&= \sup_{x' \in d(x)} \rho(x', d(y)) + \sup_{y' \in d(y)} \rho(y', d(z)).
\end{aligned}
$$

Now we have $\sup_{z' \in i(z)} \rho(i(x), z') \leq \sup_{y' \in i(y)} \rho(i(x), y') + \sup_{z' \in i(z)} \rho(i(y), z')$ and

$$\sup_{x' \in d(x)} \rho(x', d(z)) \leq \sup_{x' \in d(x)} \rho(x', d(y)) + \sup_{y' \in d(y)} \rho(y', d(z)).$$

Letting $q_1(x, y) = \sup_{y' \in i(y)} \rho(i(x), y')$ and $q_2(x, y) = \sup_{x' \in d(x)} \rho(x', d(y))$,
we get

$$q_1(x, z),\ q_2(x, z) \leq (q_1(x, y) + q_1(y, z)) \vee (q_2(x, y) + q_2(y, z))$$

which implies that

$$q(x, z) = q_1(x, z) \vee q_2(x, z) \leq (q_1(x, y) + q_1(y, z)) \vee (q_2(x, y) + q_2(y, z)).$$

Since $q_i(x, y) \leq q(x, y)$ and $q_i(y, z) \leq q(y, z)$ for each $i \in \{1, 2\}$ then

$$q_i(x, y) + q_i(y, z) \leq q(x, y) + q(y, z) \text{ for each } i \in \{1, 2\}$$

and hence

$$q(x, z) \leq (q_1(x, y) + q_1(y, z)) \vee (q_2(x, y) + q_2(y, z)) \leq q(x, y) + q(y, z).$$

Thus $q(x, z) \leq q(x, y) + q(y, z)$ and therefore q is a quasi-pseudometric on X. $\square$

We recall that a subset A of a metric space (X, ρ) is said to be ϵ-*dense* ($\epsilon > 0$) in (X, ρ) if and only if for every $x \in X$ there exists some $a \in A$ such that $\rho(x, a) < \epsilon$. Then (X, ρ) is said to be *totally bounded* if and only if for every $\epsilon > 0$ there is a finite subset $A \subseteq X$ which is ϵ-dense in (X, ρ) ([16, Section 4.3, p. 266]).
Next, we show that the quasi-pseudometric q in the above proposition captures the two topologies of the bispace $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ in the manner precisely described below.

2.4.2 Theorem

Let q be the quasi-pseudometric defined above on a metric space $(X, \rho, \leq)$ which is a C- and I-space and suppose $i(x)$ and $d(x)$ are totally bounded for every $x \in X$. Then $\mathcal{T}(q) = \mathcal{T}^{\natural}$ and $\mathcal{T}(q^{-1}) = \mathcal{T}^{\flat}$.

Proof. Recall that the quasi-pseudometric q on X is defined by

$$q(x, y) = \sup_{y' \in i(y)} \rho(i(x), y') \vee \sup_{x' \in d(x)} \rho(x', d(y)).$$

For every $\epsilon > 0$ and $x \in X$, let $B_q(x, \epsilon) = X \setminus d(X \setminus U_\epsilon(i(x))) \cap \bigcap_{x' \in d(x)} i(B(x', \epsilon))$ where $U_\epsilon(i(x)) = \{y \in X \mid (\exists x' \in i(x))(q(y, x') < \epsilon)\}$ is ρ-open.

Note that in $B_q(x, \epsilon)$, the set $X \setminus d(X \setminus U_\epsilon(i(x)))$ is open in $\mathcal{T}(\rho)$ and it is an upper set, hence it lies in $\mathcal{T}^{\natural}$. We want the same for $\bigcap_{x' \in d(x)} i(B(x', \epsilon))$. For any $x' \in d(x)$ we have $x' \in B(x', \frac{\epsilon}{2})$. Thus $d(x) \subseteq \bigcup_{x' \in d(x)} B(x', \frac{\epsilon}{2})$ which implies that $\{B(x', \frac{\epsilon}{2}) \mid x' \in d(x), \epsilon > 0\}$ is an open cover of $d(x)$. Given that $d(x)$ is totally bounded, there exists a finite set J such that $d(x) \subseteq \bigcup_{j \in J} B(x'_j, \frac{\epsilon}{2})$. For any $x' \in d(x)$, we have $x' \in B(x'_j, \frac{\epsilon}{2})$ for some $j \in J$ since $d(x)$ is covered by $\{B(x'_j, \frac{\epsilon}{2}) \mid j \in J\}$. Then $x'_j \in B(x', \frac{\epsilon}{2})$.

By the triangle inequality, $B(x'_j, \frac{\epsilon}{2}) \subseteq B(x', \epsilon)$. This is because $a \in B(x'_j, \frac{\epsilon}{2})$ implies $\rho(a, x'_j) < \frac{\epsilon}{2}$ and so $\rho(a, x') \leq \rho(a, x'_j) + \rho(x'_j, x') < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Hence $a \in B(x', \epsilon)$. Since i preserves inclusion, we obtain $i(\,B(x'_j, \frac{\epsilon}{2})\,) \subseteq i(\,B(x', \epsilon)\,)$ and then we easily get the inclusion $\bigcap_{j \in J} i(B(x'_j, \frac{\epsilon}{2})) \subseteq \bigcap_{x' \in d(x)} i(B(x', \epsilon)\,)$. This says that $\bigcap_{x' \in d(x)} i(B(x', \epsilon)\,)$ contains a neighbourhood (with respect to $\mathcal{T}^{\natural}$) of each of its points, hence it lies in $\mathcal{T}^{\natural}$. Then it follows that $B_q(x, \epsilon) \in \mathcal{T}^{\natural}$. Hence $\mathcal{T}(q) \subseteq \mathcal{T}^{\natural}$.

Conversely, given any $U \in \mathcal{T}^{\natural}$, let $x \in U$. Then $i(x) \subseteq U$ because U is an upper set. Also $U \in \mathcal{T}(\rho)$ by definition of $\mathcal{T}^{\natural}$. Hence $B(x, \epsilon) \subseteq U$ for some $\epsilon > 0$. Then $i(B(x, \epsilon)) \subseteq U$, and hence by construction of $B_q(x, \epsilon)$ we have $B_q(x, \epsilon) \subseteq i(B(x, \epsilon))$, and so $x \in B_q(x, \epsilon) \subseteq U$. Thus $\{B_q(x, \epsilon) \mid x \in X, \epsilon > 0\}$ is a base for $\mathcal{T}^\natural$. Hence $\mathcal{T}^\natural \subseteq \mathcal{T}(q)$, and therefore $\mathcal{T}(q) = \mathcal{T}^\natural$.

For $\mathcal{T}(q^{-1}) = \mathcal{T}^\flat$, recall that $q^{-1}(x, y) = q(y, x)$ and let

$$B_{q^{-1}}(x, \epsilon) = X \setminus i(X \setminus U_\epsilon(d(x))) \cap \bigcap_{x' \in i(x)} d(B(x', \epsilon)).$$

Furthermore, note that $U_\epsilon(d(x)) := \{y \in X \mid (\exists x' \in d(x))(q(y, x') < \epsilon)\}$ and $i(x)$ is totally bounded by hypothesis. Then one can easily check that $B_{q^{-1}}(x, \epsilon) \in \mathcal{T}^\flat$ so that $\mathcal{T}^\flat \subseteq \mathcal{T}(q^{-1})$. The converse is similar to the one for the dual case above. Consequently, we obtain $\mathcal{T}(q^{-1}) = \mathcal{T}^\flat$ as desired. $\square$

Remark. The above argument simply gives a uniform approach to Theorem 2.2.9, and hence the latter follows as a corollary.

2.4.3 Corollary

Let $(X, \rho, \leq)$ be a bounded metric space which is a C-space and I-space and suppose $i(x)$ and $d(x)$ are totally bounded for every $x \in X$. Then the associated bispace $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is quasi-pseudometrizable. $\square$

Chapter 3

Uniform Local Connectedness and Quasi-pseudometrizability

3.1 Introduction

In this chapter we relate quasi-pseudometrizability to the notion of uniform local connectedness of a space. The latter has been studied by many authors, for instance it was introduced into the theory of uniform spaces by P. J. Collins [14], and about thirty years later, into the theory of uniform frames by D. Baboolal [3]. In [6], A. Berarducci, D. Dikranjan and J. Pelant show that the uniformly locally connected spaces are precisely the straight metric spaces[1] studied therein. Further investigation to clarify the relation between straightness and local connectedness is carried out in [7], where various examples are given to distinguish the two concepts. Here we demonstrate that under a somewhat unexpected assumption of uniform local connectedness of an ordered metric space $(X, d, \leq)$ which is a C- and I-space, the associated bitopological space $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is quasi-pseudometrizable (here $\mathcal{T}$ is the topology generated by d).

[1]A metric space X is *straight* if, whenever X is a union of finitely many closed sets then $f \in C(X)$ is uniformly continuous if and only if its restriction to each of the closed sets is uniformly continuous.

3.2 Some Essentials on Uniformities

We recall definitions of some concepts involved here which we shall also need in the next chapter. To a large extent, we follow Fletcher and Lindgren [17].
Let X be a set. Then a non-empty collection $\mathcal{U}$ of subsets of $X \times X$ is called a *quasi-uniformity* on X if and only if $\mathcal{U}$ is a filter on $X \times X$ such that $\triangle \subseteq U$ for any $U \in \mathcal{U}$, and given any $U \in \mathcal{U}$ there exists some $V \in \mathcal{U}$ such that $V \circ V \subseteq U$. The pair $(X,\mathcal{U})$ is then called a *quasi-uniform space.*

Notation. $\triangle := \{(x,x) \mid x \in X\}$; $U^{-1} = \{(y,x) \mid (x,y) \in U\}$

$V \circ V := \{(x,y) \mid (\exists z \in X)((z,y) \in V \text{ and } (x,z) \in V)\}$

$U(x) := \{y \in X \mid (x,y) \in U\}$ for any $x \in X$

$U(A) := \bigcup_{a \in A} U(a)$ for any $A \subseteq X$.

If $\mathcal{U}$ also satisfies the condition that $U^{-1} \in \mathcal{U}$ whenever $U \in \mathcal{U}$ then $\mathcal{U}$ is called a *uniformity* on X, and $(X,\mathcal{U})$ a *uniform space.* We say that X is *uniformly locally connected* if and only if for each $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $V \subseteq U$ and $V(x)$ is connected for each $x \in X$. A topological space X is said to be *connected* if and only if there are no two open and disjoint non-empty subsets A and B of X with $X = A \cup B$. And then, X is *locally connected* if and only if every point $x \in X$ has a neighbourhood base of open connected sets ([7],[57]).

3.3 Connecting Local Connectedness and Quasi-pseudometrization

We recall the following well-known definition ([6], [7], [14]), the metric equivalent of the one given above for uniform spaces as the first lemma below it suggests.

3.3.1 Definition

A metric space (X, d) is *uniformly locally connected* if and only if for every $\epsilon > 0$ there is a $\delta > 0$ such that any two points at distance less than δ lie in a connected set of diameter less than ϵ.

3.3.2 Lemma

([6]) *A metric space (X, d) is uniformly locally connected if and only if for each $\epsilon > 0$ there is a $\delta > 0$ such that for each $x \in X$ there is a connected set K_x such that $B_\delta(x) \subseteq K_x \subseteq B_\epsilon(x)$.*

Proof. ($\Leftarrow$) This holds trivially by definition.
($\Rightarrow$) Suppose (X, d) is uniformly locally connected. Given any $\epsilon > 0$, let $\delta > 0$ be such that any two points at distance less than δ lie in a connected set of diameter less than ϵ. Fix $x \in X$. Given $y \in B_\delta(x)$, we can find a connected set C_y with $x, y \in C_y \subseteq B_\epsilon(x)$. Put $K_x = \bigcup_{y \in B_\delta(x)} C_y$. Then we have $B_\delta(x) \subseteq K_x \subseteq B_\epsilon(x)$ as desired. □

As an immediate consequence of the above lemma, we get:

3.3.3 Proposition

([6], [14]) *Any uniformly locally connected metric space is locally connected.* □

Again we remind the reader of Fox's result mentioned earlier (Proposition 1.4.7) which says that a bispace $(X, \mathcal{T}_1, \mathcal{T}_2)$ is quasi-pseudometrizable as a bispace if and only if $(X, \mathcal{T}_1, \mathcal{T}_2)$ is pairwise stratifiable and both $(X, \mathcal{T}_1)$ and $(X, \mathcal{T}_2)$ are quasi-pseudometrizable as topological spaces. Then we establish the following:

3.3.4 Theorem

If an ordered metric space $(X, \rho, \leq)$ is a C- and I-space which is uniformly locally connected then the bitopological space $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is quasi-pseudometrizable.

Proof. Suppose $(X, \rho, \leq)$ is a metric space which is a C- and I-space and uniformly locally connected. Note that here $\mathcal{T}$ is the topology generated by the metric ρ. Then by Theorem 1.4.6, the bitopological space $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is pairwise stratifiable. By Fox's result, it suffices to show that each of $(X, \mathcal{T}^\natural)$ and $(X, \mathcal{T}^\flat)$ is quasi-pseudometrizable as a topological space. We first consider $(X, \mathcal{T}^\natural)$.

As before, put $q_1(x, y) := \sup_{y' \in i(y)} \rho(i(x), y')$ and $U_\epsilon(x) = \{y \in X \mid i(y) \subseteq U_{q_1,\epsilon}(i(x))\}$ where $U_{q_1,\epsilon}(i(x)) = \{y \in X \mid (\exists\, x' \in i(x))(q_1(y, x') < \epsilon)\}$.

Note that q_1 is a quasi-pseudometric on X. We show that the sets $U_\epsilon(x)$ form a base for $\mathcal{T}^\natural$. Clearly $U_\epsilon(x)$ is an upper set. Let $O \in \mathcal{T}^\natural$ and $x \in O$. Then O is an upper set and it is open in the metric topology $\mathcal{T}(\rho)$. So there is some $\epsilon > 0$ such that $x \in B_\rho(x, \epsilon) \subseteq O$ where $B_\rho(x, \epsilon)$ is a member of a base for $\mathcal{T}(\rho)$. We should establish that $U_\epsilon(x) \subseteq O$. Take any $a \in U_\epsilon(x)$. Then $a \in X$ and $i(a) \subseteq U_{q_1,\epsilon}(i(x))$. In particular $a \in U_{q_1,\epsilon}(i(x))$. This implies that there exists some $x' \in i(x)$ such that $q_1(a, x') < \epsilon$. Hence by definition of q_1 we have $\rho(i(a), x') < \epsilon$ for such $x' \in i(x)$. In particular, $\rho(a, x') < \epsilon$, which implies that $a \in B_\rho(x, \epsilon) \subseteq O$ and thus $U_\epsilon(x) \subseteq B_\rho(x, \epsilon) \subseteq O$. Hence $\mathcal{T}(q_1) \subseteq \mathcal{T}^\natural$.

Conversely, let $G \in \mathcal{T}^\natural$ and take $x \in G$. Since G is an upper set then $i(x) \subseteq G$. Given that X is a metric space, singletons are closed in X, in particular $\{x\}$ is closed. Since X is a C-space then $i(x)$ is closed and hence $bd(i(x)) \subseteq i(x)$. Always $int(i(x)) \subseteq i(x)$. Thus $bd(i(x)) \cup int(i(x)) \subseteq G$. By Corollary 2.3.5, it follows that $bd(i(x))$ is compact. It is a known fact that for any compact subset of an open set there exists its uniform neighbourhood contained in the open set [33, Remark 2.5.4]. Hence there exists some $n \in \mathbb{N}$ such that $U_{\rho,2^{-n}}(bd(i(x))) \cup int(i(x)) \subseteq G$. (Remember that $U_{\rho,2^{-n}}(A) := \{x \in X \mid \exists\, a \in A, \rho(x, a) < 2^{-n}\}$ whenever $A \subseteq X$). Since X is uniformly locally connected there exists $m \in \mathbb{N}$ such that $U_{\rho,2^{-m}} \subseteq U_{\rho,2^{-n}}$ with $U_{\rho,2^{-m}}(x)$ connected. We may assume that $m \geq n + 1$.

We claim that $U_{\rho,2^{-m}}(i(x)) \subseteq U_{\rho,2^{-n}}(bd(i(x))) \cup int(i(x))$. Let $x' \in int(i(x))$. By connectedness of $U_{\rho,2^{-m}}(x)$ it follows that $U_{\rho,2^{-m}}(x') \cap bd(i(x)) \neq \emptyset$. So there exists some $t \in U_{\rho,2^{-m}}(x') \cap bd(i(x))$.

Let $x'' \in U_{\rho,2^{-m}}(x') \setminus i(x)$. Then we have $\rho(x',t) < 2^{-m}$ and $\rho(x',x'') < 2^{-m}$ so that

$$\begin{aligned}\rho(t,x'') &\leq \rho(t,x') + \rho(x',x'') = \rho(x',t) + \rho(x',x'') \\ &< 2^{-m} + 2^{-m} = 2(2^{-m}) = 2^{1-m} = 2^{-(m-1)},\end{aligned}$$

that is, $\rho(t,x'') \leq 2^{-(m-1)}$ which implies that $x'' \in U_{2^{-(m-1)}}(t)$.

But $t \in bd(i(x))$, so $x'' \in U_{2^{-(m-1)}}(bd(i(x)))$. Since $bd(i(x)) \subseteq i(x)$ we have $x'' \in U_{2^{-(m-1)}}(i(x))$. Hence $U_{\rho,2^{-m}}(x') \subseteq U_{2^{-(m-1)}}(i(x)) \subseteq U_n(i(x))$. But $U_{q_1,2^{-m}}(x) \subseteq U_{\rho,2^{-m}}(x) \subseteq G$. Therefore we deduce that $G \in \mathcal{T}(q_1)$. Thus $\mathcal{T}^\natural \subseteq \mathcal{T}(q_1)$. The two inclusions together yield $\mathcal{T}^\natural = \mathcal{T}(q_1)$. Therefore $(X, \mathcal{T}^\natural)$ is quasi-pseudometrizable.

Then put $q_2(x,y) := \sup_{x' \in d(x)} \rho(x', d(y))$ and $U_\epsilon(x) = \{y \in X \mid d(y) \subseteq U_{q_2,\epsilon}(d(x))\}$ where $U_{q_2,\epsilon}(d(x)) = \{y \in X \mid (\exists\, x' \in d(x))(q_2(y,x') < \epsilon)\}$.

Note that like q_1 above, q_2 is a quasi-pseudometric on X. Then $U_\epsilon(x)$ is a lower set. In the same manner as above one can show that $\mathcal{T}^\flat = \mathcal{T}(q_2)$ so that $(X, \mathcal{T}^\flat)$ is also quasi-pseudometrizable. Therefore by Fox's result, it follows that $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is quasi-pseudometrizable. $\square$

In view of the characterization of straight spaces by Berarducci, Dikranjan and Pelant [6], without further effort, we get the following:

3.3.5 Corollary

If $(X, \mathcal{T}, \leq)$ is a metrizable C- and I-space which is straight then the bitopological space $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is quasi-pseudometrizable. $\square$

One may then wonder if the assumption of Theorem 3.3.4 can be relaxed to local connectedness. In this regard, we prove the next result.

3.3.6 Proposition

If $(X, \mathcal{T}, \leq)$ is a metrizable locally connected ordered C- and I-space then the bispace $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is quasi-pseudometrizable.

Proof. Suppose $(X, \mathcal{T}, \leq)$ is a metrizable locally connected ordered C- and I-space. It suffices to show that X admits a uniformly locally connected (separated) uniformity with a countable base. To this end, assume that $\{U_n \,|\, n \in \mathbb{N}\}$ is a base for a compatible uniformity on $(X, \mathcal{T})$ such that $U_{n+1}^2 \subseteq U_n$ whenever $n \in \mathbb{N}$. Set $H_1 = U_1$. Suppose that for some $n \in \mathbb{N}$, H_n is defined as a neighborhood of the diagonal of X. Since X is paracompact, (see e.g. [40, Corollary 2]) there is a symmetric neighborhood U of the diagonal of X such that $U^4 \subseteq (H_n \cap U_{n+1})$. For each $x \in X$, find a connected neighborhood C_x of x such that $C_x \subseteq U(x)$. Set $H_{n+1} = \bigcup_{x \in X}(C_x \times C_x)$. Since $H_{n+1} \subseteq U_{n+1}$ and $H_{n+1}^2 \subseteq H_n$ whenever $n \in \mathbb{N}$, we see that $\{H_n \,|\, n \in \mathbb{N}\}$ is a countable base for a compatible uniformity $\mathcal{H}$ on $(X, \mathcal{T})$. Furthermore, $\mathcal{H}$ is uniformly locally connected, because for each $x \in X$ and $n \in \mathbb{N}$, $H_{n+1}(x)$ is connected as the union of connected sets intersecting at x (compare [14, proof of Lemma 1]). $\square$

Chapter 4

On Quasi-uniformities

4.1 Introduction

Here we provide a quasi-uniform version of the result obtained earlier, namely Corollary 2.4.3 in Chapter 2. We build up the machinery we shall use to achieve this as follows. We first construct two quasi-uniformities, $\mathcal{U}^i$ and $\mathcal{U}^d$, in terms of entourages of a uniformity $\mathcal{U}$ on a completely regular, ordered topological space $(X, \mathcal{T}, \leq)$ which is assumed to be a C- and I-space such that for any $x \in X$, $i(x)$ and $d(x)$ are totally bounded. Using these quasi-uniformities we then build up another one, denoted by $\mathcal{U}_{\Uparrow}$, which together with its conjugate provide the required quasi-uniformizability of the associated bispace $(X, (\mathcal{T}(\mathcal{U}))^{\natural}, (\mathcal{T}(\mathcal{U}))^{\flat})$.

4.2 Some Basics on Quasi-uniformities

At this point the reader may want to brush up on the definitions of a quasi-uniformity and related concepts given in Section 3.2 of Chapter 3. Furthermore, let X be a set. As in [17, Chapter 1], a non-empty subfamily $\mathcal{B}$ of a quasi-uniformity $\mathcal{U}$ on X is a base for $\mathcal{U}$ if and only if for every $U \in \mathcal{U}$ there exists $B \in \mathcal{B}$ such that $B \subseteq U$. In general, one can then show that a non-empty family $\mathcal{B}$ of subsets of $X \times X$ is a base for a quasi-uniformity $\mathcal{U}$ on X if and only if for any $U \in \mathcal{U}$ one has $\triangle \subseteq U$, and that given any $U \in \mathcal{U}$ there exists $V \in \mathcal{B}$ such that $V \circ V \subseteq U$. If, in addition, $\mathcal{B}$ is such that for any $U \in \mathcal{B}$ there exists $B \in \mathcal{B}$ with $B \subseteq U^{-1}$, then $\mathcal{B}$ is a base for a uniformity on X.

4.3 Construction of a Quasi-uniformity on an Ordered Uniform Space

In this section we give a quasi-uniform version of the construction given in Section 2.4. Let $(X, \mathcal{T})$ be a completely regular space equipped with a partial order $\leq$ and let $\mathcal{U}$ be a compatible uniformity on X. Under the assumption that $i(x)$ and $d(x)$ are totally bounded for any $x \in X$ and that $(X, \mathcal{T}, \leq)$ is a C- and I-space, we construct a quasi-uniformity $\mathcal{U}_{\Uparrow}$ on X such that $\mathcal{T}(\mathcal{U}_{\Uparrow}) = (\mathcal{T}(\mathcal{U}))^{\natural}$ and $\mathcal{T}((\mathcal{U}_{\Uparrow})^{-1}) = (\mathcal{T}(\mathcal{U}))^{\flat}$. As mentioned above, this will generalize the construction of a quasi-pseudometric in Chapter 2.

Let $U \in \mathcal{U}$ and put $U^i := \{(x, y) \in X \times X \,|\, i(y) \subseteq U(i(x))\}$ and similarly $U^d := \{(x, y) \in X \times X \,|\, d(y) \subseteq U(d(x))\}$. Furthermore, let $\mathcal{B}_{\uparrow} := \{U^i \,|\, U \in \mathcal{U}\}$ and $\mathcal{B}_{\downarrow} := \{U^d \,|\, U \in \mathcal{U}\}$. In this setting, we do not require that $\mathcal{U}$ has a countable base. Next, we use these collections to generate quasi-uniformities.

4.3.1 Lemma

The collection $\mathcal{B}_{\uparrow} = \{U^i \,|\, U \in \mathcal{U}\}$ is a base for a quasi-uniformity on X.

Proof. Let $x \in X$. For any $x' \in i(x)$ we have $x' \in U(i(x))$ so that $i(x) \subseteq U(i(x))$. This means $(x, x) \in U^i$. Hence $\triangle \subseteq U^i$ for any $U^i \in \mathcal{B}_{\uparrow}$.

It remains to show that for any $U^i \in \mathcal{B}_{\uparrow}$ there exists $V^i \in \mathcal{B}_{\uparrow}$ such that $V^i \circ V^i \subseteq U^i$. Let $U^i \in \mathcal{B}_{\uparrow}$. Since $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V \circ V \subseteq U$. Then of course $V^i \in \mathcal{B}_{\uparrow}$. Take any $(x, y) \in V^i \circ V^i$. Then there exists some $z \in X$ such that $(z, y) \in V^i$ and $(x, z) \in V^i$ which implies $i(y) \subseteq V(i(z))$ and $i(z) \subseteq V(i(x))$. To see that $(x, y) \in U^i$, it suffices to show that $V(i(z)) \subseteq U(i(x))$. Let $a \in V(i(z))$.

Then there exists $z' \in i(z)$ such that $(a, z') \in V$. But $i(z) \subseteq V(i(x))$, so $z' \in V(i(x))$ which guarantees the existence of some $x' \in i(x)$ such that $(z', x') \in V$. Now we have $(z', x'), (a, z') \in V$, hence $(a, x') \in V \circ V$. But $V \circ V \subseteq U$, so $(a, x') \in U$ which implies that $a \in U(i(x))$. Hence $V(i(z)) \subseteq U(i(x))$. Already $i(y) \subseteq V(i(z))$. So we now have $i(y) \subseteq U(i(x))$ which means $(x, y) \in U^i$. Thus $V^i \circ V^i \subseteq U^i$. Therefore $\mathcal{B}_\uparrow$ is a base for a quasi-uniformity on X. $\square$

Notation. The quasi-uniformity generated by $\mathcal{B}_\uparrow$ will be denoted by $\mathcal{U}^i$.

As expected, the following similar result holds.

4.3.2 Lemma

The collection $\mathcal{B}_\downarrow = \{U^d \,|\, U \in \mathcal{U}\}$ is a base for a quasi-uniformity on X. $\square$

Notation. We shall denote this quasi-uniformity by $\mathcal{U}^d$.

4.3.3 Proposition

Let $\mathcal{U}$ be a uniformity on X and define $\mathcal{S}$ by $\mathcal{S} := \{U^i \cap (U^d)^{-1} \,|\, U \in \mathcal{U}\}$.
Then $\mathcal{S}$ generates a quasi-uniformity on X.

Proof. We prove that $\mathcal{S}$ is a base for a quasi-uniformity on X. Recall that for each $U \in \mathcal{U}$,
$U^i = \{(x, y) \in X \times X \,|\, i(y) \subseteq U(i(x))\}$, $U^d = \{(x, y) \in X \times X \,|\, d(y) \subseteq U(d(x))\}$.
Clearly, $\triangle \subseteq U^i \cap (U^d)^{-1}$ for every $U^i \cap (U^d)^{-1} \in \mathcal{S}$. By the above two lemmas, $\mathcal{U}^i$ and $\mathcal{U}^d$ are quasi-uniformities on X so, for each $U^i \in \mathcal{U}^i$ there exists some $V^i \in \mathcal{U}^i$ such that $V^i \circ V^i \subseteq U^i$. Similarly, for any $U^d \in \mathcal{U}^d$, there exists some $V^d \in \mathcal{U}^d$ such that $V^d \circ V^d \subseteq U^d$. It remains to show that $(V^i \cap (V^d)^{-1}) \circ (V^i \cap (V^d)^{-1}) \subseteq (U^i \cap (U^d)^{-1})$.

Let $(a, b) \in (V^i \cap (V^d)^{-1}) \circ (V^i \cap (V^d)^{-1})$. Then there exists $c \in X$ such that $(c, b) \in (V^i \cap (V^d)^{-1})$ and $(a, c) \in (V^i \cap (V^d)^{-1})$ which means $(c, b) \in V^i$ and $(c, b) \in (V^d)^{-1}$, and indeed $(a, c) \in V^i$ and $(a, c) \in (V^d)^{-1}$. Now, $(a, c) \in$

V^i and $(c,b) \in V^i$ together implies that $(a,b) \in V^i \circ V^i$. But $V^i \circ V^i \subseteq U^i$, hence $(a,b) \in U^i$. In the same way we obtain $(a,b) \in (V^d)^{-1} \circ (V^d)^{-1}$. Since $(V^d)^{-1} \circ (V^d)^{-1} = (V^d \circ V^d)^{-1}$, then $(a,b) \in (V^d \circ V^d)^{-1}$ so that $(b,a) \in V^d \circ V^d$. We have already observed that $V^d \circ V^d \subseteq U^d$ so it follows that $(b,a) \in U^d$ and hence $(a,b) \in (U^d)^{-1}$. Consequently, $(a,b) \in U^i \cap (U^d)^{-1}$. Hence $\mathcal{S}$ generates a quasi-uniformity on X as asserted. $\square$

Notation. We shall denote the above quasi-uniformity by $\mathcal{U}_{\Uparrow}$, and its conjugate quasi-uniformity by $(\mathcal{U}_{\Uparrow})^{-1}$. For the next theorem we need the following facts:

4.3.4 Lemma

Let U^i and U^d be members of the bases of the quasi-uniformities $\mathcal{U}^i$ and $\mathcal{U}^d$ respectively defined on the ordered uniform space $(X, \mathcal{U}, \leq)$. Then for every $x \in X$ we have

a) $(U^d)^{-1}(x) = \bigcap_{x' \in d(x)} i(U(x'))$;

b) $(U^i)^{-1}(x) = \bigcap_{x' \in i(x)} d(U(x'))$.

Proof. We prove only part a). Part b) is obtained in a similar way.

Let $y \in X$ and U^d be a member of the base of the quasi-uniformity $\mathcal{U}^d$. Then

$y \in (U^d)^{-1}(x) \iff (x,y) \in (U^d)^{-1} \iff (y,x) \in U^d$

$\iff d(x) \subseteq U(d(y)) \iff (\forall x' \in d(x))(x' \in U(d(y)))$

$\iff (\forall x' \in d(x))(\exists y' \in d(y))(x' \in U(y'))$

$\iff (\forall x' \in d(x))(\exists y' \in d(y))((y', x') \in U)$

$\iff (\forall x' \in d(x))(\exists y' \in U(x'))(y' \leq y)$

$\iff (\forall x' \in d(x))(y \in i(U(x')))$

$\iff y \in (\bigcap_{x' \in d(x)} i(U(x')))$. Hence $(U^d)^{-1}(x) = \bigcap_{x' \in d(x)} i(U(x'))$ as desired. $\square$

Now we are ready to state and prove the quasi-uniform version of Theorem 2.4.2.

4.3.5 Theorem

Let $\mathcal{U}_{\Uparrow}$ be the quasi-uniformity defined in terms of a compatible uniformity $\mathcal{U}$ on an ordered completely regular space $(X, \mathcal{T}, \leq)$ which is a C- and I-space. If $i(x)$ and $d(x)$ are totally bounded for every $x \in X$ then $\mathcal{T}(\mathcal{U}_{\Uparrow}) = (\mathcal{T}(\mathcal{U}))^{\natural}$ and $\mathcal{T}((\mathcal{U}_{\Uparrow})^{-1}) = (\mathcal{T}(\mathcal{U}))^{\flat}$.

Proof. Suppose the hypothesis holds. We prove the equalities. Let $O \in \mathcal{T}(\mathcal{U}_{\Uparrow})$. Then there exists some $W \in \mathcal{U}_{\Uparrow}$ such that for any $x \in O$, we have $W(x) \subseteq O$. Now by construction of $\mathcal{U}_{\Uparrow}$, there exists some $U \in \mathcal{U}$ such that $U^i \cap (U^d)^{-1} \subseteq W$. Thus $(U^i \cap (U^d)^{-1})(x) \subseteq W(x) \subseteq O$ which evidently implies that $U^i(x) \cap (U^d)^{-1}(x) \subseteq O$. Since $(U^d)^{-1}(x) = \bigcap_{x' \in d(x)} i(U(x'))$, we actually have $U^i(x) \cap (\bigcap_{x' \in d(x)} i(U(x'))) \subseteq O$. Given that $d(x)$ is totally bounded, there exists a finite $D \subseteq d(x)$ such that $U^i(x) \cap (\bigcap_{x' \in D} i(U(x'))) \subseteq O$. But $\bigcap_{x' \in D} i(U(x'))$ is an upper set and it is open in $\mathcal{T}(\mathcal{U})$. Hence $\bigcap_{x' \in D} i(U(x')) \in (\mathcal{T}(\mathcal{U}))^{\natural}$. It remains to show that $U^i(x) \in (\mathcal{T}(\mathcal{U}))^{\natural}$. This holds because $U^i(x) = X \setminus d(X \setminus U(i(x)))$ and $X \setminus d(X \setminus U(i(x)))$ is clearly an open upper set. Thus $U^i(x) \cap (\bigcap_{x' \in D} i(U(x'))) \in (\mathcal{T}(\mathcal{U}))^{\natural}$, and so $O \in (\mathcal{T}(\mathcal{U}))^{\natural}$. Hence $\mathcal{T}(\mathcal{U}_{\Uparrow}) \subseteq (\mathcal{T}(\mathcal{U}))^{\natural}$. Conversely, let $O \in (\mathcal{T}(\mathcal{U}))^{\natural}$. Then $O \in \mathcal{T}(\mathcal{U})$ and O is an upper set. Observe that $O \in \mathcal{T}(\mathcal{U})$ implies that for each $t \in O$ there exists $U \in \mathcal{U}$ such that $U(t) \subseteq O$. Since O is an upper set, it follows that $i(U(t)) \subseteq O$. Since $U^i(t) \cap (\bigcap_{t' \in d(t)} i(U(t')))$ is basic open in $\mathcal{T}(\mathcal{U}_{\Uparrow})$ and $U^i(t) \cap (\bigcap_{t' \in d(t)} i(U(t'))) \subseteq i(U(t)) \subseteq O$, we deduce that $O \in \mathcal{T}(\mathcal{U}_{\Uparrow})$. Hence $(\mathcal{T}(\mathcal{U}))^{\natural} \subseteq \mathcal{T}(\mathcal{U}_{\Uparrow})$, and therefore $\mathcal{T}(\mathcal{U}_{\Uparrow}) = (\mathcal{T}(\mathcal{U}))^{\natural}$.

Let $Q \in \mathcal{T}((\mathcal{U}_{\Uparrow})^{-1})$. Then there exists $W \in (\mathcal{U}_{\Uparrow})^{-1}$ such that $W(x) \subseteq Q$ for every $x \in Q$. Note that $W = V^{-1}$ for some $V \in \mathcal{U}_{\Uparrow}$. Then by construction of $(\mathcal{U}_{\Uparrow})^{-1}$, it follows that $W \supseteq (U^i \cap (U^d)^{-1})^{-1}$, that is, $U^d \cap (U^i)^{-1} \subseteq W$ so that $U^d(x) \cap (U^i)^{-1}(x) \subseteq W(x) \subseteq Q$. Since $U^d(x) = X \setminus i(X \setminus U(d(x)))$ and

$(U^i)^{-1}(x) = \bigcap_{x' \in i(x)} d(U(x')))$ we get $U^d(x) \cap \bigcap_{x' \in i(x)} d(U(x')) \subseteq Q$. Since $i(x)$ is totally bounded, it follows that $Q \in (\mathcal{T}(\mathcal{U}))^\flat$. Hence $\mathcal{T}((\mathcal{U}_\Uparrow)^{-1}) \subseteq (\mathcal{T}(\mathcal{U}))^\flat$. Conversely, let $Q \in (\mathcal{T}(\mathcal{U}))^\flat$. Then $Q \in \mathcal{T}(\mathcal{U})$ and Q is a lower set. So for every $s \in Q$ there exists $U \in \mathcal{U}$ such that $U(s) \subseteq Q$. Then $d(U(s)) \subseteq Q$. Since $U^d(s) \cap (\bigcap_{s' \in i(s)} d(U(s')))$ is basic open in $\mathcal{T}((\mathcal{U}_\Uparrow)^{-1})$ and $U^d(s) \cap (\bigcap_{s' \in d(s)} d(U(s'))) \subseteq d(U(s)) \subseteq Q$, we deduce that $Q \in \mathcal{T}((\mathcal{U}_\Uparrow)^{-1})$. Hence $(\mathcal{T}(\mathcal{U}))^\flat \subseteq \mathcal{T}((\mathcal{U}_\Uparrow)^{-1})$. Therefore $\mathcal{T}((\mathcal{U}_\Uparrow)^{-1}) = (\mathcal{T}(\mathcal{U}))^\flat$. $\square$

4.3.6 Corollary

Let $\mathcal{U}$ be a compatible uniformity on an ordered completely regular space $(X, \mathcal{T}, \leq)$ which is a C- and I-space. If $i(x)$ and $d(x)$ are totally bounded for every $x \in X$ then the bispace $(X, (\mathcal{T}(\mathcal{U}))^\natural, (\mathcal{T}(\mathcal{U}))^\flat)$ is quasi-uniformizable.

Proof. By the above theorem, $\mathcal{U}_\Uparrow$ is the required quasi-uniformity. $\square$

4.4 A Touch on Equicontinuity

In this short section, we bring equicontinuity in perspective by simply recalling its definition and illustrate that under an assumption of an appropriate equicontinuity condition, we also get a positive result (Proposition 4.4.5).

4.4.1 Definition

([57]) Let X be a topological space and $(Y, \mathcal{V})$ a uniform space. A family $\mathcal{F}$ of continuous functions from X to Y is said to be *equicontinuous at* $x \in X$ if and only if for each $V \in \mathcal{V}$ there is a neighbourhood U of x such that $f(U) \subseteq V(f(x))$ for each $f \in \mathcal{F}$. The family $\mathcal{F}$ is said to be *equicontinuous* if it is equicontinuous at each point of X.

Remark. Note that $\mathcal{F}$ is equicontinuous with respect to the uniformity $\mathcal{V}$. The notion of equicontinuity has been used in Ascoli's theorem [16, Theorem 8.2.10] or [57, Theorem 43.15] to characterize compact function spaces in compact-open topology.[1]

We recall the definition of a topological lattice.

4.4.2 Definition

([32], [19]) A *topological lattice* is a lattice equipped with a topology such that the lattice operations join ($\vee$) and meet ($\wedge$) are continuous.

Remark. In Theorem 4.3.5, the assumption of total boundedness of $i(x)$ and $d(x)$ was essential for $\bigcap\limits_{x'\in D} i(U(x'))$ and $\bigcap\limits_{x'\in E} d(U(x'))$ (with finite $D \subseteq d(x)$ and finite $E \subseteq i(x)$) to be $\mathcal{T}^{\natural}$- and $\mathcal{T}^{\flat}$-neighbourhoods at x respectively. The next two results on equicontinuity are as good as this assumption, in a sense that they provide the same required link in this result.

4.4.3 Lemma

Let $(X, \mathcal{T}, \leq)$ be a topological lattice with a compatible uniformity $\mathcal{U}$. Furthermore, let $x \in X$ and $a \in i(x)$ then define $j_a : (X, \mathcal{T}(\mathcal{U})) \to (X, \mathcal{U})$ by $j_a(y) = a \vee y$. Suppose that the family $\mathcal{F} = \{j_a \,|\, a \in i(x)\}$ of these continuous maps is equicontinuous at x. Then for each $U \in \mathcal{U}$, $\bigcap\limits_{a\in i(x)} d(U(a))$ is a $\mathcal{T}^{\flat}$- neighbourhood at x.

Proof. Suppose $(X, \mathcal{T}, \leq)$ is a topological lattice with a compatible uniformity $\mathcal{U}$. Since any topological lattice is an I-space (see e.g. [32, p. 291]), it follows that $(X, \mathcal{T}, \leq)$ is an I-space. Let $U \in \mathcal{U}$ and $x \in X$. Since $\mathcal{F}$ is equicontinuous, there is a neighbourhood N of x such that $j_a(N) \subseteq U(j_a(x))$ whenever $a \in i(x)$. Let $y \in N$. Then $j_a(y) = a \vee y \in U(a \vee x) = U(a)$ whenever $a \geq x$. Thus $y \in d(U(a))$

[1] The compact-open topology (k-topology) on a function space $\mathcal{F} \subseteq Y^X$ is the topology having for a subbase the sets $(K, U) = \{f \in \mathcal{F} \,|\, f(K) \subseteq U\}$, where K is compact in X and U open in Y.

for all $a \in i(x)$, and so $y \in \bigcap_{a\in i(x)} d(U(a))$. Hence $N \subseteq \bigcap_{a\in i(x)} d(U(a))$ which implies that $d(N) \subseteq \bigcap_{a\in i(x)} d(U(a))$. Since X is an I-space, $d(intN)$ is open, and so $d(intN) \in \mathcal{T}^\flat$. We now have $x \in d(intN) \subseteq d(N) \subseteq \bigcap_{a\in i(x)} d(U(a))$. Therefore $\bigcap_{a\in i(x)} d(U(a))$ is a $\mathcal{T}^\flat$-neighbourhood at x. □

4.4.4 Lemma

Let $(X, \mathcal{T}, \leq)$ be a topological lattice with a compatible uniformity $\mathcal{U}$. Furthermore, let $x \in X$ and $a \in d(x)$ then define $k_a : (X, \mathcal{T}(\mathcal{U})) \to (X, \mathcal{U})$ by $k_a(y) = a \wedge y$. Suppose that the family $\mathcal{G} = \{k_a \,|\, a \in d(x)\}$ of these continuous maps is equicontinuous at x. Then for each $V \in \mathcal{U}$, $\bigcap_{a\in d(x)} i(V(a))$ is a $\mathcal{T}^\natural$- neighbourhood at x.

Proof. Suppose $(X, \mathcal{T}, \leq)$ is a topological lattice with a compatible uniformity $\mathcal{U}$. Then $(X, \mathcal{T}, \leq)$ is an I-space. Let $V \in \mathcal{U}$ and $x \in X$. Since $\mathcal{G}$ is equicontinuous, there is a neighbourhood M of x such that $k_a(M) \subseteq V(k_a(x))$ whenever $a \in d(x)$. By definition of k_a, one can easily establish that $M \subseteq \bigcap_{a\in d(x)} i(V(a))$ which implies $i(M) \subseteq \bigcap_{a\in d(x)} i(V(a))$. Since X is an I-space, $i(intM)$ is open, and so $i(intM) \in \mathcal{T}^\natural$. We now have $x \in i(intM) \subseteq i(M) \subseteq \bigcap_{a\in d(x)} i(V(a))$. Therefore $\bigcap_{a\in d(x)} i(V(a))$ is a $\mathcal{T}^\natural$-neighbourhood at x. □

Thus, in view of the remark just before the above two lemmas, we obtain the following.

4.4.5 Proposition

Let $(X, \mathcal{T}, \leq)$ be a topological lattice which is a C-space with a compatible uniformity $\mathcal{U}$. Suppose the equicontinuity conditions in Lemmas 4.4.3 and 4.4.4 hold and $\mathcal{T}$ is metrizable then the bispace $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is quasi-pseudometrizable. □

Chapter 5

Compatibility of a Uniform Structure and a Partial Order

5.1 Introduction

In this chapter we continue to investigate our problem in different but similar settings. In one case, we consider a metric space and in the other a uniform space with a partial order of a certain type in each case. In 1994, T. Richmond introduced the notion of ball transitivity of an ordered metric space (see the definition below). He established, among other results, that any ball transitive space is an I-space [53, Proposition 2]. Here we prove a similar result in the two settings introduced in the next two sections. Other results are highlighted at the beginning of each section of this chapter. However, we point out here that in the last section we study some important examples which are not restricted to the settings of this chapter.

5.2 Metric Spaces and Friendly Partial Orders

In this section we introduce a partial order $\leq$ on a metric space (X, m) which we call an m-friendly partial order (see Definition 5.2.2). This involves a natural compatibility condition between a metric and a partial order. We then show, in Lemma 5.2.4, that any metric space with such a partial order is an I-space. A known example illustrates that such a space is not a C-space in general. Furthermore, for a meaningful order of things, the fact that the bispace $(X, (\mathcal{T}(m))^\natural, (\mathcal{T}(m))^\flat)$ associated with an ordered metric space $(X, m, \leq)$ is quasi-pseudometrizable will only be established in the next section (Corollary 5.3.14). We shall also prove that a linear order on a uniformly locally order convex metric space is m-friendly (Proposition 5.2.10).

It may be necessary to recall that if A and B are binary relations on X then the composite relation $B \circ A$ on X is defined by
$B \circ A := \{(a, c) \in X \times X \mid \exists\, b \in X \text{ such that } (a, b) \in A \text{ and } (b, c) \in B\}$ on X.
We first recall the definition of ball transitivity.

5.2.1 Definition

([53, Definition 1]) Let $n \in \mathbb{N}$. A metric space (X, m) with a partial order $\leq$ is said to be $\frac{1}{n}$*-ball-transitive* if and only if $x \leq y$ implies $B(x, \frac{\epsilon}{n}) \subseteq d(B(y, \epsilon))$ and $B(y, \frac{\epsilon}{n}) \subseteq i(B(x, \epsilon))$ for any $\epsilon > 0$. We say that an ordered metric space $(X, m, \leq)$ is *ball-transitive* provided that it is $\frac{1}{n}$-ball-transitive for some $n \in \mathbb{N}$.

We now introduce friendliness.

5.2.2 Definition

Let (X, m) be a metric space and $\leq$ a partial order on X. Then we say that $\leq$ is an *m-friendly partial order* on (X, m) if and only if for each $\epsilon > 0$ there exists $\delta > 0$ such that $B_\delta(d(x)) \subseteq d(B_\epsilon(x))$ and $B_\delta(i(x)) \subseteq i(B_\epsilon(x))$ whenever $x \in X$. (Here, for a subset A of (X, m), $B_\delta(A) := \{x \in X \mid m(x, a) < \delta \text{ for some } a \in A\}$.)

5.2.3 Example

Consider the real line $\mathbb{R}$ with the usual metric m and usual order. Then the usual order is an m-friendly partial order on $(\mathbb{R}, m)$.

The following observation is similar to [53, Proposition 2].

5.2.4 Lemma

If $(X, m, \leq)$ is a metric space with an m-friendly partial order then $(X, \mathcal{T}(m), \leq)$ is an I-space.

Proof. Let G be an open subset of X. We first show that $d(G)$ is open. Let $x \in d(G)$. Then there exist some $y \in G$ and $\epsilon > 0$ such that $x \leq y$ and $B_\epsilon(y) \subseteq G$. Since $\leq$ is an m-friendly partial order, there exists $\delta > 0$ such that $B_\delta(d(y)) \subseteq d(B_\epsilon(y))$. Thus $B_\delta(d(y)) \subseteq d(B_\epsilon(y)) \subseteq d(G)$. Since $B_\delta(x) \subseteq B_\delta(d(y))$, we then get $B_\delta(x) \subseteq d(G)$. Hence $d(G)$ is open in X.

We now show that $i(G)$ is also open. Suppose $x \in i(G)$. Then there exists $y \in G$ such that $y \leq x$. Since G is open, there exists some $\epsilon > 0$ such that $B_\epsilon(y) \subseteq G$. Given that $\leq$ is m-friendly, we can find $\delta > 0$ such that $B_\delta(i(y)) \subseteq i(B_\epsilon(y))$. Now we obviously have the inclusions $B_\delta(i(y)) \subseteq i(B_\epsilon(y)) \subseteq i(G)$. Since $x \in i(y)$ we particularly have $B_\delta(x) \subseteq i(G)$, and hence $i(G)$ is open. Therefore $(X, \mathcal{T}(m), \leq)$ is an I-space. $\square$

Remark. In the above lemma, I-space cannot be replaced with C-space. So, if $(X, m, \leq)$ is a metric space with an m-friendly partial order then $(X, \mathcal{T}(m), \leq)$ need not be a C-space, as the following example shows.

5.2.5 Example

([53]) Consider $X = \mathbb{R}^2$ with its usual product topology and usual order. Let $F := \{(-n, \frac{1}{n}) \in \mathbb{R}^2 \,|\, n \in \mathbb{N}\}$. Then F is closed in $\mathbb{R}^2$. Note that for this F, $i(F) = \{(a, b) \in \mathbb{R}^2 \,|\, \exists\, n \in \mathbb{N} \text{ such that } (a, b) \geq (-n, \frac{1}{n})\}$ and observe that for any $n \in \mathbb{N}$ we have $\frac{1}{n} \nleq 0$. Thus $(0,0) \notin i(F)$ even though $(0,0) \in cl_{\mathcal{T}}(i(F))$. Hence $i(F)$ is not closed. Therefore $(\mathbb{R}^2, \mathcal{T}(m), \leq)$ is not a C-space.

The following concept is well-known and is important here.

5.2.6 Definition

([17], [46]) Let $(X, \leq)$ be an ordered set. A subset $A \subseteq X$ is said to be (*order*) *convex* if and only if $A = i(A) \cap d(A)$.

The following characterization of order convexity is important. Its proof is easy, hence is left to the leader.

5.2.7 Lemma

A subset A of X is order convex if and only if $c \in A$ whenever $a, b \in A$ and $c \in X$ such that $a \leq c \leq b$. □

Next we define uniform locally order convexity of a metric space and show that this condition is sufficient for a linear order on the space to be m-friendly.

5.2.8 Definition

([52]) An ordered metric space $(X, m, \leq)$ is called *uniformly locally order convex* if and only if for every $\epsilon > 0$ there exists $\delta > 0$ with $\delta \leq \epsilon$ such that $B_\delta(x)$ is order convex whenever $x \in X$.

5.2.9 Lemma

([52]) *Any uniformly locally order convex metric space is order convex.* □

5.2.10 Proposition

Let $(X, m, \leq)$ be a uniformly locally order convex metric space where $\leq$ is a linear order. Then $\leq$ is m-friendly.

Proof. Let $\epsilon > 0$. Since (X, m) is uniformly locally order convex, then we can choose $\delta > 0$ such that $2\delta \leq \epsilon$ and $B_\delta(x)$ is order convex whenever $x \in X$. We now show that $\leq$ satisfies the defining condition of an m-friendly partial order. We first show that $B_\delta(d(x)) \subseteq d(B_\epsilon(x))$. Take any $y \in B_\delta(d(x))$. Then there exists some $z \leq x$ such that $y \in B_\delta(z)$, and by symmetry $z \in B_\delta(y)$. We distinguish cases here.
Case 1: If $y \leq x$ then $y \in d(B_\delta(x))$ because $x \in B_\delta(x)$.
Case 2: Suppose $y \nleq x$. Since $\leq$ is linear, then the only possibility is that $x < y$. Now we have $z, y \in B_\delta(z)$ and $z \leq x < y$. Thus by order convexity of $B_\delta(z)$, we get $x \in B_\delta(z)$. Again by symmetry of the metric, it follows that $z \in B_\delta(x)$. Since we already have $y \in B_\delta(z)$ then by the triangle inequality and the assumption that $2\delta \leq \epsilon$, we get $y \in B_{2\delta}(x) \subseteq B_\epsilon(x)$. So $y \in d(B_\epsilon(x))$. Hence $B_\delta(d(x)) \subseteq d(B_\epsilon(x))$.
Next we show that $B_\delta(i(x)) \subseteq i(B_\epsilon(x))$. Let $y \in B_\delta(i(x))$. Then $y \in B_\delta(z)$ for some $z \in i(x)$. Again we distinguish cases.
Case 1: If $y \geq x$ then clearly $y \in i(B_\epsilon(x))$.
Case 2: If $x \nleq y$ then, given that the order is linear, $y < x$ and hence $y < x \leq z$ where $y, z \in B_\delta(z)$. Since $B_\delta(z)$ is order convex it follows that $x \in B_\delta(z)$ and by symmetry $z \in B_\delta(x)$. Note that $y \in B_\delta(z)$ and $z \in B_\delta(x)$ together imply $y \in B_{2\delta}(x) \subseteq B_\epsilon(x)$, and thus $y \in i(B_\epsilon(x))$. Hence $B_\delta(i(x)) \subseteq i(B_\epsilon(x))$. Therefore $\leq$ is m-friendly. $\square$

Remark. Later, we shall present a similar result (Corollary 5.3.11) which is a uniform version of the above proposition.

5.3 Uniform Spaces and Friendly Partial Orders

In this section we study the uniform version of the friendliness notion introduced for metric spaces in the previous section. We look at a compatibility condition between a uniformity and a partial order. This may be viewed as commutativity under composition of the order and the uniformity (see the remark just after the definition below). As indicated earlier, Richmond's ball transitivity in Definition 5.2.1 above is similar to this condition. However, we point out that in his definition, Richmond requires ϵ and δ to be related as expressed. Nachbin [46, p. 72, Theorem 10] investigates a variant of the condition under consideration here. As highlights of this section, after defining a $\mathcal{U}$-friendly partial order on a uniform space, we give its characterization in terms of entourages, and then show that any partial order of a uniform lattice is of this kind. Also as a consequence of this, we then establish that any linear order on a uniformly locally order convex uniform space is $\mathcal{U}$-friendly.

5.3.1 Definition

Let $(X,\mathcal{U})$ be a uniform space and $\leq$ a partial order on X. Then we say that $\leq$ is a *$\mathcal{U}$-friendly partial order* on X provided that for each $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $V(i(x)) \subseteq i(U(x))$ and $V(d(x)) \subseteq d(U(x))$ whenever $x \in X$.

Remark. The two conditions in the above definition can be expressed explicitly in terms of composition as $V\circ \leq \subseteq \leq \circ U$ and $V\circ \geq \subseteq \geq \circ U$ respectively. Next, we give a characterization of friendliness of a partial order $\leq$ on a uniform space $(X,\mathcal{U})$ in terms of entourages of $\mathcal{U}$ as follows.

5.3.2 Lemma

A partial order $\leq$ on a uniform space $(X,\mathcal{U})$ is $\mathcal{U}$-friendly if and only if the filters generated by the sets $\{\leq \circ U \,|\, U \in \mathcal{U}\}$ and $\{U \circ \leq \,|\, U \in \mathcal{U}\}$ on $X \times X$ are equal.

Proof. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be the filters generated by the sets $\{\leq \circ U \mid U \in \mathcal{U}\}$ and $\{U \circ \leq \mid U \in \mathcal{U}\}$ respectively.

($\Rightarrow$) Suppose $\leq$ is a $\mathcal{U}$-friendly partial order on X. Take $F \in \mathcal{F}_1$. Then $F \subseteq X \times X$ with $F \supseteq \leq \circ U$ for some $U \in \mathcal{U}$. Since $\leq$ is $\mathcal{U}$-friendly, then for this $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V \circ \leq \subseteq \leq \circ U \subseteq F$. Thus $V \circ \leq \subseteq F$, which implies $F \in \mathcal{F}_2$. Hence $\mathcal{F}_1 \subseteq \mathcal{F}_2$. Similarly, let $F \in \mathcal{F}_2$. Then $F \subseteq X \times X$ such that $F \supseteq U \circ \leq$ for some symmetric $U \in \mathcal{U}$. Since U is symmetric, we have $\geq \circ U \subseteq F$. By friendliness of the partial order $\leq$, there exists a symmetric $V \in \mathcal{U}$ such that $V \circ \geq \subseteq \geq \circ U \subseteq F$. Thus $V \circ \geq \subseteq F$. By symmetry of V we deduce that $\leq \circ V \subseteq F$. This implies that $F \in \mathcal{F}_1$. Hence $\mathcal{F}_2 \subseteq \mathcal{F}_1$. Therefore $\mathcal{F}_1 = \mathcal{F}_2$.

($\Leftarrow$) Suppose $\mathcal{F}_1 = \mathcal{F}_2$. Let $U \in \mathcal{U}$. Then $\leq \circ U \in \mathcal{F}_1 = \mathcal{F}_2$. So there exists some $V \in \mathcal{U}$ such that $V \circ \leq \subseteq \leq \circ U$. Then $(V \circ \leq)(x) \subseteq (\leq \circ U)(x)$ for all $x \in X$. Thus $V(i(x)) \subseteq i(U(x))$ whenever $x \in X$. For the other inclusion, take a symmetric $U \in \mathcal{U}$, then $U \circ \leq \in \mathcal{F}_2 = \mathcal{F}_1$. Then there exists a symmetric $V \in \mathcal{U}$ such that $\leq \circ V \subseteq U \circ \leq$. By symmetry of U and V we get $V \circ \geq \subseteq \geq \circ U$ which says that $V(d(x)) \subseteq d(U(x))$ for any $x \in X$. Hence $\leq$ is $\mathcal{U}$-friendly. $\square$

5.3.3 Definition

([52]) A *uniform lattice* is a triple $(L, \mathcal{U}, \leq)$ where $(L, \leq)$ is a lattice and $\mathcal{U}$ is a Hausdorff uniformity on L with respect to which the lattice operations meet and join are uniformly continuous.

Our next result is inspired by [46, Proposition 11, p. 74].

5.3.4 Proposition

The partial order $\leq$ of a uniform lattice $(X, \mathcal{U})$ is $\mathcal{U}$-friendly.

Proof. Let $(X, \mathcal{U})$ be a uniform lattice and $\leq$ the partial order on X. Also let $V \in \mathcal{U}$. By definition of a uniform lattice there is $W \in \mathcal{U}$ such that $(x', x'') \in W$

and $(y', y'') \in W$ implies that $(x' \vee y', x'' \vee y'') \in V$ and $(x' \wedge y', x'' \wedge y'') \in V$. We show that $W \circ \leq \subseteq \leq \circ V$ and $W \circ \geq \subseteq \geq \circ V$. Indeed let $(x, y) \in W \circ \leq$. Then there exists $t \in X$ such that $(x, t) \in \leq$ and $(t, y) \in W$, that is, $x \leq t$ and $(t, y) \in W$. Then $(x, x) \in W$ and $(t, y) \in W$, which implies that $(x \wedge t, x \wedge y) \in V$. Consequently, $(x, x \wedge y) \in V$ and $x \wedge y \leq y$, which implies that $(x, y) \in \leq \circ V$. This establishes the first inclusion. Similarly, let $(x, t) \in W \circ \geq$. Then there is $y \in X$ such that $x \geq y$ and $(y, t) \in W$. Then $(x, x) \in W$ and $(y, t) \in W$, which implies that $(x \vee y, x \vee t) \in V$. It follows that $(x, x \vee t) \in V$ and $x \vee t \geq t$, which implies that $(x, y) \in \leq \circ V$. Hence $(x, t) \in \geq \circ V$ so that $W \circ \geq \subseteq \geq \circ V$. Therefore $\leq$ is $\mathcal{U}$-friendly. $\square$

Before we state the next corollary, we define uniform locally order convexity for uniform spaces. This is a generalization of Definition 5.2.8.

5.3.5 Definition

([52]) We shall call an ordered uniform space $(X, \mathcal{U}, \leq)$ *uniformly locally order convex* provided that for each $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $V \subseteq U$ and $V(x)$ is order convex whenever $x \in X$.

We point out here that such uniformities have been studied by several authors under slightly different names. For instance they are called *convex* in [17, p. 84] and Redfield [52, p. 290] calls them *uniformly convex*. Here we emphasize the order on X in the name. We also recall the following concept.

5.3.6 Definition

([17], [52]) An ordered uniform space $(X, \mathcal{U}, \leq)$ is said to be *locally order convex* provided that for each $x \in X$ and $O \in \mathcal{T}(\mathcal{U})$ with $x \in O$ there exists $Q \in \mathcal{T}(\mathcal{U})$ and an order convex subset $C \subseteq X$ such that $x \in Q \subseteq C \subseteq O$.

Remark. The above definition simply says that a uniform space $(X, \mathcal{U})$ is locally order convex if and only if the set of order convex neighbourhoods of every point of X with respect to the topology induced by $\mathcal{U}$ is a base for the neighbourhood system at the point.

The following fact holds trivially and it provides us with examples.

5.3.7 Lemma

Any uniformly locally order convex uniform space is locally order convex. $\square$

The converse of the above lemma does not hold, as the following example due to Redfield [52] shows.

5.3.8 Example.

([52, Example 2.1]) Define the partial order $\leq$ on the plane $\mathbb{R}\times\mathbb{R}$ by $(x,y)\leq(s,t)$ if and only if $x\leq s$ and $y\leq t$. For any $n\in\mathbb{N}$, set

$$U_n = \triangle_{\mathbb{R}} \cup \{(x,y)\in\mathbb{R}^2 \mid (x,y)\leq(0,0) \text{ and } |y-x|\leq \tfrac{1}{n}\};$$
$$V_n = \{(x,y)\in\mathbb{Q}\times\mathbb{Q} \mid (x,y)\geq(n,n)\} \text{ and}$$
$$W_n = U_n \cup V_n.$$

Then $U_n, V_n, W_n \subseteq \mathbb{R}\times\mathbb{R}$. By construction of W_n, we have $\triangle_{\mathbb{R}} \subseteq W_n$, and it is easy to see that $W_n^{-1} = W_n$ for all $n\in\mathbb{N}$. Note that $W_{2n} = U_{2n}\cup V_{2n}$. Take any $(x,y),(y,z)\in W_{2n}$. If $x=y$ or $y=z$ then $(x,z)\in W_{2n}$. But $W_{2n}\subseteq W_n$, so $(x,z)\in W_n$. If not then $x\neq y$ and $y\neq z$. If $(x,y)\in U_{2n}$ then by definition of U_n we have $(x,y)\leq(0,0)$ and $|y-x|\leq\frac{1}{2n}$. This implies that $y\leq 0$ and hence $(y,z)\notin V_{2n}$. Consequently, $(y,z)\in U_{2n}$ which means $(y,z)\leq(0,0)$ and $|z-y|\leq\frac{1}{2n}$. It follows that $(x,z)\leq(0,0)$ and then

$$|z-x|\leq|z-y|+|y-x|\leq\tfrac{1}{2n}+\tfrac{1}{2n}=\tfrac{1}{n}.$$

Hence $(x,z)\in U_n\subseteq W_n$. If $(x,y)\in V_{2n}$ then $(x,y)\geq(2n,2n)>(0,0)$ and hence by a similar argument as above, we get $(y,z)\in V_{2n}$. Since $x,y,z\in\mathbb{Q}$, we deduce that $(x,z)\in V_{2n}\subseteq W_n$. This says that $W_{2n}\circ W_{2n}\subseteq W_n$. Thus the set

$\{W_n \,|\, n \in \mathbb{N}\}$ is a filter base for a uniformity $\mathcal{U}$ on $\mathbb{R}$.

We now show that $\mathcal{U}$ is locally order convex. Let $x \in X$ and $O \in \mathcal{T}(\mathcal{U})$ with $x \in O$. Then there exists $W_n \in \mathcal{U}$ such that $W_n(x) \subseteq O$. Obviously, $x \in W_n(x)$ and $W_n(x)$ is open. To see that $W_n(x)$ is also order convex, let $a, b \in W_n(x)$ and $c \in \mathbb{R}$ such that $a \leq c \leq b$. Then $(a, x), (b, x) \in W_n$. By definition of W_n, we have $(a, x), (b, x) \in U_n$ or $(a, x), (b, x) \in V_n$. In the former case, it follows that $(c, x) \leq (0, 0)$ and $|x - c| \leq \frac{1}{n}$ because $c \leq b \leq 0$. Hence $(c, x) \in U_n$ so that $c \in W_n(x)$. In the latter case, $(a, x), (b, x) \in \mathbb{Q} \times \mathbb{Q}$ with $(a, x), (b, x) \geq (n, n)$. But then $b \geq c \geq a \geq n$ which implies $c \geq n$. Thus $(c, x) \geq (n, n)$ so that $(c, x) \in V_n$. Hence $c \in W_n(x)$. Therefore $W_n(x)$ is order convex.
Moreover, one can then show that $\mathcal{U}$ is Hausdorff but not uniformly order convex.□

In view of [46, Proposition 9] which simply says that the topology of every uniform preordered space is locally convex, we get:

5.3.9 Proposition

If $(X, \mathcal{U})$ is a uniform ordered space then $(X, \mathcal{T}(\mathcal{U}))$ is locally order convex. □

The following result, which is due to Redfield, is important in this work as we will shortly use it. For convenience of the reader, we include its detailed proof.

5.3.10 Proposition

([52]) *Let $(X, \mathcal{U}, \leq)$ be a uniformly locally order convex uniform space such that $\leq$ is linear order on X. Then $(X, \mathcal{U}, \leq)$ is a uniform lattice.*

Proof. Suppose $(X, \mathcal{U}, \leq)$ is a uniformly locally order convex uniform space such that $\leq$ is linear order on X. Given any $U \in \mathcal{U}$, take a symmetric $W \in \mathcal{U}^s$ such that $W \circ W \subseteq U$. Here, $\mathcal{U}^s$ is the uniformity generated by $\{U \cap U^{-1} \,|\, U \in \mathcal{U}\}$, the coarsest symmetric uniformity finer than $\mathcal{U}$ [17, p. 2]. Since $(X, \mathcal{U}, \leq)$ is uniformly locally order convex, there exists some $V \in \mathcal{U}$ such that $V \subseteq W$ and $V(x)$ is order convex for every $x \in X$. Set $V^* = \{(a, x, b, y) \subseteq X^4 \,|\, (a, b), (x, y) \in V\}$. Then obviously $V^* \subseteq X^2 \times X^2$. Showing that $(\bigvee \times \bigvee)V^* \subseteq U$ and $(\bigwedge \times \bigwedge)V^* \subseteq U$ will complete the proof.

For the first inclusion, let $(a, x, b, y) \in V^*$. If $a \geq x$ and $b \geq y$ or $a \leq x$ and $b \leq y$ then $(a \vee x, b \vee y)$ is just (a, b) or (x, y) which lies in V. But $V \subseteq U$ since $V \subseteq W$ implies that $V \subseteq W \circ W \subseteq U$. So $(a \vee x, b \vee y) \in U$.

What can also happen is that $x \leq a$ and $y \geq b$ or $a \leq x$ and $y \leq b$. Suppose $x \leq a$ and $y \geq b$. Then $(\bigvee \times \bigvee)(a, x, b, y) = (a, y)$. We distinguish the two cases:

Case 1. Suppose $y \geq a$. Then $y \geq a \geq x$. Since $V(x)$ is order convex and $x, y \in V(x)$, it follows that $a \in V(x)$ so that $(x, a) \in V$ which implies $(a, x) \in V^{-1}$. Given that $(x, y) \in V$, we deduce that $(a, y) \in V \circ V^{-1} \subseteq W \circ W^{-1} = W \circ W \subseteq U$, that is, $(a, y) \in U$.

Case 2. Suppose $y < a$. This yields $b \leq y < a$. Since $(a, b) \in V$ we have $b \in V(a)$. Always $a \in V(a)$. By order convexity of $V(a)$, we deduce that $y \in V(a)$. The latter says $(a, y) \in V \subseteq U$. Hence $(a, y) \in U$.

Now suppose $a \leq x$ and $y \leq b$. Then $(\bigvee \times \bigvee)(a, x, b, y) = (x, b)$. We show that $(x, b) \in U$.

Case 1. Suppose $b \geq x$. Then we have $b \geq x \geq a$. Since $a, b \in V(a)$ and $V(a)$ is order convex, we deduce that $x \in V(a)$ so that $(a, x) \in V$ and thus $(x, a) \in V^{-1}$. Thus $(x, b) \in V^{-1} \circ V \subseteq W^{-1} \circ W = W \circ W \subseteq U$, that is, $(x, b) \in U$.

Case 2. Suppose $b < x$. In this case we have $y \leq b < x$ and hence by convexity of $V(x)$ we get $b \in V(x)$ so that $(x, b) \in V \subseteq U$. Thus $(x, b) \in U$. Now we have shown that $(\bigvee \times \bigvee)V^* \subseteq U$.

For the other inclusion, let $(a, x, b, y) \in V^*$. If $a \geq x$ and $b \geq y$, or $a \leq x$ and $b \leq y$, then $(a \wedge x, b \wedge y)$ is (x, y) or (a, b) which lies in $V \subseteq U$, hence in U. Otherwise, $x \leq a$ and $y \geq b$ or $a \leq x$ and $y \leq b$. Suppose $a \leq x$ and $y \leq b$. Then $(\bigwedge \times \bigwedge)(a, x, b, y) = (a \wedge x, b \wedge y) = (a, y)$.

Case 1. If $a \geq y$ then $x \geq a \geq y$ and by order convexity of $V(y)$ we get $(a, y) \in V \subseteq U$.

Case 2. If $a < y$ then $a < y \leq b$, and by convexity of $V(a)$ we deduce that $(a, y) \in V \subseteq U$. Now suppose $x \leq a$ and $b \leq y$. Then $(\bigwedge \times \bigwedge)(a, x, b, y) =$

$(a \wedge x, b \wedge y) = (x, b)$. One can then easily check that a similar argument shows that in any case $(x, b) \in U$. Hence $(\bigwedge \times \bigwedge)V^* \subseteq U$. Therefore $(X, \mathcal{U}, \leq)$ is a uniform lattice. $\square$

5.3.11 Corollary

Let $(X,\mathcal{U})$ be a uniformly locally order convex uniform space equipped with a linear order $\leq$ (on X). Then $\leq$ is a $\mathcal{U}$-friendly partial order.

Proof. Let $(X,\mathcal{U})$ be a uniformly locally order convex uniform space equipped with a linear order $\leq$ on X. Then by Proposition 5.3.10, it follows that $(X,\mathcal{U})$ is a uniform lattice. But according to Proposition 5.3.4, the partial order of a uniform lattice is $\mathcal{U}$-friendly. Therefore $\leq$ is $\mathcal{U}$-friendly. $\square$

Remark. Fletcher and Lindgren [17, Theorem 4.20] show that under the assumptions of the above corollary, there is a unique quasi-uniformity $\mathcal{V}$ which determines X and its symmetrization $\mathcal{V}^s$ coincides with the given uniformity $\mathcal{U}$.

Given that the property of friendliness is productive, we deduce from the above corollary that the usual product order $\leq$ on $\mathbb{R}^2$ is $\mathcal{U}_e$-friendly where $\mathcal{U}_e$ is the Euclidean product uniformity on $\mathbb{R}^2$. However, as observed above, the corresponding ordered space is not a C-space.

The following result is similar to [53, Proposition 2], and it generalizes Lemma 5.2.4 above. It advocates that a $\mathcal{U}$-friendly partial order on a uniform space guarantees that the underlying ordered topological space is an I-space.

5.3.12 Lemma

If $(X,\mathcal{U})$ is a uniform space with a $\mathcal{U}$-friendly partial order $\leq$ then $(X, \mathcal{T}(\mathcal{U}), \leq)$ is an I-space.

Proof. Let G be $\mathcal{T}(\mathcal{U})$-open and $x \in d(G)$. Then there exist some $y \in G$ and $U \in \mathcal{U}$ such that $x \leq y$ and $U(y) \subseteq G$. By friendliness of the partial order $\leq$, there is $V \in \mathcal{U}$ such that $V(d(y)) \subseteq d(U(y))$. Then $V(d(y)) \subseteq d(U(y)) \subseteq d(G)$, and thus $V(x) \subseteq d(G)$. Therefore $d(G)$ is $\mathcal{T}(\mathcal{U})$-open in X. Similarly, $i(G)$ is $\mathcal{T}(\mathcal{U})$-open in X whenever G is $\mathcal{T}(\mathcal{U})$-open in X. Hence X is an I-space. $\square$

Among other tools, we use the above lemma to prove the following:

5.3.13 Proposition

If $(X,\mathcal{U})$ is a uniform space with a $\mathcal{U}$-friendly partial order $\leq$ on X, then the bispace $(X, (\mathcal{T}(\mathcal{U}))^{\natural}, (\mathcal{T}(\mathcal{U}))^{\flat})$ is quasi-uniformizable by a quasi-uniformity of a weight smaller than or equal to the weight of $\mathcal{U}$.

Proof. Let $(X,\mathcal{U})$ be a uniform space with a $\mathcal{U}$-friendly partial order $\leq$ on X. We show that the filter $\mathcal{U}_{\Uparrow}$ on $X \times X$ which is generated by the base $\{U_{\Uparrow} := \bigcup_{x\in X}(\{x\} \times i(U(x))) \,|\, U \in \mathcal{U}\}$ is a quasi-uniformity on X. It is easy to see that each described relation is reflexive. Fix $U \in \mathcal{U}$. Then, there exists some $V \in \mathcal{U}$ such that $V \circ V \subseteq U$. Furthermore, by $\mathcal{U}$-friendliness of $\leq$, there is $W \in \mathcal{U}$ such that $W \subseteq V$ and for each $y \in X$, we have $W(i(y)) \subseteq i(V(y))$. Consequently, $i(W(i(W(x)))) = \bigcup_{y\in W(x)} i(W(i(y))) \subseteq i(i(V(W(x)))) \subseteq i(U(x))$ whenever $x \in X$. Hence we conclude that $\mathcal{U}_{\Uparrow}$ is a quasi-uniformity on X. In addition, by Lemma 5.3.12, $(X, \mathcal{T}(\mathcal{U}), \leq)$ is an I-space. Then it follows from the definition of $\mathcal{U}_{\Uparrow}$ that $\mathcal{T}(\mathcal{U}_{\Uparrow}) = (\mathcal{T}(\mathcal{U}))^{\natural}$. It can readily checked that the family of all relations $\bigcup_{x\in X}(\{x\} \times U(d(x)))$ (where $U \in \mathcal{U}$) generates the conjugate quasi-uniformity $(\mathcal{U}_{\Uparrow})^{-1}$ on $X \times X$. Then $\mathcal{T}((\mathcal{U}_{\Uparrow})^{-1}) \subseteq (\mathcal{T}(\mathcal{U}))^{\flat}$, because (by friendliness of $\leq$) for any symmetric $U \in \mathcal{U}$ there is a symmetric $V \in \mathcal{U}$ such that $V\circ \leq \subseteq \leq \circ U$. Hence, by conjugation, we get $d(V(x)) \subseteq U(d(x))$ for each $x \in X$. Therefore $U(d(x))$ is a $\mathcal{T}^{\flat}$-neighborhood at x for each $x \in X$.

Let us set $\mathcal{U}_{\Updownarrow} := \mathcal{U}_{\Uparrow} \vee (\mathcal{U}_{\Downarrow})^{-1}$. Here, $\mathcal{U}_{\Downarrow}$ denotes the quasi-uniformity on X generated by the family of all relations $U_{\Downarrow} := \bigcup_{x\in X}(\{x\} \times d(U(x)))$ where $U \in \mathcal{U}$. By an argument similar to the one given above, $\mathcal{T}(\mathcal{U}_{\Downarrow}) = (\mathcal{T}(\mathcal{U}))^{\flat}$ and $\mathcal{T}((\mathcal{U}_{\Downarrow})^{-1}) \subseteq (\mathcal{T}(\mathcal{U}))^{\natural}$, and so we conclude that $\mathcal{T}(\mathcal{U}_{\Updownarrow}) = (\mathcal{T}(\mathcal{U}))^{\natural}$. Similarly, $\mathcal{T}((\mathcal{U}_{\Updownarrow})^{-1}) = (\mathcal{T}(\mathcal{U}))^{\flat}$. Hence we have shown that the bitopological space

$(X, (\mathcal{T}(\mathcal{U}))^\natural, (\mathcal{T}(\mathcal{U}))^\flat)$ is quasi-uniformizable by $\mathcal{U}_\Updownarrow$. If $\mathcal{U}$ has a base of cardinality κ, then so does $\mathcal{U}_\Updownarrow$. $\square$

5.3.14 Corollary

If (X, m) is a metric space with a $\mathcal{U}_m$-friendly partial order $\leq$, then the bispace $(X, (\mathcal{T}(m))^\natural, (\mathcal{T}(m))^\flat)$ is quasi-pseudometrizable.

Proof. Let (X, m) be a metric space and $\mathcal{U}_m$ the uniformity induced by the metric m. Then $(X, \mathcal{U}_m, \leq)$ is a uniform space with a $\mathcal{U}_m$-friendly partial order, and hence by the above proposition $(X, (\mathcal{T}(\mathcal{U}_m))^\natural, (\mathcal{T}(\mathcal{U}_m))^\flat)$ is quasi-uniformizable. But $\mathcal{T}(\mathcal{U}_m)$ is precisely $\mathcal{T}(m)$ and $\mathcal{U}_m$ has a countable base. Therefore the bispace $(X, (\mathcal{T}(m))^\natural, (\mathcal{T}(m))^\flat)$ is quasi-pseudometrizable. $\square$

Remark. In the proof of Proposition 5.3.13 above, if $\mathcal{U}$ is a uniformly locally order convex uniformity, then $\mathcal{U} = (\mathcal{U}_\Updownarrow)^s$. To see why this holds, consider any $U \in \mathcal{U}$ such that $U(x)$ is order convex whenever $x \in X$. Then for any $x \in X$, we have $(U_\Uparrow \cap U_\Downarrow)(x) = i(U(x)) \cap d(U(x)) = U(x)$, because $U(x)$ is order convex, and so $U_\Uparrow \cap U_\Downarrow = U$. Hence $[U_\Uparrow \cap (U_\Downarrow)^{-1}] \cap [(U_\Uparrow)^{-1} \cap U_\Downarrow] = (U_\Uparrow \cap U_\Downarrow) \cap (U_\Uparrow \cap U_\Downarrow)^{-1} = U \cap U^{-1}$. Since $\mathcal{U}$ has a base consisting of such entourages U then the statement follows. $\square$

5.4 Some Examples

In this section we study some interesting examples. First, we construct a quasi-metrizable ordered topological space $(X, \mathcal{T}, \leq)$ which is a C- and I-space, such that the upper topology $\mathcal{T}^\natural$ is not quasi-pseudometrizable. In the light of Kofner's theorem [30, p. 334], which says that a first countable, closed continuous image of a quasi-metric space is a quasi-metric space, our construction illustrates that the analogy between closed mappings and C-spaces breaks down here. We then employ Fox's result (Proposition 1.4.7) to deduce that the bispace $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is not quasi-pseudometrizable. This work takes up the next five pages. Secondly, like in Example 5.2.5 above, we provide another subspace of the plane which is not a C-space, and then one which is both a C-space and I-space (Example 5.4.8).

5.4.1 Some Basics

We begin by recalling some definitions (see for instance [48] and [57]). Let X be a topological space. A subset $S \subseteq X$ is said to be *nowhere dense* in X if and only if $int_X(cl_X S) = \emptyset$. We say that S is of *first category* in X if and only if $S = \bigcup_{n=1}^{\infty} S_n$ where each S_n is nowhere dense in X. Any set that is not of first category in X is said to be of *second category* in X.

5.4.2 The Construction

We partition the closed unit interval $[0,1]$ into two sets say A and B such that for any non-empty open interval $J \subseteq [0,1]$, the sets $A \cap J$ and $B \cap J$ are of second category ([12]). Without loss of generality, we may assume that $0, 1 \notin A$. Then note that $B = [0,1] \setminus A$. Consider the usual linear order and the usual Euclidean topology τ on $[0,1]$ and isolate the points of A. Then by definition of an isolated point each singleton in A is open. Then $\{\{a\} \,|\, a \in A\} \cup \tau$ is a subbase for the topology $\mathcal{T}$. Note that $\mathcal{T}$ is finer than the usual topology τ. Since the usual topology is metrizable, it has a σ-locally finite base, hence the new topology $\mathcal{T}$ has a σ-interior-preserving base and this makes it quasi-metrizable.

The following lemma gives us more properties of the space $([0,1], \mathcal{T}, \leq)$.

5.4.3 Lemma

The ordered topological space $([0,1], \mathcal{T}, \leq)$ defined above is a C- and I-space.

Proof. We first show that it is an I-space. Let G be an open set in $[0,1]$. If $i(G) = \emptyset$ then it is open. Also, if $i(G)$ is the whole space $[0,1]$ then it is again open. Suppose $i(G) \neq \emptyset$ and $i(G) \neq [0,1]$. We consider the two cases $\inf G \in G$

and $\inf G \notin G$.
Case 1: If $\inf G \in G$ then $\inf G \in A$ because if it were in B then there would be some $\epsilon > 0$ such that $(\inf G - \epsilon, \inf G + \epsilon) \subseteq G$. But this implies that $\inf G - \frac{\epsilon}{2} \in G$, contradicting that $\inf G$ is the greatest lower bound of G. So $\inf G \in A$, and hence $\{\inf G\}$ is open in τ. Given that $(\inf G, 1]$ is open in the usual topology, it is open in τ and thus $[\inf G, 1] = \{\inf G\} \cup (\inf G, 1]$ is open in $\mathcal{T}$.
We claim that $i(G) = [\inf G, 1]$. For any $x \in i(G)$ there exists $g \in G$ such that $g \leq x$. But $\inf G \leq g$ for all $g \in G$ so $\inf G \leq x$. Since no point of $[0, 1]$ is greater that 1 we get $x \in [\inf G, 1]$. Thus $i(G) \subseteq [\inf G, 1]$. Conversely, take any $x \in [\inf G, 1]$ then $\inf G \leq x \leq 1$. Since $\inf G \in G$, it follows that $x \in i(G)$. This shows that $i(G) \subseteq [\inf G, 1]$. Hence $i(G) = [\inf G, 1]$.
Case 2: Suppose $\inf G \notin G$. We show that $i(G) = (\inf G, 1]$. Let $x \in i(G)$. Then $x \geq g$ for some $g \in G$ and thus $x \geq \inf G$ because $g \geq \inf G$ for all $g \in G$. Since $\inf G \notin G$, we have $\inf G \neq g$ for all $g \in G$ and hence $1 \geq x \geq g > \inf G$. The latter simply says $x \in (\inf G, 1]$. So $i(G) \subseteq (\inf G, 1]$. For the other inclusion, take any $x \in (\inf G, 1]$. Then $\inf G < x \leq 1$. Since $\inf G < x$ means that x is not a lower bound of G, there exists some $g_0 \in G$ such that $g_0 < x$ which yields $g_0 \leq x$. Thus $x \in i(G)$. Hence $(\inf G, 1] \subseteq i(G)$. Together, $i(G) = (\inf G, 1]$. Given that $\mathcal{T}$ is finer than the usual topology and $(\inf G, 1]$ open in the usual topology, $i(G)$ is open in $\mathcal{T}$. This shows that $i(G)$ is open whenever G is open.

Next we show that $d(G)$ is open whenever G is open. Let G be an open set in $[0, 1]$. As above, we have two cases to consider, namely $\sup G \in G$ and $\sup G \notin G$.
Case 1. Suppose $\sup G \notin G$. We show that in this case $d(G) = [0, \sup G)$. Let $x \in d(G)$. Then there exists $g \in G$ such that $x \leq g$. Thus $x \leq g \leq \sup G$. Since $\sup G \notin G$ then $\sup G \neq g$ for all $g \in G$. Hence $x \leq g < \sup G$, which implies $x \in [0, \sup G)$. This says $d(G) \subseteq [0, \sup G)$. Conversely, take any $x \in [0, \sup G)$ then $0 \leq x < \sup G$. The strict inequality $x < \sup G$ says that x is not an upper bound of G, so there exists some $g \in G$ with $x < g$ so that $x \leq g$. The latter says $x \in d(G)$. Thus $[0, \sup G) \subseteq d(G)$. Hence $d(G) = [0, \sup G)$. Since $[0, \sup G)$ is open in the usual topology τ, it is open in $\mathcal{T}$.

Case 2. Suppose $\sup G \in G$. Then $\sup G \in A$; otherwise $\sup G \in B$, and since G is open then there exists some $\epsilon > 0$ that $\sup G + \frac{\epsilon}{2} \in G$. But $\sup G + \frac{\epsilon}{2} > \sup G$ hence we have a contradiction. We claim that $d(G) = [0, \sup G]$. Let $t \in d(G)$. Then $t \leq g$ for some $g \in G$. But $g \leq \sup G$ for all $g \in G$ so $t \leq g \leq \sup G$. Obviously no point of $[0,1]$ is below 0, hence $t \in [0, \sup G]$. This shows that $d(G) \subseteq [0, \sup G]$.
Conversely, let $t \in [0, \sup G]$ then $0 \leq t \leq \sup G$ where $\sup G \in G$. Thus $t \in d(G)$, giving us $[0, \sup G] \subseteq d(G)$. Hence $d(G) = [0, \sup G]$. Then note that $\sup G \in A$ implies $\{\sup G\}$ is open. To this end, $[0, \sup G] = [0, \sup G) \cup \{\sup G\}$ is a union of two open sets hence open. In any case, $d(G)$ is open whenever G is open. Now we have established that $i(G)$ and $d(G)$ are open whenever G is open. Therefore $([0,1], \mathcal{T}, \leq)$ is an I-space.

It remains to show that X is a C-space. We begin with a closed subset F of X and prove that $i(F)$ and $d(F)$ are closed in $\mathcal{T}$. So take any closed subset $F \subseteq X$. Since $\emptyset$ and $[0,1]$ are obviously closed, we suppose that $i(F) \neq \emptyset$ and $i(F) \neq [0,1]$. As above, we distinguish cases.
Case 1. If $\inf F \in F$ then one can easily show that $i(F) = [\inf F, 1]$ using mainly the definitions of $i(F)$ and $\inf F$. Since $[\inf F, 1] = [0,1] \setminus [0, \inf F)$ is closed, $i(F)$ is closed.
Case 2. If $\inf F \notin F$ then $\inf F \in A$. It is easy to see that $i(F) = (\inf F, 1]$. Given that $\inf F \in A$, the set $\{\inf F\}$ is open. Now $(\inf F, 1] = [0,1] \setminus [0, \inf F]$ where $[0, \inf F] = [0, \inf F) \cup \{\inf F\}$ which is open. Hence $i(F) = (\inf F, 1]$ is closed. Analogously, one can show that $d(F)$ is closed whenever F is closed. Hence $([0,1], \mathcal{T}, \leq)$ is a C-space. $\square$

In the remaining part of the example, we need, among others, the following facts about sets of second category.

5.4.4 Lemma

([48], [54, p. 74]) *If A is a set of second category in the closed unit interval and $A \subseteq \bigcup_{n\in\mathbb{N}} A_n$ then there exists a non-empty open interval $J \subseteq [0,1]$ and at least one subset A_m of A such that $J \subseteq \overline{A_m}$.* $\square$

5.4.5 Lemma

([48]) *Any superset of a second category set in a space X is itself of second category in the space X.*

Proof. Let $M, N \subseteq X$ such that $M \subseteq N$ and M is of second category in X. We show that N is of second category. Assume the contrary, that is, N is of first category. Then $N = \bigcup_{n=1}^{\infty} N_n$ where N_n is nowhere dense for each $n \in \mathbb{N}$. Since $M \subseteq N$ then $M = M \cap N = M \cap (\bigcup_{n=1}^{\infty} N_n) = \bigcup_{n=1}^{\infty}(M \cap N_n)$. Also, $int(\overline{M \cap N_n}) \subseteq int(\overline{M} \cap \overline{N_n}) = int(\overline{M}) \cap int(\overline{N_n}) = int(\overline{M}) \cap \emptyset = \emptyset$, hence $int(\overline{M \cap N_n}) = \emptyset$. This says $M \cap N_n$ is nowhere dense for each $n \in \mathbb{N}$. Hence M of first category, contradicting our hypothesis. Thus, the assumption that N is of first category is wrong. Therefore N is of second category as asserted. $\square$

The remaining part of this example serves to show that the bispace $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is not quasi-pseudometrizable.

5.4.6 Proposition

The space $([0,1], \mathcal{T}^\natural)$ with $\mathcal{T}$ as defined in 5.4.2 is not quasi-pseudometrizable.

Proof. (*by contradiction*) Suppose that d is a compatible quasi-pseudometric on $([0,1], \mathcal{T}^\natural)$. Let $B_{2^{-n}} = \{(x,y) \in X \times X \mid d(x,y) < 2^{-n}\}$ whenever $n \in \mathbb{N}$. Recall that $[0,1]$ is partitioned into two sets A and B such that for any non-empty open interval $J \subseteq [0,1]$, the sets $A \cap J$ and $B \cap J$ are of second category. Put $A_n = \{x \in A \mid B_{2^{-n}}(x) = [x,1]\}$ for each $n \in \mathbb{N}$. Obviously $A_n \subseteq A$. Then by the above lemma, A is also of second category. Hence there exist $n_0 \in \mathbb{N}$ and a nonempty open interval J of $[0,1]$ such that $J \subseteq \overline{A_{n_0}}$. In addition, for each $m \in \mathbb{N}$, put

$$C_m = \{x \in B \cap J \,|\, (x - 2^{-m}, 1] \subseteq B_{2^{-(n_0+1)}}(x)\}.$$

Then $C_m \subseteq B \cap J$. Given that $B \cap J$ is of second category, there exist some $m_0 \in \mathbb{N}$ and a nonempty open interval $J' \subseteq \overline{C_{m_0}}$. Now suppose that $J' \cap A_{n_0} = \emptyset$. Then $A_{n_0} \subseteq [0,1] \setminus J'$ where $[0,1] \setminus J'$ is closed. Since $\overline{A_{n_0}}$ is the smallest closed set containing A_{n_0} we have $\overline{A_{n_0}} \subseteq [0,1] \setminus J'$ so that $J' \cap \overline{A_{n_0}} = \emptyset$. Given that $J \subseteq \overline{A_{n_0}}$ it follows that $J \cap J' = \emptyset$. Since $C_{m_0} \subseteq J$ we deduce that $J' \cap C_{m_0} = \emptyset$ and hence $J' \cap \overline{C_{m_0}} = \emptyset$. This contradicts the fact that J' is a nonempty subset of $\overline{C_{m_0}}$. So our assumption, $J' \cap A_{n_0} = \emptyset$, is false. Thus $J' \cap A_{n_0} \neq \emptyset$. Take $a \in J' \cap A_{n_0}$. With $J' \subseteq \overline{C_{m_0}}$ there exists $\delta > 0$ such that $(a, a+\delta) \subseteq \overline{C_{m_0}}$. There is a strictly decreasing sequence $(c_k)_{k\in\mathbb{N}}$ of elements of C_{m_0} converging to a with respect to the Euclidean topology. The latter says that there exists some $N \in \mathbb{N}$ such that for all $k \geq N$, $|c_k - a| < \epsilon$. Since $a \in A_{n_0}$ we get $c_k \in [a, 1] = B_{2^{-(n_0+1)}}(a)$ whenever $k \in \mathbb{N}$. Also, given that $c_k \in C_{m_0}$, we have $(c_k - 2^{-m_0}, 1] \subseteq B_{2^{-(n_0+1)}}(c_k)$. Then it follows that $(c_k - 2^{-m_0}, 1] \subseteq B_{2^{-n_0}}(a) = [a, 1]$ whenever $k \in \mathbb{N}$. Thus $c_k - 2^{-m_0} \to a - 2^{-m_0} < a$. But this contradicts the convergence of $(c_k)_{k\in\mathbb{N}}$ to a with respect to the Euclidean topology on $[0,1]$. Hence a compatible quasi-pseudometric d on $(X, \mathcal{T}^{\natural})$ cannot exist. Therefore $(X, \mathcal{T}^{\natural})$ is not quasi-pseudometrizable. □

Applying Fox's idea, Proposition 1.4.7, we eventually obtain the following result.

5.4.7 Corollary

The bispace $([0,1], \mathcal{T}^{\natural}, \mathcal{T}^{\flat})$ associated with $([0,1], \mathcal{T})$ constructed in Subsection 5.4.2 above is not quasi-pseudometrizable. □

In view of the open question of which subspaces of the plane are C-spaces or I-spaces, we study some examples below.

5.4.8 The Sector Example.

a) We show that the open sector $S_1 = \{(x,y) \in \mathbb{R}^2 \,|\, 0 < \frac{1}{2}x < y < 2x\}$ is not a C-space. We need to provide a closed subset K of S_1 such that $d(K)$ or $i(K)$ is not closed. Let $K = \{(x_n, y_n) \,|\, x_n = 1 - \frac{1}{2n}$ and $y_n = 2 - \frac{2}{n}, n \in \mathbb{N}, n \geq 3\}$. Then observe that $K \subseteq S_1$. Note that $(a,b) \in d(K)$ if and only if $(a,b) \leq (s,t)$ for some $(s,t) \in K$, that is, $s = 1 - \frac{1}{2n}$ and $t = 2 - \frac{2}{n}$ for some $n \geq 3$. We claim that $\bigcup\limits_{n\in\mathbb{N}, n\geq 3} d\{(x_n, y_n)\} = \{(x,y) \in S_1 \,|\, x < 1\}$. If $(x,y) \in \bigcup\limits_{n\in\mathbb{N}, n\geq 3} d\{(x_n, y_n)\}$ then $(x,y) \in d\{(x_n, y_n)\}$ for some $n \in \mathbb{N}, n \geq 3$ so that $x \leq 1 - \frac{1}{2n}$ and hence $x < 1$ as desired. Conversely, given any $(x,y) \in S_1$ with $x < 1$ then $\frac{1}{2}x < y < 2x$. But $x < 1$ implies that $y < 2$ so that $y \leq 2 - \frac{2}{n}$ for some $n \in \mathbb{N}, n \geq 3$. Thus $\{(x,y) \in S_1 \,|\, x < 1\} \subseteq \bigcup\limits_{n\in\mathbb{N}, n\geq 3} d\{(x_n, y_n)\}$. Hence equality holds.

b) Let $S_2 = \{(x,y) \in \mathbb{R}^2 \,|\, 0 \leq \frac{1}{2}x \leq y \leq 2x\}$ be a closed sector in $\mathbb{R}^2$. We claim that S_2 is a C- and I-space. We first verify that S_2 is a C-space. Let F be a closed subset of S_2. We need to show that $d_{S_2}(F)$ and $i_{S_2}(F)$ are closed.
Case 1. If F is bounded then it is compact and hence $d_{S_2}(F)$ and $i_{S_2}(F)$ are closed in a T_2-ordered space ([17, Proposition 4.3] or [46, Proposition 4, p.44]).
Case 2. If F is unbounded then $d_{S_2}(F) = S_2$ which is closed by construction. Now, choose $(x,y) \in F$. We claim that $i_{S_2}(F)$ is a union of two closed sets $i_{S_2}(x,y)$ and $i_{S_2}(F \cap P)$ where $P = \{(a,b) \in S_2 \,|\, a \leq x$ or $b \leq y\}$. Here, note that $i(F \cap P) =$ is closed because $F \cap P$ is bounded. Take any $(m,n) \in i_{S_2}(F)$. Then $(m,n) \in S_2$ with $m \geq x$ and $n \geq y$ for some $(x,y) \in F$. Thus $i_{S_2}(F) \subseteq \bigcup\limits_{(x,y)\in F} i_{S_2}(x,y) \cup i_{S_2}(F \cap P)$. Conversely, $(m,n) \in i_{S_2}(x,y) \cup i_{S_2}(F \cap P)$ implies $(m,n) \in i_{S_2}(x,y)$ or $(m,n) \in i_{S_2}(F \cap P)$. But $(x,y) \in F$ and $F \cap P \subseteq F$, so in any case we obtain $(m,n) \in i_{S_2}(F)$. Hence $i_{S_2}(x,y) \cup i_{S_2}(F \cap P) \subseteq i_{S_2}(F)$. And then equality follows. Consequently, S_2 is a C-space.

To see that S_2 is an I-space, let G be its open subset and take any $x \in i(G)$. Then there exists some $g \in G$ such that $x \geq g$. Since G is open we can find some $\epsilon > 0$ such that $B_\epsilon(g) \subseteq G$. Then $i(B_\epsilon(g)) \subseteq i(G)$. We want an open ball centered at

x which is contained in $i(G)$. Note that $B_\epsilon(g) \subseteq i(B_\epsilon(g))$, and that the partial order $\leq$ on S_2 is friendly. Hence it follows that $B_\epsilon(x) \subseteq i(G)$. Thus $i(G)$ is open. Similarly, given any $x \in d(G)$, there exists some $g \in G$ such that $x \leq g$. Since G is open, we can find an $\epsilon > 0$ such that $B_\epsilon(g) \subseteq G$ and hence $d(B_\epsilon(g)) \subseteq d(G)$. But $B_\epsilon(x) \subseteq d(B_\epsilon(g))$ so $d(G)$ is also open. Thus S_2 is an I-space. $\square$

Chapter 6

Conclusion

6.1 Introduction

We round off this piece of work by briefly reflecting on, and emphasizing the heart of the investigation carried out here. We also suggest some direction of further research on the subject herein treated. Hence this conclusion kick starts the next stage of investigation along the same line. As already seen in Chapter 1, stratifiable (M_3-) spaces which were introduced by E. Michael's student J. Ceder [13] in 1961 play a great role in this work but they are also generally important as generalized metric spaces. It is known that all M_1-spaces are M_2-spaces, which are in turn stratifiable. In their independent works, G. Gruenhage and H. Junnila have shown that stratifiable spaces are precisely the M_2-spaces. It is then natural to wonder if the bitopological version of this result holds. More formally, we ask
Question 1. *Are the pairwise stratifiable spaces precisely the pairwise M_2-spaces*?

Herein, we shall soon pose related questions on stratifiable spaces in a bitopological setting. Furthermore, for future research which may be a direct continuation of the current work, we shall phrase a generalized version of the problem we have studied and point out a connection to some old work on generalized metric spaces. Again, the main problem of study in this project was the following:

If $(X, \mathcal{T}, \leq)$ is an ordered topological C- and I-space which is pseudometrizable, is the associated bispace $(X, \mathcal{T}^{\natural}, \mathcal{T}^{\flat})$ quasi-pseudometrizable?

Although we could not construct a counterexample to disprove the conjecture in its generality, the modified problem in which the topology $\mathcal{T}$ we begin with is only quasi-metrizable has a negative solution. We illustrated this by constructing a quasi-metrizable ordered topological space which is a C- and I-space but the upper topology $\mathcal{T}^{\natural}$ is not quasi-pseudometrizable (see Section 5.4).

On the positive side of things, we have given certain conditions under which one gets an affirmative answer to the problem. We showed in Lemma 1.2.3 that if one assumes that the topology $\mathcal{T}$ is separable then the bispace $(X, \mathcal{T}^{\natural}, \mathcal{T}^{\flat})$ is quasi-pseudometrizable. Then we demonstrated the importance of stratifiable spaces in this investigation. For instance, showing that if $(X, \mathcal{T}, \leq)$ is a stratifiable ordered C-space then $(X, \mathcal{T}^{\natural}, \mathcal{T}^{\flat})$ is pairwise stratifiable was an important step which enabled us to eventually prove, with the help of Fox's result (which says that a bitopological space $(X, \mathcal{T}_1, \mathcal{T}_2)$ is quasi-pseudometrizable provided that it is pairwise stratifiable and each of the topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ admits a local quasi-uniformity with a countable base [18, 31]) that given an ordered topological space $(X, \mathcal{T}, \leq)$ which is a C-space such that $\mathcal{T}$ is metrizable then the bitopological space $(X, \mathcal{T}^{\natural}, \mathcal{T}^{\flat})$ is quasi-pseudometrizable if and only if each of the topologies $\mathcal{T}^{\natural}$ and $\mathcal{T}^{\flat}$ is quasi-pseudometrizable.

In addition, we established a partial analogue of the Hanai-Morita-Stone Theorem. This stipulates the equivalence of first countability of both the upper topology $\mathcal{T}^{\natural}$ and lower topology $\mathcal{T}^{\flat}$ to compactness of boundaries of $i(y)$ and $d(y)$ whenever y is a point in an ordered metrizable C-space $(X, \mathcal{T}, \leq)$. The rest of the results give conditions like uniform local connectedness under which the problem has a positive solution in different settings. More precisely, if $(X, \mathcal{T}, \leq)$ is a metrizable C- and I-space which is locally connected then $(X, \mathcal{T}^{\natural}, \mathcal{T}^{\flat})$ is quasi-pseudometrizable. In the friendly setting, a similar result, Corollary 5.3.14, also emerged in several steps. It says that the bispace $(X, (\mathcal{T}(m))^{\natural}, (\mathcal{T}(m))^{\flat})$ is quasi-pseudometrizable for any metric space (X, m) with a $\mathcal{U}_m$-friendly partial order $\leq$. The following generalization of the problem studied in this thesis remains open.

Problem A. *Give a characterization of the pairwise completely regular bispaces* $(X, \mathcal{T}_1, \mathcal{T}_2)$ *for which there exists a metrizable ordered* C*- and* I*-space* $(X, \mathcal{T}, \leq)$ *such that* $\mathcal{T}_1 = \mathcal{T}^\natural$ *and* $\mathcal{T}_2 = \mathcal{T}^\flat$.

The characterization of metrizable spaces by Bing-Nagata-Smirnov [57, Theorem 23.9] in terms of σ-locally finite bases, which are known to be closure-preserving, motivated J. G. Ceder to study spaces with σ-closure preserving bases (see Definition 6.2.1 below). Recall that a topological space is said to be an M_1*-space* if it is regular and has a σ-closure preserving base. In 1961, Ceder [13] got researchers on their feet with the question whether stratifiable spaces are M_1. In 1973, following Ceder's efforts [13, Theorem 7.6, p. 117], F. G. Slaughter, Jr established that if f is a closed continuous mapping from a metric space X onto a topological space Y then Y is an M_1-space. Then in 2000, T. Mizokami and N. Shimane introduced a technical property (P) and showed that every stratifiable space with (P) is an M_1-space whose every closed subset has a closure-preserving open neighbourhood base [45]. In the same line, but in the current setting we make some effort below.

6.2 A Look at Closure-preserving Collections

Here we briefly prepare for the next section by establishing some facts on closures and closure-preserving collections. As usual, $\overline{A}$ or $cl(A)$, and $cl_{\mathcal{T}^\natural} A$ denote the closure of A in $(X, \mathcal{T})$, and in $\mathcal{T}^\natural$ respectively.

6.2.1 Definition

Let $\mathcal{C}$ be a collection of subsets of a topological space X. Then $\mathcal{C}$ is said to be *closure-preserving* if and only if $\bigcup_{C \in \mathcal{C}'} cl(C) = cl(\bigcup_{C \in \mathcal{C}'})$ for any $\mathcal{C}' \subseteq \mathcal{C}$. Furthermore, as with any σ-property, we say that $\mathcal{C}$ is a *σ-closure-preserving collection* if and only if $\mathcal{C} = \bigcup_{n \in \mathbb{N}} \mathcal{C}_n$ where each $\mathcal{C}_n$ is a closure-preserving subcollection of $\mathcal{C}$.

6.2.2 Lemma

If $(X, \mathcal{T}, \leq)$ is an ordered topological C-space and $A \subseteq X$ then $cl_{\mathcal{T}^\natural} A = d(\overline{A})$ and hence $d(\overline{A}) = d(\overline{d(A)})$.

Proof. Since $\overline{A}$ is closed and X is a C-space, $d(\overline{A})$ is closed. Always, $\overline{A} \subseteq d(\overline{A})$ and $A \subseteq \overline{A}$ hence $A \subseteq d(\overline{A})$. Thus $d(\overline{A})$ is a closed set containing A. But then $cl_{\mathcal{T}^\natural} A \subseteq d(\overline{A})$. Conversely, $cl_{\mathcal{T}^\natural} A$ is a complement of an open set in $\mathcal{T}^\natural$. But these open sets are upper sets, so $cl_{\mathcal{T}^\natural} A$ is a lower set which is closed. So $cl_{\mathcal{T}^\natural} A \supseteq \overline{A}$ and so $cl_{\mathcal{T}^\natural} A \supseteq d(\overline{A})$. Hence $cl_{\mathcal{T}^\natural} A = d(\overline{A})$. For the other identity, $A \subseteq d(A)$ implies $\overline{A} \subseteq \overline{d(A)}$ and so $d(\overline{A}) \subseteq d(\overline{d(A)})$. Conversely, note that $d(A) \subseteq d(\overline{A})$ and $d(\overline{A})$ is closed. Thus $\overline{d(A)} \subseteq d(\overline{A})$. But $d(\overline{d(A)})$ is the smallest lower set containing $\overline{d(A)}$. Hence $d(\overline{d(A)}) \subseteq d(\overline{A})$. Therefore $d(\overline{A}) = d(\overline{d(A)})$. $\square$

A similar argument proves the following result.

6.2.3 Lemma

If $(X, \mathcal{T}, \leq)$ is an ordered topological C-space and $A \subseteq X$ then $cl_{\mathcal{T}^\flat} A = i(\overline{A})$ and hence $i(\overline{A}) = i(\overline{i(A)})$. $\square$

The above facts are related to Theorem 1 of McCartan in [44], and they enable us to prove the following result.

6.2.4 Proposition

Let $(X, \mathcal{T}, \leq)$ be an ordered topological C- and I-space. If $\mathcal{B}$ is an open and closure-preserving collection in $(X, \mathcal{T})$ then $\mathcal{B}_d = \{d(B) \mid B \in \mathcal{B}\}$ is open in $(X, \mathcal{T}^\flat)$ and it is closure-preserving in $(X, \mathcal{T}^\natural)$.

Proof. Let $\mathcal{B}$ be an open and closure-preserving collection in $(X, \mathcal{T})$. Since X is an I-space then for each $B \in \mathcal{B}$, $d(B)$ is an open lower set and hence $\mathcal{B}_d$ is

open in $(X, \mathcal{T}^\flat)$. Next we use Lemma 6.2.2 to show that $\mathcal{B}_d$ is closure-preserving in $(X, \mathcal{T}^\natural)$. Let $\mathcal{B}' \subseteq \mathcal{B}$. Then $\bigcup\limits_{B \in \mathcal{B}'} cl_{\mathcal{T}^\natural} d(B) = \bigcup\limits_{B \in \mathcal{B}'} d(\overline{d(B)}) = \bigcup\limits_{B \in \mathcal{B}'} d(\overline{B}) = cl_{\mathcal{T}^\natural}(\bigcup\limits_{B \in \mathcal{B}'} d(B))$. The above first two equalities clearly hold by Lemma 6.2.2. The last one holds because
$\bigcup\limits_{B \in \mathcal{B}'} d(\overline{B}) = \bigcup\limits_{B \in \mathcal{B}'} d(\overline{d(B)}) = d(\bigcup\limits_{B \in \mathcal{B}'} \overline{d(B)}) \subseteq d(\overline{\bigcup\limits_{B \in \mathcal{B}'} d(B)}) = cl_{\mathcal{T}^\natural}(\bigcup\limits_{B \in \mathcal{B}'} d(B))$, and for the reverse inclusion: Given that $\mathcal{B}$ is $\mathcal{T}$-closure-preserving, we know that for each $\mathcal{B}' \subseteq \mathcal{B}$, we have $\bigcup\limits_{B \in \mathcal{B}'} \overline{B} = \overline{\bigcup\limits_{B \in \mathcal{B}'} B}$. Then $\bigcup\limits_{B \in \mathcal{B}'} d(\overline{B}) = d(\bigcup\limits_{B \in \mathcal{B}'} \overline{B}) = d(\overline{\bigcup\limits_{B \in \mathcal{B}'} B})$. By the fact that X is a C-space, this implies that $\bigcup\limits_{B \in \mathcal{B}'} d(\overline{B})$ is a closed lower set, and so $\bigcup\limits_{B \in \mathcal{B}'} d(B) \subseteq \bigcup\limits_{B \in \mathcal{B}'} d(\overline{B})$ implies $cl_{\mathcal{T}^\natural}(\bigcup\limits_{B \in \mathcal{B}'} d(B)) \subseteq \bigcup\limits_{B \in \mathcal{B}'} d(\overline{B})$. Therefore $\bigcup\limits_{B \in \mathcal{B}'} cl_{\mathcal{T}^\natural} d(B) = cl_{\mathcal{T}^\natural}(\bigcup\limits_{B \in \mathcal{B}'} d(B))$, which means $\mathcal{B}_d$ is $\mathcal{T}^\natural$-closure-preserving. □

As expected, the following similar result also holds.

6.2.5 Proposition

Let $(X, \mathcal{T}, \leq)$ be an ordered topological C- and I-space. If $\mathcal{B}$ is an open and closure-preserving collection in $(X, \mathcal{T})$ then $\mathcal{B}_i = \{i(B) \,|\, B \in \mathcal{B}\}$ is open in $(X, \mathcal{T}^\natural)$ and it is closure-preserving in $(X, \mathcal{T}^\flat)$.

Proof. Let $\mathcal{B}$ be an open and closure-preserving collection in $(X, \mathcal{T})$. Since X is an I-space then $i(B)$ is open for each $B \in \mathcal{B}$ and hence $\mathcal{B}_i$ is an open collection in $(X, \mathcal{T}^\flat)$. In view of Lemma 6.2.3, we now establish that $\mathcal{B}_i$ is, as desired, $\mathcal{T}^\flat$-closure-preserving. Let $\mathcal{B}' \subseteq \mathcal{B}$. Then by similar arguments as in the above proof, we get $\bigcup\limits_{B \in \mathcal{B}'} cl_{\mathcal{T}^\flat} i(B) = \bigcup\limits_{B \in \mathcal{B}'} i(\overline{i(B)}) = \bigcup\limits_{B \in \mathcal{B}'} i(\overline{B}) = cl_{\mathcal{T}^\flat}(\bigcup\limits_{B \in \mathcal{B}'} i(B))$. Hence $\bigcup\limits_{B \in \mathcal{B}'} cl_{\mathcal{T}^\flat} i(B) = cl_{\mathcal{T}^\flat}(\bigcup\limits_{B \in \mathcal{B}'} i(B))$, so that $\mathcal{B}_i$ is closure-preserving in $(X, \mathcal{T}^\flat)$. □

6.3 On Pairwise M_1- versus Pairwise stratifiable bispaces

In their Spanish paper [22] of 1986, A. Gutierrez and S. Romaguera introduced the concepts of pairwise M_i-spaces into the theory of bispaces as a generalization of Ceder's M_i-spaces (i=1,2,3). Recall that a collection $\mathcal{Q}$ of subsets of X is called a *quasi-base* for the space $(X, \mathcal{T})$ if and only if for any $O \in \mathcal{T}$ and $x \in O$, there exists $Q \in \mathcal{Q}$ such that $x \in int_{\mathcal{T}} Q \subseteq Q \subseteq O$. For the remaining part of this section, we need to remember the following.

6.3.1 Definition

([22]) A bispace $(X, \mathcal{T}_1, \mathcal{T}_2)$ is said to be $\mathcal{T}_1$-M_1 *with respect to* $\mathcal{T}_2$ if and only if it is $\mathcal{T}_1$-regular with respect to $\mathcal{T}_2$ [1] and there exists a quasi-base of $\mathcal{T}_1$ which is $\mathcal{T}_2$-σ-closure preserving. A $\mathcal{T}_2$-M_1 *with respect to* $\mathcal{T}_1$ bispace is defined similarly.

6.3.2 Definition

([22]) A bispace $(X, \mathcal{T}_1, \mathcal{T}_2)$ is said to be *pairwise* M_1 if and only if it is $\mathcal{T}_1$-M_1 with respect to $\mathcal{T}_2$ and $\mathcal{T}_2$-M_1 with respect to $\mathcal{T}_1$.

We now obtain a new result on pairwise M_1-spaces which is similar to Proposition 1.4.3, the corresponding one for stratifiable spaces.

6.3.3 Theorem

If $(X, \mathcal{T}, \leq)$ is a metrizable ordered topological C- and I-space then the bispace $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is pairwise M_1.

Proof. Let $(X, \mathcal{T}, \leq)$ be an ordered metrizable C- and I-space. Since any pseudometrizable space is completely regular then so is $(X, \mathcal{T}, \leq)$. Thus by [10, Corol-

[1] This simply means that in the bispace $(X, \mathcal{T}_1, \mathcal{T}_2)$, the topology $\mathcal{T}_1$ is regular with respect to $\mathcal{T}_2$ as defined in Section 0.3 in Chapter 0.

lary 6.2] we deduce that $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is pairwise completely regular, and hence pairwise regular. The latter implies that the bispace is $\mathcal{T}^\natural$-regular with respect to $\mathcal{T}^\flat$ and $\mathcal{T}^\flat$-regular with respect to $\mathcal{T}^\natural$. Furthermore, as pointed out by Ceder [13, Theorem 2.2], any metrizable space is M_1 so $(X, \mathcal{T}, \leq)$ is M_1. This means that $\mathcal{T}$ has a σ-closure-preserving base, say $\mathcal{B}$. Let $\mathcal{B} = \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$ where each $\mathcal{B}_n$ is $\mathcal{T}$-closure-preserving. Now we need to produce a σ-closure-preserving basis for the upper topology and another for the lower topology.
Let $\mathcal{D}_n = \{d(B) \mid B \in \mathcal{B}_n\}$. By Lemma 1.2.3, the collection $\mathcal{D}_n$ is a base for $\mathcal{T}^\flat$. Put $\mathcal{D} = \bigcup_{n\in\mathbb{N}} \mathcal{D}_n$. By Proposition 6.2.4, we know that $\mathcal{D}$ is a $\mathcal{T}^\natural$-σ-closure-preserving base for $\mathcal{T}^\flat$. Similarly, let $\mathcal{I}_n = \{i(B) \mid B \in \mathcal{B}_n\}$. Again by Lemma 1.2.3, it follows that $\mathcal{I}_n$ is a base for $\mathcal{T}^\natural$. Finally, put $\mathcal{I} = \bigcup_{n\in\mathbb{N}} \mathcal{I}_n$. By Proposition 6.2.5, we deduce that $\mathcal{I}$ is a $\mathcal{T}^\flat$-σ-closure-preserving base for $\mathcal{T}^\natural$. Thus, the bispace $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is $\mathcal{T}^\natural$-M_1 with respect to $\mathcal{T}^\flat$ and $\mathcal{T}^\flat$-M_1 with respect to $\mathcal{T}^\natural$ and hence, by definition of pairwise M_1-bispace, we conclude that $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is pairwise M_1 as desired. $\square$

6.3.4 Corollary

If $(X, \mathcal{T}, \leq)$ is an ordered topological C- and I-space and M_1-space then the bitopological space $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is pairwise M_1. $\square$

As mentioned above, F. G. Slaughter, Jr proved that if f is a closed continuous mapping of a metric space X onto the space Y, then Y is an M_1-space [55, Theorem 6]. His proof employs Lašnev's decomposition theorem [38, Theorem 1, p. 1504]. It is therefore natural to wonder whether an appropriate variant of this theorem might be relevant to our question. Hence in the same vein we ask

Question 2. *Can I-space be deleted in the above theorem? That is, if $(X, \mathcal{T}, \leq)$ is a metrizable ordered topological C-space is the bispace $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ already pairwise M_1?*

Remark. According to Gutierrez and Romaguera [22, Proposition 1], any pairwise M_1-bispace is pairwise stratifiable. Thus the above theorem implies Proposition 1.4.6. In view of Ceder's question whether the implication $M_1 \Rightarrow M_2 \Rightarrow M_3$ is reversible, the following problem for bispaces remains open.

Problem B. *Is any pairwise stratifiable bispace pairwise* M_1?

In the light of the research done on the connection between stratifiable spaces and M_1-spaces by the Japanese mathematicians, T. Mizokami and N. Shimane [45], we pose the following problem.

Problem C. *If* $(X, \mathcal{T}, \leq)$ *is an ordered topological* C*- and* I*-space such that* $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ *is pairwise stratifiable and doubly first countable, is* $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ *pairwise* M_1?

Without emphasizing order and order related concepts involved in this study, we close this chapter and hence the thesis with the following general stimulus.

Problem D. *Is every pairwise stratifiable and doubly first countable bispace pairwise* M_1?

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