Topics in Stringy Cosmology

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Declaration

The work presented in this thesis is partly based on collaborations with Marco Bruni (Institute of Cosmology and Gravitation, Portsmouth University, UK), Alan Coley (Department of Mathematics and Statistics, Dalhousie University, Halifax, Nova Scotia, Canada) Peter K. S. Dunsby, (Department of Mathematics and Applied Mathematics, University of Cape Town), Matthew Kleban (Institute for Advanced Study, Princeton, USA) and Leonard Susskind (Korea Institute for Advanced Study and Department of Physics, Stanford University, USA).

The list below identifies sections or paragraphs which are partially based on the listed publications:

- Chapter 3 and Appendix A:

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I hereby declare that this thesis has not been submitted, either in the same or different form, to this or any other university for a degree and that it represents my own work.

Naureen Goheer

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Abstract

This thesis is composed of two distinct research topics. The first topic concerns Randall-Sundrum cosmological models, in particular their isotropization properties at early times. We first study the state space of inflationary braneworld models with exponential potential and identify the past and future attractors within the homogeneous state space. Our main result is that the isotropic high energy BDL model is the generic past attractor in the state space of FLRW braneworld models. In the state space of Bianchi I models both the BDL model and the inflationary anisotropic Kasner model are past attractors. We then address the question of isotropization using a different approach by studying the evolution of general inhomogeneous and anisotropic perturbations around the spatially flat FLRW braneworld background for three different energy regimes. Here we find that for the high energy braneworld background, the physically relevant anisotropic, inhomogeneous perturbations decay when approaching the initial singularity. We confirm the standard results for the low energy limit: as in General Relativity, the anisotropy here grows in the past. We conclude that the initial singularity appears to be generically homogeneous and isotropic in the braneworld context.

The second focus of this thesis is on classical cosmological spacetimes with horizons and their relation to non-classical concepts such as Horizon Complementarity and Holography. We first assume the de Sitter space version of black hole complementarity, which states that a single causal patch of de Sitter space is described as an isolated finite temperature cavity bounded by a horizon which allows no loss of information. After discussing how the de Sitter symmetries should be implemented, we prove that the finiteness of the horizon entropy stands in contradiction with the symmetry group of de Sitter space. We conclude that we must either give up the finiteness of de Sitter entropy or the exact symmetry of the classical space. We discuss the implications; in particular we argue that the lifetime of a de Sitter phase can not exceed the Poincaré recurrence time. We find that the same arguments hold for 1+1 dimensional Rindler space. This means that compactifications of String Theory to maximally symmetric 1+1 dimensional Minkowski space are in conflict with Holography and expected to be unstable.

Contents

Part I

Introduction

Chapter 1

Why Stringy Cosmology?

1.1 The Standard Model of Cosmology

Einstein's theory of General Relativity has lead to a well-established "Standard Model of Cosmology", describing the universe as a Friedman-Lemaître-Robertson-Walker (FLRW) model: a homogeneous isotropic model on large scales with specific matter contents. By extrapolating the model into the past, it appears that at a time $t = 0$ our observed universe was "born" in a **Hot Big Bang**, a singularity at which the size of the universe was "zero" and general relativistic invariants such as energy density, temperature and curvature diverged. Following the initial singularity the universe starts in an expanding, hot and super-dense phase, cooling down as it is growing in size. After 10^{-43} seconds the universe has cooled down to the Planck temperature [1] of 10^{19} GeV and has a curvature scale of the order of the Planck length. This energy regime is not expected to be described by classical General Relativity, since it must have been dominated by extreme quantum effects. As the universe keeps expanding and cooling down, high energy particle physics become important. Finally, after around 10^{-2} seconds, the universe is cool enough to be reliably described by General Relativity. The universe is now a hot plasma filled with radiation, dominated by pair creation of the elementary particles. The Standard Model of Cosmology starts around $t = 1s$ with a universe filled with a "soup" of elementary particles – neutrinos, photons, electrons, positrons, protons and neutrons – in thermal equilibrium [2]. With dropping temperature the neutrinos decouple, the proton-neutron conversion freezes, and electrons and positrons are converted into photons. Three minutes after the big bang, **nucleosynthesis** takes place: protons and neutrons form hydrogen and helium. The predicted abundance of

[1] When describing the early universe, temperatures are often given in particle physics units. They relate to the thermodynamic scale via $1GeV \simeq 10^{13}K$.

[2] The absence of anti-baryons must be explained by a different model – it could e.g. be due to a slight asymmetry between matter and anti-matter.

helium with respect to the other light elements seems to agree extremely well with observations, and is one of the most important predictions supporting this model. The major success of this model however may be its accurate description of the period of **recombination**: after around 10^5 years, at a temperature of around $3000K$, the universe is so cold that the remaining electrons and protons combine to form hydrogen. The photons that had scattered off the ions before now decouple from the remaining matter. These photons have been travelling mostly freely till the present day, cooling down as the universe continues to expand. They can now be observed as Black Body radiation with a temperature of $2.78K$ surrounding us from all angles in the sky. This radiation, called the Cosmic Microwave Background (CMB), is almost homogeneous and isotropic, and presents arguably the best piece of evidence supporting the Standard Model of Cosmology.

After recombination, the decoupled radiation ceases to prevent gravitational instabilities, and **galaxies** are formed due to the growth of density fluctuations. The universe becomes dominated by ordinary matter not radiation. Approaching the present time of around 15 billion years after the Big Bang, the universe has cooled down to $T = 2.78K$, the temperature observed in the microwave background.

Despite its success, this model is plagued by various problems. First of all, there are problems relating to the initial conditions. It appears that our universe is **homogeneous**, **isotropic** and (almost) **flat**. However, during the initial quantum regime the curvature of the universe is expected to violently fluctuate. We would therefore expect a highly inhomogeneous and anisotropic universe to enter into the lower energy regimes. Furthermore, why is the universe flat? Even worse, the universe looks the same in different regions that are not even causally connected in the past. How can these different regions of space share the same properties without **violating causality**?

Another problem is related to the **formation of the large scale structure** of the universe. As explained above, it is believed that initial density fluctuations act as seeds that, after recombination, gravitationally collapse to form early galaxies. However, these fluctuations cannot be random, but must instead have a very special spectrum in order to seed the structure we observe (see e.g. Peebles [163]). What is the **origin of these fluctuations**?

This set of questions can be answered by assuming that the very early universe, at around $t = 10^{-35}s$, underwent an **inflationary** phase, during which the universe expanded at a super-luminal rate. The rapid expansion serves to smooth out all inhomogeneities and inflate the universe so that it is almost flat. Inflation can furthermore provide fluctuations in the initial energy density that can produce the observed large scale structures.

The inflationary scenario however poses a new set of questions [33]. What is the origin of inflation? It is believed that a light or massless scalar field called the "inflaton field" caused the universe to inflate. Once again one must ask what the origin of this field is, or why we have not observed the corresponding particle.

The second set of problems of the standard model might even be more disturbing, since they concern the present day universe. Several independent and highly refined experiments, using both the CMB and type Ia supernovae (see e.g. Bahcall et al [9] and references therein) converge to the fact that the present day universe is filled to almost 30% with "cold dark matter" (CDM) and 70% with "dark energy". Since CDM can not be composed of ordinary baryonic matter or neutrinos, it is unclear what the origin of CDM is. Dark energy, causing a late time acceleration of the universe, is equally unaccounted for. A possible candidate is a very small but non-zero energy density of the vacuum, or a cosmological constant Λ. This possibility however raises fundamental questions such as why Λ would be so highly fine-tuned, and why the fine-tuned value can be stable against quantum corrections. In summary, there is substantial experimental evidence supporting the "concordance" or "ΛCDM" model, implying that the present day universe is dominated by forms of matter and energy whose origin and physics are unknown to us [181].

1.2 The search for a Quantum Theory of Gravity

It is well-known that there is a fundamental problem with our present understanding of theoretical physics. On the one hand we have General Relativity (GR) describing the long-range gravitational interactions - a theory well-tested from cosmological scales down to millimeter scales. On the other hand we have quantum field theory, the theory that underlies the Standard Model of Particle Physics describing the other three fundamental forces. This theory is well-tested on the scales at which these three forces dominate, which is from submillimeter scales down to 10^{-19} meters. What we are missing however is a quantum theory of gravity. This presents a serious problem for several reasons. First of all, physicists have long been looking for a aesthetically appealing unified "theory of everything", or Grand Unified Theory (GUT), containing the Standard Model of Particle Physics and General Relativity in appropriate limits. Secondly, there are unresolved problems in each of the established theories, and there is hope that a quantum theory of gravity will resolve problems such as the existence of singularities in GR and provide a natural cutoff for the quantum field theories of the Standard Model. Most importantly however, it is known that the gravitational field *must* be quantized in order to avoid contradictions such as the violation of the uncertainty principle. This means that a quantum theory of gravity is required not only for aesthetical reasons such as the search for a GUT, but also in order to preserve the intrinsic consistency of the existing theories. The role of spacetime in such a unified theory is expected to be altered: spacetime itself will be quantized and might loose its role as a fundamental underlying structure altogether.

There are many different approaches to quantizing gravity. The famous "two best candidates" are String Theory and Loop Quantum Gravity (often also called Quantum Geometry). Detailed reviews of the progress in each field are published regularly; we here just briefly summarize the underlying concept of both programs partly following the lines of [109]. See also [43] for further references and a very detailed review of progress and problems in both fields.

The first candidate, String Theory, is built on the assumption that all the fundamental particles are not point-like, but excitations of one-dimensional open or closed strings. Initially, these fundamental strings could only be consistently quantized and studied when propagating in a ten-dimensional background spacetime, which lead to perturbative String Theory. The theory aims to provide a unified theory of all particles and forces upon introduction of the fermionic sector as the super-symmetric partners of the bosonic strings [3]. This perturbative theory itself is considered an immense success, since it is a Lorentz and diffeomorphism invariant, perturbatively finite quantum theory of gravity.

However, the perturbation expansion does not converge. Furthermore there exists not just one, but *five different* string theories, differing in their fundamental gauge groups. These problems have recently been turned into arguments in favor of the existence of an underlying, non-perturbative theory that contains the different string theories and 11-dimensional super-gravity in appropriate limits. The existence of such a single 11-dimensional theory would explain the dualities of String Theory that link the weak coupling limit of one String Theory to the strong coupling limit of another one. One of the main building blocks of non-perturbative String Theory is Maldacena's AdS/CFT conjecture [151], which states that String Theory on a ten-dimensional space that asymptotically approaches the product of Anti de Sitter (AdS) space and a compact space is equivalent to a super-conformal field theory (CFT) on the boundary of AdS (see [1] for a review) [4]. The conjecture must remain unproven until an independent description of non-perturbative String Theory has been found – however, it can and has been successfully tested in special cases (see for example [1, 110]). At the present date, most string theorists believe that the AdS/CFT correspondence is indeed correct, and use the "dictionary" translating between the dual theories in order to gain deeper insights into less understood aspects of each theory. One of the main achievements using this approach is the resolution of the black hole information paradox at least for the very special case of a (small) black hole in asymptotically $\mathrm{AdS}_5 \times S^5$ spacetimes [152]. Another implication is that if the conjecture is true, and the entire String Theory in the bulk can be described by a CFT on the boundary, then String Theory must have fewer degrees of freedom than expected from perturbation theory. This argument is taken as further support for the existence of a non-perturbative theory

In summary, evidence for the existence of a non-perturbative description of String Theory encompassing the different perturbative string theories seems to be building up. This non-perturbative theory is often dubbed "M-Theory" [5], and string theorists hope that this

[3]This means that String Theory "predicts" a huge number of fields that are not currently observed. However, String Theory is still far from making experimental predictions, and these extra particles could be hidden in some compactification or be forbidden by non-perturbative effects.

[4]This conjecture is an example of Holography as discussed in section 1.5.2.

[5]Some people prefer to use the term M-theory for the 11D super-gravity limit; we will not do so here.

theory will in fact be the "Theory of Everything".

Quantum Geometry is in a sense less ambitious: it aims to provide a quantum theory of gravity and just that. It does not attempt to unify all fundamental forces and particles. The basic idea of Quantum Geometry is to rewrite the General Relativity in terms of a tetrad and a connection and replace the usual Lorentz connection by one that is self-dual. Imposing diffeomorphism and local SU(2) invariance as in General Relativity yields Hamiltonian, momentum and Gauss constraints, one obtains a background independent constrained theory. Following an approach initiated by Dirac, the theory is first quantized ignoring the constraints. Then quantum operators corresponding to the constraints must be found, and the physical states finally correspond to the solutions of the quantum constraints. It turns out that the unconstrained system can be quantized in a diffeomorphism independent way, and all but the Hamiltonian (or scalar) constraint are well-understood. This means that there exists a well-defined theory of Quantum Geometry with a Hilbert space of states that can be viewed as the kinematical Hilbert space of the gravitational field. However, despite tremendous work and progress towards understanding the dynamics contained in the Hamiltonian constraint, ambiguities concerning regularization issues of the constraint are still not resolved. For a detailed review of quantum geometry and references see e.g. [6] or [7].

It should be pointed out that there exist several alternative attempts to quantize gravity, including amongst others the twistor theory program [164]. A theory that has recently received considerable attention is non-commutative geometry (NCG), where it is assumed that spacetime coordinates themselves do not commute (see e.g. [56] for an introduction). These models are now of particular interest, after it has been observed that in certain limits of String Theory, the entire string dynamics can be described by a gauge theory on a non-commutative space [179]. This means that on a certain level String Theory automatically has some degree of spacetime-non-commutativity built-in.

What all these theories have in common is that even if they are promising candidates for a quantum theory of gravity, it is hard to tell whether they accurately describe the world we live in. In a physicists' perfect world, *the* quantum theory of gravity would uniquely determine the fundamental forces and their respective strengths in our 4-dimensional world, and the physical description of our world would be obtained by finding the (unique) ground state of the theory. However, since all these candidate theories are formulated to address problems at the Planck energy scale, finding *any* low energy limit is generally not trivial, not to mention the problem of uniqueness. In fact, for a long time finding *any* low energy limit of quantum geometry was a major problem. However, there has been recent progress on that matter mainly due to Bojowald (see e.g. [27]), enabling cosmologists to deduce the first cosmological signatures of quantum geometry [203].

From now on however, we will concentrate on string-inspired cosmological models. In String Theory, finding the low-energy limit is fairly straightforward: we just have to compactify the extra dimensions on a Calabi-Yau (CY) manifold so that we are left with an effective field theory in a 4-dimensional spacetime (see [76] and [116] for lectures on string compactifications). Since String Theory aims to be the "Theory of Everything", the ground state of the theory should not only predict the exact form of classical gravitational interactions, but also the entire Standard Model of Particle Physics or whatever else would replace it. The mechanism is simple: in String Theory all the parameters of the Standard Model of Particle Physics are turned into dynamical variables or moduli, and by finding the ground state of the theory one should be able to predict the parameter values uniquely.

Unfortunately, there are a great number of different CY manifolds and ways to compactify the theory, corresponding to a vast number of vacua often allowing for continuous ranges of moduli values.

There has been great progress in constructing more realistic 4-dimensional models from 10-dimensional String Theory with fixed moduli: Under very specific conditions, it has been possible to construct metastable de Sitter vacua [118] and even inflating models [117] in type IIB String Theory. However, there is no clear motivation why String Theory should select these very special and only metastable ground states from the vast number of possible vacua, and in case of the inflationary models, very restrictive assumptions have to be made in order for the models to inflate.

A further complicating matter is the fact that String Theory contains a large number of fields that are not observed in experiments. Any chosen compactification would have to get rid off all these additional fields in the low-energy-limit.

There might be a solution to this question within string theory. Alternatively one can resurrect an anthropic principle in order to chose the right vacuum state [194]. Either way, String Theory seems far from predicting a realistic cosmological model purely based on the aesthetic top-down approach.

In this absence of concrete String/M-Theory cosmological models, cosmologists prefer to build simple String/M-Theory inspired models and study their cosmological properties. It is hoped that some light can be shed on some of the cosmological properties of String/M-Theory by studying these simple yet highly unusual cosmologies including key features such as extra dimensions, branes and/or warping. If nothing else, cosmology can serve as a testing field for any new model [208]: if a seemingly preferred compactification does not agree with the universe we live in, more work on the issue is needed.

1.3 String cosmology

In what follows we will attempt to outline the basic ideas of the different models without going into technical details. For a more details on technical aspects see the excellent review by Easson [62] and references therein.

The first cosmological model inspired by String Theory is presumably the Brandenberger-Vafa model [35] of the 1980s, which used the 1980s understanding of String Theory in order to explain why we live in four dimensions. The basic idea was to start with a 9-dimensional box with each dimension being of the same length R of the order of the Planck length. The box was filled with a gas of strings, and after inserting periodic boundary conditions, T-duality, adiabaticity and some more simplifying assumptions it was shown that the largest number of spatial dimensions under which the system stays in thermal equilibrium is three. This simple model has been modified to incorporate certain p-dimensional branes of M-Theory [2], and it was shown that the main results remain the same, but there seems to be reasonable evidence for the existence of an intermediate 5-dimensional stage of the early universe in this setup. For a more detailed discussion of this toy model and the problems it has to address see [62].

The possibility of a 5-dimensional state of the early universe is in agreement with the pioneering work of Hořava and Witten [107, 108], who showed in 1995 that eleven-dimensional M-Theory compactified on a S^1/Z_2 orbifold is equivalent to a strongly coupled heterotic String Theory, each with appropriate gauge groups [6]. Soon afterwards it was shown by Witten [209] that this M-theory limit can be consistently compactified on a deformed Calabi-Yau threefold, leading to a super-symmetric $\mathcal{N} = 1$ theory in four dimensions. It turns out that the S^1/Z_2 orbifold must be larger than the Calabi-Yau radius in order for the phenomenological gravitational and grand-unified couplings to match at tree level [209, 13]. This means that there seems to be a regime where the universe appears five-dimensional.

Inspired by this result much work has since then been carried out on the search for possible 5-dimensional cosmologies. Most importantly Lukas, Ovrut, Stelle and Waldram derived the five-dimensional effective action for Hořava-Witten theory and found exact cosmological domain wall solutions [139] called LOW models. The existence of these solutions is exciting since it shows that it is possible to construct a more or less realistic cosmological model from String/M-Theory. However, there are significant unresolved problems with these models, most importantly the fact that they are vacuum solutions (without matter or radiation), there is no effective 4-dimensional cosmological constant, no inflation and no natural mechanism providing super-symmetry breaking on the domain wall. All these properties seem to disagree with our observed universe. Furthermore, the usual problems of string compactifications remain: the CY-manifold that the theory is compactified on is chosen by hand, and choosing any other one would most certainly result in a very different theory. There is no mechanism explaining why this particular model should be preferred over the huge amount of other possible compactifications.

Around the same time, Arkani-Hamed, Dimopoulos and Dvali (ADD) suggested that Hierarchy problem of Particle Physics can be solved using large extra dimensions [4]: the relative weakness of gravity can be explained if there existed n large[7] flat extra dimensions of equal size in which gravity can propagate while the standard model fields are confined

[6] Again, we refer to e.g. [62] for details.
[7] "Large" here means much larger than the weak scale.

to our 1+3 dimensional "brane".

The idea of employing extra dimensions to unify the fundamental forces originated in the 1920s, when Kaluza [120] and Klein [125] showed that electromagnetism can be derived from General Relativity in five dimensions after compactification of the fifth dimension on a circle. However, the compactification scale had to be of the order of the Planck length in order to avoid massive excitations of gravity, so it is essentially impossible to experimentally verify the theory, and furthermore even in the 1920s it was questionable whether General Relativity was still valid on these small scales.

If however the standard model fields - which are what we do our experiments with - were confined to our 1+3 dimensional brane, then the only way to see the extra dimensions is with gravitational interactions, which are much less constrained on small scales. This is exactly what the ADD model makes use of, and the setup is natural when embedding the model into String Theory [3].

Agreement with experiments however requires the number of extra dimensions to be larger or equal than two, so the model is not in agreement with the results of Hořava-Witten theory. Moreover, even though the model eliminates the hierarchy between the weak scale and the Planck scale, it does introduce a new hierarchy between the weak scale (relating to the 5-dimensional "fundamental" Planck scale) and the compactification scale. And yet the biggest problem with this model is of cosmological origin: the model suffers from primordial Kaluza-Klein (KK)gravitons, which would destroy the primordial deuterium and hence violate bounds obtained from big bang nucleosynthesis. In order to ensure none of these KK modes are present in the model one needs a fairly low reheating temperature after inflation, too low in fact for electroweak baryogenesis [21]. So altogether the ADD model seems to face major challenges concerning its cosmology (see also [47]).

1.4 The Randall-Sundrum cosmological models

Inspired by the ADD model for solutions of the hierarchy problem and the newly found physical motivation for studying 5-dimensional cosmologies with the fifth dimension compactified on the orbifold S^1/Z_2 from [209], Randall and Sundrum shortly afterwards found a simple warped solution to the 5-d Einstein field equations: a 5-dimensional AdS spacetime with the fifth dimension compactified on a circle with Z_2-symmetry - the S^1/Z_2 orbifold. This model (RS1), containing two three-branes (each of spacetime dimensionality four) on the opposing endpoints of the orbifold, has similarities to the LOW model, but has the new feature that the two branes have opposite non-zero tension or vacuum energies and it does not contain any super-symmetry. Furthermore, it can be shown that on the negative tension brane, the Hierarchy problem is naturally solved if the distance between the branes is of order of the Planck length. Unlike in the ADD model, the hierarchy between the 5-dimensional Planck scale and the compactification scale is only of order 50, so no new hierarchy is introduced. The mechanism for solving the Hierarchy problem in the RS1 model is so effective because it does not simply "dilute" gravity by allowing it to spread in more dimensions than the other three standard model forces as in the ADD model. What weakens gravity in the RS1 model down to the level of the standard model forces is the fact that gravity is localized on the second brane, so the observers on the first (negative tension) brane essentially only catch the tail end of the gravitational potential. As a consequence, the model furthermore does not suffer from light KK modes which would violate experimental bounds, even though only one extra dimension is required.

It can be further shown that the distance between the branes is stabilized at least in a first order approximation [88], so that no fine-tuning seems to be required in order to keep the branes at the separation required for solving the Hierarchy Problem.

However, it was shown that the energy densities on a negative tension brane must be negative and gravity repulsive [48, 55, 183], so the cosmology on the brane solving the Hierarchy Problem is "wrong".

On the positive tension brane, things are how we want them to be: the energy densities are positive and gravity is attractive. This was one of the motivations for the second model of Randall and Sundrum called RS2 [167], where it was now assumed that the standard model fields are confined to the positive tension brane and negative tension brane is moved off to infinity. This means that the model now has a non-compact extra dimension which is not periodic but still possesses Z_2-symmetry as in RS1. The RS2 model does not attempt to solve the Hierarchy Problem, since the brane on which this could be done is removed from the system. However, a combination of the two RS models may be able to address this problem as we will outline below. Since we now effectively have gravity in five infinite dimensions, one might wonder whether this model still agrees with Newtonian gravity. Well it does actually, and surprisingly well. It turns out that when doing the KK reduction of the general linearized 5-d gravitational (tensor) fluctuations in terms of the 4-d KK modes, the spectrum contains a single normalizable bound state and continuum of higher states. There is no mass gap (unlike in factorizable models like ADD), and while the ground state yields the usual Newtonian potential, the continuum of KK modes give rise to non-newtonian corrections. However, even though these modes can be arbitrarily light, they reach their maximal amplitude arbitrarily far away from the brane, which means that the corrections are extremely suppressed and hence have very little impact on gravity on the brane.

It is this model that has evoked the interest of a huge number of cosmologists, and which we will study in part II of this thesis. We emphasize that this model is not of interest to us because we think this is *the* "model of everything" – it simply presents a tangible string inspired cosmological model that does not seem to violate well-tested pillars of cosmology like Newtonian gravity, nucleosynthesis and late time cosmology. This model makes for an extremely interesting playground to explore methods developed in standard General Relativity in order to determine the cosmological properties such as gravitational wave production, isotropization properties and so on.

Beyond that, the model has interesting theoretical potential, both for model building as well as a testing field for some ideas from String Theory.

First of all, it is interesting to note that this model indicates that the occurrence of runaway moduli in String Theory, which is usually seen as a problem, is not necessarily a problem at all: here the modulus describing the brane separation of the RS1 system is taken to infinity to obtain the RS2 system, yet the RS2 model is perfectly well-defined.

Secondly, since RS2 geometrically describes a slice of AdS_5, the brane can be interpreted as a infrared cutoff for gravity in AdS_5 (see [74]), which means that it makes sense to apply the ADS/CFT conjecture to this model. It turns out that when calculating the first order correction to Newtonian gravity in the dual CFT picture, one obtains exactly the same term as obtained from calculating the massive KK modes in the 4-d effective theory. This result can be viewed as one of the few tests of the AdS/CFT conjecture in a setup without super-symmetry.

Finally, as mentioned above, it would be nice if we could combine the cosmological features of the RS2 model with the solution of the Hierarchy problem as in the RS1 setup. This might be possible: in [140] the authors introduced a model consisting of two positive tension branes separated by certain distance in the fifth dimension. Since both branes have positive tension there are no un-physical negative total energy densities. By localizing gravity on one brane called the Planck brane, the Hierarchy Problem can be solved on the other brane just like in the RS1 model. It seems to make sense to interpret this second brane as a probe of the geometry determined by the Planck brane. This can be done only if the tension of the probe brane on which the observers (including the standard model fields) are located is negligible compared to the brane tension of the Planck brane thus introducing the fine-tuning condition of almost vanishing vacuum energy on the probe brane.

Under these conditions, the model can be described by the RS2 metric. One can calculate the (classical) corrections to the Newtonian potential both using approximations to calculate the massive KK modes in the effective 4-d theory and using the CFT dual. These corrections are still within the experimental bounds, so gravity on the standard model brane is effectively 4-dimensional even though there is an infinite fifth dimension. This model seems to nicely combine the best of the two (Randall- Sundrum-brane-)worlds. However, it is not clear whether this model is stable against quantum corrections.

To end this introduction we would like to briefly mention the possibilities of experimentally testing or constraining the RS scenarios, and furthermore the possibility to differentiate between the warped RS scenarios and the ADD model with flat and possibly more than one extra dimension.

There are two ways to so: On the one hand side there are *experiments* on earth like laboratory and accelerator experiments. On the other hand one can use the information obtained from *observing* the sky such as astrophysical data from pulsars, supernovae (SNs) etc., and from analyzing the Cosmic Microwave Background (CMB).

On the experimental side, the RS models predict both changes in the gravitational inverse square law on sub-millimeter distances and specific new high energy collider signatures.

This has firstly lead to new efforts in testing the inverse square law, which now seems to hold down to separations of around a hundred micrometers [111] (see also [157]). These results constrain the number of extra dimensions in the ADD model to more than two, but do not yet constrain the RS models.

Secondly, experimentalists have been looking for new collider signatures such as the Higgs-like radion excitation mode and "normal" new particle effects due to Nambu-Goldstone modes corresponding to inhomogeneous fluctuations of the brane in the bulk. Furthermore, the massive KK coupling to ordinary matter should give rise to new effects [8]. The absence of any new signals [44] can be used to further constrain the ADD models. Again, only slight – if any – constraints are put on the RS models, partly because the couplings of all but the zero KK modes to matter are strongly suppressed [167] and may only be accessible in future experiments [45].

On the observational side things are looking up: excitingly, it should be possible in the near future to directly detect massive KK modes produced in the early universe using gravitational wave detectors both on earth (GEO600, LIGO, VIRGO, TAMA) and in the sky (LISA). These detectors should be able to furthermore detect the tensor modes of radion and Nambu-Goldstone modes produced in the early universe [99].

Additionally, the recent data from WMAP on the cosmic microwave background, results from LRG (SDSS+2dF) surveying galaxy redshift and luminosity, type Ia supernovae (SN Ia) surveys and astrophysical observations can be used to test the alternative cosmological model. In [202] several RS2 inflationary models with different potentials were constrained in the high energy regime using recent data from WMAP, 2dF and SDSS.

In [82] it was suggested that data from type Ia supernovae imposes strong constraints on RS models with matter on the brane in the form of dust. The same data has been used in [185], where it was shown that modified RS models including a scalar curvature term in the action can explain both the SN Ia and the WMAP data very well - even in a model with vanishing cosmological constant on the brane [205].

In [59] the authors found that a simple brane model with a 5-dimensional Minkowski bulk is experimentally excluded using observations of the binary pulsar PSR 1913+16. It was further claimed that the same result holds for any model with compact extra dimensions, hence excluding the RS1 model and the ADD models; no statement about the RS2 model could be made though.

Anti-proton emission from braneworld primordial black holes (PBHs) furthermore can put upper limits on the size of the curvature radius of the RS2 model [180].

Finally, the authors of [149] proposed that the entire class of braneworld models should be constrained covariantly by only assuming the existence of extra dimensions, confinement of standard model fields on the brane and propagation of gravity in the bulk described by the Einstein-Hilbert action for the bulk geometry. It was found that various constraints relax when not imposing Z_2-symmetry or increasing the number of extra dimensions.

1.5 Cosmological Spacetimes with Horizons

In the third part of this thesis we change focus. After studying the Randall-Sundrum model, a very special realization of string cosmology, in part II, we take a step back and concentrate on more general, conceptual issues: in part III we will study some general aspects of cosmological spaces with finite temperature horizons. The physics of horizons are still far from being well-understood as recently pointed out by 't Hooft [102]. However, a great deal of insight has been obtained by studying Black Hole physics. Possibly two of the most important concepts underlying Black Hole physics are called "Black Hole Complementarity" and "Holography". Both principles are thought to be fundamental, and to underlie any quantum gravity theory. They have thus been generalized from the context of Black Holes to general spaces with horizons. We will here briefly outline the basic ideas.

1.5.1 Horizon Complementarity

The principle of **Black Hole Complementarity** [195] simply states that no observer on either side of the horizon sees a violation of the laws of physics. This sounds like a fairly redundant statement – why should we expect anything else? To answer this question, we should first of all note that the horizon appears very different to different observers. An observer freely falling towards the horizon feels nothing special when crossing the horizon until he hits the singularity where he will get crushed to death. An external observer on the other hand sees a horizon: a thermal "membrane" of finite temperature. Any object falling into the horizon, including the freely falling observer, appears to be absorbed, thermalized and re-emitted by the horizon. It is now clear that these fundamentally different observations could indeed lead to violations of the known laws of physics.

One very interesting problem described in [24] is related to the "Quantum Xerox Problem": according to the laws of quantum mechanics it is not possible to duplicate quantum information. Now let us consider a bit of information thrown into a Black Hole. An observer passing the horizon with the information will localize the information at some point behind the horizon. Let us call this copy 1. The external observer E on the other hand will see the information absorbed and re-emitted by the horizon as copy 2. Thus the bit of information seems to be localized on both sides of the horizon, either violating locality or the Quantum Xerox principle. It is clear that the E could never see copy 1 since this copy is located behind the horizon. However, E could travel through the horizon once she has received the re-emitted copy 2. Having passed the horizon, the first observer could send her a signal with copy 1, and she would now have two copies of the same bit of information in violation of the Quantum Xerox principle.

The resolution of this paradox lies in the fact that there is very little time for the information exchange to occur before either of the observers hits the singularity. The originally infalling observer will have to send the signal to observer E in a very short period of time. According to the laws of quantum mechanics, this will require a very large amount of energy. It can be shown that the energy required to get the signal to observer E in time is in fact larger than the energy contained in the Black Hole, which leads to a contradiction. It is thus not possible to consistently to violate the Quantum Xerox principle.

This example is a particular realization of the principle of Black Hole Complementarity. Mathematically, this principle can be translated into the statement that the two observers use the same Hilbert space, but measure mutually non-commuting observables in their quantum theory.

This concept has been generalized to any cosmological space with horizon and in its general form is often called Observer Complementarity [162] or **Horizon Complementarity**. The principle states again that any two observers in a space with a horizon never see a violation of the known laws of physics. This translates into the mathematical statement that the quantum (field) theory of observers who are classically causally disconnected use (independent tensor factors of) the same Hilbert space but measure mutually non-commuting observables. Observer Complementarity implies that the semi-classical theory of a space with a horizon can be described by an external observer outside the horizon. It explains for example why the Hawking radiation of Black Holes cannot be described by a free field theory that is defined on both sides of the horizon: one must instead describe the Black Hole by a strongly interacting theory restricted to the spacetime region outside the horizon [196].

1.5.2 The Holographic Principle

The concept of Holography goes back to 1993, when 't Hooft [100] studied Black Hole formation via gravitational collapse. It is well-known that the maximum entropy in any given region of space, being a measure for the number of possible quantum states, must be proportional to the boundary not the volume of the region. If this was not the case Black Hole formation would cause the second law of thermodynamics to be violated as first demonstrated by Bekenstein [20].

In his pioneering paper [100], 't Hooft suggested that Black Hole physics can be reconciled with quantum mechanics if one adapts a highly unconventional view of spacetime. He extended his reasoning from Black Holes to quantum gravity theories in *any* spacetime, in line with Bekenstein's argument. We will here briefly summarize his main arguments.

The notion of cutoffs in quantum field theory naturally suggests that at the Planck energy scale, our macroscopically 4-dimensional spacetime should be replaced by a 3-dimensional lattice evolving in time, where the lattice spacing relates to the cutoff scale and is expected to be of the order of the Planck scale. 't Hooft however suggested that at the Planck scale, spacetime should be a 2-dimensional lattice evolving in time. He argued that the Black Hole entropy must be attributed not just to the degrees of freedom of the spacetime metric, but instead to the degrees of freedom of *all quantized fields* in the neighborhood of the horizon. This means that the number of degrees of freedom existing in the neighborhood of the Black Hole horizon is finite and proportional to the horizon area.

In analogy, 't Hooft suggested that any region of space inside a closed surface should be sufficiently described by the boundary: "all physical degrees of freedom can be projected onto the boundary", and "one Boolean variable per Planckian surface element should suffice" to describe all the physics in the inside [100]. It is now clear why this concept is aptly named "The Holographic principle". He further considered the limit where the boundary region becomes indefinitely large, and concluded that in fact any 3-dimensional space can be projected onto a 2-dimensional surface as long as this surface ranges to infinity. This means that on a Planckian level, the spatial degrees of freedom in a three-space must be somehow highly non-locally correlated; on this fundamental scale the degrees of freedom are reduced to the degrees of freedom of a 2-dimensional lattice.

Susskind [192] showed in 1994 that this concept has great similarities with the light front lattice string model by Klebanov and Susskind [124], and suggested that String Theory could present an example of a holographic quantum theory of gravity. It seems like this intuition was correct: Holography is now thought to be a fundamental concept underlying non-perturbative String Theory [110]. The above-mentioned AdS/CFT conjecture [151] is one particular implementation of the Holographic Principle in String Theory.

The holographic principle has also been used to impose entropy bounds on flat FLRW geometries [71] and more general spacetimes [28]. In particular Bousso has formulated a more general version of the original idea, supplying a concrete implementation of the Holographic Principle for any region of a general spacetime by introducing a method to construct the relevant boundary surface based on the construction of null-surfaces called "light-sheets".

According to the holographic principle, a quantum gravity theory in a given (cosmological) background spacetime must have a dual description as a field theory on some well-defined asymptotic boundary of the spacetime. Again we refer to the AdS/CFT correspondence [151] as an example.

Similarly there should be a holographic dual to quantum gravity in other cosmologically relevant spacetimes. Even if this dual theory is unknown, it must inherit some of the properties of the original theory such as its symmetry properties. This knowledge provides us with a setup in which we can try to constrain possible backgrounds in String Theory or, in fact, any consistent theory that includes gravity and obeys horizon complementarity and holography. It is this question that we will address in part III.

Part II

Randall-Sundrum cosmology

Chapter 2

Randall-Sundrum basics

2.1 Introduction

In this part of this thesis we study the Randall-Sundrum model with one effective brane (RS2). In this chapter we briefly summarize the cosmological properties for the effective one brane system, keeping in mind that this is not a rigorous derivation, just a summary of results.

The effective action for the system containing one brane, on which the matter fields are confined, embedded in a slice of AdS_5 with Z_2-symmetry is given by

$$S = S_{gravity} + S_{brane}\,, \tag{2.1}$$

where

$$S_{gravity} = \frac{1}{\kappa_{(5)}^2}\int d^4x \int dy\sqrt{-^{(5)}g}[^{(5)}R - 2\Lambda_{(5)}] \tag{2.2}$$

is the Einstein-Hilbert action describing gravity in five dimensions, reflecting the fact that gravity can propagate in the bulk. As usual, $\kappa_{(5)}^2$ is the five-dimensional gravitational coupling constant, $^{(5)}g$ is the determinant of the five-dimensional metric $^{(5)}g_{AB}$, $^{(5)}R$ is the five-dimensional Ricci scalar constructed from $^{(5)}g_{AB}$, and $\Lambda_{(5)} < 0$ is the bulk cosmological constant.

The matter on the brane is described by

$$S_{brane} = \int d^4x\sqrt{-g_{brane}}[\mathcal{L}_{brane} - 2\lambda]\,, \tag{2.3}$$

where $\mathcal{L}_{brane}$ is the Lagrangian describing matter on the brane, and λ is proportional to the vacuum energy of the brane. The exact form of the 3-brane Langrangian is not relevant for the classical five-dimensional metric in the ground state.

The five-dimensional Einstein field equations are given by

$$G^{(5)}_{AB} = {}^{(5)}R_{AB} - \frac{1}{2}{}^{(5)}R^{(5)}g_{AB} = \kappa^2_{(5)}T^{(5)}_{AB}\,. \tag{2.4}$$

The five-dimensional energy-momentum tensor $T^{(5)}_{AB}$ should be proportional to the bulk vacuum energy $\Lambda_{(5)}$. In addition, we allow for a brane energy momentum T^{brane}_{AB} which is only defined on the brane, and which contains the matter fields described by T_{AB} and the brane vacuum energy λ. This means that the total energy momentum tensor $T^{(5)}_{AB}$ is of the form

$$T^{(5)}_{AB} = \frac{1}{\kappa^2_{(5)}}\Lambda_{(5)}{}^{(5)}g_{AB} + \delta(y)T^{brane}_{AB}\,, \tag{2.5}$$

with

$$T^{brane}_{AB} = -\lambda g_{AB} + T_{AB}\,. \tag{2.6}$$

Note that we have to defined the 5-d energy-momentum tensor to include the 5-d cosmological constant in agreement with [183], but in contrast to e.g. [144].

A solution to the field equations satisfying 4-dimensional Poincaré invariance [166] is given by

$${}^{(5)}ds^2 = e^{-2\sqrt{-\Lambda_{(5)}/6}\,|y|}\,\eta_{\mu\nu}dx^\mu dx^\nu + dy^2\,, \tag{2.7}$$

where $\eta_{\mu\nu}(x^\alpha)$ is the 4-dimensional metric. The Z_2-symmetry of the bulk is reflected in the absolute value of y in the metric (3.7). This solution represents a Minkowski brane embedded in a slice of AdS_5. It only holds when the bulk cosmological constant $\Lambda_{(5)}$ is related to the brane cosmological term λ by

$$\kappa^4_{(5)}\lambda^2 = -6\Lambda_{(5)}\,, \tag{2.8}$$

which effectively sets the 4-dimensional cosmological constant Λ to zero.

Subsequently, various generalizations of this scenario have been studied. In particular, an exact solution for a FLRW brane embedded in AdS_5 with appropriate junction conditions [25, 72, 78, 91, 113, 119, 158] has been found, and it has been shown, by finding an explicit coordinate transformation [159], that this solution is in fact the same as the solution describing a domain wall moving in a 5-dimensional Schwarzschild-Anti-de Sitter spacetime [32, 113, 130].

It has also been observed that if the fine-tuning condition (2.8) is violated just by a tiny amount, then we obtain an effective 4-dimensional cosmological constant

$$\Lambda = \frac{1}{2}\left(\Lambda_{(5)} + \frac{1}{6}\kappa^4_{(5)}\,\lambda^2\right)\,, \tag{2.9}$$

which will cause late time acceleration of the observable universe [25], in agreement with observations (see for example [170]).

The fine-tuning condition (2.8) is the brane equivalent to the cosmological constant problem (see e.g. [66, 161, 174, 204] for a current review and [190] for a historical review) in General Relativity.

Finally, it can be seen that if the bulk is exactly Anti-de Sitter, then the matter on the brane must be spatially homogeneous [183]. Turning this argument around, this means that we need to study models where the bulk deviates from exact AdS if we are interested in inhomogeneous perturbations of the brane. This can be done by projecting the general bulk field equations onto the brane, assuming nothing but the fact that the bulk is 5-dimensional with Z_2-symmetry and confinement of the matter on the brane under appropriate junction conditions. In this way we obtain a relation between the bulk Weyl tensor (which is zero if the bulk is exactly AdS due to the fact that exact AdS is conformally flat) and the brane energy momentum tensor. We will present this mechanism in the following section, following the approach by Shiromizu et. al. [183]. We will then be able to decompose the projected equations with respect to the comoving matter velocity as in GR, but we will have correction terms arising from the extrinsic curvature of the brane.

2.2 Geometric formulation

Close to the Minkowski or FLRW brane, we can choose Gaussian normal coordinates X^μ, Y such that the metric takes the form

$$^{(5)}ds^2 = g_{\mu\nu}dX^\mu dX^\nu + dY^2 \,. \tag{2.10}$$

X^μ ($\mu = 0, ..., 4$) are coordinates on the brane, and the fifth Gaussian coordinate Y is an affine parameter measuring the proper distance from the brane. The brane is located at $Y = 0$ without loss of generality.

With this choice of coordinates we can define

$$n^A \equiv \frac{dX^A}{dy} \tag{2.11}$$

to be the unit normal to the brane. The induced metric on the brane is given by

$$g_{AB} = {}^{(5)}g_{AB} - n_A n_B \,, \tag{2.12}$$

and the extrinsic curvature of slices of constant Y is

$$K_{AB} = \frac{1}{2}\pounds_n {}^{(5)}g_{AB} = g^C{}_A {}^{(5)}\nabla n_B \,, \tag{2.13}$$

$$K_{[AB]} = 0 = K_{AB}n^B \,. \tag{2.14}$$

The change of K_{AB} along surfaces of $\{Y = const\}$ is given by the Codacci equation

$$g^B{}_C \nabla_B K^C{}_A - g^B{}_A \nabla_B K = {}^{(5)}R_{CD}n^C g^D{}_A \,. \tag{2.15}$$

The 4-dimensional Riemann curvature tensor R_{ABCD} can be expressed in terms of the 5-dimensional curvature ${}^{(5)}R_{ABCD}$ and the extrinsic curvature K_{AB}:

$$R^{A}{}_{BCD} = {}^{(5)}R^{E}{}_{FGH}\; g^{A}_{\ E}\, g^{F}_{\ B}\, g^{G}_{\ C}\, g^{H}_{\ D} + K^{A}{}_{C}K_{BD} - K^{A}{}_{D}K_{BC}\,. \tag{2.16}$$

This equation is called the Gauss equation. On contracting (2.16), we obtain the 4-dimensional Ricci tensor R_{AB} and Ricci scalar R in terms of the extrinsic curvature and the 5-dimensional curvature invariants, which in turn yield the field equations for the 4-dimensional Einstein tensor G_{AB}. These are given by

$$\begin{aligned} G_{AB} &= \left({}^{(5)}R_{CD} - \frac{1}{2}{}^{(5)}R\,{}^{(5)}g_{CD}\right) g^{C}_{\ A}\, g^{D}_{\ B} + {}^{(5)}R_{CD}\, n^{C}\, n^{D}\, g_{AB} - K^{C}{}_{A}\, K_{BC} \\ &+ KK_{AB} - \frac{1}{2}g_{AB}(K^2 - K^{CD}\, K_{CD}) - {}^{(5)}R^{C}{}_{DFG}\, n_C\, n^F\, g^{D}_{\ A}\, g^{G}_{\ B}\,, \end{aligned} \tag{2.17}$$

where $K = K^{A}_{\ A}$.

The 5-dimensional Riemann tensor ${}^{(5)}R_{ABCD}$ can be split into scalar curvature ${}^{(5)}R$, Ricci tensor ${}^{(5)}R_{AB}$ and Weyl curvature ${}^{(5)}C_{ABCD}$,

$${}^{(5)}R_{ABCD} = \frac{2}{3}\left({}^{(5)}g_{A[C}{}^{(5)}R_{D]B} - {}^{(5)}g_{B[C}{}^{(5)}R_{D]A}\right) - \frac{1}{6}{}^{(5)}g_{A[C}{}^{(5)}g_{D]B}{}^{(5)}R + {}^{(5)}C_{ABCD}\,. \tag{2.18}$$

Using this decomposition and the 5-dimensional field equations (2.4), we find that the 4-dimensional field equations take the form

$$\begin{aligned} G_{\mu\nu} &= \frac{2\kappa_{(5)}^2}{3}\left({}^{(5)}T_{AB}g^{A}{}_{\mu}g^{B}{}_{\nu} + \left({}^{(5)}T_{AB}n^A n^B - \frac{1}{4}{}^{(5)}T\right) g_{\mu\nu}\right) \\ &+ KK_{\mu\nu} - K_{\mu}{}^{\sigma}K_{\nu\sigma} - \frac{1}{2}g_{\mu\nu}\left(K^2 - K^{\alpha\beta}K_{\alpha\beta}\right) - {}^{(5)}E_{\mu\nu}\,, \end{aligned} \tag{2.19}$$

where $T = T^{A}_{\ A}$, and

$${}^{(5)}E_{AB} \equiv {}^{(5)}C^{C}_{\ DEF}n_C n^E\, g_{A}^{\ D}\, g_{B}^{\ F} \tag{2.20}$$

is the electric part of the 5-dimensional Weyl tensor. We now impose Israel's junction conditions [114]:

$$[g_{\mu\nu}] = 0\,, \tag{2.21}$$

$$[K_{\mu\nu}] = -\kappa_{(5)}^2\left(T^{brane}_{\mu\nu} - \frac{1}{3}g_{\mu\nu}T^{brane}\right) \tag{2.22}$$

$$= -\kappa_{(5)}^2\left(T_{\mu\nu} - \frac{1}{3}g_{\mu\nu}(T-\lambda)\right), \tag{2.23}$$

where $[X] := \lim_{Y\to+0} X - \lim_{Y\to-0} X = X^+ - X^-$, and (2.23) is obtained on inserting the explicit form of the brane energy-momentum tensor (2.6).

Because of the bulk Z_2-symmetry with the brane at $Y = 0$ fixed, (2.23) determines the extrinsic curvature. On the brane, $K_{\mu\nu}$ is given by

$$K_{\mu\nu} = -\frac{1}{2}\kappa_{(5)}^2\left(T_{\mu\nu} - \frac{1}{3}g_{\mu\nu}(T-\lambda)\right). \tag{2.24}$$

Substituting (2.24) into (2.19), and using the explicit form of the energy momentum tensor (2.5), we find that the effective field equations on the brane are given by

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} + \kappa^2 T_{\mu\nu} + \frac{6\kappa^2}{\lambda}S_{\mu\nu} - {}^{(5)}E_{\mu\nu}\,, \tag{2.25}$$

where

$$S_{\mu\nu} = -\frac{1}{4}T_{\mu\alpha}T_{\nu}{}^{\alpha} + \frac{1}{12}TT_{\mu\nu} + \frac{1}{8}g_{\mu\nu}T_{\alpha\beta}T^{\alpha\beta} - \frac{1}{24}g_{\mu\nu}T^2\,. \tag{2.26}$$

The 4-dimensional gravitational coupling constant is given by

$$\kappa^2 = \frac{\kappa_{(5)}^4\,\lambda}{6}\,, \tag{2.27}$$

and the 4-dimensional cosmological constant Λ is the mismatch of the bulk cosmological constant $\Lambda_{(5)}$ and the brane tension λ defined in (2.9).

Note that the electric part of the Weyl tensor ${}^{(5)}E_{\mu\nu}$ is not defined on the brane, but only in the limit $Y \to \pm 0$,

$${}^{(5)}E_{\mu\nu} = \frac{1}{2}\left[{}^{(5)}E_{\mu\nu}\right]\,. \tag{2.28}$$

General Relativity is obtained when taking the 5-d gravitational coupling constant $\kappa_{(5)}$ to zero while keeping the 4-d gravitational coupling constant κ constant, i.e. in the general relativistic limit the brane tension λ is taken to infinity.

If we rewrite (2.25) as

$$\begin{aligned} G_{\mu\nu} &= -\Lambda g_{\mu\nu} + \kappa^2 T^{\rm tot}_{\mu\nu}, \\ T^{\rm tot}_{\mu\nu} &= T_{\mu\nu} + \tfrac{6}{\lambda}S_{\mu\nu} - \tfrac{1}{\kappa^2}{}^{(5)}E_{\mu\nu}\,, \end{aligned} \tag{2.29}$$

we can see that the effective 4-d equations are very similar to the general relativistic Einstein equations, but with two types of correction terms: the tensor $S_{\mu\nu}$ is quadratic in the brane energy momentum $T_{\mu\nu}$ and entails the effects of the extrinsic curvature of the brane, and the tensor ${}^{(5)}E_{\mu\nu}$ carries the non-local gravitational effects of the bulk.

Substituting the explicit form of $K_{\mu\nu}$ (2.24) into the Codacci equation (2.15), we find that the energy-momentum tensor is conserved on the brane,

$$\nabla^{\nu}T_{\mu\nu} = 0\,. \tag{2.30}$$

Using this and the 4-d contracted Bianchi identities $\nabla^\nu G_{\mu\nu} = 0$ one obtains the additional constraint

$$\nabla^\nu \left({}^{(5)}E_{\mu\nu} \right) = \tfrac{6\kappa^2}{\lambda} \nabla^\nu S_{\mu\nu} \,, \tag{2.31}$$

which shows how the non-local bulk effects are sourced by the evolution and spatial inhomogeneity of the brane matter content.

2.3 Covariantly decomposed Kinematics

After projecting the 5-d equations onto the brane, we are effectively left with a set of modified 4D field equations with non-linear corrections due to the bulk effects.

Since the matter fields are confined to the brane, we can define the 4-velocity u^μ to be the velocity of an observer on the brane who is comoving with the matter. As in standard cosmology, we can then define the **projection tensor** onto the 3-spaces orthogonal to the fluid flow vector, which correspond to the instantaneous rest-spaces of comoving observers:

$$h_{\mu\nu} \equiv g_{\mu\nu} + u_\mu u_\nu \,. \tag{2.32}$$

For future reference, we also define the **fully projected covariant derivative** operator D_μ where for any tensor $X^{\alpha\ldots\beta}{}_{\gamma\ldots\delta}$

$$\mathrm{D}_\mu X^{\alpha\ldots\beta}{}_{\gamma\ldots\delta} = h^\alpha{}_\sigma \cdots h^\beta{}_\tau \; h^\kappa{}_\gamma \cdots h^\lambda{}_\delta \; h^\rho{}_\mu \; \mathrm{D}_\rho X^{\sigma\ldots\tau}{}_{\kappa\ldots\lambda} \,. \tag{2.33}$$

The choice of u^μ also defines a **proper time derivative** along the flow lines, often called the "dot derivative":

$$\dot{X}^{\alpha\ldots\beta}{}_{\gamma\ldots\delta} = u^\mu \; \nabla_\mu X^{\alpha\ldots\beta}{}_{\gamma\ldots\delta} \,. \tag{2.34}$$

The kinematics of the matter flow are fully contained in the covariant derivative of the flow vector u^μ, which can be covariantly and uniquely split into its irreducible parts:

$$\nabla_\nu u_\mu = \mathrm{D}_\nu u_\mu - \dot{u}_\mu u_\nu = \tfrac{1}{3}\Theta + \sigma_{\mu\nu} + \omega_{\mu\nu} - A_\mu u_\nu \,, \tag{2.35}$$

where

$$\Theta \equiv \mathrm{D}_\mu u^\mu , \;\; \sigma_{\mu\nu} \equiv \mathrm{D}_{<\mu} u_{\nu>} , \;\; \omega_{\mu\nu} \equiv \mathrm{D}_{[\mu} u_{\nu]} , \;\; A_\mu \equiv \dot{u}_\mu \,. \tag{2.36}$$

As usual round (square) brackets denote the **(anti-)symmetric part** of a tensor, and angled brackets denote the **orthogonally projected symmetric trace-free part**

$$X^{<\mu\nu>} = \left(h^{(\mu}{}_\alpha h^{\nu)}{}_\beta - \tfrac{1}{3} h^{\mu\nu} h_{\alpha\beta} \right) X^{\alpha\beta} \,. \tag{2.37}$$

We also define the **covariant spatial curl** for any vector X^μ or tensor $X^{\mu\nu}$ via

$$\mathrm{curl}\, X_\mu \equiv \epsilon_{\mu\alpha\beta} \mathrm{D}^\alpha X^\beta \,, \quad \mathrm{curl}\, X_{\mu\nu} \equiv \epsilon_{\alpha\beta(\mu} \mathrm{D}^\alpha X^\beta_{\nu)} \,, \tag{2.38}$$

where $\epsilon^{\mu\alpha\beta}$ is the three-volume element.
The definitions (2.36) correspond to the following physical quantities:

- The trace Θ is the **volume rate of expansion** of the fluid, and related to the Hubble parameter H by $\Theta = 3H$. From now on we will use H instead of Θ, but the two quantities are interchangeable up to a factor of 3.

- The trace-free and symmetric part is the **shear** tensor ($\sigma_{\mu\nu} = \sigma_{(\mu\nu)}$, $\sigma_{\mu\nu}u^\nu = 0$, $\sigma^\alpha{}_\alpha = 0$) and describes the distortion of the fluid.

- The skew-symmetric (hence necessarily trace-free) part $\omega_{\mu\nu}$ is the **vorticity** tensor ($\omega_{\mu\nu} = \omega_{[\mu\nu]}$, $\omega_{\mu\nu}u^\nu = 0$), describing the rotation of the fluid relative to a non-rotating Fermi propagated frame.

It is often useful to define the magnitudes

$$\sigma^2 = \tfrac{1}{2}\sigma_{\mu\nu}\sigma^{\mu\nu}, \ \omega^2 = \tfrac{1}{2}\omega_{\mu\nu}\omega^{\mu\nu} \tag{2.39}$$

and the vorticity vector

$$\omega_\mu = \frac{1}{2}\epsilon^{\mu\alpha\beta}\omega_{\alpha\beta}\,. \tag{2.40}$$

If the vorticity vanishes ($\omega_{\mu\nu} = 0$), then the projection tensor (2.32) satisfies

$$h^\mu{}_\nu h^\nu{}_\lambda = h^\mu{}_\lambda, \ h_{\mu\nu}u^\nu = 0 \tag{2.41}$$

and serves as a metric on the rest spaces orthogonal to u^μ.[1]
A general energy-momentum tensor $T_{\mu\nu}$ can be covariantly decomposed as

$$T_{\mu\nu} = \rho u_\mu u_\nu + p h_{\mu\nu} + q_\mu u_\nu + q_\nu u_\mu + \pi_{\mu\nu}\,, \tag{2.42}$$

where

$$q_\mu u^\mu = 0, \ \pi^\mu{}_\mu = 0, \ \pi_{\mu\nu} = \pi_{(\mu\nu)}, \ \pi_{\mu\nu}u^\nu = 0\,. \tag{2.43}$$

This uniquely defines the following quantities:

- $\rho = T_{\mu\nu}u^\mu u^\nu$ is the relativistic **energy density** relative to u^μ,

- $q^\mu = -T_{\alpha\beta}u^\alpha h^{\beta\mu}$ is the **energy flux** relative to u^μ,

- $p = \frac{1}{3}T_{\mu\nu}h^{\mu\nu}$ is the **isotropic pressure** and

- $\pi_{\mu\nu} = T_{\alpha\beta}h^\alpha_{<\mu}h^\beta_{\nu>}$ is the **anisotropic stress**.

[1] If $\omega_{\mu\nu} \neq 0$, then this statement still holds locally, but not globally.

In analogy to the decomposition of the energy-momentum tensor, we can decompose the projected 5-D Weyl tensor defined by (2.18) as

$$-\tfrac{1}{\kappa^2}{}^{(5)}E_{\mu\nu} = \rho^*(u_\mu u_\nu + \tfrac{1}{3}h_{\mu\nu}) + q^*_\mu u_\nu + q^*_\nu u_\mu + \pi^*_{\mu\nu}\,, \tag{2.44}$$

where we define

$$q^*_\mu u^\mu = 0\,, \quad \pi^*_{(\mu\nu)} = \pi^*_{\mu\nu}\,, \quad \pi^{*\,\alpha}_{\alpha} = 0\,, \quad \pi^*_{\mu\nu}u^\nu = 0\,. \tag{2.45}$$

Effectively, the projected 5-D Weyl tensor has the form of a trace-free energy momentum tensor with energy density ρ^*, energy flux q^*_μ and anisotropic stress $\pi^*_{\mu\nu}$.

ρ^* appears as the energy density of a barotropic perfect fluid with equation of state $p = \frac{1}{3}\rho^*$, which is the equation of state of radiation. For this reason ρ^* is referred to as the **"dark" radiation density** of the Weyl fluid.

q^*_μ is a spatial tensor and $\pi^*_{\mu\nu}$ is a spatial, symmetric and trace-free tensor. q^*_μ and $\pi^*_{\mu\nu}$ are analogous to the usual energy flux vector q^μ and anisotropic stress tensor $\pi_{\mu\nu}$ in General Relativity.

For the remainder of this thesis, we will assume that the matter on the brane is of perfect fluid form [2]

$$T_{\mu\nu} = \rho u_\mu u_\nu + p h_{\mu\nu} \;\Leftrightarrow\; q^\mu = \pi_{\mu\nu} = 0\,, \tag{2.46}$$

with an equation of state $p = p(\rho)$. The most important cases can be summarized using a γ-law equation of state:

$$p = (\gamma - 1)\rho\,, \; \gamma = const\,, \tag{2.47}$$

where $0 \le \gamma \le 2$ is required in order to satisfy the dominant energy condition and to preserve causality. This barotropic equation of state includes

- $\gamma = 0$: a false vacuum or cosmological constant state with equation of state $p = -\rho$,
- $\gamma = 1$: pressure free matter, often called "dust" or "cold dark matter" (CDM) with $p = 0$,
- $\gamma = 4/3$: a perfect Fermi gas with $p = \frac{1}{3}\rho$, often simply called "radiation",
- $\gamma = 2$: the stiffest possible equation of state [3], $p = \rho$, often referred to as "stiff matter".

The non-linear matter correction tensor $S_{\mu\nu}$ then simplifies to

$$S_{\mu\nu} = \frac{1}{12}\rho^2 u_\mu u_\nu + \frac{1}{12}(\rho + 2p)h_{\mu\nu}\,, \tag{2.48}$$

and on using (2.29), the effective energy density, pressure, flux and anisotropic stress become

$$\rho_{\text{tot}} \;=\; \rho + \frac{\rho^2}{2\lambda} + \rho^*\,, \tag{2.49}$$

[2] For the general case with $q^\mu, \pi_{\mu\nu} \neq 0$ see [141].

$$\begin{aligned} p_{\rm tot} &= p + \frac{\rho}{2\lambda}(2p+\rho) + \frac{\rho^*}{3}\,, & (2.50)\\ q_\mu^{\rm tot} &= q_\mu^*\,, & (2.51)\\ \pi_{\mu\nu}^{\rm tot} &= \pi_{\mu\nu}^*\,. & (2.52) \end{aligned}$$

We can see that the effective energy density carries the bulk effects in the two correction terms $\rho^2/2\lambda$ and ρ^*. There are three different energy regimes to be considered. At early times or at late times when approaching a big-bang or big-crunch singularity, the quadratic term dominates $\rho^2/\lambda \gg \rho,\ \rho^*$. We refer to this as the *high energy limit.*
If the dark radiation term is the dominant energy contribution, $\rho^* \gg \rho,\ \rho^2/\lambda$, then we speak of the *dark radiation regime.*
At late times, we expect that the fluid energy density is the dominating term in (2.49), $\rho \gg \rho^2/\lambda,\ \rho^*$. In this *low-energy limit* we recover the general relativistic effective energy density ρ.

It is important to note that the non-local bulk effects generally turn the perfect fluid into an imperfect fluid form, as can be directly seen from (2.51) and (2.52).

We also observe that the effective equation of state changes, since

$$\gamma_{\rm tot} \equiv \frac{p_{\rm tot}}{\rho_{\rm tot}} + 1 = \frac{\gamma + 2\gamma\rho/2\lambda + \rho^*/3\rho}{1 + \rho/2\lambda + \rho^*/\rho}\,. \tag{2.53}$$

At low energies $\rho \ll \lambda$ and $\rho^* = 0$, we recover the usual equations of state $\gamma_{\rm tot} = \gamma$. At very high energies $\rho \gg \lambda$ on the other hand, when we can generally neglect the dark radiation term ρ^*, the effective equation of state is stiffened: $\gamma_{\rm tot} \to 2\gamma$. This can have important consequences close to big bang and big crunch singularities. In particular, the correction terms change the constraint for the onset of inflation (see section 2.4), and have the interesting effect that collapsing pressure-free matter ($\gamma = 1$) feels an effective non-zero pressure ($\gamma_{\rm tot} = 2$) [79].

2.3.1 1+3 Covariant Propagation and Constraint equations on the brane

The projected Einstein field equations (2.29) can be covariantly decomposed in order to obtain propagation and constraint equations for the covariantly defined kinematic quantities (2.36), the local and non-local fluid components defined in (2.46) and (2.44), and the free gravitational field on the brane [141].

We will decompose the 4-d Riemann tensor into scalar curvature R, Ricci tensor $R_{\mu\nu}$ and Weyl curvature $C_{\mu\nu\alpha\beta}$,

$$R_{\alpha\beta\gamma\delta} = \frac{2}{3}\left(g_{\alpha[\gamma}R_{\delta]\beta} - g_{\beta[\gamma}R_{\delta]\alpha}\right) - \frac{1}{6}g_{\alpha[\gamma}g_{\delta]\beta}R + C_{\alpha\beta\gamma\delta}\,. \tag{2.54}$$

The Weyl tensor $C_{\alpha\beta\gamma\delta}$ characterizing the free gravitational field on the brane can be split into the electric part $E_{\mu\nu}$ and the magnetic part $H_{\mu\nu}$ as

$$E_{\mu\nu} = C_{\mu\alpha\nu\beta}u^\alpha u^\beta = E_{\langle\mu\nu\rangle}\,, \quad H_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\alpha\beta}C^{\alpha\beta}{}_{\nu\gamma}u^\gamma = H_{\langle\mu\nu\rangle}\,.$$

We emphasize that $E_{\mu\nu}$ is the electric part of the *brane* Weyl tensor and must not be confused the electric part of the *bulk* Weyl tensor ${}^{(5)}E_{\mu\nu}$.

There are three sets of equations resulting from the Ricci identities and the Bianchi identities.

Ricci identities

The Ricci identities for the vector field u_μ are given by

$$\nabla_{[\mu}\nabla_{\nu]}u_\alpha = \frac{1}{2}R_{\alpha\nu\mu\beta}u^\beta\,.$$

When substituting $\nabla_\mu u_\nu$ from (2.35), we find the 1+3-covariant propagation and constraint equations for the kinematic quantities $3H$, ω_μ, $\sigma_{\mu\nu}$. We can see that the bulk terms, arranged on the left-hand-side of the equations below, act as additional source terms.

Firstly there is the **Raychaudhuri equation**

$$\begin{aligned} &3\dot{H} + 3H^2 + \sigma_{\mu\nu}\sigma^{\mu\nu} - 2\omega_\mu\omega^\mu - \mathrm{D}^\mu A_\mu + A_\mu A^\mu + \frac{\kappa^2}{2}(\rho+3p) - \Lambda \\ &= -\frac{\kappa^2}{2}(2\rho+3p)\frac{\rho}{\lambda} - \kappa^2\rho^*\,, \end{aligned} \qquad (2.55)$$

which is the basic equation of gravitational attraction.
The **vorticity propagation** equation is given by

$$\dot{\omega}_{\langle\mu\rangle} + 2H\omega_\mu + \frac{1}{2}\mathrm{curl}\,A_\mu - \sigma_{\mu\nu}\omega^\nu = 0\,. \qquad (2.56)$$

Together with (2.60) below this is the only equations that does not include any bulk terms.
The **shear propagation** equation

$$\dot{\sigma}_{\langle\mu\nu\rangle} + 2H\sigma_{\mu\nu} + E_{\mu\nu} - \mathrm{D}_{\langle\mu}A_{\nu\rangle} + \sigma_{\alpha\langle\mu}\sigma_{\nu\rangle}{}^\alpha + \omega_{\langle\mu}\omega_{\nu\rangle} - A_{\langle\mu}A_{\nu\rangle} = \frac{\kappa^2}{2}\pi^*_{\mu\nu} \qquad (2.57)$$

shows how the non-local gravitational field $E_{\mu\nu}$ as well as the non-local shear $\pi^*_{\mu\nu}$ induce shear of the brane.
The three corresponding constraint equations are the **shear constraint**

$$\mathrm{D}^\nu\sigma_{\mu\nu} - \mathrm{curl}\,\omega_\mu - 2\mathrm{D}_\mu H + 2\varepsilon_{\mu\nu\alpha}\omega^\nu A^\alpha = -\kappa^2 q^*_\mu\,, \qquad (2.58)$$

the **vorticity constraint**

$$\mathrm{D}^{\mu}\omega_{\mu} - A^{\mu}\omega_{\mu} = 0 \tag{2.59}$$

and the $H_{\mu\nu}$-equation or **gravito-magnetic constraint**

$$\operatorname{curl}\sigma_{\mu\nu} + \mathrm{D}_{\langle\mu}\omega_{\nu\rangle} - H_{\mu\nu} + 2A_{\langle\mu}\omega_{\nu\rangle} = 0\,. \tag{2.60}$$

Once-contracted Bianchi identities

A second set of equations arises from the Bianchi identities

$$\nabla_{[\alpha}R_{\beta\gamma\delta]\epsilon} = 0\,.$$

When substituting the decomposed form of the Riemann tensor (2.54) into the once-contracted Bianchi identities

$$\nabla_{[\alpha}R^{\beta}_{\gamma]\beta\epsilon} = 0\,,$$

we find propagation and equations for the gravito-electric and gravito-magnetic fields on the brane.
The **gravito-electric propagation** equation is given

$$\begin{aligned}
&\dot{E}_{\langle\mu\nu\rangle} + 3HE_{\mu\nu} - \operatorname{curl}H_{\mu\nu} + \frac{\kappa^2}{2}(\rho+p)\sigma_{\mu\nu} - 2A^{\alpha}\varepsilon_{\alpha\beta(\mu}H_{\nu)}{}^{\beta} - 3\sigma_{\alpha\langle\mu}E_{\nu\rangle}{}^{\alpha} + \omega^{\alpha}\varepsilon_{\alpha\beta\langle\mu}E_{\nu\rangle}{}^{\beta}\\
&= -\frac{\kappa^2}{2}(\rho+p)\frac{\rho}{\lambda}\sigma_{\mu\nu} - \frac{\kappa^2}{6}\Big[4\rho^*\sigma_{\mu\nu} + 3\dot{\pi}^*_{\langle\mu\nu\rangle} + 3H\pi^*_{\mu\nu} + 3\mathrm{D}_{\langle\mu}q^*_{\nu\rangle}\\
&\quad + 6A_{\langle\mu}q^*_{\nu\rangle} + 3\sigma^{\alpha}{}_{\langle\mu}\pi^*_{\nu\rangle\alpha} + 3\omega^{\alpha}\varepsilon_{\alpha\beta(\mu}\pi^*_{\nu)}{}^{\beta}\Big]
\end{aligned} \tag{2.61}$$

and the gravito-magnetic field obeys the **gravito-magnetic propagation** equation

$$\begin{aligned}
&\dot{H}_{\langle\mu\nu\rangle} + 3HH_{\mu\nu} + \operatorname{curl}E_{\mu\nu} - 3\sigma_{\alpha(\mu}H_{\nu)}{}^{\alpha} + \omega^{\alpha}\varepsilon_{\alpha\beta(\mu}H_{\nu)}{}^{\beta} + 2A^{\alpha}\varepsilon_{\alpha\beta(\mu}E_{\nu)}{}^{\beta}\\
&= \frac{\kappa^2}{2}\Big[\operatorname{curl}\pi^*_{\mu\nu} - 3\omega_{\langle\mu}q^*_{\nu\rangle} + \sigma_{\alpha(\mu}\varepsilon_{\nu)}{}^{\alpha\beta}q^*_{\beta}\Big]\,.
\end{aligned} \tag{2.62}$$

The constraint equations are the **gravito-electric divergence** equation

$$\begin{aligned}
&\mathrm{D}^{\nu}E_{\mu\nu} - \frac{\kappa^2}{3}\mathrm{D}_{\mu}\rho - \varepsilon_{\mu\nu\alpha}\sigma^{\nu}{}_{\beta}H^{\alpha\beta} + 3H_{\mu\nu}\omega^{\nu}\\
&= \frac{\kappa^2}{3}\frac{\rho}{\lambda}\mathrm{D}_{\mu}\rho + \frac{\kappa^2}{6}\Big(2\mathrm{D}_{\mu}\rho^* - 6Hq^*_{\mu} - 3\mathrm{D}^{\nu}\pi^*_{\mu\nu} + 3\sigma_{\mu}{}^{\nu}q^*_{\nu} - 9\varepsilon_{\mu}{}^{\nu\alpha}\omega_{\nu}q^*_{\alpha}\Big)
\end{aligned} \tag{2.63}$$

and the **gravito-magnetic divergence** equation

$$\begin{aligned}
&\mathrm{D}^{\nu}H_{\mu\nu} - \kappa^2(\rho+p)\omega_{\mu} + \varepsilon_{\mu\nu\alpha}\sigma^{\nu}{}_{\beta}E^{\alpha\beta} - 3E_{\mu\nu}\omega^{\nu}\\
&= \kappa^2(\rho+p)\frac{\rho}{\lambda}\omega_{\mu} + \frac{\kappa^2}{6}\Big(8\rho^*\omega_{\mu} - 3\operatorname{curl}q^*_{\mu} - 3\varepsilon_{\mu}{}^{\nu\alpha}\sigma_{\nu}{}^{\beta}\pi^*_{\alpha\beta} - 3\pi^*_{\mu\nu}\omega^{\nu}\Big)\,.
\end{aligned} \tag{2.64}$$

If the fluid is non-rotating ($\omega_\mu = 0$), then the auxiliary **Gauss-Codazzi equations** on the brane

$$ {}^3R_{\langle\mu\nu\rangle} + \dot{\sigma}_{\langle\mu\nu\rangle} + 3H\sigma_{\mu\nu} - \mathrm{D}_{\langle\mu}A_{\nu\rangle} - A_{\langle\mu}A_{\nu\rangle} = \kappa^2\pi^*_{\mu\nu}\,, \tag{2.65} $$

$$ {}^3R + 6H^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} - 2\kappa^2\rho - 2\Lambda = \kappa^2\frac{\rho^2}{\lambda} + 2\kappa^2\rho^* \tag{2.66} $$

relate the kinematic quantities (2.36) to the curvature of the 3-spaces. Here ${}^3R_{\mu\nu}$ is the Ricci tensor for 3-surfaces orthogonal to u^μ on the brane and ${}^3R = {}^3R_{\mu\nu}h^{\mu\nu}$ is the Ricci scalar of the 3-spaces.

Twice-contracted Bianchi identities

The twice contracted Bianchi identities

$$ \nabla^\nu G_{\mu\nu} = 0 $$

finally lead to the conservation of the energy momentum tensor $T_{\mu\nu}$ on the brane (2.30) and to the constraint equation (2.31) for ${}^{(5)}E_{\mu\nu}$. Projecting the perfect fluid energy-momentum tensor (2.46) parallel and orthogonal to the fluid 4-velocity u_μ, we obtain the usual **energy and momentum conservation** equations

$$ \dot{\rho} + 3H(\rho + p) = 0\,, \tag{2.67} $$

$$ \mathrm{D}_\mu p + (\rho + p)A_\mu = 0\,, \tag{2.68} $$

where a dot denotes $u^b\nabla_b$ and $H = \dot{a}/a$ is simply the Hubble expansion when considering FLRW models.
We similarly decompose (2.46) and obtain **propagation equations for the non-local quantities** ρ^* **and** q^*_μ

$$ \dot{\rho^*} + 4H\rho^* + \mathrm{D}^\mu q^*_\mu + 2A^\mu q^*_\mu + \sigma^{\mu\nu}\pi^*_{\mu\nu} = 0\,, \tag{2.69} $$

$$ \begin{aligned} &\dot{q}^*_{\langle\mu\rangle} + 4Hq^*_\mu + \tfrac{1}{3}\mathrm{D}_\mu\rho^* + \tfrac{4}{3}\rho^* A_\mu + \mathrm{D}^\nu\pi^*_{\mu\nu} + A^\nu\pi^*_{\mu\nu} + \sigma_\mu{}^\nu q^*_\nu - \epsilon_\mu{}^{\nu\alpha}\omega_\nu q^*_\alpha \\ &= -\frac{(\rho+p)}{\lambda}\mathrm{D}_\mu\rho\,. \end{aligned} \tag{2.70} $$

There is no propagation equation for the non-local anisotropic stress $\pi^*_{\mu\nu}$. This means that the equations on the brane are not closed, and the exact form of $\pi^*_{\mu\nu}$ can only be obtained from the full 5-dimensional equations. This presents a problem when studying perturbations around the exact FLRW solution, since $\pi^*_{\mu\nu}$ acts as a source for many of the kinematic quantities as can be seen e.g. for the shear from equation (2.57).

In special cases we can neglect $\pi^*_{\mu\nu}$, e.g. when studying (scalar) density or curvature perturbations [90] on large scales. When translating the curvature perturbations into metric perturbations in order to calculate the Sachs-Wolfe effect however, knowledge about the form of $\pi^*_{\mu\nu}$ is required [132]. When calculating large angle CMB anisotropies and tensor perturbations (or gravitational waves), the non-local anisotropic stress cannot be neglected either, since these effects are partly sourced by $\pi^*_{\mu\nu}$ (see e.g. [144]).

It follows that in general, one has to impose constraints on $\pi^*_{\mu\nu}$ by hand, and we must make sure that these assumptions are at least self-consistent. The validity of these assumptions can only be proven in a full treatment of the 5-dimensional bulk theory.

2.4 Scalar Field Dynamics on the Brane

Just as in General Relativity, we are often interested in the case where the matter is described by a scalar field on the brane. The main goal of this section is to show that there is a natural 4-velocity u^μ associated with the scalar "fluid", which allows us to covariantly decompose the equations with respect to u^μ as described above.

We will restrict ourselves to a minimally coupled self-interacting scalar field ϕ confined to the brane. The Lagrangian describing such a field is

$$\mathcal{L} = -\sqrt{-g}\ \left(\tfrac{1}{2}\nabla^\mu\phi\nabla_\mu\phi + V(\phi)\right)\,, \tag{2.71}$$

where $V(\phi)$ is a general potential describing the self-interaction of ϕ.

Covariant Scalar Field Dynamics

The equation of motion can be derived from $\mathcal{L}$ using the Euler-Lagrange equations and is given by the Klein-Gordon equation

$$\nabla^\mu\nabla_\mu\phi - \frac{\partial V}{\partial\phi} = 0\,. \tag{2.72}$$

On using Noether's Theorem, the energy-momentum tensor has the form

$$T_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - g_{\mu\nu}\left(\tfrac{1}{2}\nabla^\alpha\nabla_\alpha\phi + V(\phi)\right)\,, \tag{2.73}$$

and on using (2.72), we find that this energy-momentum tensor is conserved:

$$\nabla_\nu T^{\mu\nu} = 0\,. \tag{2.74}$$

Assuming that the momentum density $\nabla_\mu\phi$ is time-like

$$\nabla_\mu\phi\nabla^\mu\phi < 0\,, \tag{2.75}$$

ϕ is not constant and so the surfaces $\{\phi = constant\}$ are well-defined space-like surfaces in space-time[4]. We can now define the unique time-like vector field u^μ by

$$u^\mu = -\psi^{-1}\nabla^\mu\phi \text{ where } \psi \equiv -(\nabla_\mu\phi\nabla^\mu\phi)^{1/2}\,. \tag{2.76}$$

This unit magnitude vector field ($u^\mu u_\mu = -1$) is parallel to the normals of the surface $\{\phi = constant\}$ and presents the scalar field analogue of the fundamental 4-velocity comoving with the ordinary matter fields in a given space-time[5].

With this choice of u^μ we can define the projection tensor $h_{\mu\nu}$, the fully projected derivative D_μ and the time derivative along u^μ as in (2.32)-(2.34). We can then identify the normalization factor ψ as the momentum density magnitude, or simply the momentum of the field $\psi = \dot\phi$. The kinematic quantities (2.36) can be defined as usual and are in general functions of $\dot\phi$ and $\nabla^\mu\phi$.

Using the form of the energy-momentum tensor (2.73), it is easy to see that we can represent the minimally coupled scalar field as a perfect fluid

$$T_{\mu\nu} = \rho u_\mu u_\nu + p h_{\mu\nu} \tag{2.77}$$

if we define energy density ρ and pressure p of the scalar field fluid as

$$\begin{aligned} \rho &= \tfrac{1}{2}\dot\phi^2 + V(\phi)\,, \\ p &= \tfrac{1}{2}\dot\phi^2 - V(\phi)\,. \end{aligned} \tag{2.78}$$

Energy-momentum conservation implies the usual energy and momentum conservation equations (2.67) and (2.68). Using the explicit form of ρ and p (2.78), the energy conservation equation (2.67) takes the form

$$\ddot\phi + 3H\dot\phi + V'(\phi) = 0\,, \tag{2.79}$$

which is the covariant form of the Klein-Gordon equation, valid for any brane space-time with a minimally coupled scalar field on the brane and the 4-velocity defined by (2.76).

It is convenient to relate p and ρ by the *index* γ:

$$p = (\gamma - 1)\rho \quad\Leftrightarrow\quad \gamma = \frac{p+\rho}{\rho} = \frac{\dot\phi^2}{\rho}\,. \tag{2.80}$$

This index is constant in the case of a simple one - component barotropic fluid, but in the case of a scalar field it varies with time:

$$\dot\gamma = 3H\gamma(\gamma - 2) - 2\gamma\frac{dV}{d\phi}(\dot\phi)^{-1}\,. \tag{2.81}$$

[4]If ϕ were constant, then energy conservation implies $V = constant$, hence we would be dealing with an effective cosmological constant instead of a dynamical scalar field. For more details see [38].

[5]u^μ is in fact the unique time-like eigenvector of the energy-momentum tensor.

Of particular interest later (see chapter 3) is the limit $\gamma \to 0$. Intuitively, we expect that this limit corresponds to slow-roll behavior $\dot{\phi}^2 \to 0$, which seems to imply that the second term proportional to $\frac{1}{\dot{\phi}}\frac{dV}{d\phi}$ in (2.81) diverges in this limit. However, a more careful analysis shows that $\dot{\gamma}$ is well-defined if $\gamma \to 0$. To show this, we first invert (2.78) to obtain

$$\dot{\phi}^2 = \gamma\rho\,, \tag{2.82}$$

$$2V = (2-\gamma)\rho\,. \tag{2.83}$$

This means that the limit $\gamma \to 0$ is equivalent to

$$\frac{\dot{\phi}^2}{\rho} \to 0 \text{ and } \frac{V}{\rho} \to 1\,. \tag{2.84}$$

Since V/ρ approaches a constant value as $\gamma \to 0$, l'Hospital's rule implies that the quotient of the derivatives with respect to ϕ (briefly denoted V' and ρ') cannot diverge as $\gamma \to 0$, i.e.

$$\frac{V'}{\rho'} \sim \frac{V'\dot{\phi}}{H\gamma\rho} \to \text{const} \tag{2.85}$$

as $\gamma \to 0$. Therefore

$$\frac{dV}{d\phi}\cdot\frac{1}{\dot{\phi}} \equiv V'\cdot\frac{1}{\dot{\phi}} \sim \frac{V'\dot{\phi}}{H\gamma\rho}\cdot\frac{H\gamma\rho}{\dot{\phi}^2} = \frac{V'\dot{\phi}}{H\gamma\rho}\cdot H \to \text{const}\cdot H\,, \tag{2.86}$$

showing that equation (2.81) is still valid in the limit $\gamma \to 0$ as long as the Hubble expansion rate H is finite.

2.4.1 Braneworld inflation

As in General Relativity, a scalar field can cause the brane to inflate. The Randall-Sundrum model by definition has empty bulk, so inflation must be caused by a scalar field confined to the brane. There do however exist generalized braneworld models in which a bulk scalar field can cause the brane to inflate (see e.g. [73, 98, 126]).

The condition for accelerated expansion $\ddot{a} > 0$ can be obtained from the Raychaudhuri equation (2.55) as usual. For a flat FLRW brane with $\rho^* = 0$ this condition becomes

$$\ddot{a} > 0 \;\Leftrightarrow\; \gamma < 1 - \frac{1}{3}\left[\frac{1+2\rho/\lambda}{1+\rho/\lambda}\right]. \tag{2.87}$$

At low energies $\rho/\lambda \to 0$, we recover the usual condition for inflation $\gamma < 2/3$, but when approaching higher energies, we find that the equation of state must be more negative for inflation to occur, and when $\rho/\lambda \to \infty$ we find that inflation only occurs for $\gamma < 1/3$.

The high energy term does on the other hand increases the Hubble rate as can be seen from the generalized Friedmann equation (2.66). The increased Hubble 'friction' damps the kinetic energy of the scalar field, allowing for condition (2.87) to be reached earlier. This argument can be made more formal by introducing slow roll parameters ϵ, η as in general relativity [144]:

$$\epsilon \equiv -\frac{\dot{H}}{H^2} \sim \left(\frac{V'}{V}\right)^2 \left[\frac{1+V/\lambda}{(1+V/2\lambda)^2}\right], \tag{2.88}$$

$$\eta \equiv -\frac{\ddot{\phi}}{H\dot{\phi}} \sim \left(\frac{V''}{V}\right) \left[\frac{1}{1+V/2\lambda}\right]. \tag{2.89}$$

We can see that at high energies, these parameters become

$$\epsilon \approx \epsilon_{\rm gr} \left[\frac{4\lambda}{V}\right], \ \eta \approx \eta_{\rm gr} \left[\frac{2\lambda}{V}\right], \tag{2.90}$$

where $\epsilon_{\rm gr}, \eta_{\rm gr}$ are the standard slow-roll parameters used in General Relativity. When $\rho/\lambda \to \infty$, the terms in square brackets become very small, which allows inflation even for potentials too steep to cause inflation in general relativity [53, 112, 137, 146, 150, 160, 175]. When V drops below λ inflation comes to a natural end. Interestingly, the quadratic matter term ρ^2/λ remains dominant at high energies even when including anisotropy [145]. This means that, unlike in General Relativity, even highly anisotropic models can inflate. For the particular case of inflationary Bianchi I models with quadratic potential and a conformally flat bulk with $\rho^* = 0$, it can be explicitly shown that the initial singularity is generically dominated by the non-linear contribution to the energy-momentum tensor, and not – as in general relativity – by the shear. Furthermore numerical integration indicates in this case that the anisotropy even *always* disappears in a fixed interval of time after a transient anisotropy dominated regime [145] – unlike in general relativity [156].

The experimental signatures of RS2 braneworld inflation can differ significantly from the general relativistic ones and may be used to constrain the different braneworld inflationary models. First of all, the number of e-folds between two given inflaton values for a given potential is larger in the braneworld model than in the general relativistic model due to the increased Hubble friction in the braneworld model. The main testing ground for any inflationary model however is the spectrum of perturbations caused by quantum fluctuations of the scalar or inflaton field. The increased Hubble friction in the RS inflationary models can lead to deviations from scale invariance of the spectrum differing the prediction for the general relativistic inflationary analogue. In particular the quadratic and quartic potentials are under increasing pressure from experiments like WMAP and 2dF (see [136]). Furthermore, the steep inflationary models mentioned above generically have a blue spectrum of primordial gravitational waves due to the fact that they are kinetically dominated at the end of inflation, which implies further constraints in order to agree with nucleosynthesis

[53, 112, 137, 146, 150, 160, 175]. The blue tilt of the gravitational wave spectrum can possibly be directly detected by gravitational wave experiments such as LIGO II and LISA [175].

Chapter 3

Dynamics of inflationary models with exponential potential

In this chapter, we will study the dynamics of certain types of inflationary braneworlds. We will assume that the FLRW braneworld is filled with a minimally coupled self-interacting scalar field with exponential potential $V(\phi) = \exp(\kappa b\phi)$. These potentials are particularly interesting, because they arise naturally in the 4-dimensional effective Kaluza-Klein theories obtained from compactification of higher dimensional supergravity or superstring theories [201]. They also have the attractive feature that they correspond to past attractors in the state space of FLRW models, meaning that different models with a wide set of initial conditions undergo inflation [39, 92, 168].

It is particularly interesting to study this class of inflationary models in the braneworld context, since the high energy corrections – arising from the bulk curvature – allow inflation on the brane even for potentials that are too steep for inflation in General Relativity [53, 112, 137, 146, 150, 160]. In the analysis presented in this chapter however, we are not interested in the conditions for the onset of inflation. Instead, we are interested in the local stability of different cosmological models within the state space of homogeneous cosmologies. Using dynamical systems methods first introduced by Goliath and Ellis [89], we study the complete dynamics of Friedmann - Lemaître - Robertson - Walker (FLRW) and Bianchi type I cosmological models, where the matter in each case is described by a dynamical scalar field on the brane with an exponential potential. This work is closely related to the work of Campus and Sopuerta, who have studied the complete dynamics of FLRW and Bianchi type I and V cosmological models with a barotropic equation of state in the braneworld context [41, 42]. Their analysis led to the discovery of new critical points m corresponding to the Binétruy - Deffayet - Langlois (BDL) models [1] [26], representing the

[1]In the literature, these models are often also denoted $\mathcal{F}_b$; in order to be consistent with the according literature we will denote the BDL model by m in this chapter and by $\mathcal{F}_b$ in the following chapter.

dynamics at very high energies, where effects due to the extra dimension become dominant. These solutions appear to be a generic feature of the state space of more general cosmological models. They also showed that for models satisfying all the ordinary energy conditions and causality requirements, the anisotropy is negligible near the initial singularity, a result first demonstrated by Maartens *et. al.* in the context of braneworld inflation [145].

In this chapter, we extend the analysis in [41, 42] to the case where the matter is described by a dynamical scalar field ϕ with exponential potential. We will show that our model has similar features as the perfect fluid model. In particular, we also obtain the set of high-energy past attractors m, but here they represent high energy *steep inflationary* models in which inflation ends naturally as the energy drops below the brane tension and the slow roll conditions for inflation no longer hold (see section 2.4.1). We will also relate the results of this analysis to the work of Burd and Barrow [39] and Halliwell [92], who have studied the dynamics of FLRW and Bianchi I models with an exponential potential in General Relativity, and discuss in detail the non-general-relativistic effects caused by the terms due to the bulk curvature.

One particularly interesting question is the nature of the initial singularity in the braneworld context. It has been argued that the initial singularity is generically isotropic in the RS2 braneworld scenario [49, 50, 51] – unlike in General Relativity, where only a set of very specific initial conditions provide a scenario smooth enough to give rise to inflation [156]. We will here provide support for this conjecture by showing that the isotropic BDL model is a generic past attractor in the – albeit very special – classes of FLRW and anisotropic Bianchi I models.

This chapter is organized as follows. We will first introduce the basic concepts of dynamical systems in 3.1, followed by a summary of the covariant equations characterizing the FLRW and Bianchi I models in 3.2. In sections 3.3 and 3.4 we analyze the dynamical system of the class of FLRW and Bianchi I models respectively: we find the equilibrium points and their eigenvalues, from which we deduce the dynamical character of the equilibrium points, and discuss the corresponding state spaces. The main results are summarized in section 3.5.

3.1 Dynamical Systems

We will here give a brief and very informal introduction to dynamical systems. By no means do we attempt to present a complete review of the field, we merely try to overview some of the main concepts. We do not give many technical definitions and theorems, instead we try to motivate the results intuitively. For a complete formal introduction see [5, 207], and see [89] for applications to cosmology.

To most physical systems can be associated an abstract state space X, such that each possible state of the system at a time t corresponds to an element $\mathbf{x}(t)$ of the state space.

We will restrict ourselves to the case where the state space is a subset of $\mathbf{R}^n$, so that each element of the state space is of the form $\mathbf{x} = (x_1, ..., x_n)$. Moreover, we will assume that the evolution of the system is governed by an **autonomous differential equation**

$$\dot{\mathbf{x}} \equiv \frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}). \tag{3.1}$$

The function $\mathbf{f}: \mathbf{R}^n \to \mathbf{R}^n$ can be interpreted as a vector field on $\mathbf{R}^n$,

$$\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), ..., f_n(\mathbf{x})). \tag{3.2}$$

Equation (3.1) then reduces to a system of n ordinary differential equations.

It can be difficult or impossible to solve a given system (3.1) explicitly, and even if the system can be solved for each initial condition $\mathbf{x}(t_0)$, we are often not interested in the actual solution. Instead, we want to know about the qualitative dynamics of the solution: what happens if we slightly change the initial conditions, will the solution evolve similarly or will the system evolve to a drastically different state? This question can be answered – without knowing the exact solutions – when using dynamical systems methods.

Particulary interesting points of the full state space are points $\mathbf{x_0}$ for which $\mathbf{f}(\mathbf{x_0}) = 0$, because at these points the system does not change in time, since all the derivatives at $\mathbf{x} = \mathbf{x_0}$ are zero. This means that if the system is in the state $\mathbf{x} = \mathbf{x_0}$ at some time t_0, then it will stay there forever. For this reason such a point is called **equilibrium point** or fixed point. An equilibrium point can be classified as a repeller (or source), an attractor (or sink) and a saddle point as will be explained below.

It is also useful to find the **invariant sets** of a given system of differential equations. These are subsets S of the entire state space, with the property that each element in S can only evolve to another state $\mathbf{x} \in S$. In this sense, invariants subsets of the state space are similar to subspaces of vector spaces.

Note that if one imposes k additional constraints

$$C_i(\mathbf{x}) = 0\,, \;\; i = 1, ..., k \tag{3.3}$$

on the system (3.1), each of the constraints $C_i = 0$ will give rise to the new constraint $\dot{C}_i = 0$, and one must make sure that each of the new constraints is satisfied and propagated itself. This means that each constraint can give rise to an infinite chain of new constraints, unless each new constraint is a linear combination of the old constraints:

$$\dot{C}_i = \alpha_i{}^j C_j\,, \tag{3.4}$$

where $\alpha_i{}^j$ are functions of $\mathbf{x}$.

However, in our study there is only one constraint, namely the Friedmann constraint, which is an integral of the Raychaudhuri equation and hence naturally propagated. We will comment on this in the analysis below.

If the state space is not compact, it is often useful to compactify the state space by introducing new variables, in order to study the behavior of the system at the asymptotic boundaries.

3.1.1 Linear systems

let us first study a system of n linear differential equations with constant coefficients in order to introduce the main concepts. Such a system is given by

$$\begin{aligned} \dot{x}_1 &= a_{11}x_1 + \cdots + a_{1n}x_n \\ &\vdots \\ \dot{x}_n &= a_{n1}x_1 + \cdots + a_{nn}x_n \,, \end{aligned} \tag{3.5}$$

and can be conveniently written as a matrix equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \,, \tag{3.6}$$

where $\mathbf{x} = (x_1, \ldots, x_n)$, $(\mathbf{A})_{ij} = a_{ij} \in \mathbf{R}$ for $i, j = 1, ..., n$. The solutions of (3.6) are given by

$$\mathbf{x}(t) = \sum_{i=1}^{n} b_i e^{\lambda_i (t - t_0)} \mathbf{v}_i \,, \tag{3.7}$$

where λ_i are the eigenvalues of the matrix $\mathbf{A}$ with corresponding eigenvectors $\mathbf{v}_i$, and b_i are constants. Note that even if $\mathbf{A}$ is real, the eigenvalues need not be real. An equilibrium point is called **hyperbolic** if all eigenvalues have non-zero real parts. If the eigenvectors $\mathbf{v}_i$ are linearly independent, i.e. if the eigenvectors span $\mathbf{R}^n$, then any point in the state space can be expressed in the form of (3.7).

The equilibrium points of the system can be classified using their eigenvalues; in fact it is the real parts of the eigenvalues that determine the stability of an equilibrium point. This can be seen by evaluating (3.7) at some time $t_0 + \epsilon$, where ϵ is a taken to be positive. If we decompose each eigenvalue λ_i into real part α_i and complex part β_i, then the solution (3.7) can be decomposed as

$$\mathbf{x}(t_0 + \epsilon) = \sum_{i=1}^{n} \left(e^{\alpha_i \epsilon}\right) a_i e^{i\beta_i \epsilon} \mathbf{v}_i \,. \tag{3.8}$$

One can see that a positive real part α_i means that at $\mathbf{x}(t_0 + \epsilon)$ grows as time evolves ($t_0 \rightarrow t_0 + \epsilon$), whereas $\mathbf{x}(t_0 + \epsilon)$ decreases in time if α_i is negative. The complex part β_i on the other hand does not influence the magnitude of the perturbation, but causes the orbit to oscillate. A more formal discussion can be found in [207]. For our purpose, it is enough to classify hyperbolic equilibrium points in the following way:

- if all eigenvalues have positive real parts, then the equilibrium point is called a **source** or repeller, because arbitrarily small deviations away from that point will move the system away from the equilibrium state,
- if the real parts of all eigenvalues are negative, the point is called a **sink** or attractor, since small perturbations away from the point will decay in time, and the system will return to the equilibrium state,
- if some of the real parts of the eigenvalues are positive and some are negative, then the point is called a **saddle** point.

Finally, if one (or $k \leq n$) of the eigenvalues has vanishing real part, i.e. if the equilibrium point is non-hyperbolic, then we are dealing with a **line (or a k-dimensional set) of equilibrium points** of the linear system, which can be classified just like equilibrium points:

- if all non-zero eigenvalues have positive real parts, then the set is called a source,
- if the real parts of all non-zero eigenvalues are negative the set is called a sink,
- if some of the real parts of the non-zero eigenvalues are positive and some are negative, then the set is called a saddle.

3.1.2 Local analysis of non-linear systems

Let

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \;\; \mathbf{f} = (f_1, ..., f_n) \tag{3.9}$$

be a system of n non-linear differential equations.
By definition, each equilibrium point $\mathbf{x_0}$ of this system satisfies the condition $\mathbf{f}(\mathbf{x_0}) = \mathbf{0}$. In order to study the local stability properties of an equilibrium point, we define in the variable

$$\boldsymbol{\xi} \equiv \mathbf{x} - \mathbf{x_0} \tag{3.10}$$

measuring the displacement from equilibrium. In the vicinity of the equilibrium point, we can now Taylor-expand each of the non-linear functions f_i, and find that

$$\mathbf{f}(\mathbf{x}) = \mathbf{J} \cdot (\mathbf{x} - \mathbf{x_0}) + \mathbf{h}(\mathbf{x} - \mathbf{x_0}), \tag{3.11}$$

where

$$(\mathbf{J})_{ij} \equiv \frac{\partial f_i}{\partial x^j} |_{\mathbf{x}=\mathbf{x_0}} \tag{3.12}$$

is the Jacobian of $\mathbf{f}$ and the set of functions $\mathbf{h}(\mathbf{x}-\mathbf{x_0})$ is at least of second order in $\mathbf{x}-\mathbf{x_0}$. This means that in the vicinity of an equilibrium point $\mathbf{x_0}$, the non-linear system (3.9) can be approximated in linear order by the system

$$\dot{\boldsymbol{\xi}} = \mathrm{J}\boldsymbol{\xi} \tag{3.13}$$

which is linear in the displacement variable $\boldsymbol{\xi}$.

We expect that at least some aspects of the local dynamics of the full non-linear system in the vicinity of the equilibrium points can be studied by linearizing the system around each equilibrium point and analyzing the linear system – a simple system of linear differential equations with constant coefficients given by (3.13) and (3.12). The Hartman-Grobman theorem (see e.g. [207]) in fact states that if an equilibrium point is *hyperbolic*, the orbits of the linear system can locally be continuously transformed into the orbits of the full non-linear system.

We will however often deal with sets of equilibrium points, which have at least one vanishing eigenvalue, hence are *not hyperbolic.* If the set of equilibrium points has only one eigenvalue equal to zero and the remaining eigenvalues have non-zero real parts, then the set is called **normally hyperbolic** and linearization is still expected to yield the correct qualitative results [207].

However, whenever there are doubts about the adequacy of the linearization, the full non-linear system should be studied near the equilibrium point in question. The same has to be done if the Jacobian diverges in the vicinity of an equilibrium point. We will study one particular example in great detail in appendix A, and refer to the methods introduced there for any of the non-hyperbolic equilibrium points that occur in the analysis below.

3.2 The governing equations

In the following we study in detail the dynamics of (a) FLRW models and (b) the anisotropic Bianchi I models in the braneworld context, each filled with a minimally coupled scalar field ϕ with an exponential potential of the form

$$V(\phi) = \exp(\kappa b\phi)\,.$$

Here κ is the 4-dimensional gravitational coupling constant, and the parameter $b \leq 0$ determines the steepness of the potential. As explained in section 2.4, the scalar field can be treated like a perfect fluid $p = (\gamma - 1)\rho$, but with a dynamical equation of state parameter γ governed by the evolution equation

$$\dot{\gamma} = 3H\gamma(\gamma-2) + sgn(\dot{\phi})\sqrt{\gamma}(\gamma-2)b\kappa\sqrt{\rho}\,. \tag{3.14}$$

We will assume that the effective 4-dimensional cosmological constant Λ is zero; the inflationary models are recovered if $p = -\rho \Leftrightarrow \gamma = 0$ which occurs when the kinetic energy of the scalar field is negligible: $\dot{\phi}^2 \ll V(\phi)$. If $b = 0$, then the potential is constant. This case corresponds to a massless scalar field and an effective cosmological constant as can be seen from the covariant Klein-Gordon equation (2.79).

When studying the FLRW models in section 3.3, our setup will be totally general, i.e. we include all the high energy correction terms. In the analysis of the Bianchi I models in section 3.4 however, we will have to make assumptions on the form of the non-local stress $\pi^*_{\mu\nu}$, since the projected field equations on the brane are not closed as discussed at the end of section 2.3.

3.2.1 FLRW models

The class of homogeneous isotropic FLRW models is covariantly characterized by the following statements:

$$E_{\mu\nu} = H_{\mu\nu} = 0 \tag{3.15}$$

$$\Leftrightarrow \quad \sigma_{\mu\nu} = \omega_{\mu\nu} = A_\mu = 0 \tag{3.16}$$

$$\Rightarrow \quad \mathrm{D}_\mu\rho = \mathrm{D}_\mu p = \mathrm{D}_\mu H = 0\,. \tag{3.17}$$

The statements above hold everywhere and mean that these models are conformally flat (3.15), locally isotropic (3.16) and spatially homogeneous (3.17) respectively.

In comoving coordinates, the metric of the FLRW models can be written as

$$ds^2 = -dt^2 + a^2(t)\left(dr^2 + f^2(r)d\theta^2 + sin^2\theta d\phi^2\right)\,,\ u^\mu = \delta^\mu{}_0\,. \tag{3.18}$$

We can identify $a(t)$ as the scale factor for distances between any pair of fundamental observers. The 3-spaces $\{t = const\}$ have constant curvature k/a^2, and we can normalize k to ± 1 if it is not zero. The function $f(r)$ can be found to be

$$f(r) = \begin{cases} \sin r & \text{for } k = +1 \\ r & \text{for } k = 0 \\ \sinh r & \text{for } k = -1\,. \end{cases} \tag{3.19}$$

In these coordinates, the time derivative along the fundamental world lines is simply

$$\dot{} = u^\mu \nabla_\mu = d/dt\,.$$

The only non-trivial equations out of the full set of conservation, propagation and constraint equations (2.54)-(2.70) are the Raychaudhuri equation (2.54),

$$\dot{H} = -H^2 - \frac{3\gamma - 2}{6}\kappa^2\rho\left[1 + \frac{3\gamma - 1}{3\gamma - 2}\frac{\rho}{\lambda}\right] - \frac{\kappa^2\rho^*}{3}\,, \tag{3.20}$$

the Friedman constraint (2.66) which simplifies to give

$$H^2 = \frac{\kappa^2\rho}{3} + \frac{\kappa^2\rho^2}{6\lambda} - \frac{1}{6}{}^3R + \frac{\kappa^2\rho^*}{3}\,, \tag{3.21}$$

and energy (2.67) and non-local energy conservation (2.69), which for a perfect fluid become

$$\dot\rho + 3\gamma H\rho = 0\,, \tag{3.22}$$

$$\dot\rho^* + 4H\rho^* = 0\,. \tag{3.23}$$

We have defined ${}^3R = 6k/a^2$.

Since the FLRW models only have constant scalar curvature ${}^3R$, the projected trace-free Ricci tensor ${}^3R_{<\mu\nu>}$ is zero, and the remaining equations are identically satisfied if

$$q^*_\mu = \pi^*_{\mu\nu} = 0\,, \tag{3.24}$$

which, via (2.70), implies

$$\mathrm{D}_\mu\rho^* = 0 \Leftrightarrow \rho^* = \rho^*(t)\,. \tag{3.25}$$

3.2.2 Bianchi I models

The Bianchi type I models are the simplest generalization of the FLRW models, containing the flat FLRW space-time as a subset. These models are anisotropic, but the surfaces orthogonal to the fluid flow u^μ are flat and homogeneous. This means that these non-tilted models are **spatially flat** and **spatially homogeneous**:

$${}^3R_{\mu\nu} = \mathrm{D}_\mu f = 0 \tag{3.26}$$

for all physically defined scalars f. The fluid flow is irrotational and geodesic, and the magnetic part of the Weyl tensor vanishes:

$$\omega_\mu = A_\mu = H_{\mu\nu} = 0\,. \tag{3.27}$$

Since the models are non-tilted, the total fluid flow $q^{\rm tot}_\mu = 0$ relative to u^μ must be zero, which implies upon using (2.51), that

$$q^*_\mu = 0\,. \tag{3.28}$$

Equations (3.26)-(3.27) covariantly characterize the Bianchi I brane. However, it is useful to present a metric in order to understand the geometry of these models. The line element of the Bianchi I models can be written in terms of comoving coordinates as

$$ds^2 = -dt^2 + \sum_{i=1}^{3}\left(a_i(t)dx^i\right)^2\,,\ u^\mu = \delta^\mu{}_0\,. \tag{3.29}$$

With the metric in this form, we can easily see that the Bianchi I models are spatially flat, since at any time $t = t_0$ all the coefficients $a_i(t_0)$, $i = 1, 2, 3$ are constants. The anisotropy is characterized by the different scale factors a_i in orthogonal directions . If

$$a_i(t) = t^{p_i} \text{ and } \sum_{i=1}^{3} p_i = \sum_{i=1}^{3} {p_i}^2 = 1\,, \tag{3.30}$$

we obtain the anisotropic **Kasner** space-time. The case $a_1(t) = a_2(t) = a_3(t)$ corresponds to the isotropic **FLRW limit**.

Since we have chosen comoving coordinates, the time derivative along the fluid flow lines is simply

$$\dot{} = u^\mu \nabla_\mu = d/dt\,.$$

We can define the **average scale factor**

$$a(t) = (a_1 a_2 a_3)^{1/3} \tag{3.31}$$

and the **average Hubble rate**

$$H(t) = \frac{\dot{a}}{a}\,. \tag{3.32}$$

The conservation equations (2.67)-(2.70) for the Bianchi I models then become

$$\begin{aligned} &\dot{\rho} + 3H(\rho + p) = 0\,, \\ &\dot{\rho}^* + 4H\rho^* + \sigma^{\mu\nu}\pi^*_{\mu\nu} = 0\,, \\ &\mathrm{D}^\nu \pi^*_{\mu\nu} = 0\,. \end{aligned} \tag{3.33}$$

There is no propagation equation for the non-local anisotropic stress $\pi^*_{\mu\nu}$, since the projected equations on the brane do not close as explained in section 2.3.1. This is not a problem when studying the FLRW models, since the Gauss-Codazzi constraint (2.65) implies $\pi^*_{\mu\nu} = 0$ for these models. For the Bianchi I models however, none of the propagation and constraint equations (2.55)-(2.66) constrain $\pi^*_{\mu\nu}$, and we will have to constrain $\pi^*_{\mu\nu}$ by hand.

In the following, we are interested in the evolution of the shear, not the Weyl tensor characterized by the electric part $E_{\mu\nu}$. For this reason, we will try to find a closed subsystem of (2.55)-(2.66) that does not involve $E_{\mu\nu}$. This can be done by using equations (2.55), (2.65) and (2.66), which for the Bianchi I models take the form

$$\dot{H} = -H^2 - \frac{3\gamma - 2}{6}\kappa^2\rho - (3\gamma - 1)\frac{\kappa^2\rho^2}{6\lambda} - \frac{2}{3}\sigma^2 - \frac{\kappa^2\rho^*}{3}\,, \tag{3.34}$$

$$\dot{\sigma}_{\langle\mu\nu\rangle} = -3H\sigma_{\mu\nu} + \kappa^2\pi^*_{\mu\nu}\,, \tag{3.35}$$

$$C \equiv H^2 - \frac{1}{3}\sigma^2 - \frac{\kappa^2\rho}{3} - \frac{\kappa^2\rho^2}{6\lambda} - \frac{\kappa^2\rho^*}{3} = 0\,. \tag{3.36}$$

In General Relativity, one can integrate the GR analogue of (3.35) after contraction with $\sigma_{\mu\nu}$. The same can be done here if we additionally assume that

$$\sigma^{\mu\nu}\pi^*_{\mu\nu} = 0\,. \tag{3.37}$$

This assumption may be motivated by the fact that it follows naturally from (3.33) if the dark radiation ρ^* is zero [145], but for general ρ^* it is an assumption that has been put in by hand.

Using (3.37), the integral of the shear propagation equation (3.35) yields an evolution equation for the magnitude σ^2 of the shear tensor:

$$(\sigma^2)^{\cdot} + 6H\sigma^2 = 0\,. \tag{3.38}$$

3.3 Analysis of FLRW models with exponential potentials

We now want to construct a dynamical system that describes the class of FLRW braneworld models. As in General Relativity, we introduce appropriately expansion normalized variables via the Friedmann equation (2.66), ensuring that they define a compact state space.

In General Relativity, one has to distinguish between the case ${}^3R \leq 0$ and ${}^3R \geq 0$ when introducing appropriate expansion normalized coordinates for the FLRW models (see for example [5, 89, 207]). We here include an additional quantity, the non-local energy density ρ^* originating from the bulk Weyl tensor, which can be positive or negative. Therefore, we have to consider four different subcases when studying the FLRW models, and each one will have to be normalized by different quantities in order to obtain a compact state space. The different cases are given by

$$\begin{aligned}
&(\mathrm{A})\ \rho^* \geq 0 \text{ and } {}^3R \leq 0\,,\\
&(\mathrm{B})\ \rho^* \geq 0 \text{ and } {}^3R \geq 0\,,\\
&(\mathrm{C})\ \rho^* \leq 0 \text{ and } {}^3R \leq 0\,,\\
&(\mathrm{D})\ \rho^* \leq 0 \text{ and } {}^3R \geq 0.
\end{aligned}$$

The total state space is composed of these 4 sectors, which are disconnected, since trajectories cannot leave one sector and enter another.

In the following four sections 3.3.1-3.3.4, we will analyze the dynamics in each sector, and then combine the results in section 3.3.5 to discuss the dynamics of the full state space.

3.3.1 Sector A: $\rho^* \geq 0$ and ${}^3R \leq 0$

As explained by several authors (see e.g. [89, 42]), a compact state space is obtained by using the dimensionless general-relativistic variables

$$\Omega_\rho \equiv \frac{\kappa^2\rho}{3H^2}\,, \quad \Omega_k \equiv -\frac{{}^3R}{6H^2}\,, \tag{3.39}$$

together with the following new variables:

$$\Omega_\lambda \equiv \frac{\kappa^2\rho^2}{6\lambda H^2}\,, \quad \Omega_{\rho^*} \equiv \frac{\kappa^2\rho^*}{3H^2}\,. \tag{3.40}$$

The variables represent the fractional contributions of the ordinary energy density, the curvature, the brane tension, and the dark radiation energy to the total energy density, normalized with respect to the Hubble parameter $H(t)$.

In order to decouple the evolution equation for H (3.20) from the evolution equations for the density parameters Ω_i, we introduce the dimensionless time derivative

$$' \equiv \epsilon H^{-1} d/dt\,, \tag{3.41}$$

where $\epsilon = sgn(H)$ is the sign of H.

Using the variables (3.39) and (3.40), the Friedmann constraint (3.21) becomes

$$C \equiv \Omega_\rho + \Omega_k + \Omega_\lambda + \Omega_{\rho^*} - 1 = 0\,. \tag{3.42}$$

Note that this constraint is propagated, since the time derivative of the Friedmann constraint is proportional to the Raychaudhuri equation, which we use to derive the evolution equations for the dynamical quantities (3.39)-(3.40). Alternatively one can explicitly calculate $\dot{C}$ and show, using (3.20)-(3.23), that $\dot{C} = 0$ follows.

Note that each of the variables Ω_ρ, Ω_k, Ω_λ, Ω_{ρ^*} takes values in the interval $[0, 1]$, since all of the terms in the sum (3.42) are non-negative.

Using the Friedmann constraint, we may eliminate any of the five variables defined in (3.39), (3.40) in order to keep the dimensionality of the state space as low as possible; we here choose to eliminate Ω_ρ.

The dynamics of the scalar field are fully determined by the evolution of the equation of state parameter γ, which is dimensionless and bounded by causality requirements: $0 \leq \gamma \leq 2$.

This means that the minimal set of variables describing the full dynamics, given by $(\gamma,\ \Omega_k,\ \Omega_\lambda,\ \Omega_{\rho^*})$, defines a compact state space $[0, 2] \times [0, 1]^4$.

Inserting the Friedmann constraint (3.42) into (2.81), the evolution equation for γ becomes

$$\gamma' = \epsilon\sqrt{3\gamma}(\gamma - 2)[\sqrt{3\gamma} + \epsilon\ sgn(\dot{\phi}) b\sqrt{1 - \Omega_k - \Omega_\lambda - \Omega_{\rho^*}}\,]\,. \tag{3.43}$$

We can see from (3.43) that we only need to consider the case $\dot{\phi} \geq 0$, since the case $\dot{\phi} \leq 0$ can be recovered from the former by time reversal: simultaneously changing the sign of $\dot{\phi}$ and H results in an overall change of sign of γ'. From (3.44) below we can see that this transformation also changes the sign of Ω'_k, Ω'_λ, Ω'_{ρ^*}, which means that $\dot{\phi} \to -\dot{\phi}$ corresponds to time reversal $\tau \to -\tau$.

Using the definitions (3.39)-(3.40) and (3.20)-(3.23), it follows that the dynamics of the open and flat models with positive ρ^* are described by the following equations:

$$\begin{aligned} \gamma' &= \epsilon\sqrt{3\gamma}(\gamma-2)[\sqrt{3\gamma}+\epsilon\, b\sqrt{1-\Omega_k-\Omega_\lambda-\Omega_{\rho^*}}\,]\,, \\ \Omega_k' &= \epsilon\,[(3\gamma-2)(1-\Omega_k)+3\gamma\,\Omega_\lambda+(4-3\gamma)\,\Omega_{\rho^*}]\,\Omega_k\,, \\ \Omega_\lambda' &= \epsilon\,[3\gamma(\Omega_\lambda-\Omega_k-1-\Omega_{\rho^*})+2\,\Omega_k+4\,\Omega_{\rho^*}]\,\Omega_\lambda\,, \\ \Omega_{\rho^*}' &= \epsilon\,[(3\gamma-4)(1-\Omega_{\rho^*})+(2-3\gamma)\Omega_k+3\gamma\,\Omega_\lambda\,]\,\Omega_{\rho^*}\,. \end{aligned} \tag{3.44}$$

The evolution equation for H decouples and needs not be considered in our analysis. We can immediately identify the following invariant subsets of (3.44): the cosmological constant-like subset $\gamma = 0$, the stiff-matter-subset $\gamma = 2$, the flat subset $\Omega_k = 0$, the brane-tension-less subset $\Omega_\lambda = 0$ and the dark-radiation-free subset $\Omega_{\rho^*} = 0$. The vacuum subset $\Omega_\rho = 1 - \Omega_k + \Omega_\lambda + \Omega_{\rho^*} = 0$ is another invariant subset. This can not directly be seen from (3.44), since we suppress the propagation equation for Ω_ρ, but it is clear when realizing that the Friedmann constraint (3.42) is propagated.

Setting the left hand side of (3.44) equal to zero, we obtain the equilibrium points of the system, and after linearizing the system, we obtain the eigenvalues of the hyperbolic equilibrium points.

There are two cases that require special attention: firstly, the **Jacobian can diverge** and hence be ill-defined for certain parameter values, e.g. if $\gamma = 0$ or $\Omega_k + \Omega_\lambda + \Omega_{\rho^*} = 1$. This simply means that the dynamical system can not be linearized around these points in state space, and instead we have to use the full non-linear equations and study small perturbations away from these equilibrium points. We describe this method in appendix A for a specific equilibrium point; the other points are treated in the same way.

The second situation that requires additional attention is the case of **vanishing eigenvalues**. When studying systems of linear differential equations, zero valued eigenvalues indicate lines of equilibrium in the direction of the corresponding eigenvector. When studying non-linear equations on the other hand, a vanishing eigenvalue only means that the first order perturbation away from the equilibrium point in question is zero. Only if *all* the higher order perturbations vanish do we have a line of equilibrium points, otherwise the stability of the equilibrium point is determined by the sign of the lowest order non-zero perturbation. We again refer to appendix A for details.

All the equilibrium points of the dynamical system (3.44), their coordinates in state space and their eigenvalues - if finite - are given in table 3.1 below. We follow the notation in [42], where the capital letter stands for the solution, for example F for the flat FLRW model. The subscript ϵ differentiates between expanding ($\epsilon = +1$) and collapsing ($\epsilon = -1$) solutions, and the superscript represents the equation of state parameter γ of the specific solution.

The equilibrium points corresponding to general relativistic models are

Table 3.1: This table gives the coordinates and eigenvalues of the isotropic critical points with $\rho^* \geq 0$ and ${}^3R \leq 0$. We have defined the non-negative quantities $\psi = -b^{-1}\sqrt{8-3b^2}$ and $\zeta = -b^{-1}\sqrt{64-15b^2}$. The eigenvalues marked with a dagger only hold if $\Omega_\lambda \neq 1$. The infinite eigenvalue in row 5 actually reads $\lim_{\gamma\to 0}\left(-6\epsilon - \sqrt{3}b\sqrt{\frac{1-\Omega_\lambda}{\gamma}}\right) = \infty$.

Model	Coordinates	Eigenvalues
F_ϵ^2	$(2,0,0,0)$	$(6\epsilon+\sqrt{6}b, 4\epsilon, -6\epsilon, 2\epsilon)$
M_ϵ^0	$(0,1,0,0)$	-
M_ϵ^2	$(2,1,0,0)$	$2\epsilon(3,-2,-5,-1)$
$\mathrm{m}_\epsilon^0(\Omega_\lambda)$ *for* $b=0$	$(0,0,\Omega_\lambda,0)$	$-2\epsilon(3,1,0,2)^\dagger$
$\mathrm{m}_\epsilon^0(\Omega_\lambda)$ *for* $b\neq 0$	$(0,0,\Omega_\lambda,0)$	$(\infty,-2\epsilon,0,-4\epsilon)^\dagger$
m_ϵ^2	$(2,0,1,0)$	$2\epsilon(3,5,3,4)$
R_ϵ^0	$(0,0,0,1)$	-
R_ϵ^2	$(2,0,0,1)$	$2\epsilon(3,1,-4,-1)$
$\mathrm{F}_+^{b^2/3}$ for $b^2\in[0,6]$	$(\frac{b^2}{3},0,0,0)$	$(\frac{b^2}{2}-3, b^2-2, -b^2, b^2-4)$
$\mathrm{X}_+^{2/3}(b)$ for $b^2\in[2,\infty[$	$(\frac{2}{3},1-\frac{2}{b^2},0,0)$	$(-1-\psi,-1+\psi,-2,-2)$
$\mathrm{A}_+^{4/3}(b)$ for $b^2\in[4,\infty[$	$(\frac{4}{3},0,0,1-\frac{4}{b^2})$	$(-\frac{1}{2}(1+\zeta),2,-4,-\frac{1}{2}(1-\zeta))$

- the flat FLRW universe F_ϵ^2 with stiff matter ($\gamma=2$, $k=\lambda^{-1}=\rho^*=0$) and $a(t)=t^{1/3}$,
- the points M_ϵ^0 and M_ϵ^2 representing the same physical model: the Milne universe M_ϵ, a vacuum model ($\rho=\lambda^{-1}=\rho^*=0$) with $a(t)=t$,
- the flat FLRW universe $F_+^{b^2/3}$ with $\gamma=b^2/3$ ($k=\lambda^{-1}=\rho^*=0$) and $a(t)=t^{2/b^2}$,
- a set of universe models $X_+^{2/3}(b)$ with $\gamma=2/3$ and curvature $\Omega_k=1-2/b^2$ depending on the value of b ($\lambda^{-1}=\rho^*=0$).

The models with vanishing dark radiation but non-zero brane tension are

- the flat non-general-relativistic model m_ϵ^2 with $\gamma=2$, ($k=\rho^*=0$) and scale factor $a(t)=t^{1/6}$, which has been discussed in detail in [41],
- a line of non-general-relativistic critical points with $\gamma=0$ ($k=\rho^*=0$) which extends in Ω_λ-direction. This set is denoted by $m_\epsilon^0(\Omega_\lambda)$; it contains the flat general relativistic FLRW model F_ϵ^0 with $\gamma=k=\lambda^{-1}=\rho^*=0$, which is why this point does not appear separately in table 3.1.

In addition to these equilibrium points with vanishing ρ^*, we find the following points with $\rho^* > 0$:

- the points R_ϵ^0 and R_ϵ^2 represent the same model: a vacuum model ($\rho = k = \lambda^{-1} = 0$) with the same metric $a(t) = t^{1/2}$ as the flat radiation FLRW model,

- the point $A_+^{4/3}(b)$ describing a flat expanding model with $\gamma = 4/3$ and vanishing brane-tension ($\lambda^{-1} = k = 0$), but which has in general non-vanishing energy density and non-local energy density contributions ($\Omega_\rho,\ \Omega_{\rho^*} > 0$). For $b = 0$, this model has maximal positive ρ^* ($\Omega_{\rho^*} = 1,\ \Omega_\rho = 0$), whereas for $b^2 = 4$ the model coincides with the flat FLRW model with $\gamma = 4/3$ ($\Omega_\rho = 1,\ \Omega_{\rho^*} = 0$). The scale factor of this model is proportional to $t^{1/2}$ for all values of b.

Note that in this sector of state space, the model $F_+^{b^2/3}$ only occurs for $0 \leq b^2 \leq 6$, $X_+^{2/3}(b)$ occurs for $b^2 \geq 2$, and $A_+^{4/3}(b)$ occurs only if $b^2 \geq 4$. All three points move in state space as the parameter value b is varied, but irrespective of the value of b, they only occur in the expanding sector $\epsilon = +1$.

The model m_+^2

We observe that the scale factor of the BDL model m_+^2 cannot be directly obtained from the energy conservation equation (3.23), since for this model both

$$\frac{\kappa^2\rho}{H^2} \to 0 \quad \text{and} \quad \frac{\kappa^2\rho^2}{\lambda H^2} \to 1\,. \tag{3.45}$$

Instead, we first observe that if $\gamma = 2$ and $k = \rho^* = 0$, the scale factor for general Ω_ρ, Ω_λ is given by

$$a(t) = (t - t_{BB})^{1/6}(t + t_{BB})^{1/6}\,, \tag{3.46}$$

where $t_{BB} = 1/\sqrt{6\kappa^2\lambda}$ is the Big Bang time. When approaching the high energy model m_+^2 with $\rho^2/\lambda \gg \rho$, we approach the initial singularity $t \to t_{BB}$, and therefore obtain the scale factor of the BDL model:

$$a(t) = (t - t_{BB})^{1/6}\,. \tag{3.47}$$

The models $m_\epsilon^0(\Omega_\lambda)$

The models $m_\epsilon^0(\Omega_\lambda)$ with $\gamma = 0$ are de Sitter like: their scale factor can be obtained from the energy conservation equation (3.23) and is given by

$$a(t) = \begin{cases} \frac{3}{\tilde{\Lambda}}\cosh\left[\epsilon\frac{\tilde{\Lambda}}{3}(t - t_0)\right] & \text{for } k = +1 \\ e^{\epsilon\frac{\tilde{\Lambda}}{3}(t-t_0)} & \text{for } k = 0 \\ \frac{3}{\tilde{\Lambda}}\sinh\left[\epsilon\frac{\tilde{\Lambda}}{3}(t - t_0)\right] & \text{for } k = -1\,, \end{cases} \tag{3.48}$$

where

$$\tilde{\Lambda} \equiv \frac{\kappa^2 \rho}{3} + \frac{\kappa^2 \rho^2}{6\lambda} \tag{3.49}$$

acts as an effective cosmological constant.

These models have vanishing eigenvalues; they form a set of sinks (sources) if $b = 0$ and $\epsilon = +1$ ($\epsilon = -1$) and a set of saddle points if $b \neq 0$. This statement can be verified by inspecting the full non-linear system (3.44).

The infinite eigenvalue for $b \neq 0$ is obtained by a limiting process:

$$\lim_{\gamma \to 0} \left(-6\epsilon - \sqrt{3} b \sqrt{\frac{1 - \Omega_\lambda}{\gamma}} \right) = \infty \, .$$

Although an infinite eigenvalue is mathematically not well-defined, it does show that perturbations in the vicinity of the equilibrium point in γ-direction do grow, which is all we want to know. We could make this statement rigorous by using perturbative methods as described in appendix A.

The vacuum models with $\gamma = 0$

Another important point is that the Jacobian of the dynamical system in the vicinity of the vacuum models with $\gamma = 0$ – M^0_ϵ, $m^0_\epsilon(\Omega_\lambda = 1)$ and R^0_ϵ – diverges as mentioned above. In this case we have to use perturbative methods in order to analyze the nature of these models. We explain the dynamics in the vicinity of M^0_ϵ in great detail in appendix A; the other points can be analyzed in exactly the same way.

3.3.2 Sector B: $\rho^* \geq 0$ and ${}^3R \geq 0$

Generalizing the procedure in General Relativity, we will use the variables γ, Q, $\tilde{\Omega}_\rho$, $\tilde{\Omega}_\lambda$, $\tilde{\Omega}_{\rho^*}$, where we define

$$Q \equiv \frac{H}{D}, \; D^2 \equiv H^2 + \tfrac{1}{6} {}^3R \, , \tag{3.50}$$

and the variables with a tilde are the analogues of those in (3.39) and (3.40) but normalized with respect to D instead of H.

Introducing the time derivative

$$' \equiv D^{-1} d/dt \, , \tag{3.51}$$

the evolution equation for D decouples from the evolution equations for $\tilde{\Omega}_i$. The Friedmann equation (3.21) becomes

$$\tilde{\Omega}_\rho + \tilde{\Omega}_\lambda + \tilde{\Omega}_{\rho^*} = 1 \, , \tag{3.52}$$

from which it can be seen that the quantities $\tilde{\Omega}_i$ take values in $[0,1]$, whereas Q takes values in $[-1,1]$. Hence these coordinates define a compact state space. As explained in section 3.3.1, the Friedmann constraint is propagated since we ensure that its first integral, the Raychaudhuri equation (3.20) is satisfied.

From equation (2.81), we find that the evolution of γ is governed by the equation

$$\gamma' = \sqrt{3\gamma}(\gamma-2)[\sqrt{3\gamma}Q + sgn(\dot{\phi})b\sqrt{1-\tilde{\Omega}_\lambda-\tilde{\Omega}_{\rho^*}}]\,, \tag{3.53}$$

and again we can easily see from (3.53) and (3.54) below that $\dot{\phi} \to -\dot{\phi}$ corresponds to time reversal $\tau \to -\tau$. Therefore we restrict ourselves to the case $\dot{\phi} \geq 0$ in the following analysis.

Using (3.23)-(3.21) and (3.52), we find that the dynamics of the models in this sector are described by the following system of non-linear differential equations:

$$\begin{aligned}
\gamma' &= \sqrt{3\gamma}(\gamma-2)[\sqrt{3\gamma}Q + b\sqrt{1-\tilde{\Omega}_\lambda-\tilde{\Omega}_{\rho^*}}]\,,\\
Q' &= [1-\frac{3}{2}\gamma(1+\tilde{\Omega}_\lambda-\tilde{\Omega}_{\rho^*})-2\tilde{\Omega}_{\rho^*}](1-Q^2)\,,\\
\tilde{\Omega}'_\lambda &= [3\gamma(\tilde{\Omega}_\lambda-1-\tilde{\Omega}_{\rho^*})+4\tilde{\Omega}_{\rho^*}]Q\,\tilde{\Omega}_\lambda\,,\\
\tilde{\Omega}'_{\rho^*} &= [(3\gamma-4)(1-\tilde{\Omega}_{\rho^*})+3\gamma\tilde{\Omega}_\lambda]Q\,\tilde{\Omega}_{\rho^*}\,.
\end{aligned} \tag{3.54}$$

We can immediately identify the invariant sets $\gamma = 0, 2$, the flat subset $Q = \pm 1$, the brane-tension-less subset $\tilde{\Omega}_\lambda = 0$, the dark-radiation-free subset $\tilde{\Omega}_{\rho^*} = 0$. As argued in section 3.3.1, the vacuum subset $\tilde{\Omega}_\lambda + \tilde{\Omega}_{\rho^*} = 1$ is another invariant subset.

The equilibrium points of this system, their coordinates in state space and their eigenvalues are given in table 3.2.

Naturally, all the equilibrium points of sector A that corresponded to flat models are recovered in this sector. In particular, we find the equilibrium point $A_+^{4/3}(b)$ for $b^2 \geq 4$, which for $b^2 \neq 4$ describes a model with non-vanishing non-local energy density ρ^*.

The static models E

In addition to these equilibrium points on the boundary of sectors A and B, we find a 1-parameter set of static ($Q=0$) models E extending into the interior of sector B. These models occur for any fixed values of coordinates $(\gamma^*, 0, \tilde{\Omega}^*_\lambda, \tilde{\Omega}^*_{\rho^*})$, subject to the constraints (3.52) and

$$1-\frac{3}{2}\gamma^*(1+\tilde{\Omega}^*_\lambda-\tilde{\Omega}^*_{\rho^*})-2\tilde{\Omega}^*_{\rho^*} = 0 \tag{3.55}$$

as well as

$$\sqrt{\gamma^*}(\gamma^*-2)b\sqrt{1-\tilde{\Omega}^*_\lambda-\tilde{\Omega}^*_{\rho^*}} = 0\,. \tag{3.56}$$

Notice that the constraint (3.56) comes from the γ'-equation in (3.54), hence from the dynamics of the scalar field, and does not occur in the studies of a similar model with

Table 3.2: This table gives the coordinates and eigenvalues of the isotropic critical points with $\rho^* \geq 0$ and ${}^3R \geq 0$. We have defined the real quantities $\chi = \frac{1}{\sqrt{2}}\sqrt{8-3b^2}$, $\xi = \frac{3\gamma-2}{\sqrt{2}\gamma} b\sqrt{3\gamma(1-\tilde{\Omega}_{\rho^*}) + 2\tilde{\Omega}_{\rho^*} - 1}$ and $\alpha = \sqrt{9\gamma^2(\tilde{\Omega}_{\rho^*} - 1) + 6\gamma(1 - 2\tilde{\Omega}_{\rho^*}) + 4(1 - \tilde{\Omega}_{\rho^*})}$. Notice that α is a real positive quantity within the allowed parameter range. The eigenvalues marked with a dagger only hold if $\tilde{\Omega}_\lambda \neq 1$. The infinite eigenvalue in row 3 actually reads $\lim_{\gamma\to 0}\left(-6\epsilon - \sqrt{3}b\sqrt{\frac{1-\tilde{\Omega}_\lambda}{\gamma}}\right) = \infty$.

Model	Coordinates	Eigenvalues
[1ex] F^2_ϵ	$(2, \epsilon, 0, 0)$	$(6\epsilon + \sqrt{6}b, 4\epsilon, -6\epsilon, 2\epsilon)$
$\mathrm{m}^0_\epsilon(\tilde{\Omega}_\lambda)$ *for* $b = 0$	$(0, \epsilon, \tilde{\Omega}_\lambda, 0)$	$-2\epsilon(3, 1, 0, 2)^\dagger$
$\mathrm{m}^0_\epsilon(\tilde{\Omega}_\lambda)$ *for* $b \neq 0$	$(0, \epsilon, \tilde{\Omega}_\lambda, 0)$	$(\infty, -2\epsilon, 0, -4\epsilon)^\dagger$
m^2_ϵ	$(2, \epsilon, 1, 0)$	$2\epsilon(3, 5, 3, 4)$
R^0_ϵ	$(0, \epsilon, 0, 1)$	-
R^2_ϵ	$(2, \epsilon, 0, 1)$	$2\epsilon(3, 1, -4, -1)$
E	$(\gamma^\bullet, 0, \tilde{\Omega}^\bullet_\lambda, \tilde{\Omega}^\bullet_{\rho^*})$	$(\xi, \alpha, 0, -\alpha)$
$\mathrm{F}^{b^2/3}_+$ for $b^2 \in [0, 6]$	$(\frac{b^2}{3}, 1, 0, 0)$	$(\frac{b^2}{2} - 3, b^2 - 2, -b^2, b^2 - 4)$
$\mathrm{X}^{2/3}_+(b)$ for $b^2 \in [0, 2]$	$(\frac{2}{3}, -\frac{b}{\sqrt{2}}, 0, 0)$	$(\frac{b}{\sqrt{2}} - \chi, \frac{b}{\sqrt{2}} + \chi, \sqrt{2}b, \sqrt{2}b)$
$\mathrm{A}^{4/3}_+(b)$ for $b^2 \in [4, \infty[$	$(\frac{4}{3}, 1, 0, 1 - \frac{4}{b^2})$	$(-\frac{1}{2}(1+\zeta), 2, -4, -\frac{1}{2}(1-\zeta))$

constant equation of state [41, 42]. For constant potential ($b = 0$), equation (3.56) is automatically fulfilled for all values of $\gamma^\bullet$, $\tilde{\Omega}^\bullet_\lambda$, $\tilde{\Omega}^\bullet_{\rho^*}$, and E degenerates into a 2-dimensional surface containing static models for all values of $\gamma \in [0, 2/3]$. For $b \neq 0$, the only solution to (3.52), (3.55) and (3.56) is

$$\gamma^\bullet = 0, \;\; \tilde{\Omega}^\bullet_{\rho^*} = 1/2, \;\; \tilde{\Omega}^\bullet_\rho + \tilde{\Omega}_\lambda = 1/2\,. \tag{3.57}$$

This is still a very interesting result, since in the general relativistic analogue, static models only exist if the potential is flat ($b = 0$).

Note that unlike dynamical models (where $Q \neq 0$), the static models E do not admit solutions with $\tilde{\Omega}_\rho = 0$ and $\tilde{\Omega}_\lambda \neq 0$. This is due to the fact that if $Q = 0$, then $a = a_\bullet$ and $\rho = \rho_\bullet$ are constants, and therefore $\tilde{\Omega}_\rho = 0$ implies $\rho_\bullet = 0$, which leads to $\tilde{\Omega}_\lambda = 0$. The static model denoted by $E^{1/3}$ in [83] is therefore un-physical, since this model has $Q = 0$, $\tilde{\Omega}_\rho = 0$ and $\tilde{\Omega}_\lambda = 1$.

The models E have the same line-element as the Einstein universe ($H = 0,\; k = +1$), but in general they also have non-zero brane tension λ and positive nonlocal energy density

ρ^*. Each point in E, corresponding to a static cosmological model, is characterized by a (generally different) constant scale factor $a_\bullet$ and a constant values $\rho_\bullet$, $\rho^*_\bullet$ of energy density and non-local energy density. Requiring positive energy density $\rho_\bullet$, we find from (3.21) that the constants $a_\bullet, \rho^*_\bullet$ must satisfy

$$\rho^*_\bullet \leq \frac{\lambda\kappa^2}{2}\frac{1}{a^2_\bullet}. \tag{3.58}$$

Let us now discuss the stability properties of E. This set is a one-dimensional invariant set, as can be seen from the full non-linear system (3.54). The eigenvector corresponding to the vanishing eigenvalue is tangential to the surface E, so the vanishing eigenvalue corresponds to the direction the invariant set extends into. It is important to note that the invariant set must be identified as such using the full *non-linear system* (3.54), not just the eigenvalues, since the eigenvalues of non-hyperbolic equilibrium points only characterize the *linearized system* in the vicinity of each equilibrium point in E.
The remaining eigenvalues, being of opposite signs, indicate that the E is an unstable invariant subset of the state space. Note that the Jacobian of the dynamical system (3.54) is in general not well-defined in the vicinity of E if $\gamma^\bullet = 0$ [2]. In that case a perturbative study along the lines of appendix A confirms that the points $E\ |_{\gamma^\bullet=0}$ are indeed saddle points in state space.

We conclude that even though the set E presents a geometrically interesting object in the state space, it is not of interest to our stability analysis, since all these equilibrium points are unstable saddle points.

The vacuum models with $\gamma = 0$

Finally, we point out that the Jacobian of the dynamical system (3.54) diverges in the vicinity of the models $m^0_\epsilon(\Omega_\lambda = 1)$ and R^0_ϵ similarly to section 3.3.1. A perturbative analysis analogous to appendix A was performed to study the nature of these models; the results are presented in 3.3.5.

3.3.3 Sector C: $\rho^* \leq 0$ and ${}^3R \leq 0$

In this case we obtain a compact state space by introducing the dynamical variables Z, $\bar{\Omega}_\rho$, $\bar{\Omega}_k$, $\bar{\Omega}_\lambda$, where

$$Z \equiv \frac{H}{N}, \quad N^2 \equiv H^2 - \frac{\kappa^2\rho^*}{3}, \tag{3.59}$$

and the variables $\bar{\Omega}_i$ are defined as in (3.39) and (3.40), but normalized with respect to N instead of H. Using these variables, the Friedmann constraint, being propagated as

[2] For $\gamma^\bullet = b = 0$, the Jacobian is well-defined, and the non-zero eigenvalues of E are given by $\pm\alpha\ |_{\gamma^\bullet=0} = \pm 2\sqrt{1-\bar{\Omega}_{\rho^*}}$.

explained in section 3.3.1, reads

$$\bar{\Omega}_\rho + \bar{\Omega}_k + \bar{\Omega}_\lambda = 1\,. \tag{3.60}$$

As before, all the terms in that sum are non-negative by definition, hence the variables $\bar{\Omega}_\rho$, $\bar{\Omega}_k$, $\bar{\Omega}_\lambda$ take values in the interval $[0,1]$. Since $\gamma \in [0,2]$ and $Z \in [-1,1]$, we find that the state space is again compact.

Introducing the time derivative

$$' \equiv N^{-1} d/dt\,, \tag{3.61}$$

the evolution of N decouples from the rest of the variables. We obtain

$$\begin{aligned} \gamma' &= \sqrt{3\gamma}\,(\gamma-2)[\sqrt{3\gamma}Z + b\sqrt{1-\bar{\Omega}_\lambda - \bar{\Omega}_k}]\,, \\ Z' &= -[\frac{3}{2}\gamma\,(1+\bar{\Omega}_\lambda) + (1-\frac{3}{2}\gamma)\,\bar{\Omega}_k - 2]\,(1-Z^2)\,, \\ \bar{\Omega}_k' &= [(3\gamma-2)(1-\bar{\Omega}_k) + 3\gamma\,\bar{\Omega}_\lambda]\,Z\,\bar{\Omega}_k\,, \\ \bar{\Omega}_\lambda' &= [2\,\bar{\Omega}_k - 3\gamma(1+\bar{\Omega}_k - \bar{\Omega}_\lambda)]\,Z\,\bar{\Omega}_\lambda\,. \end{aligned} \tag{3.62}$$

This system has the usual invariant sets $\gamma = 0, 2$, the dark-radiation-free subset $Z = \pm 1$, the flat subset $\bar{\Omega}_k = 0$ and the brane-tension-less subset $\bar{\Omega}_\lambda = 0$, as well as the vacuum subset $\bar{\Omega}_k + \bar{\Omega}_\lambda = 1$.

The equilibrium points of this system, their coordinates in state space and their eigenvalues are given in table 3.3.

We recover the models with $\rho^* = 0$ that we have obtained in the previous two sections. Notice that the point $A_+^{4/3}(b)$ now only occurs for $b^2 \in [0,4]$. This is due to the fact that the point represents a model with nonlocal energy density depending on the value of the parameter b. For $b^2 < 4$, the point corresponds to a model with negative nonlocal energy density ρ^*, whereas for $b^2 > 4$ it represents a model with positive ρ^*. For $b^2 = 4$, the point coincides with the expanding FLRW model $F_+^{b^2/3}\,|_{b^2=4}$ with $\rho^* = 0$ and $\gamma = 4/3$. Thus the point $A_+^{4/3}(b)$ is leaving the sector describing models with $\rho^* \leq 0$ and entering the sector of models with $\rho^* \geq 0$ at the parameter value $b^2 = 4$ $(b = -2)$.

The static models S

In addition to these points, we find the set S of static models ($H = 0$), whose coordinates $(\gamma^\bullet,\ 0,\ \bar{\Omega}_k^\bullet,\ \bar{\Omega}_\lambda^\bullet)$ satisfy the constraints

$$2 - \frac{3}{2}\gamma^\bullet(1 + \bar{\Omega}_\lambda^\bullet - \bar{\Omega}_k^\bullet) - \bar{\Omega}_k^\bullet = 0 \tag{3.63}$$

Table 3.3: This table gives the coordinates and eigenvalues of the equilibrium points with $\rho^* \leq 0$ and ${}^3R \leq 0$. We have defined $\psi = -b^{-1}\sqrt{8-3b^2}$, $\xi = \sqrt{64-15b^2}$, $\eta = \frac{3\gamma-2}{\sqrt{2\gamma}} b\sqrt{3\gamma(1-\bar{\Omega}_k)+\bar{\Omega}_k-2}$ and $\Gamma = \sqrt{9\gamma^2(\bar{\Omega}_k-1)+6\gamma(2-\bar{\Omega}_k)+4(1-\bar{\Omega}_k)}$. Notice that Γ is a real positive quantity within the allowed range of variables. The eigenvalues marked with a dagger only hold if $\bar{\Omega}_\lambda \neq 1$. The infinite eigenvalue in row 5 actually reads $\lim_{\gamma\to 0}\left(-6\epsilon - \sqrt{3}b\sqrt{\frac{1-\bar{\Omega}_\lambda}{\gamma}}\right) = \infty$.

Model	Coordinates	Eigenvalues
F^2_ϵ	$(2, \epsilon, 0, 0)$	$(6\epsilon + \sqrt{6}b, 2\epsilon, 4\epsilon, -6\epsilon)$
M^0_ϵ	$(0, \epsilon, 1, 0)$	-
M^2_ϵ	$(2, \epsilon, 1, 0)$	$2\epsilon(3, -1, -2, -5)$
$\mathrm{m}^0_\epsilon(\bar{\Omega}_\lambda)$ *for* $b = 0$	$(0, \epsilon, 0, \bar{\Omega}_\lambda)$	$-2\epsilon(3, 2, 1, 0)^\dagger$
$\mathrm{m}^0_\epsilon(\bar{\Omega}_\lambda)$ *for* $b \neq 0$	$(0, \epsilon, 0, \bar{\Omega}_\lambda)$	$(\infty, -4\epsilon, -2\epsilon, 0)^\dagger$
m^2_ϵ	$(2, \epsilon, 0, 1)$	$2\epsilon(3, 4, 5, 3)$
S	$(\gamma^\bullet, 0, \bar{\Omega}^\bullet_k, \bar{\Omega}^\bullet_\lambda)$	$(\eta, \Gamma, 0, -\Gamma, 0)$
$\mathrm{F}^{b^2/3}_+$ for $b^2 \in [0, 6]$	$(\frac{b^2}{3}, 1, 0, 0)$	$(\frac{b^2}{2} - 3, b^2 - 4, b^2 - 2, -b^2)$
$\mathrm{X}^{2/3}_+(b)$ for $b^2 \in [2, \infty[$	$(\frac{2}{3}, 1, 1 - \frac{2}{b^2}, 0)$	$(-1 - \psi, -2, -1 + \psi, -2)$
$\mathrm{A}^{4/3}_+(b)$ for $b^2 \in [0, 4]$	$(\frac{4}{3}, -\frac{b}{2}, 0, 0)$	$(\frac{1}{4}(b - \xi), \frac{1}{4}(b + \xi), -b, 2b)$

and

$$\sqrt{\gamma^\bullet}(\gamma^\bullet - 2)b\sqrt{1 - \bar{\Omega}^\bullet_k - \bar{\Omega}^\bullet_\lambda} = 0\,. \tag{3.64}$$

Requiring non - negative energy density $\rho_\bullet$, we also find from (3.21) that the constants $a_\bullet, \rho^*_\bullet$ must satisfy

$$-\rho^*_\bullet \geq -\frac{\lambda\kappa^2}{2}\frac{k}{a^2_\bullet}\,. \tag{3.65}$$

It is important to note that each element of this one-parameter set of static models in general has negative curvature ($k = -1$), which stands in strong contrast to General Relativity, where static models only exist if $k = +1$.

For $b = 0$, the 1 - dimensional set degenerates into a 2 - dimensional surface; in this special case we find that static models occur for all values of $\gamma^\bullet \geq 2/3$. If $b \neq 0$, equations (3.60), (3.63) and (3.64) can be satisfied if

$$\gamma^\bullet = 2,\ \bar{\Omega}^\bullet_k = \frac{1}{2} + \frac{3}{2}\bar{\Omega}^\bullet_\lambda,\ \bar{\Omega}^\bullet_\rho = \frac{1}{2} - \frac{5}{2}\bar{\Omega}^\bullet_\lambda\,, \tag{3.66}$$

which lies within the state space if $\bar{\Omega}^\bullet_\lambda < 1/5$.

Dynamically these models again are not very interesting, since they all represent saddle points in state space.

3.3.4 Sector D: $\rho^* \leq 0$ and ${}^3R \geq 0$

As explained in [42], we need to take into account that the Friedmann equation (3.21) has two non-positive terms in this sector. It turns out that we can obtain a compact state space by introducing the dimensionless dynamical variables W, $\hat{\Omega}_\rho$, $\hat{\Omega}_\lambda$, $\hat{\Omega}_{\rho^*}$, where

$$W \equiv \frac{H}{P}\,, \quad P^2 \equiv H^2 + \frac{1}{6}{}^3R - \frac{\kappa^2\rho^*}{3}\,, \tag{3.67}$$

and the variables with a hat are defined as those in (3.39) and (3.40), but normalized with respect to P instead of H. The Friedmann constraint now reduces to

$$\hat{\Omega}_\rho + \hat{\Omega}_\lambda = 1\,, \tag{3.68}$$

and using the time derivative

$$' \equiv P^{-1}d/dt\,, \tag{3.69}$$

we can verify that the Friedmann constraint (3.68) is propagated as usual.

It is important to note that $\hat{\Omega}_{\rho^*}$ does not appear in the Friedmann equation (3.68). Indeed, it can be seen from its definition that $\hat{\Omega}_{\rho^*}$ is negative and belongs to the interval $[-1, 0]$. As before, we can see from (3.68) that $\hat{\Omega}_\rho$, $\hat{\Omega}_\lambda \in [0, 1]$. Together with $W \in [-1, 1]$ and $\gamma \in [0, 2]$, we find that these variables define a compact state space.

The dynamics of the state space are governed by the evolution equations for γ and the dimensionless variables W, $\hat{\Omega}_\rho$, $\hat{\Omega}_\lambda$, $\hat{\Omega}_{\rho^*}$, which are derived by using (3.23)-(3.21). They are given by

$$\begin{aligned}
\gamma' &= \sqrt{3\gamma}(\gamma - 2)[\sqrt{3\gamma}\,W + b\sqrt{1 - \hat{\Omega}_\lambda}]\,, \\
W' &= [1 - \frac{3}{2}\gamma\,(1 + \hat{\Omega}_\lambda)](1 - W^2) - \hat{\Omega}_{\rho^*}\,, \\
\hat{\Omega}'_\lambda &= 3\gamma\,(\hat{\Omega}_\lambda - 1)\,W\,\hat{\Omega}_\lambda\,, \\
\hat{\Omega}'_{\rho^*} &= [3\gamma\,(1 + \hat{\Omega}_\lambda) - 4]\,W\,\hat{\Omega}_{\rho^*}\,.
\end{aligned} \tag{3.70}$$

The obvious invariant sets $\gamma = 0, 2$, $\hat{\Omega}_\lambda = 0$ and $\hat{\Omega}_{\rho^*} = 0$ can be easily identified. The vacuum subset $\hat{\Omega}_\rho + \hat{\Omega}_\lambda = 1$ is another invariant set. It is more difficult to see that the flat subset is an invariant set too. We first note that, on using definition (3.67), that flat models are characterized by $W^2 = 1 + \hat{\Omega}_{\rho^*}$. Using (3.70), we can then explicitly verify that $(W^2 - 1 - \hat{\Omega}_{\rho^*})' = 0$, which means that flat models can never evolve to non-flat models: the flat subset is invariant.

Table 3.4: This table gives the coordinates and eigenvalues of the isotropic critical points with $\rho^* \leq 0$ and ${}^3R \geq 0$. We have defined $\chi = \frac{1}{\sqrt{2}}\sqrt{8-3b^2}$ and $\xi = \sqrt{64-15b^2}$, $\delta = \sqrt{6}b\sqrt{1-\hat{\Omega}_\lambda}$, $\beta = \sqrt{9\gamma^2\hat{\Omega}_\lambda + 4}$. Notice that β is a real positive quantity within the allowed range of variables. The eigenvalues marked with a dagger only hold if $\hat{\Omega}_\lambda \neq 1$. The infinite eigenvalue in row 3 actually reads $\lim_{\gamma\to 0}\left(-6\epsilon - \sqrt{3}b\sqrt{\frac{1-\hat{\Omega}_\lambda}{\gamma}}\right) = \infty$.

Model	Coordinates	Eigenvalues
F^2_ϵ	$(2,\epsilon,0,0)$	$(6\epsilon+\sqrt{6}b, 4\epsilon, -6\epsilon, 2\epsilon)$
$\mathrm{m}^0_\epsilon(\hat{\Omega}_\lambda)$ *for* $b=0$	$(0,\epsilon,\hat{\Omega}_\lambda,0)$	$-2\epsilon(3,1,0,2)^\dagger$
$\mathrm{m}^0_\epsilon(\hat{\Omega}_\lambda)$ *for* $b\neq 0$	$(0,\epsilon,\hat{\Omega}_\lambda,0)$	$(\infty,-2\epsilon,0,-4\epsilon)^\dagger$
m^2_ϵ	$(2,\epsilon,1,0)$	$2\epsilon(3,5,3,4)$
E	$(\gamma^\bullet,0,\hat{\Omega}^\bullet_\Lambda,\hat{\Omega}^\bullet_\lambda,\hat{\Omega}^\bullet_{\rho^*})$	$(\delta,\beta,0,-\beta,0)$
$\mathrm{F}^{b^2/3}_+$	$(\frac{b^2}{3},1,0,0)$	$(\frac{b^2}{2}-3, b^2-2, -b^2, b^2-4)$
$\mathrm{X}^{2/3}_+(b)$ for $b^2\in[0,2]$	$(\frac{2}{3},-\frac{b}{\sqrt{2}},0,0)$	$(\frac{b}{\sqrt{2}}-\chi, \frac{b}{\sqrt{2}}+\chi, \sqrt{2}b, \sqrt{2}b)$
$\mathrm{A}^{4/3}_+(b)$ for $b^2\in[0,4]$	$(\frac{4}{3},-\frac{b}{2},0,\frac{b^2}{4}-1)$	$(\frac{1}{4}(b-\xi), \frac{1}{4}(b+\xi), -2b, 2b)$

The equilibrium points of this system, their coordinates in state space and the finite eigenvalues are given in 3.4.
Again, we recover the equilibrium points that represent flat models obtained in the previous subsection.

The static models E

In addition, we find another set of equilibrium points representing static models denoted by E. These models correspond to the Einstein - universe - like models found in sector B, since they have the same line element as the Einstein universe ($H = 0$, $k = +1$). They also have non - zero brane tension and non - local energy density ρ^*, except now $\rho^* \leq 0$.

Their coordinates ($\gamma^\bullet$, 0, $\hat{\Omega}^\bullet_\lambda$, $\hat{\Omega}^\bullet_{\rho^*}$) satisfy (3.68) and the additional constraints

$$1 - \frac{3}{2}\gamma^\bullet(1+\hat{\Omega}^\bullet_\lambda) - \hat{\Omega}^\bullet_{\rho^*} = 0 \tag{3.71}$$

and

$$\sqrt{\gamma^\bullet}(\gamma^\bullet - 2)b\sqrt{1-\hat{\Omega}^\bullet_\lambda} = 0\,. \tag{3.72}$$

If $b \neq 0$, there are no solutions to (3.68), (3.71) and (3.72). For $b = 0$, we find static models for all $\gamma \in [1/3, 4/3]$. However, these models are unstable as indicated in table 3.4.

3.3.5 The structure of the state space

The full state space is obtained by matching sectors A-D. From the definitions of the different sectors it is clear that we must match sector A and sector B at the flat invariant subsets $k = 0$, and sector C must be matched with sector D along the flat invariant subset. Note that in sector D, flat models are characterized by $W^2 = 1 + \hat{\Omega}_{\rho^*}$. This means that we have the identification

$$\begin{array}{ccc} A|_{\Omega_k=0} & \longleftrightarrow & B|_{Q^2=1} \\ C|_{\tilde{\Omega}_k=0} & \longleftrightarrow & D|_{W^2=1+\hat{\Omega}_{\rho^*}} \,. \end{array}$$

Similarly, sector A and C as well as sector B and D respectively match for $\rho^* = 0$:

$$\begin{array}{ccc} A|_{\Omega_{\rho^*}=0} & \longleftrightarrow & C|_{Z^2=1} \\ B|_{\tilde{\Omega}_{\rho^*}=0} & \longleftrightarrow & D|_{\hat{\Omega}_{\rho^*}=0} \,. \end{array}$$

When matching the different sectors of state space, we must of course ensure that we match regions of state space describing expanding (contracting) models with regions of state space describing expanding (contracting) models. This is not explicit in the symbolic notation above; more precisely, there are 5 disconnected sectors: the expanding part of sector A, the collapsing part of sector A and sectors B, C, and D.

It is hard to visualize the four-dimensional, topologically non-trivial state space. We will not aim to give a graphic illustration of the entire state space here. Instead we will first discuss the dynamics of the general relativistic subspace in detail, emphasizing the relation to the original work [92, 39] and the importance of a compact state space. We will illustrate the dynamics of this 2-dimensional set embedded in the 3-dimensional subspace with non-zero brane tension. We will then discuss the dynamics of the full 4-dimensional state space, presenting the results in tables without graphical representations of the state space.

Since the eigenvalues of the equilibrium points in general depend on the parameter b, the nature of these critical points varies with b. Furthermore, some of the critical points ($F_+^{b^2/3}$, $X_+^{2/3}$) will also move in state space as b^2 increases, since their coordinates are also functions of b. We also find critical points, which only appear for $b = 0$ and then disappear, as we increase b^2.

For certain values of b, some of the eigenvalues pass through zero. At these values of b, the state space is "torn" in that it undergoes topological changes. In contrast to [42], the appearance of such bifurcations in our model does not coincide with the appearance of lines of critical points. This is a consequence of the fact that in our inflationary model, the critical points are moving in state space. Bifurcations appear, when two of the equilibrium points merge, which occurs at $b^2 = 0, 2, 4, 6$.

The GR - subspace

As discussed above, the general relativistic subsets Ω_λ, $\Omega_{\rho^*} = 0$ and $\tilde{\Omega}_\lambda$, $\tilde{\Omega}_{\rho^*} = 0$ are invariant subsets of sector A and B respectively. In the following, we will refer to them as the *open sector* and the *closed sector* respectively. We need not consider sectors C and D, since these match A and B if the dark radiation is zero. It is important to note that it only makes sense to study the dynamics in the general relativistic subsets because these sets are *invariant.*

The GR - submanifold has first been discussed by Halliwell [92], who analyzed the dynamics of a relativistic scalar field with exponential potential in a FLRW spacetime. This work was extended by Burd and Barrow [39] to 1+N dimensional FLRW models and 1+3 dimensional Bianchi I models. In both works the equilibrium points here denoted as $X_+^{2/3}(b)$ and $F_+^{b^2/3}$ were found to be the only equilibrium points of the state space of the 1+3 dimensional FLRW spacetime.

Their analysis however was somewhat incomplete, since the state space considered was non - compact, with the undesired consequence that the authors could not discuss the equilibrium points at the boundary of our compactified state space.

The variables chosen to describe the inflationary dynamics of homogeneous isotropic models were [92]

$$x = \frac{d\alpha}{d\tau}, \quad y = \frac{d\phi}{d\tau}.$$

Here the potential has been absorbed into the time derivative by rescaling the time variable $t \to \tau$ by

$$\frac{d}{d\tau} = V^{-1/2}\frac{d}{dt},$$

and α relates to the scale factor $a(\tau)$ by

$$a(\tau) = e^{\alpha(\tau)}.$$

We find that these coordinates relate to our expansion normalized variables via the following transformations:

$$\begin{aligned} x^2 &= \frac{2}{3}\frac{1}{(2-\gamma)(1-\Omega_k)}, \\ y^2 &= \frac{2\gamma}{2-\gamma} \end{aligned}$$

for $\gamma \neq 2$, $\Omega_k \neq 1$. We can see the boundaries of our state space correspond to Halliwell's variables "at infinity":

$$\begin{aligned} x \to \infty \quad &\Leftrightarrow \quad \Omega_k \to 1 \ \Leftrightarrow \ \Omega_\rho \to 0\,. \\ y \to \infty \quad &\Leftrightarrow \quad \gamma \to 2\,. \end{aligned} \tag{3.73}$$

This means, that we have compactified the state space by mapping $y \in [0, \infty] \longrightarrow \gamma \in [0, 2]$ and $x \in [0, \infty] \longrightarrow \Omega_k \in [0, 1]$. In this way we have extended the work that has been done on inflationary models with exponential potentials in the general relativistic context. We find that $X_+^{2/3}$, $F_+^{b^2/3}$ correspond to the critical points I, II in [39]. In addition, we obtain the new critical points F_ϵ^2 and M_ϵ corresponding to a FLRW universe with stiff matter and the Milne universe respectively. These models lie on the boundary of our compact state space, which is why they cannot be found or analyzed in the non-compact state space discussed in [92, 39]: they correspond to $x,\ y \rightarrow \infty$. This shows how important it is to work with compactified state spaces in order to include *all* the equilibrium points of the system.

Notice that we also find the new equilibrium point F_ϵ^0, which does not lie on the boundary of our state space and therefore is expected to be found in [92]. It is however not an equilibrium point in the state space (x, y). The reason is that we are using γ to describe the dynamics of the scalar field ϕ, which essentially is a function of $\dot{\phi}^2$, whereas Halliwell essentially used $\dot{\phi}$. Therefore our $\dot{\gamma}$-equation is proportional to $\dot{\phi}$, unlike the dynamical equation for $\dot{\phi}$. It might seem surprising, that we find a critical point at $\dot{\phi}^2 = 0$, whereas there is no equilibrium point at $\dot{\phi} = 0$ in [39, 92]. In order to explain this discrepancy one needs to understand that we are describing different physical quantities when using γ or $\dot{\phi}$ in order to study the dynamics of the scalar field. Although the equation of state does not change (that means $\dot{\gamma} = 0$) as $\dot{\phi} \rightarrow 0$, the rate of change of the scalar field, $\dot{\phi}$, does change as $\dot{\phi} \rightarrow 0$ since $\ddot{\phi}, x' \neq 0$.

We will now proceed to describe the dynamics of the general relativistic state space. We will graphically present the dynamics of the general relativistic models including effects arising from the quadratic energy corrections, but not including the non-local effects due to the bulk Weyl tensor (see 2.29) in figures 3.1-3.9. This is an interesting subspace, since the terms quadratic in the energy density are expected to be dominant at high energies. In this discussion we will focus on the general relativistic subspace, i.e. the bottom planes of each figure where $\Omega_\lambda = 0$. For a discussion of the dynamics including bulk effects, we refer to the following subsection.

The dynamics of the collapsing open sector are not very interesting, since they do not depend on the parameter value b: the flat FLRW model with constant energy density is the past attractor, and the flat FLRW model with stiff matter is the future attractor for all values of b (see figure 3.1).

Having said that, we will now discuss the more complicated dynamics of the closed and the expanding open models for the different ranges of the parameter value b.
At $b = 0$, we find a bifurcation of the state space, since for this value of b, the equilibrium points E and $X_+^{2/3}(b)$ coincide, as well as F_+^0 and $F_+^{b^2/3}$ coincide. All expanding open models are attracted to F_+^0. All closed models will evolve into either F_-^2 or F_+^0, depending on the initial conditions. The Einstein universe appears as an unstable saddle point (see figure 3.2).

As b^2 increases, the Einstein universe disappears. Instead, we find the saddle point $X_+^{2/3}(b)$, which moves in Q-direction as b^2 is increases, remaining a saddle until it reaches the flat sector $Q = 1$ at $b^2 = 2$. There it merges with expanding FLRW model $F_+^{b^2/3}$. For $0 < b^2 < 2$, $F_+^{b^2/3}$ is a sink moving along the γ-axis. All open and some of the closed models are attracted to this solution (see figure 3.3).

At $b^2 = 2$, the models $X_+^{2/3}(b)$ and $F_+^{b^2/3}$ merge, which causes a bifurcation of the state space. The nature of the two merging critical points is significantly changed at this value of b. The flat solution $X_+^{2/3} = F_+^{2/3}$ is an attractor for all open or flat models, but a repeller for all closed models (see figure 3.4). This behavior has been observed by [39]. At this value of b^2, the two merging critical points swap their nature: for all $b^2 \geq 2$, $X_+^{2/3}(b)$ is a sink, while $F_+^{b^2/3}$ is a saddle (see figures 3.5 - 3.9).

As b is further increased ($2 < b^2 < 6$), $F_+^{b^2/3}$ moves further along the γ-axis. It is now a saddle for all models. All flat models are attracted, whereas all open or closed models are repelled. $X_+^{2/3}(b)$ has now entered the open sector and moves further to the $\Omega_k = 1$ boundary. $X_+^{2/3}(b)$ is a node sink for all $2 < b^2 < 8/3$. For all $b^2 \geq 2$, F_-^2 is the future attractor for all closed models and $X_+^{2/3}(b)$ the future attractor for all expanding open models (see figures 3.5 - 3.7).

At $b^2 = 8/3$, there is a bifurcation when the node sink $X_+^{2/3}(b)$ turns into a spiral sink for all $8/3 < b^2 < \infty$.

At $b^2 = 6$, we find another bifurcation of the state space. The two points $F_+^{b^2/3}$ and F_+^2 merge. This turns F_+^2 from a source into a saddle (see figure 3.8).

For $b^2 \geq 6$, all open or closed models are still repelled from F_+^2, but all flat models are now attracted. In the limit $b^2 \to \infty$, the future attractor for the open sector, $X_+^{2/3}(b)$ approaches the vacuum solution $\Omega_k = 1$, $\Omega_\rho \to 0$ with $\gamma = 2/3$ (see figure 3.9).

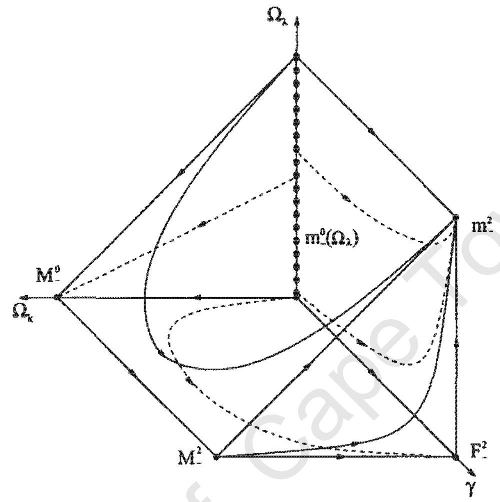

Figure 3.1: State space for the collapsing open FLRW models, $\epsilon = -1$, ${}^3R \leq 0$. The bottom plane $\Omega_\lambda = 0$ corresponds to general relativity. The line joining M^0_- and M^2_- corresponds to vacuum $\rho = 0$ and represent one single model: the collapsing general-relativistic Milne model. The remaining equilibrium points F^2_-, m^2_-, $m^0_-(\Omega_\lambda)$ the flat FLRW model with stiff matter, the non-general-relativistic BDL model with stiff matter and a line of non-general-relativistic models with constant energy density (including the flat FLRW model with $\gamma = 0$). We are only drawing the trajectories in the invariant planes, from which the whole dynamics can be deduced. The structure of this part of the whole state space does not change with the parameter value b. The full state space can be obtained by matching this section to figures 3.2-3.9.

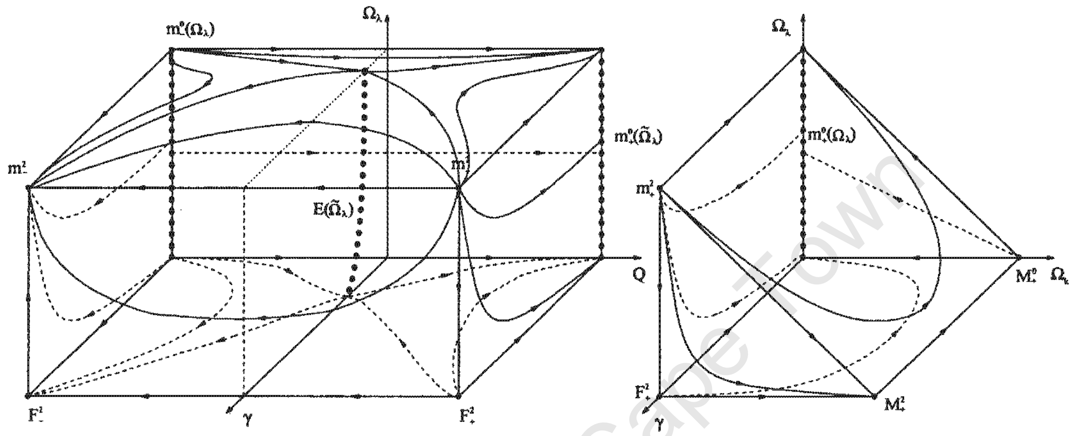

Figure 3.2: State space for the FLRW models with non-negative spatial curvature ${}^3R \geq 0$, $\epsilon = \pm 1$ (on the left) and the expanding FLRW models with non-positive spatial curvature, ${}^3R \leq 0$, $\epsilon = 1$ (on the right) for $b = 0$ (a bifurcation). In the left part of the figure, which describes the closed models, the plane $Q = 0$ differentiates between the expanding sector $Q \geq 0$, $\epsilon = 1$ and the collapsing sector $Q \leq 0$, $\epsilon = -1$. As in figure 3.1, the bottom plane corresponds to GR, and the line joining M_+^0 and M_+^2, corresponds to vacuum $\rho = 0$, and represents one single model: the expanding general-relativistic Milne model. We only give the trajectories on the invariant planes, from which the whole dynamics can be deduced. The critical points F_ϵ^2, m_ϵ^2, $m_\epsilon^0(\tilde{\Omega}_\lambda)$ correspond to the flat FLRW model with stiff matter, the non-general-relativistic BDL model with stiff matter and a line of non-general relativistic model with constant energy density (including the flat FLRW model with $\gamma = 0$) respectively. $E(\tilde{\Omega}_\lambda)$ is a line of static Einstein universes with $1/3 \leq \gamma < 2/3$. The vacuum Einstein model with $\Omega_\lambda = 1$ and $\gamma = 2/3$ must be excluded as explained in section 3.3.2.

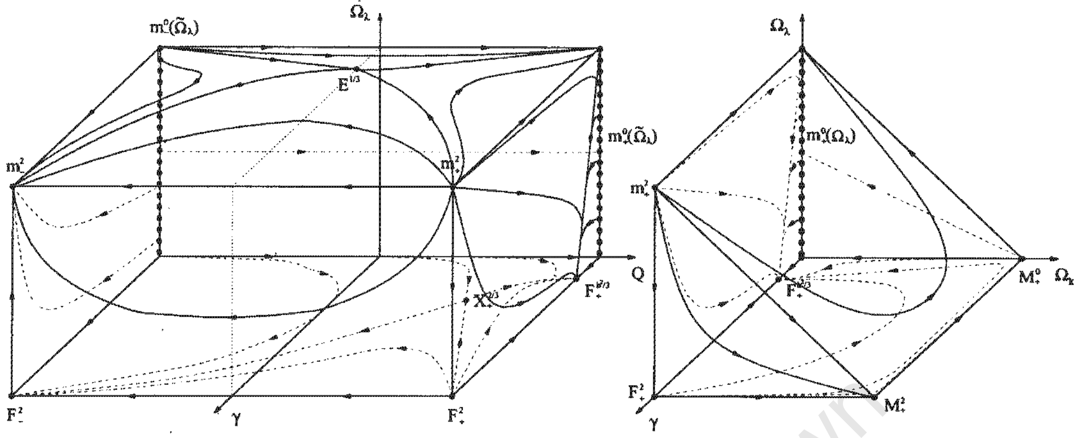

Figure 3.3: State space for the FLRW models with $0 < b^2 < 2$. The line of Einstein universes in the closed sector of the state space has collapsed into the un-physical non-general-relativistic Einstein model $E^{1/3}$ and the general-relativistic model $X_+^{2/3}$, which is expanding and moving towards the expanding flat subspace $Q = 1$, $(\epsilon = 1)$ as b^2 is increasing. The equilibrium point $F_+^{b^2/3}$ corresponds to a flat FLRW model with $\gamma = b^2/3$. See the captions of figure 3.1 and figure 3.2 for more details.

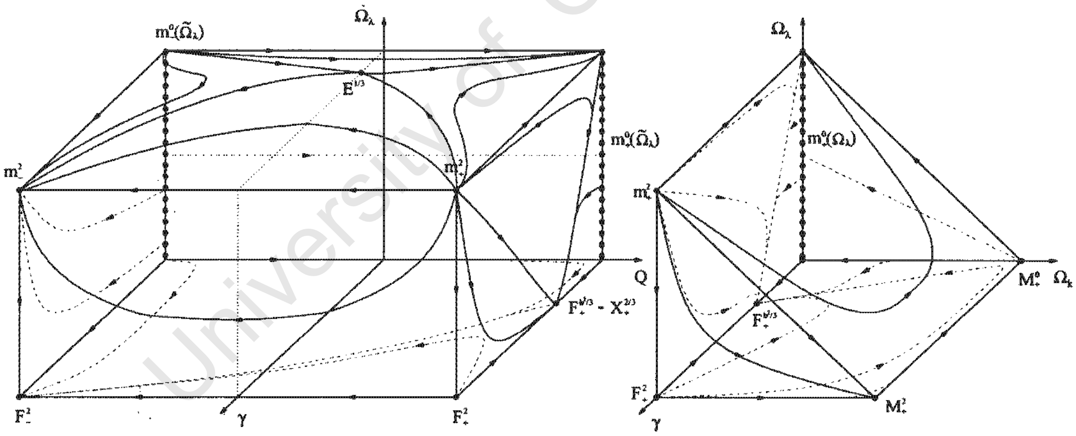

Figure 3.4: State space for the FLRW models with $b^2 = 2$ (a bifurcation). The equilibrium point $X_+^{2/3}$ has reached the flat subspace, where it merges with $F_+^{b^2/3}$. This causes the bifurcation; the nature of the two critical points will be swapped as they are moving on (see figures 3.5-3.9). All open and flat models are attracted to this point $X_+^{2/3} = F_+^{b^2/3}$, whereas all closed models are repelled. See the captions of figures 3.1, 3.2 for more details.

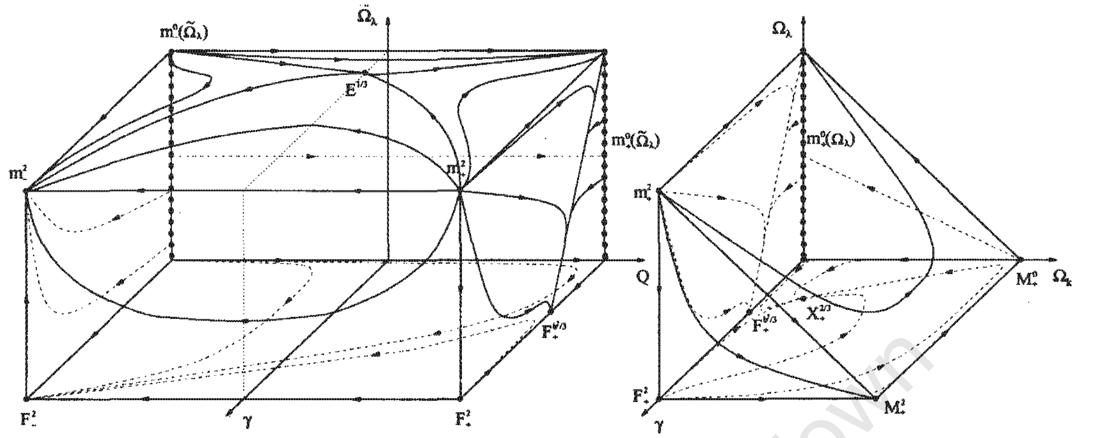

Figure 3.5: State space for the FLRW models with $2 < b^2 < 8/3$. $X_+^{2/3}$ has entered the open sector and turned into a node sink. $F_+^{b^2/3}$ is now an unstable saddle for all models. See the captions of figures 3.1, 3.2 for more details.

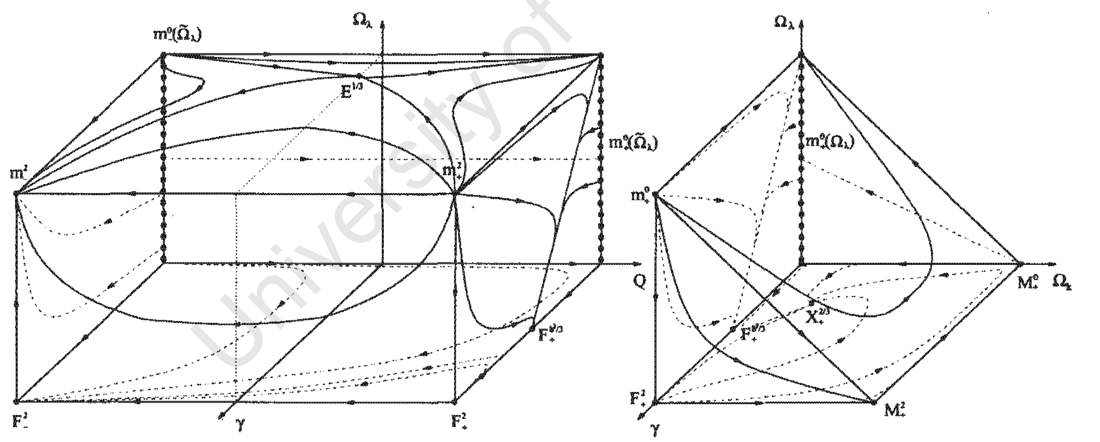

Figure 3.6: State space for the FLRW models with $b^2 = 8/3$. The node sink $X_+^{2/3}$ is turning into a spiral sink; the structure of the state space has not changed. See the captions of figures 3.1, 3.2 for more details.

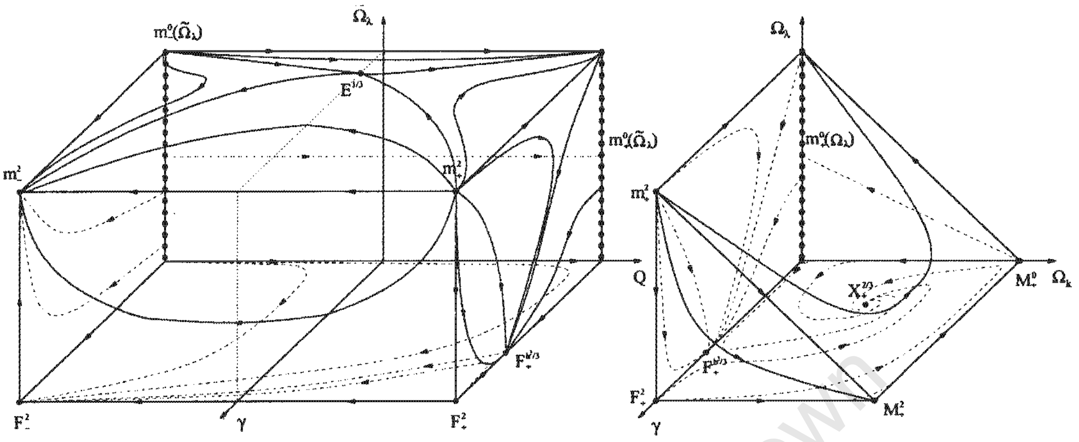

Figure 3.7: State space for the FLRW models with $8/3 < b^2 < 6$. $X_+^{2/3}$ is now a spiral sink; the structure of the state space is essentially the same as for $2 < b^2 \leq 8/3$. See the captions of figures 3.1, 3.2 for more details.

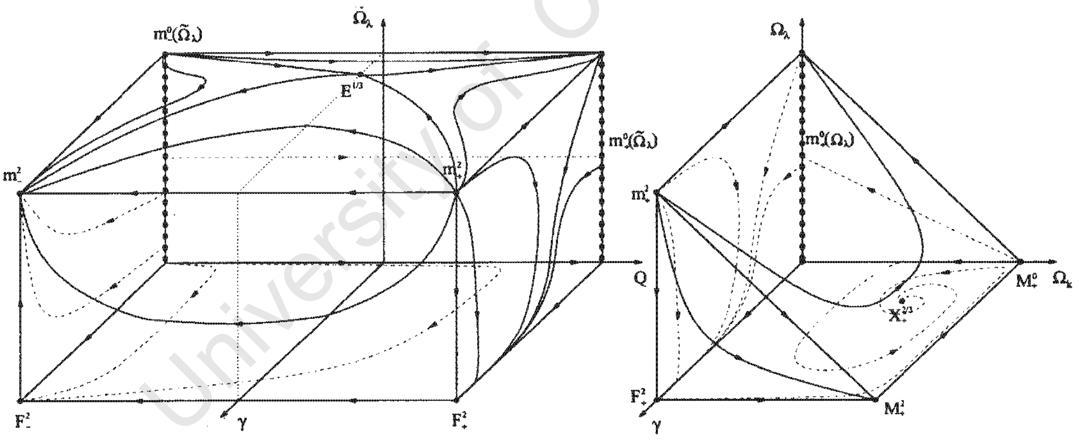

Figure 3.8: State space for the FLRW models with $b^2 = 6$ (a bifurcation). The equilibrium point $F_+^{b^2/3}$ has merged with the equilibrium point F_+^2, which causes the bifurcation. The topology of the state space is changing at this value of b, since the point F_+^2 is turning from an attractor in γ-direction into a repeller in that direction (compare to figures 3.7, 3.9). See the caption of figures 3.1, 3.2 for more details.

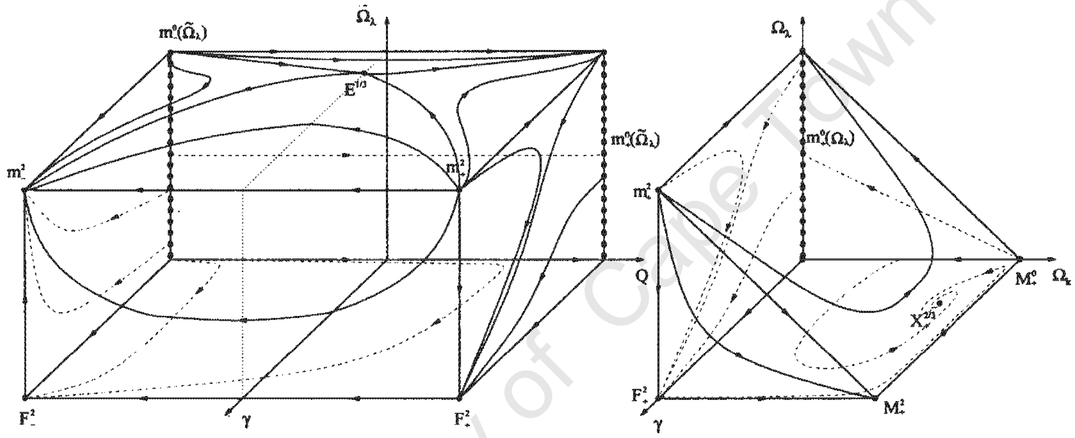

Figure 3.9: State space for the FLRW models with $b^2 > 6$. The model $F_+^{b^2/3}$ has moved out of state space, while $X_+^{2/3}$ is approaching the open vacuum boundary of the GR-subspace. See the captions of figures 3.1, 3.2 for more details.

Higher dimensional effects

We now turn our attention to the nature of the equilibrium points of the full state space. In sections 3.3.1-3.3.4 we have studied the dynamics in the different sectors. When combining this information, we obtain the dynamical character of each equilibrium point. The results are obtained in table 3.3.5 below. The question we will discuss is how the behavior of

Table 3.5: Dynamical character of the critical points of the class of FLRW models.

Model	$b=0$	$0<b^2<2$	$b^2=2$	$2<b^2<4$	$b^2=4$	$4<b^2<6$	$b^2=6$	$b^2>6$
$\mathrm{m}_+^0(\Omega_\lambda)$	**sink**	saddle	saddle	saddle	saddle	saddle	saddle	saddle
$\mathrm{m}_-^0(\Omega_\lambda)$	**source**	**source**	**source**	**source**	**source**	**source**	**source**	**source**
E	saddle	saddle	saddle	saddle	saddle	saddle	saddle	saddle
S	saddle	saddle	saddle	saddle	saddle	saddle	saddle	saddle
F_ϵ^2	saddle	saddle	saddle	saddle	saddle	saddle	saddle	saddle
m_+^2	**source**	**source**	**source**	**source**	**source**	**source**	**source**	**source**
m_-^2	**sink**	**sink**	**sink**	**sink**	**sink**	**sink**	**sink**	**sink**
M_ϵ	saddle	saddle	saddle	saddle	saddle	saddle	saddle	saddle
R_ϵ	saddle	saddle	saddle	saddle	saddle	saddle	saddle	saddle
$\mathrm{F}_+^{b^2/3}$	**sink**	**sink**	saddle	saddle	saddle	saddle	saddle	-
$\mathrm{X}_+^{2/3}(b)$	saddle	saddle	saddle	**sink**	**sink**	**sink**	**sink**	**sink**
$\mathrm{A}_+^{4/3}(b)$	saddle	saddle	saddle	saddle	saddle	saddle	saddle	saddle

the dynamical system describing General Relativity is changed within the brane-world context. In particular, do the additional degrees of freedom Ω_λ, Ω_{ρ^*} change the stability of the general relativistic models, and are there new non-general relativistic stable equilibrium points?

We will first answer the second question. In addition to the general relativistic points, we have found the non-general-relativistic equilibrium points m_ϵ, R_ϵ and $A_+^{4/3}(b)$. The latter two models are dynamically not very interesting, since they are saddle points in the state space. The models m_ϵ on the other hand are of great interest: For all values of b, the unique expanding past attractor in the state space of FLRW models is the non-general-relativistic high-energy m_+^2 with $\gamma=2$, that means $V << \dot{\phi}^2$. This point corresponds to the BDL model first found in [26] (see also [25, 72]) and is characterized by $\Omega_\lambda \gg \Omega_\rho$, Ω_k and $a(t) \sim t^{1/6}$ as explained in 3.3.1. The set of collapsing inflationary ($\dot{\phi}^2 << V$) non-general relativistic models $m_+^0(\Omega_\lambda)$ is another past attractor.

The future attractors are the contracting BDL model m_-^2 with $\gamma=2$ and $a(t) \sim (-t)^{1/6}$ (obtained from the solution for m_+^2 by time-reversal) and the general-relativistic models $F_+^{b^2/3}$ and $X_+^{2/3}(b)$. To be precise, the flat FLRW model $F_+^{b^2/3}$ is an attractor for $b^2<2$ and the model $X_+^{2/3}(b)$ for $b^2 > 2$. This means that the future evolution of the class of FLRW models is either towards ever expanding general relativistic models or towards collapsing non-general relativistic models. If the potential is flat, then the expanding de Sitter like models $m_+^0(\Omega_\lambda)$ and F_+^0 are additional future attractors.

In summary, we find that at high energies, when the terms quadratic in the energy density become dominant, the BDL models take the place of the general-relativistic models as the attractor (repeller) in the state space: in the full state space, m_-^2 instead of F_-^2 is the future attractor for all collapsing open models and some of the closed models. This means that within the brane-world context, the FLRW with stiff matter changes from a stable solution into a unstable saddle when including the bulk term Ω_λ, Ω_{ρ^*}. The collapsing FLRW model F_-^0 remains a past attractor for the collapsing open sector and parts of the closed sector. However, unlike in GR, it is now merely a subset of the line of equilibrium points $m_-^0(\Omega_\lambda)$. This set $m_-^0(\Omega_\lambda)$ extends in Ω_λ direction and presents a set of sources. Similarly, for $b = 0$, the future attractor of the expanding open and some of the closed models is not necessarily F_+^0, but instead any of the points $m_+^0(\Omega_\lambda)$ depending on the initial conditions.

We note that for $b > 0$ the models $m_+^0(\Omega_\lambda)$ represent high energy inflationary models with exponential potentials (which are too steep to inflate in GR) [54]. The fact that they are all unstable (saddle points) reflects the fact that *steep inflation* ends naturally, since as the energy drops below the brane tension the condition for inflation no longer holds.

The last issue we want to address here is the existence of static universes. In General Relativity, the scenario that we are describing here (matter described by a dynamical scalar field with exponential potential) only admits a static universe in the special case of a constant potential ($b = 0$), and this model has the same line element as the Einstein universe ($H = 0,\ k = +1$) and only occurs at $\gamma = 2/3$.

If we allow for non-zero brane tension but neglect the non-local bulk effects ($\rho^* = 0$), this condition is relaxed: we find static universe models characterized by $H = 0,\ k = +1$ for all values of $1/3 < \gamma \leq 2/3$ for $b = 0$ [83]. For a non-flat potential ($b \neq 0$) there are still no physical static equilibrium points [3].

When taking into account the non-local bulk effects, we find that there are not only static Einstein universe like models ($k = +1$), but also static saddle points that are flat ($k = 0$) or even negatively curved ($k = -1$), and these models occur even if $b \neq 0$: we find Einstein static models with $\gamma = 0$ in the $\rho^* \geq 0$-sector and for $\gamma = 2$ there are open and flat models in the $\rho^* \leq 0$-sector.

Altogether, the conditions for allowing for static models in general relativity are changed dramatically when considering the terms corresponding to brane tension and non-local energy density. Instead of finding only the static Einstein universe with $\gamma = 2/3$, and this only in the special case of a flat potential, we now find static models for all values of b.

[3] As explained above, the static model $E^{1/3}$ in [83] with $\tilde{\Omega}_\rho = 0$ and $\tilde{\Omega}_\lambda = 1$ must be excluded.

3.4 Bianchi I models with an exponential potentials

We now turn our attention to the dynamics of Bianchi type I models in the brane-world scenario with an exponential potential. Again, we follow very closely the analysis done in [42] where the perfect fluid case was studied, the only difference being that we have the additional equation (3.14) describing the dynamics of the scalar field. As we have discussed above, the non-zero contributions from the five-dimensional Weyl tensor are ρ^* and $\pi^*_{\mu\nu}$ but since the second one has no evolution equation we will assume for simplicity that $\pi^*_{\mu\nu}\sigma^{\mu\nu} = 0$. This assumption is consistent in the sense that it leads to an integrability condition for $\pi^*_{\mu\nu}$ that is consistent on the brane [145]. The condition $\pi^*_{\mu\nu}\sigma^{\mu\nu} = 0$ also follows naturally from the non-local energy conservation equation 3.33 in the case $\rho^* = 0$, and since the non-local energy can be neglected at very early times, this seems to be a reasonable assumption. Furthermore, since $\pi^*_{\mu\nu}$ corresponds to gravitational waves, it should not affect the dynamics close to the singularity [103, 132] [4].

When introducing appropriate variables, we have to consider two different cases:

$$\text{(A) } \rho^* \geq 0\,,$$
$$\text{(B) } \rho^* \leq 0\,.$$

Again, we will first discuss the dynamics in each sector, and then combine the results in 3.4.3, where we discuss the dynamics of the full state space of the class of the Bianchi I models.

3.4.1 Sector E: $\rho^* \geq 0$

This case is formally very similar to the analysis of sector A. We obtain a compact state space using the dimensionless variables γ, Ω_ρ, Ω_σ, Ω_λ, Ω_{ρ^*}, where

$$\Omega_\sigma \equiv \frac{\sigma^2}{3H^2}\,. \tag{3.74}$$

defines a normalized variable for the shear contribution, and the remaining variables are defined by (3.39) and (3.40). The Friedmann constraint (3.36) reduces to

$$\Omega_\rho + \Omega_\sigma + \Omega_\lambda + \Omega_{\rho^*} = 1\,, \tag{3.75}$$

and is propagated as before.
Using the time derivative

$$' \equiv \epsilon H^{-1} d/dt\,, \tag{3.76}$$

[4]However, evidence has recently built up suggesting that the non-local stress may not be neglected at early times [40, 105]. A final answer to this issue can only be given using the much more complicated bulk field equations instead of the projected brane equations.

the system of dynamical equations is given by

$$
\begin{aligned}
\gamma' &= \epsilon\sqrt{3\gamma}(\gamma-2)[\sqrt{3\gamma}+\epsilon\, b\sqrt{1-\Omega_\sigma-\Omega_\lambda-\Omega_{\rho^*}}]\,, \\
\Omega'_\sigma &= \epsilon\,[(3\gamma-6)(1-\Omega_\sigma)+3\gamma\,\Omega_\lambda+(4-3\gamma)\,\Omega_{\rho^*}]\,\Omega_\sigma\,, \\
\Omega'_\lambda &= \epsilon\,[3\gamma(\Omega_\lambda-1)+(6-3\gamma)\,\Omega_\sigma+(4-3\gamma)\,\Omega_{\rho^*}]\,\Omega_\lambda\,, \\
\Omega'_{\rho^*} &= \epsilon\,[(3\gamma-4)(1-\Omega_{\rho^*})+(6-3\gamma)\,\Omega_\sigma+3\gamma\,\Omega_\lambda]\,\Omega_{\rho^*}\,.
\end{aligned}
\tag{3.77}
$$

The invariant sets of this dynamical system are $\gamma=0$, $\gamma=2$, the isotropic subset $\Omega_\sigma=0$, the brane-tension-less subset $\Omega_\lambda=0$ and the dark-radiation-free subset $\Omega_{\rho^*}=0$ and the vacuum subset $\Omega_k+\Omega_\Lambda+\Omega_\lambda+\Omega_{\rho^*}=1$.

The equilibrium points of this system, their coordinates in state space and their eigenvalues – if well-defined – are given in table 3.6. The only new equilibrium points that we

Table 3.6: This table gives the coordinates and eigenvalues of the critical points with $\rho^*\geq 0$. We have defined $\zeta=-b^{-1}\sqrt{64-15b^2}$. The eigenvalues marked with a dagger only hold if $\Omega_\lambda\neq 1$. The infinite eigenvalue in row 4 actually reads $\lim_{\gamma\to 0}(-6\epsilon-\sqrt{3}b\sqrt{\frac{1-\Omega_\lambda}{\gamma}})=\infty$.

Model	Coordinates	Eigenvalues
K^0_ϵ	$(0,1,0,0)$	-
$\mathrm{K}^2_\epsilon(\Omega_\sigma)$	$(2,\Omega_\sigma,0,0)$	$(6\epsilon+\sqrt{6}b\sqrt{1-\Omega_\sigma},0,-6\epsilon,2\epsilon)$
$\mathrm{m}^0_\epsilon(\Omega_\lambda)$ *for* $b=0$	$(0,0,\Omega_\lambda,0)$	$-2\epsilon(3,3,0,2)^\dagger$
$\mathrm{m}^0_\epsilon(\Omega_\lambda)$ *for* $b\neq 0$	$(0,\Omega_\Lambda,0,\Omega_\lambda,0)$	$(\infty,-6\epsilon,0,-4\epsilon)^\dagger$
m^2_ϵ	$(2,0,1,0)$	$2\epsilon(3,3,3,4)$
R^0_ϵ	$(0,0,0,1)$	-
R^2_ϵ	$(2,0,0,1)$	$2\epsilon(3,-1,-4,-1)$
$\mathrm{F}_+^{b^2/3}$ for $b^2\in[0,6]$	$(\frac{b^2}{3},0,0,0)$	$(\frac{b^2}{2}-3,b^2-6,-b^2,b^2-4)$
$\mathrm{A}_+^{4/3}(b)$ for $b^2\in[4,\infty[$	$(\frac{4}{3},0,0,1-\frac{4}{b^2})$	$(-\frac{1}{2}(1+\zeta),-2,-4,-\frac{1}{2}(1-\zeta))$

obtain in this subsection compared to the class of FLRW models studied above are the anisotropic models K. These have been identified in [42] as the general relativistic vacuum Kasner models with line element (3.30). The vacuum models K^0_ϵ and $K^2_\epsilon(\Omega_\sigma=1)$ must be identified, since the equation of state parameter is irrelevant if $\rho=0$.

3.4.2 Sector F: $\rho^* \leq 0$

In analogy to subsection 3.3.3, we consider the variables γ, Z, $\bar{\Omega}_\rho$, $\bar{\Omega}_\sigma$, $\bar{\Omega}_\lambda$, where $\bar{\Omega}_\sigma$ is defined as Ω_σ in equation (3.74), but normalized with respect to N defined in (3.59), and the remaining variables are defined as in subsection 3.3.3. With these variables the Friedmann constraint (3.36) - which is propagated as usual - reads

$$\bar{\Omega}_\rho + \bar{\Omega}_\sigma + \bar{\Omega}_\lambda = 1 \,. \tag{3.78}$$

With respect to the time derivative $' \equiv N^{-1}d/dt$, the dynamical system becomes

$$\begin{aligned} \gamma' &= \sqrt{3\gamma}(\gamma-2)[\sqrt{3\gamma}Z + b\sqrt{1-\bar{\Omega}_\lambda-\bar{\Omega}_\sigma}] \,, \\ Z' &= [(2-\frac{3}{2}\gamma) + (\frac{3}{2}\gamma - 3)\,\bar{\Omega}_\sigma - \frac{3}{2}\,\gamma\bar{\Omega}_\lambda](1-Z^2) \,, \\ \bar{\Omega}'_\sigma &= [(3\gamma-6)(1-\bar{\Omega}_\sigma) + 3\gamma\,\bar{\Omega}_\lambda]\,Z\,\bar{\Omega}_\sigma \,, \\ \bar{\Omega}'_\lambda &= [(6-3\gamma)\,\bar{\Omega}_\sigma + 3\gamma\,(\bar{\Omega}_\lambda - 1)]\,Z\,\bar{\Omega}_\lambda \,. \end{aligned} \tag{3.79}$$

Similarly to sector C in the state space of FLRW models, this system has the invariant sets $\gamma = 0, 2$, the dark-radiation-free subset $Z = \pm 1$, the isotropic subset $\bar{\Omega}_\sigma = 0$ and the brane-tension-less subset $\bar{\Omega}_\lambda = 0$, as well as the vacuum subset $\bar{\Omega}_k + \bar{\Omega}_\lambda = 1$. The equilibrium points of this system, their coordinates in state space and their eigenvalues are given in table 3.7.

The static models E

In addition to the anisotropic Kasner models, we also obtain the set of saddle points nE, whose coordinates $(\gamma^\bullet, 0, \bar{\Omega}^\bullet_\sigma, \bar{\Omega}^\bullet_\lambda)$ satisfy

$$2 - \frac{3}{2}\gamma^\bullet(1 + \bar{\Omega}^\bullet_\lambda - \bar{\Omega}^\bullet_\sigma) - 3\bar{\Omega}^\bullet_\sigma = 0 \tag{3.80}$$

and

$$\sqrt{\gamma^\bullet}(\gamma^\bullet - 2)b\sqrt{1 - \bar{\Omega}^\bullet_\lambda - \bar{\Omega}^\bullet_\sigma} \,. \tag{3.81}$$

Furthermore, the constants $\rho^*_\bullet, \sigma^2_\bullet$ must satisfy the condition

$$-\rho^*_\bullet \geq \frac{\lambda\kappa^2}{6}\sigma^2_\bullet \tag{3.82}$$

in order for the energy density $\rho_\bullet$ to be non-negative.

The set of static models nE forms a 1-parameter set of equilibrium points which for $b = 0$ degenerates into a 2-dimensional surface. Keeping in mind that there are no static

Table 3.7: This table gives the coordinates and eigenvalues of the critical points with $\rho^* \leq 0$. We have defined $\xi = \sqrt{64 - 15b^2}$, $\phi = \frac{3\gamma-2}{\sqrt{2}\gamma} b\sqrt{3\gamma(1-\bar{\Omega}_\sigma) + 3\bar{\Omega}_\sigma - 2}$ and $\varphi = \sqrt{9\gamma^2(\bar{\Omega}_\sigma - 1) - 18\gamma\bar{\Omega}_\sigma + 12\gamma + 4}$. Notice that φ is a real positive quantity within the allowed range of variables. The eigenvalues marked with a dagger only hold if $\bar{\Omega}_\lambda \neq 1$. The infinite eigenvalue in row 4 actually reads $\lim_{\gamma\to 0}(-6\epsilon - \sqrt{3}b\sqrt{\frac{1-\bar{\Omega}_\lambda}{\gamma}}) = \infty$.

Model	Coordinates	Eigenvalues
K_ϵ^0	$(0, \epsilon, 1, 0)$	-
$\mathrm{K}_\epsilon^2(\bar{\Omega}_\sigma)$	$(2, \epsilon, \bar{\Omega}_\sigma, 0)$	$(6\epsilon + \sqrt{6}b\sqrt{1-\bar{\Omega}_\sigma}, 2\epsilon, 0, -6\epsilon)$
$\mathrm{m}_\epsilon^0(\bar{\Omega}_\lambda)$ *for* $b = 0$	$(0, \epsilon, 0, \bar{\Omega}_\lambda)$	$-2\epsilon(3, 2, 3, 0)^\dagger$
$\mathrm{m}_\epsilon^0(\bar{\Omega}_\lambda)$ *for* $b \neq 0$	$(0, \epsilon, 0, \bar{\Omega}_\lambda)$	$(\infty, -4\epsilon, -6\epsilon, 0)^\dagger$
m_ϵ^2	$(2, \epsilon, 0, 1)$	$2\epsilon(3, 4, 3, 3)$
nE	$(\gamma^\bullet, 0, \bar{\Omega}_\sigma^\bullet, \bar{\Omega}_\lambda^\bullet)$	$(\phi, \varphi, -\varphi, 0)$
$\mathrm{F}_+^{b^2/3}$ for $b^2 \in [0, 6]$	$(\frac{b^2}{3}, 1, 0, 0)$	$(\frac{b^2}{2} - 3, b^2 - 4, b^2 - 6, -b^2)$
$\mathrm{A}_+^{4/3}(b)$ for $b^2 \in [0, 4]$	$(\frac{4}{3}, -\frac{b}{2}, 0, 0)$	$(\frac{1}{4}(b - \xi), \frac{1}{4}(b + \xi), b, 2b)$

vacuum ($\bar{\Omega}_\rho = 0$) solutions with $\bar{\Omega}_\lambda \neq 0$, we find that (3.78), (3.80) and (3.81) has the unique solution

$$\gamma^\bullet = 0,\ \bar{\Omega}_\sigma^\bullet = 2/3,\ \bar{\Omega}_\rho + \bar{\Omega}_\lambda = 1/3\,. \tag{3.83}$$

These models are the same as the ones found in [42]. Their scale factors are given by

$$a_i(t) = \mathrm{e}^{q_i t}\,,$$

where q_i are constants which satisfy

$$\sum_{i=1}^{3} q_i = 0 \quad \text{and} \quad \sum_{i=1}^{3} q_i^2 = 2\sigma_\bullet^2\,.$$

We point out that although $H = 0$, the scale factors $a_i(t)$ are in general not static. It is only the overall scale factor $\prod_{i=1}^3 a_i(t) = 1$ which remains constant. The matter can still expand or contract along the principal shear axes, forming a pancake singularity if one q_i is negative and a cigar type singularity if two of them are negative.

The set nE is an invariant subset of the state space, as can be seen from the full non-linear system (3.79). It is an unstable subset, because it has eigenvalues are of opposite signs. As before, we point out that the Jacobian of the dynamical system (3.79) in general diverges when approaching nE if $\gamma^\bullet = 0$ [5]. In that case a perturbative study confirms that

[5] For $\gamma = b = 0$ the Jacobian is well defined and the non - zero eigenvalues of nE are given by $\pm\varphi\mid_{\gamma=0} = \pm 2$.

Table 3.8: Dynamical character of the critical points of the class of Bianchi I models.

Model	$b=0$	$0<b^2<4$	$b^2=4$	$4<b^2<6$	$b^2=6$	$b^2>6$
K^0_+	saddle	**source**	**source**	**source**	**source**	**source**
K^0_-	saddle	saddle	saddle	saddle	saddle	saddle
$K^2_\epsilon(\Omega_\sigma)$	saddle	saddle	saddle	saddle	saddle	saddle
$m^0_+(\Omega_\lambda)$	**sink**	saddle	saddle	saddle	saddle	saddle
$m^0_-(\Omega_\lambda)$	**source**	**source**	**source**	**source**	**source**	**source**
m^2_+	**source**	**source**	**source**	**source**	**source**	**source**
m^2_-	**sink**	**sink**	**sink**	**sink**	**sink**	**sink**
R_ϵ	saddle	saddle	saddle	saddle	saddle	saddle
nE	saddle	saddle	saddle	saddle	saddle	saddle
$F^{b^2/3}_+$	**sink**	**sink**	saddle	saddle	saddle	-
$A^{4/3}_+(b)$	saddle	saddle	saddle	**sink**	**sink**	**sink**

the points $nE\mid_{\gamma^*=0}$ are also saddle points in state space.

3.4.3 The full Bianchi I state space

As explained in detail in section 3.3.5, the full state space is obtained by matching the sectors describing models with positive and negative non-local energy density ρ^* respectively along the invariant subset describing models with vanishing ρ^*. There are three disconnected sectors: the expanding sector E, the collapsing sector E and sector F.

The dynamical character of the equilibrium points on the $\rho^* = 0$-boundary is then obtained by combining the information gained in the two different sectors, presented in tables 3.6 and 3.7.

The models R_ϵ and $A^{4/3}_+$ for $b^2 \neq 4$ are of course fully described by sector E alone, since they do not lie on the boundary $\rho^* = 0$. Similarly, the static models nE, having non-zero negative dark radiation, are fully characterized by the dynamics in sector F alone.

The dynamical character of the equilibrium points of the state space describing the Bianchi I models is presented in table 3.8 below.

The main question we want to address here is whether the initial singularity is isotropic or not when restricting our analysis to Bianchi I models. We obtain the following result.

For $b = 0$ we find that the only equilibrium points of the dynamical system that are sources and therefore present stable initial configurations are the models m^2_+ representing expanding non-general relativistic BDL model with $\gamma = 2$, and the line of the flat and collapsing models with $\gamma = 0$ denoted $m^0_-(\Omega_\lambda)$ including the flat FLRW model and the maximally non-general relativistic BDL model. All these models are isotropic.

If $b < 0$, the dynamical system possesses in addition to the sources above one further source denoted by K_+^0. This equilibrium point represents the anisotropic Kasner model with $\gamma = 0$.

This means that if $b = 0$, the initial singularity must be isotropic, since the anisotropic models are not stable at early times. If on the other hand the potential $V(\phi)$ is not flat, i.e. $b \neq 0$, we find that the initial singularity could be anisotropic, since the anisotropic Kasner model is stable at early times.

3.5 Discussion and Conclusion

We end this chapter with a comparison of our results with work previously done in this area. In particular, we focus on two issues: Firstly we discuss the occurrence and stability of equilibrium points in the dynamical systems analysis of FLRW and Bianchi I models with and without an exponential potential and compare our work with the results obtained by Campos and Sopuerta [41, 42] for the perfect fluid case. Secondly we discuss the issue of isotropization in the past; here we mainly comment on the early results by Coley *et al* in ([50, 49]).

Addressing the first issue let us state the differences between this work and [41, 42] where the scalar field free analog of the scenario was studied. Formally, the new features of our work is contained in an additional evolution equation for γ, which was considered to be a constant in [41, 42]. This means in particular that we should expect to obtain no more than the equilibrium points already obtained in [42], and the stability of these equilibrium points is expected to be altered since the additional dynamical equation yields an additional eigenvalue in the dynamical systems analysis which can potentially destabilize the equilibrium points. In fact we found that in general we only recover the equilibrium points that Campos et al found for any linear barotropic equation of state for the special values of $\gamma = 0, 2$ which correspond to $\dot{\phi} << V$ and $V << \dot{\phi}$ respectively. It is worthwhile mentioning that the "new" equilibrium points $F_+^{b^2/3}$, $X_+^{2/3}(b)$ and $A_+^{4/3}(b)$ - which are moving in state space as the steepness of the potential characterized by the value of b is increased - are in fact not new. The point $F_+^{b^2/3}$ simply corresponds to the flat FLRW model with equation of state parameter γ depending on the value of b. More interestingly, the points X and A correspond to the bifurcations occurring in [42] for $\gamma = 2/3$, $4/3$ respectively. For each value of b the model $X_+^{2/3}(b)$ corresponds to one point of the line of equilibrium points that occurs for $\gamma = 2/3$ in [41] (see figure 4 in that paper). Similarly, for each value of b the model $A_+^{4/3}(b)$ corresponds to one of the points on the line of equilibrium points joining the models denoted by F_+ and R_+ in [42] for $\gamma = 4/3$ (see figure 2 in that paper). In [41, 42] the stability of the equilibrium points forming the bifurcations was not discussed in much detail. We find here that for a sufficiently steep potential ($b^2 > 4$) the model $A_+^{4/3}(b)$ is a stable sink in the state space of the Bianchi I models, and in fact is the only non-collapsing future attractor for these values of b. In other words: for sufficiently steep potential, all non-singular Bianchi I models asymptotically evolve towards the vacuum model $A_+^{4/3}(b)$ with the scale factor $a(t) = t^{1/2}$ of pure radiation. This model is however unstable when one includes non-vanishing spatial curvature, as shown when studying the class of FLRW models, so we do not expect this model to be an attractor in a more general class of anisotropic models.

We confirm that non-general-relativistic BDL model $m_\pm$ dominates the dynamics at high energies (near the Big Bang and Big Crunch), supporting the claim that this solution is a generic feature of the state space of more general cosmological models in the brane-world scenario [25, 26, 42, 72].

The second important issue we want to address here is the issue of isotropization. In [50] it was claimed that the initial singularity in the braneworld scenario is generically isotropic. It was shown that for $\gamma \geq 1$ the non-general relativistic BDL-model m_+ with scale factor $a(t) = t^{1/3\gamma}$ is a source in the state space of all Bianchi IX models, and it was also claimed that this is the *only* source with physically relevant values $\gamma \geq 1$. In the work presented in this chapter we have considered the more general situation of $\gamma \in [0, 2]$. We have found *all* equilibrium points of the state space and identified the past attractors without imposing any constraints on γ. We confirm that the BDL model with $\gamma = 2$ is a generic past attractor in both the FLRW case and the Bianchi I case.

We point out that in the analysis of the Bianchi I models the anisotropic Kasner model K_+^0 with $\gamma = 0$ is another past attractor unless the potential is flat ($b = 0$). This model has the equation of state $p = -\rho$ and corresponds to a slow rolling scalar field $\dot{\phi} << V$. The stability of this model is in line the results obtained in [145], where it was shown that a general anisotropic inflationary braneworld model undergoes a period of growing anisotropy, which in turn supports further inflation, causing the anisotropy to decay as in General Relativity. Our result has since been confirmed in various different approaches. It has been shown that under certain assumptions, the BDL model is the only expanding local source in the state space of Bianchi IX braneworld models. Here the matter on the brane is assumed to be a combination of a perfect fluid and a minimally coupled scalar field with exponential potential, and the argument has been extended to a scalar field with general potential [103]. The same result holds for the class of Bianchi II models filled with a perfect fluid, a minimally scalar field and a non-negative non-local energy [106]. In [94], it is shown that the initial singularity is isotropic for all Bianchi type space-times filled with a non-zero pressure barotropic fluid.

Chapter 4

Inhomogeneous Perturbations

4.1 Introduction

In this chapter we will investigate in a more general framework whether the initial singularity is generically isotropic in the braneworld scenario. As mentioned in the previous chapter, one of the well known problems of cosmology is to explain the very high degree of isotropy observed in the CMB. In General Relativity, isotropy is a special rather than generic feature of cosmological models, and we need a dynamical mechanism able to produce isotropy. Inflation was proposed, among other reasons, as a way to isotropize the universe. Inflation is effective in this sense, but it needs homogeneous enough initial data in order for inflation to begin [128]. Although one could perhaps adopt the point of view that one smooth enough patch in an otherwise non-smooth initial universe is all that is needed to explain observations, this may not be satisfactory [138].

We here analyze the stability of flat FLRW brane models with respect to general, inhomogeneous and anisotropic perturbations. Since we are interested in early time physics, we restrict ourselves to models with vanishing cosmological constant ($\Lambda = 0$) and background dark radiation ($\rho^* = 0$). We can further restrict our analysis to very large scales, and assume that the physical scales λ of any perturbations are much larger than the Hubble radius H^{-1}, which is equivalent to neglecting Laplacian terms in the evolution equations for the perturbations below. This may at first glance seem (too) restrictive, but this is not the case for the non-inflationary perfect fluid models that are relevant to our discussion. Indeed, it is well known that any wavelength $\lambda < H^{-1}$ at a given time becomes much larger than H^{-1} at early enough times. Because of this crucial property of perturbations for non-inflationary models, our analysis is completely general at early times, i.e. valid for perturbations of any scale λ, as $a \to 0$.

Our aim is to show that at very high energies, where the quadratic matter corrections become dominant, the perturbations decay in the past, supporting the conjecture that the initial singularity is generically homogeneous and isotropic in the RS2 braneworld scenario [41, 42, 49, 50, 52, 83, 84, 103, 106, 145, 176]. We will thus argue that RS2 braneworld cosmology has the very attractive feature of having isotropy built in, and although inflation in this context would still be the most likely way of producing the fluctuations seen in the CMB, there would be no need for special initial conditions for it to start.

This chapter is organized as follows: in section 4.2 we will introduce the methods and notation, motivate the choice of variables corresponding to the physically relevant quantities and present the underlying propagation and constraint equations. In section 4.3 we decompose these equations into scalar, vector and tensor parts. We then evaluate and solve the harmonically decomposed perturbation equations in the low energy, dark radiation and the high energy limit in sections 4.4-4.6. We summarize our results in section 4.7 and conclude in section 4.8.

4.2 Cosmological perturbations

The theory of linear perturbations has been well-established in General Relativity. There are mainly two different approaches: the metric based approach (see for example [34] for a review) and the 1+3 covariant approach (see [65] and references therein). Both formalisms have been extended to the RS2 scenario. The braneworld metric approach has been developed amongst others in [36, 169]. We will here however concentrate on the braneworld extension of the 1+3 formalism [134, 141, 143], where the perturbations are projected onto the rest-spaces orthogonal to the worldlines of the fundamental observers comoving with the matter. These perturbations, being gauge-invariant (GI) because they vanish in the background, are then decomposed into scalar, vector and tensor modes. Since this approach builds on the projected field equations developed in chapter 2, the above mentioned problems with the unknown form of the non-local anisotropic stress $\pi^*_{\mu\nu}$ due to the lack of propagation equations for this quantity remain.

In certain special cases, e.g. when studying scalar (density) or curvature perturbations on very large scales, the brane perturbation equations decouple from the bulk equations and can therefore be explicitly solved [90, 132, 141]. In general however, simplifying assumptions on the form of $\pi^*_{\mu\nu}$ have to be made [134], an ansatz for $\pi^*_{\mu\nu}$ has to be chosen by hand [19, 135] or approximation schemes have to be developed (see e.g. references [103]-[108] in [144]).

The results obtained in this chapter are strictly speaking only true for vanishing non-local anisotropic stress. Since we will be focusing on super-hubble scales, our results are expected to remain valid even for non-vanishing $\pi^*_{\mu\nu}$, since the evolution of $\pi^*_{\mu\nu}$ decouples from the brane perturbations in this special case [90, 141]. Our results for the vector and tensor modes on the other hand are subject to the assumption that the non-local anisotropic stress can be neglected. This assumption might not be legitimate as indicated in [40, 105].

4.2.1 The setup

In the following sections we will present a complete covariant description of general large-scale inhomogeneous perturbations for several different flat ($K = 0$) brane FLRW background models with vanishing cosmological constant on the brane ($\Lambda = 0$). These models are conformally flat or equivalently covariantly characterized by the vanishing of shear, vorticity and acceleration of the fluid flow

$$\sigma_{\mu\nu} = \omega_{\mu\nu} = A_\mu = 0 \tag{4.1}$$

as explained in section 3.2.1. In particular, all spatial gradients of scalars f vanish in the exact background:

$$\mathrm{D}_\mu f = 0\,. \tag{4.2}$$

We describe the matter on the brane by a perfect fluid with barotropic equation of state $p = (\gamma - 1)\rho$. We restrict our analysis to values of $0 < \gamma \leq 2$. The case $\gamma = 0$ can be treated similarly to our analysis below, but using different variables. We do not study this special case here, but refer the reader to [38] for details on how to treat this case.

Since there is no evolution equation for the non-local anisotropic stress π^*_{ab} we restrict our analysis to the case $\pi^*_{ab} = 0$.

This class of background models defines a 3-dimensional state space parameterized by the dimensionless quantities

$$\Omega_\rho \equiv \frac{\kappa^2\rho}{3H^2}, \quad \Omega_\lambda \equiv \frac{\kappa^2\rho^2}{6\lambda H^2}, \quad \Omega_{\rho^*} \equiv \frac{\kappa^2\rho^*}{3H^2}\,. \tag{4.3}$$

Here Ω_ρ is the general relativistic dimensionless density parameter, and Ω_λ, Ω_{ρ^*} correspond to the non-GR contributions to the Friedmann equation. In this way we can classify the various background solutions by their coordinates $(\Omega_\rho\,\Omega_\lambda, \Omega_{\rho^*})$ in the phase space of FLRW models similarly to chapter three.

The point $(1, 0, 0)$ corresponds to the flat GR Friedmann model with $\lambda^{-1} = \rho^* = 0$ and $a(t) \sim t^{2/3\gamma}$. The point $(0, 1, 0)$ corresponds to the high energy model $\mathcal{F}_b$ with $\rho^* = 0$ and $\rho \ll \rho^2/\lambda$; the scale factor is given by $a(t) \sim t^{1/3\gamma}$, which can be found by a limiting process. Finally the point $(0, 0, 1)$ corresponds to a model (R) with scale factor evolution $a(t) \sim t^{1/2}$, satisfying $\rho = \lambda^{-1} = 0$.

In what follows, we develop the perturbation equations for a general background model $(\Omega_\rho\,\Omega_\lambda, \Omega_{\rho^*})$ and decompose the propagation and constraint equations into their respective scalar, vector and tensor contributions in the usual way. We then evaluate these equations for the three backgrounds described above and interpret the results.

The key difference between this analysis and the work in the previous chapter is that we here study *general inhomogeneous* perturbations. In other words, we discuss perturbations *outside* the state space of homogeneous isotropic models in this chapter.

4.2.2 Dimensionless first order quantities

We will now introduce the basic gauge-invariant variables characterizing an almost-FLRW model. We recall the perturbations characterizing an almost-FLRW model in general depend on how we choose to map the background FLRW model on the perturbed (realistic) almost-FLRW model. In particular, we can always choose maps such that e.g. the energy density in the perturbed model equals the energy density in the unperturbed model [63]. This is obviously an undesired property, and we will therefore require that our variables be invariant under the choice of mapping from the background to the realistic model. The only quantities satisfying this conditions are, as stated in the Stewart and Walker lemma [188], (1) scalars that are constant in the background model, (2) tensors that vanish in the background model and (3) constant linear combinations of products of Kronecker deltas. It follows that the basic gauge-invariant quantities for an almost-FLRW model are the kinematic quantities $\sigma_{\mu\nu}$, $\omega_{\mu\nu}$, A_μ , the components of the Weyl tensor $E_{\mu\nu}$, $H_{\mu\nu}$ and the matter tensor components q_μ, $\pi_{\mu\nu}$ as well as the non-local fluid components q^*_μ, $\pi^*_{\mu\nu}$. We will assume that the fluid is an exact perfect fluid even in first order, so that q_μ, $\pi_{\mu\nu} = 0$, and since there is no propagation equation for the non-local anisotropic stress $\pi^*_{\mu\nu}$ we will also assume that $\pi^*_{\mu\nu} = 0$ even in first order.

Furthermore, the orthogonal spatial gradients of the energy density ρ, the fluid expansion H and the dark radiation ρ^* define another set of gauge-invariant variables: $\mathrm{D}_\mu\rho$, $\mathrm{D}_\mu H$, $\mathrm{D}_\mu\rho^*$. Since we only consider a barotropic equation of state, the gradient of the fluid pressure p is simply proportional to the gradient of the fluid energy density:

$$\mathrm{D}_\mu p = (\gamma - 1) D_\mu \rho \tag{4.4}$$

and need not be considered. The physically interesting quantities related to these gradients are the comoving fractional density gradients defined below, but we need to carefully consider how to normalize these quantities in the different energy regimes, since the total fluid is composed of the three different components defined in (2.49). The same is true for the quantity q^*_μ corresponding to the non-local Weyl flux: the relevant quantity is the fractional energy flux obtained by normalizing with respect to the dominant energy contribution in (2.49).

We will now introduce the relevant gauge-invariant quantities. Instead of using the standard kinematic quantities above we define dimensionless expansion normalized variables by

$$\Sigma_{ab} \equiv \frac{\sigma_{ab}}{H}\,,\; W_a \equiv \frac{\omega_a}{H}\,,\; \mathcal{E}_{ab} \equiv \frac{E_{ab}}{H^2}\,,\; \mathcal{H}_{ab} \equiv \frac{H_{ab}}{H^2}\,. \tag{4.5}$$

We emphasize that E_{ab} and the expansion normalized quantity $\mathcal{E}_{ab}$ must not be confused with the 5-D Weyl tensor $E^{(5)}_{ab}$. It turns out that using the dimensionless vorticity variable

$$W^*_a \equiv aHW_a \tag{4.6}$$

simplifies the calculations below, however it should be noted that W_a and not W_a^* is the physically relevant quantity.

Density perturbations are physically characterized by the comoving fractional density gradient defined by [63]:

$$\Delta_a \equiv \frac{a}{\rho}\mathrm{D}_a\rho\,. \tag{4.7}$$

In addition, it is convenient to define the following dimensionless gradients, describing inhomogeneity in the expansion rate H, the non-local energy density ρ^* and the non-local flux q_a^*:

$$Z_a^* \equiv \frac{3a}{H}\mathrm{D}_a H\,,\ U_a^* \equiv \frac{\kappa^2 a}{H^2}\mathrm{D}_a\rho^*\,, Q_a^* \equiv \frac{\kappa^2 a}{H}q_a^*\,. \tag{4.8}$$

Because of the multi-component form of the total fluid energy density in equation (2.49) it makes sense to define specially tailored inhomogeneity variables by normalizing the gradients of energy density, non-local energy density and the non-local energy flux with respect to the dominant energy density term in the Friedmann equation.

For the low-energy limit we use the usual dimensionless perturbation quantities:

$$\Delta_a^{(\mathrm{LE})} \equiv \frac{a}{\rho}\mathrm{D}_a\rho,\ U_a^{(\mathrm{LE})} \equiv \frac{a}{\rho}\mathrm{D}_a\rho^*,\quad Q_a^{(\mathrm{LE})} \equiv \frac{1}{\rho}q_a^*\,. \tag{4.9}$$

However, since ρ_* is the dominant term in the *dark radiation* dominated regime, the appropriate inhomogeneity variables are

$$\Delta_a^{(\mathrm{DR})} \equiv \frac{a}{\rho^*}\mathrm{D}_a\rho,\ U_a^{(\mathrm{DR})} \equiv \frac{a}{\rho^*}\mathrm{D}_a\rho^*,\quad Q_a^{(\mathrm{DR})} \equiv \frac{1}{\rho^*}q_a^*\,. \tag{4.10}$$

In the high-energy limit we define

$$\Delta_a^{(\mathrm{HE})} \equiv \frac{\lambda a}{\rho^2}\mathrm{D}_a\rho\,,\ U_a^{(\mathrm{HE})} \equiv \frac{\lambda a}{\rho^2}\mathrm{D}_a\rho^*\,,\quad Q_a^{(\mathrm{HE})} \equiv \frac{\lambda}{\rho^2}q_a^*\,. \tag{4.11}$$

since the leading energy density term is proportional to ρ^2/λ, which becomes our normalization factor in this case.

Note that although Δ_a is not defined for the exact dark radiation background (R) where $\rho = 0$, it is well-defined in a neighborhood of (R) and since (R) is a saddle point in the phase space of the background homogeneous models [42], this solution can never be exactly attained. It is therefore sufficient to study arbitrarily small but non-zero inhomogeneous perturbations of this model. In other words we may evaluate the perturbation equations *arbitrarily close* to the background (R), but not *on* the exact background itself.

When decomposing the equations into harmonics we will have to deal with the curls of some of the variables. One approach would be to eliminate the curls by deriving second and higher order equations, or alternatively introduce new spatial harmonics corresponding to the curls of the original harmonics. Instead, we find it more convenient to define new variables corresponding to the curls of the original quantities and derive propagation and constraint equations for them. In this way we obtain a complete closed set of linear differential equations which can be easily solved. Note that *all* of the additional propagation and constraint equations have to be satisfied, since these equations are necessary to close the system. We denote the curl of a quantity with an overbar. The key kinematic variables of this type are

$$\bar{W}^*_a \equiv \frac{1}{H}\mathrm{curl}\, W^*_a\,, \bar{\Sigma}_{ab} \equiv \frac{1}{H}\mathrm{curl}\,\Sigma_{ab}\,, \bar{\mathcal{E}}_{ab} \equiv \frac{1}{H}\mathrm{curl}\,\mathcal{E}_{ab}\,, \bar{\mathcal{H}}_{ab} \equiv \frac{1}{H}\mathcal{H}_{ab} \tag{4.12}$$

and the quantity corresponding to the curl of the non-local flux gradient is given by

$$\bar{Q}^*_a \equiv \frac{1}{H}\mathrm{curl}\, Q^*_a\,. \tag{4.13}$$

Finally, we will also use the dimensionless time derivative defined as before:

$${}' = \frac{d}{d\tau} = \frac{d}{d\,ln(a)} = \frac{1}{H}\frac{d}{dt} \tag{4.14}$$

in order to decouple the evolution of the scale factor from the evolution equation for the first-order quantities defined above.

4.2.3 Dimensionless Linearized Propagation and Constraint Equations

The complete set of propagation and constraint equations for the kinematic and non-local quantities on the brane were presented in 2 (2.67)-(2.70). Here we linearize these equations around the flat homogeneous isotropic background models with $\Lambda = 0$, and rewrite these equations in terms of the dimensionless variables defined by (4.5-4.6). We will then use these equations to derive similar propagation and constraint equations for the curls defined in (4.12)-(4.13).

We begin with the linearized Raychaudhuri equation (2.54)

$$-\frac{H'}{H} = (1+q) + \tfrac{\gamma-1}{\gamma}\tfrac{1}{3aH^2}\mathrm{D}^a\Delta_a, \tag{4.15}$$

where $q = \frac{3}{2}\gamma\Omega_\rho + 3\gamma\Omega_\lambda + 2\Omega_{\rho^*} - 1$ is the usual deceleration parameter, and the conservation equations (2.67)- (2.70) which become

$$\dot{\rho} + 3\gamma H\rho = 0\,, \tag{4.16}$$

$$(\gamma-1)\mathrm{D}_a\rho + \gamma\rho A_a = 0\,, \tag{4.17}$$

$$\dot{\rho}^* + 4H\rho^* + \mathrm{D}^a q^*_a = 0\,, \tag{4.18}$$

$$\dot{q}^*_a + 4Hq^*_a + \tfrac{1}{3}\mathrm{D}_a\rho^* + \tfrac{4}{3}\rho^* A_a = -\tfrac{\rho}{\lambda}\gamma\mathrm{D}_a\rho\,. \tag{4.19}$$

The generalized Friedmann equation on the brane for a flat, homogeneous isotropic background with vanishing 4-D cosmological constant Λ is obtained from the Gauss-Codazzi equation (2.66) and given by

$$H^2 = \frac{\kappa^2\rho}{3} + \frac{\kappa^2\rho^2}{6\lambda} + \frac{\kappa^2\rho^*}{3}\,. \tag{4.20}$$

The remaining propagation equations are given by

$$\begin{aligned}
W^{*\prime}_a &= (3\gamma-4)W^*_a\,,\\
\Sigma'_{ab} &= (q-1)\Sigma_{ab} - \mathcal{E}_{ab} - \tfrac{\gamma-1}{\gamma}(aH)^{-2}\mathrm{D}_{<a}\Delta_{b>}\,,\\
\mathcal{E}'_{ab} &= (2q-1)\mathcal{E}_{ab} - (q+1)\Sigma_{ab} - \tfrac{1}{2aH^2}\mathrm{D}_{<a}Q^*{}_{b>} + \bar{\mathcal{H}}_{ab}\,,\\
\mathcal{H}'_{ab} &= (2q-1)\mathcal{H}_{ab} - \bar{\mathcal{E}}_{ab}\,,
\end{aligned} \tag{4.21}$$

which are subject to following dimensionless constraints:

$$\begin{aligned}
a\mathrm{D}^bW^*_b &= 0\,,\\
a\mathrm{D}^b\Sigma_{ab} &= \bar{W}^*_a + \tfrac{2}{3}Z^*_a - Q^*_a\,,\\
H\bar{\Sigma}_{ab} &= -\tfrac{1}{aH}\mathrm{D}_{<a}W^*_{b>} + H\mathcal{H}_{ab}\,,\\
a\mathrm{D}^b\mathcal{E}_{ab} &= (\Omega_\rho + 2\Omega_\lambda)\Delta_a + \tfrac{1}{3}U^*_a - Q^*_a\,,\\
a\mathrm{D}^b\mathcal{H}_{ab} &= 2(q+1)W^*_a - \tfrac{1}{2}\bar{Q}^*_a\,.
\end{aligned} \tag{4.22}$$

Equations for the inhomogeneity variables Δ_a, Z^*_a, Q^*_a and Q^*_a can be derived using the conservation equations (4.16)- (4.19). They are given by

$$\begin{aligned}
\Delta_a{}' &= (3\gamma-3)\Delta_a - \gamma Z^*_a\,,\\
Z^{*\prime}_a &= (q-1)Z^*_a - \tfrac{3}{2}[\,\Omega_\rho + (6\gamma+2)\Omega_\lambda - \tfrac{4\gamma-4}{\gamma}\Omega_{\rho^*}\,]\,\Delta_a - U^*_a - 6(\gamma-1)\bar{W}_a - \tfrac{\gamma-1}{\gamma}\tfrac{1}{H^2}\mathrm{D}^2\Delta_a\,,\\
Q^{*\prime}_a &= (q-2)Q^*_a - \tfrac{1}{3}U^*_a + (4\tfrac{\gamma-1}{\gamma}\Omega_{\rho^*} - 6\gamma\Omega_\lambda)\Delta_a\,,\\
U^{*\prime}_a &= (2q-2)U^*_a - 4\Omega_{\rho^*}\,Z^*_a + \tfrac{12\gamma-12}{\gamma}\Omega_{\rho^*}\,\Delta_a - \tfrac{1}{H^2}\mathrm{D}_a(\mathrm{D}^bQ^*_b)\,.
\end{aligned} \tag{4.23}$$

Finally using the definitions (4.12) and (4.13) we obtain equations for the curls of the original quantities:

$$\begin{aligned}
\bar{W}^{*\prime}_a &= (q+3\gamma-4)\bar{W}^*_a\,,\\
\bar{\Sigma}'_{ab} &= (2q-1)\bar{\Sigma}_{ab} - \bar{\mathcal{E}}_{ab}\,,\\
\bar{\mathcal{E}}'_{ab} &= (3q-1)\bar{\mathcal{E}}_{ab} - (q+1)\bar{\Sigma}_{ab} + \tfrac{3}{2H^2}\mathrm{D}_{<a}\mathrm{D}^c\mathcal{H}_{b>c} - \tfrac{1}{4aH^2}\mathrm{D}_{<a}\bar{Q}^*_{b>} - \tfrac{1}{H^2}\mathrm{D}^2\mathcal{H}_{ab}\,,\\
\bar{\mathcal{H}}'_{ab} &= (3q-1)\bar{\mathcal{H}}_{ab} - \tfrac{3}{2H^2}\mathrm{D}_{<a}\mathrm{D}^c\mathcal{E}_{b>c} + \tfrac{1}{H^2}\mathrm{D}^2\mathcal{E}_{ab}\,,\\
\bar{Q}^{*\prime}_a &= (2q-2)\bar{Q}^*_a + 4[(6\gamma-8)\Omega_{\rho^*} - 9\gamma^2\Omega_\lambda]W^*_a\,,
\end{aligned} \tag{4.24}$$

which are subject to the following constraints, obtained by taking the curls of (4.22):

$$\begin{aligned}
a\mathrm{D}^b\bar{W}^*_b &= 0\,,\\
a\mathrm{D}^b\bar{\Sigma}_{ab} &= 2(q+1)W^*_a - \tfrac{1}{2}\bar{Q}^*_a - \tfrac{1}{2H^2}\mathrm{D}^2 W^*_a\,,\\
\bar{\mathcal{H}}_{ab} &= \tfrac{1}{2aH}\mathrm{D}_{<a}\bar{W}^*_{b>} - \tfrac{1}{H^2}\mathrm{D}^2\Sigma_{ab} + \tfrac{3}{2H^2}\mathrm{D}_{<a}\mathrm{D}^c\Sigma_{b>c}\,,\\
a\mathrm{D}^b\bar{\mathcal{E}}_{ab} &= 2(q+1)W^*_a - \tfrac{1}{2}\bar{Q}^*_a\,,\\
a\mathrm{D}^b\bar{\mathcal{H}}_{ab} &= (q+1)\bar{W}^*_a + \tfrac{1}{4H^2}(\mathrm{D}^2 Q^*_a - \mathrm{D}_a(\mathrm{D}^b Q^*_b))\,,\\
a\mathrm{D}^a\bar{Q}^*_a &= 0\,.
\end{aligned} \tag{4.25}$$

We emphasize that one must check whether the constraints 4.22 and 4.25 are propagated, i.e. whether the derivative of each constraint is a linear combination of the other ones. If this is not the case, a constraint can give rise to a possibly infinite chain of new constraints as explained in the previous chapter (see section 3.1). This would indicate that the linearized system is inconsistent, and an alternative linearization scheme must be found.

The system obtained above is in fact consistent: the harmonically decomposed constraints are propagated, as can be easily seen from equations 4.28, 4.30 and 4.33.

4.3 Harmonic Decomposition

In order to solve these equation we employ the standard approach [37] of expanding the variables in these equations in terms of scalar (S), vector (V) and tensor (T) harmonics Q. This yields a covariant and gauge invariant splitting into three sets of evolution and constraint equations for scalar, vector and tensor modes.

Thus a scalar X, vector X^a (orthogonal to u^a) and tensor X^{ab} (orthogonal to u^a) can be expanded as follows

$$\begin{aligned}
X &= X^S Q^S\,,\\
X_a &= k^{-1}X^S Q^S_a + X^V Q^V_a\,,\\
X_{ab} &= k^{-2}X^S Q^S_{ab} + k^{-1}X^V Q^V_{ab} + X^T Q^T_{ab}\,.
\end{aligned} \tag{4.26}$$

In what follows we drop the subscripts S, V, T and also restrict our analysis to the long wavelength limit defined by $\frac{k^2}{a^2H^2} << 1$. For more details on the definition and properties of the harmonics see appendix B.

Scalar Perturbations

In the long wavelength limit the scalar evolution equations for the kinematics which follow after expanding (4.21) in of scalar harmonics are given by

$$\begin{aligned}
\Sigma' &= (q-1)\Sigma - \mathcal{E}\,,\\
\mathcal{E}' &= (2q-1)\mathcal{E} - (q+1)\Sigma\,,
\end{aligned} \tag{4.27}$$

and these are subject to the following constraints (which follow from (4.22):

$$\begin{aligned} W &= \bar{W} = \mathcal{H} = \bar{\mathcal{H}} = \bar{\Sigma} = \bar{\mathcal{E}} = \bar{Q} = 0 \\ 2Z_* &= 2\Sigma + 3Q_* \,, \\ 3(\Omega_\rho + 2\Omega_\lambda)\Delta &= 2\mathcal{E} + 3Q_* - U_* \,. \end{aligned} \tag{4.28}$$

In addition we have scalar evolution equations for the inhomogeneity variables which follow from (4.23) after harmonic analysis

$$\begin{aligned} \Delta' &= (3\gamma - 3)\Delta - \gamma Z_* \,, \\ Z_*' &= (q-1)Z_* - [\tfrac{3}{2}\Omega_\rho + (9\gamma+3)\Omega_\lambda - 6\tfrac{\gamma-1}{\gamma}\Omega_{\rho^*}]\,\Delta - U_* \,, \\ Q_*' &= (q-2)Q_* - \tfrac{1}{3}U_* + (4\tfrac{\gamma-1}{\gamma}\Omega_{\rho^*} - 6\gamma\Omega_\lambda)\,\Delta \,, \\ U_*' &= (2q-2)U_* - 4\Omega_{\rho^*}\, Z_* + 12\tfrac{\gamma-1}{\gamma}\Omega_{\rho^*}\,\Delta \,. \end{aligned} \tag{4.29}$$

Vector Perturbations

Expanding equations (4.21) and (4.24) in terms of vector harmonics we obtain the following evolution equations for the kinematic and non-local quantities together with their curls:

$$\begin{aligned} {W_*}' &= (3\gamma - 4)W_* \,, \\ \Sigma' &= (q-1)\Sigma - \mathcal{E} - 2(\gamma-1)\bar{W}_* \,, \\ \mathcal{E}' &= (2q-1)\mathcal{E} - (q+1)\Sigma + \bar{\mathcal{H}} \,, \\ \mathcal{H}' &= (2q-1)\mathcal{H} - \bar{\mathcal{E}} \,, \\ \bar{W}_*' &= (3\gamma - 4 + q)\bar{W}_* \,, \\ \bar{\Sigma}' &= (2q-1)\bar{\Sigma} - \bar{\mathcal{E}} \,, \\ \bar{\mathcal{E}}' &= (3q-1)\bar{\mathcal{E}} - (q+1)\bar{\Sigma} \,, \\ \bar{\mathcal{H}}' &= (3q-1)\bar{\mathcal{H}} \,, \\ \bar{Q}_*' &= (2q-2)\bar{Q}_* + 4[(6\gamma-8)\Omega_{\rho^*} - 9\gamma^2\Omega_\lambda]W_* \,. \end{aligned}$$

These equations are subject to the following constraints which are obtained from (4.22) and (4.25):

$$\begin{aligned} 4Z_* &= 3\Sigma + 6Q_* - 6\bar{W}_* \,, \\ 6(\Omega_\rho + 2\Omega_\lambda)\Delta &= 3\mathcal{E} + 6Q_* - 2U_* \,, \\ \bar{\Sigma} &= \mathcal{H} \,, \\ \mathcal{H} &= 4(q+1)W_* - \bar{Q}_* \\ \bar{\mathcal{H}} &= 2(q+1)\bar{W}_* = 0 \,, \\ \bar{\Sigma} &= 4(q+1)W_* - \bar{Q}_* \,, \\ \bar{\mathcal{E}} &= 4(q+1)W_* - \bar{Q}_* \,. \end{aligned} \tag{4.30}$$

Equations for the vector parts of the inhomogeneity variables follow from (4.23) and are given by

$$\begin{aligned}
\Delta' &= (3\gamma-3)\Delta - \gamma Z_* \,, \\
{Z_*}' &= (q-1)Z_* - [\tfrac{3}{2}\Omega_\rho + 3(3\gamma+1)\Omega_\lambda - 6\tfrac{\gamma-1}{\gamma}\Omega_{\rho^*}\,]\,\Delta - U_* - 6(\gamma-1)\bar{W}_* \,, \\
{Q_*}' &= (q-2)Q_* - \tfrac{1}{3}U_* + (4\tfrac{\gamma-1}{\gamma}\Omega_{\rho^*} - 6\gamma\Omega_\lambda)\Delta \,, \\
{U_*}' &= (2q-2)U_* - 4\Omega_{\rho^*}\,Z_* + 12\tfrac{\gamma-1}{\gamma}\Omega_{\rho^*}\,\Delta \,.
\end{aligned} \tag{4.31}$$

Tensor Perturbations

Finally the long wavelength behavior of tensor perturbations are obtained by expanding (4.21) and (4.24) in terms of tensor harmonics:

$$\begin{aligned}
\Sigma' &= (q-1)\Sigma - \mathcal{E} \,, \\
\mathcal{E}' &= (2q-1)\mathcal{E} - (q+1)\Sigma \,, \\
\mathcal{H}' &= (2q-1)\mathcal{H} - \bar{\mathcal{E}} \,, \\
\bar{\Sigma}' &= (2q-1)\bar{\Sigma} - \bar{\mathcal{E}} \,, \\
\bar{\mathcal{E}}' &= (3q-1)\bar{\mathcal{E}} - (q+1)\bar{\Sigma} \,,
\end{aligned} \tag{4.32}$$

subject to the following constraints

$$\bar{\mathcal{H}} = 0 \,, \;\; \bar{\Sigma} = \mathcal{H} \,. \tag{4.33}$$

4.4 Low Energy Limit: The GR Background

We begin with perturbations in the low - energy limit, defined by $\rho \gg \rho^2/\lambda$, ρ^* or $\Omega_\rho \gg \Omega_\lambda, \Omega_{\rho^*}$. We therefore evaluate the perturbation equations (4.28-4.32) in the limit $(\Omega_\rho, \Omega_\lambda, \Omega_{\rho^*}) \to (1,0,0)$. Using the energy conservation equation (2.67), the Friedmann equation (4.20) can be solved to give the background scale factor a and the Hubble parameter H:

$$a(t) = (t/t_0)^{2/3\gamma} \,, \quad H = H_0 a^{-3\gamma/2} \,, \tag{4.34}$$

where we fix an arbitrary initial condition by choosing $a_0 = a(t_0) = 1$. The deceleration parameter is given by

$$q = \tfrac{3}{2}\gamma - 1 \,, \tag{4.35}$$

and the energy density scales as usual,

$$\rho = \rho_0 a^{-3\gamma} \,. \tag{4.36}$$

The constants of integration are related via the Friedmann constraint as $H_0^2 = \frac{\kappa^2}{3}\rho_0$.

Scalar Perturbations

In this case the propagation equations for the kinematic quantities are

$$\begin{aligned} \Sigma' &= (\tfrac{3}{2}\gamma - 2)\Sigma - \mathcal{E}\,, \\ \mathcal{E}' &= (3\gamma - 3)\mathcal{E} - \tfrac{3}{2}\gamma\Sigma\,, \end{aligned} \tag{4.37}$$

while the constraints are given by

$$\begin{aligned} W &= \bar{W} = \mathcal{H} = \bar{\mathcal{H}} = \bar{\Sigma} = \bar{\mathcal{E}} = \bar{Q} = 0\,, \\ 2Z_* &= 2\Sigma + 3Q_*\,, \\ 3\Delta &= 2\mathcal{E} + 3Q_* - U_*\,, \end{aligned} \tag{4.38}$$

and equations for the inhomogeneity variables become

$$\begin{aligned} \Delta' &= (3\gamma - 3)\Delta - \gamma Z_*\,, \\ {Z_*}' &= (\tfrac{3}{2}\gamma - 2)Z_* - \tfrac{3}{2}\Delta - U_*\,, \\ {Q_*}' &= (\tfrac{3}{2}\gamma - 3)Q_* - \tfrac{1}{3}U_*\,, \\ {U_*}' &= (3\gamma - 4)U_*\,. \end{aligned} \tag{4.39}$$

The above equations can be easily solved to give

$$\begin{aligned} \Sigma &= \Sigma_0 a^{3\gamma-2} + \Sigma_1 a^{\frac{3}{2}\gamma-3}\,, \\ \mathcal{E} &= -\tfrac{3}{2}\gamma\Sigma_0 a^{3\gamma-2} + \Sigma_1 a^{\frac{3}{2}\gamma-3}\,, \end{aligned} \tag{4.40}$$

and

$$\begin{aligned} \Delta &= -\gamma\Sigma_0 a^{3\gamma-2} + (\tfrac{2}{3}\Sigma_1 + Q_0^*)a^{\frac{3}{2}\gamma-3} - \tfrac{\gamma}{3\gamma-2}U_0^* a^{3\gamma-4}\,, \\ Z_* &= \Sigma_0 a^{3\gamma-2} + (\Sigma_1 + \tfrac{3}{2}Q_0^*)a^{\frac{3}{2}\gamma-3} - \tfrac{1}{3\gamma-2}U_0^* a^{3\gamma-4}\,, \\ Q_* &= Q_0^* a^{\frac{3}{2}\gamma-3} - \tfrac{2}{3}\tfrac{1}{3\gamma-2}U_0^* a^{3\gamma-4}\,, \\ U_* &= U_0^* a^{3\gamma-4} \end{aligned} \tag{4.41}$$

for $\gamma \neq \frac{2}{3}$, and

$$\begin{aligned} \Delta &= -\tfrac{2}{3}\Sigma_0 + (\tfrac{2}{3}\Sigma_1 + Q_0^* - \tfrac{1}{3}U_0^*)a^{-2} - \tfrac{1}{3}U_0^* \ln a\; a^{-2}\,, \\ Z_* &= \Sigma_0 + (\Sigma_1 + \tfrac{3}{2}Q_0^*)a^{-2} - \tfrac{1}{2}U_0^* \ln a\; a^{-2}\,, \\ Q_* &= Q_0^* a^{-2} - \tfrac{1}{3}U_0^* \ln a\; a^{-2}\,, \\ U_* &= U_0^* a^{-2} \end{aligned} \tag{4.42}$$

Table 4.1: Large scale contributions of the different modes to the geometric and kinematic quantities for the low energy background. We assume $0 < \gamma \leq 2$, $\gamma \neq \frac{2}{3}$, and we omit non-zero constant coefficients. The first line in this table should be read as $\Sigma = \alpha\,\Sigma_0\,a^{3\gamma-2} + \beta\,\Sigma_1\,a^{\frac{3}{2}\gamma-3}$, where α, β are some non-zero constants, for all scalar, vector and tensor modes. General Relativity is recovered when $Q_0^* = U_0^* = 0$.

harmonic	scalar				vector				tensor			
mode	$a^{3\gamma-2}$	$a^{\frac{3}{2}\gamma-3}$	$a^{3\gamma-4}$	$a^{\frac{9}{2}\gamma-5}$	$a^{3\gamma-2}$	$a^{\frac{3}{2}\gamma-3}$	$a^{3\gamma-4}$	$a^{\frac{9}{2}\gamma-5}$	$a^{3\gamma-2}$	$a^{\frac{3}{2}\gamma-3}$	$a^{3\gamma-4}$	$a^{\frac{9}{2}\gamma-3}$
Σ	Σ_0	Σ_1	-	-	Σ_0	Σ_1	-	-	Σ_0	Σ_1	-	-
$\mathcal{E}$	Σ_0	Σ_1	-	-	Σ_0	Σ_1	-	-	Σ_0	Σ_1	-	-
$\mathcal{H}$	-	-	-	-	-	-	$\mathcal{H}_0$	-	-	-	$\mathcal{H}_0$	$\mathcal{H}_1$
W	-	-	-	-	-	-	-	W_0^*	-	-	-	-
Δ	Σ_0	$2\Sigma_1 + 3Q_0^*$	U_0^*	-	Σ_0	$\Sigma_1 + 2Q_0^*$	U_0^*	-	-	-	-	-
Z^*	Σ_0	$2\Sigma_1 + 3Q_0^*$	U_0^*	-	Σ_0	$\Sigma_1 + 2Q_0^*$	U_0^*	-	-	-	-	-
$Q^{(\mathrm{LE})}$	-	-	Q_0^*	U_0^*	-	-	Q_0^*	U_0^*	-	-	-	-
$U^{(\mathrm{LE})}$	-	-	U_0^*	-	-	-	U_0^*	-	-	-	-	-

if $\gamma = \frac{2}{3}$.

Here Σ_0, Σ_1, Q_0^*, U_0^* are arbitrary constants of integration, corresponding to the four independent modes.

Finally the solutions for Q_*, U_* into can be converted into the physical quantities $Q^{(\mathrm{LE})}$, $U^{(\mathrm{LE})}$ (which correspond to the scalar modes of Q_a, U_a defined in (4.9)) using the background solutions for H and ρ:

$$\begin{aligned} Q^{(\mathrm{LE})} &= \frac{1}{3H_0} Q_0^* a^{3\gamma-4} - \frac{2}{9H_0}\frac{1}{3\gamma-2} U_0^* a^{\frac{9}{2}\gamma-5}\,, \\ U^{(\mathrm{LE})} &= \frac{1}{3} U_0^* a^{3\gamma-4} \end{aligned} \tag{4.43}$$

for $\gamma \neq \frac{2}{3}$ and

$$\begin{aligned} Q^{(\mathrm{LE})} &= \frac{1}{3H_0} Q_0^* a^{-2} - \frac{1}{9H_0} U_0^* \ln a\, a^{-2}\,, \\ U^{(\mathrm{LE})} &= \frac{1}{3} U_0^* a^{-2}\,. \end{aligned} \tag{4.44}$$

if $\gamma = \frac{2}{3}$.

Vector Perturbations

For vector perturbations the complete set of propagation equations for the kinematic and non-local quantities are given by the ten-dimensional system

$$W_*{}' = (3\gamma - 4) W_*\,,$$

$$
\begin{aligned}
\Sigma' &= (\tfrac{3}{2}\gamma - 2)\Sigma - \mathcal{E}\,,\\
\mathcal{E}' &= (3\gamma - 3)\mathcal{E} - \tfrac{3}{2}\gamma\Sigma\,,\\
\mathcal{H}' &= (3\gamma - 3)\mathcal{H} - \bar{\mathcal{E}}\,,\\
\bar{\Sigma}' &= (3\gamma - 3)\bar{\Sigma} - \bar{\mathcal{E}}\,,\\
\bar{\mathcal{E}}' &= (\tfrac{9}{2}\gamma - 4)\bar{\mathcal{E}} - \tfrac{3}{2}\gamma\bar{\Sigma}\,,\\
\bar{Q}_*' &= (3\gamma - 4)\bar{Q}_*\,.
\end{aligned}
\tag{4.45}
$$

subject to the following constraints

$$
\begin{aligned}
4Z_* &= 3\Sigma + 6Q_*\,,\\
6\Delta &= 3\mathcal{E} + 6Q_* - 2U_*\,,\\
\mathcal{H} &= \bar{\Sigma} = \bar{\mathcal{E}} = 6\gamma W_* - \bar{Q}_*\,,\\
\bar{\mathcal{H}} &= \bar{W}_* = 0\,,
\end{aligned}
\tag{4.46}
$$

while the propagation equations for the inhomogeneity variables are

$$
\begin{aligned}
\Delta' &= (3\gamma - 3)\Delta - \gamma Z_*\,,\\
Z_*{}' &= (\tfrac{3}{2}\gamma - 2)Z_* - \tfrac{3}{2}\Delta - U_*\,,\\
Q_*{}' &= (\tfrac{3}{2}\gamma - 3)Q_* - \tfrac{1}{3}U_*\,,\\
U_*{}' &= (3\gamma - 4)U_*\,.
\end{aligned}
\tag{4.47}
$$

Solutions can again be easily obtained and are given by

$$
\begin{aligned}
\Sigma &= \Sigma_0 a^{3\gamma-2} + \Sigma_1 a^{\frac{3}{2}\gamma-3}\,,\\
\mathcal{E} &= -\tfrac{3}{2}\gamma\Sigma_0 a^{3\gamma-2} + \Sigma_1 a^{\frac{3}{2}\gamma-3}\,,\\
\mathcal{H} &= \bar{\Sigma} = \bar{\mathcal{E}} = \mathcal{H}_0 a^{3\gamma-4}\,,\\
W &= (aH)^{-1}W_* = \tfrac{3}{2}\gamma W_0^* a^{\frac{9}{2}\gamma-5}\,,\\
\bar{Q}_* &= (6\gamma W_0^* - \mathcal{H}_0)a^{3\gamma-4}\,,
\end{aligned}
\tag{4.48}
$$

and

$$
\begin{aligned}
\Delta &= -\tfrac{3}{4}\gamma\Sigma_0 a^{3\gamma-2} + (\tfrac{1}{2}\Sigma_1 + Q_0^*)a^{\frac{3}{2}\gamma-3} - \tfrac{\gamma}{3\gamma-2}U_0^* a^{3\gamma-4}\,,\\
Z_* &= \tfrac{3}{4}\Sigma_0 a^{3\gamma-2} + \tfrac{3}{4}(\Sigma_1 + 2Q_0^*)a^{\frac{3}{2}\gamma-3} - \tfrac{1}{3\gamma-2}U_0^* a^{3\gamma-4}\,,\\
Q^{(\mathrm{LE})} &= \tfrac{1}{3H_0}Q_0^* a^{3\gamma-4} - \tfrac{2}{9H_0}\tfrac{1}{3\gamma-2}U_0^* a^{\frac{9}{2}\gamma-5}\,,\\
U^{(\mathrm{LE})} &= \tfrac{1}{3}U_0^* a^{3\gamma-4}
\end{aligned}
\tag{4.49}
$$

for $\gamma \neq \frac{2}{3}$, and

$$\begin{aligned}
\Delta &= -\tfrac{1}{2}\Sigma_0 + (\tfrac{1}{2}\Sigma_1 + Q_0^* - \tfrac{1}{3}U_0^*)a^{-2} - \tfrac{1}{3}U_0^* \ln a\; a^{-2}\,, \\
Z_* &= \tfrac{3}{4}\Sigma_0 + (\tfrac{3}{4}\Sigma_1 + \tfrac{3}{2}Q_0^*)a^{-2} - \tfrac{1}{2}U_0^* \ln a\; a^{-2}\,, \\
Q^{(\mathrm{LE})} &= \tfrac{1}{3H_0}Q_0^* a^{-2} - \tfrac{1}{9H_0}U_0^* \ln a\; a^{-2}\,, \\
U^{(\mathrm{LE})} &= \tfrac{1}{3}U_0^* a^{-2}
\end{aligned} \tag{4.50}$$

if $\gamma = \frac{2}{3}$. Again, we have converted the solutions for Q_*, U_* into the physical quantities $Q^{(\mathrm{LE})}$, $U^{(\mathrm{LE})}$. There are six independent modes corresponding to the constants of integration Σ_0, Σ_1, W_0, $\mathcal{H}_0$, Q_0^* and U_0^*.

Tensor Perturbations

For tensor perturbations the propagation equations in the long wavelength limit are

$$\begin{aligned}
\Sigma' &= (\tfrac{3}{2}\gamma - 2)\Sigma - \mathcal{E}\,, \\
\mathcal{E}' &= (3\gamma - 3)\mathcal{E} - \tfrac{3}{2}\gamma\Sigma\,, \\
\mathcal{H}' &= (3\gamma - 3)\mathcal{H} - \bar{\mathcal{E}}\,, \\
\bar{\Sigma}' &= (3\gamma - 3)\bar{\Sigma} - \bar{\mathcal{E}}\,, \\
\bar{\mathcal{E}}' &= (\tfrac{9}{2}\gamma - 4)\bar{\mathcal{E}} - \tfrac{3}{2}\gamma\bar{\Sigma}\,,
\end{aligned} \tag{4.51}$$

subject to the following constraints:

$$\bar{\mathcal{H}} = 0, \quad \bar{\Sigma} = \mathcal{H}\,. \tag{4.52}$$

The solutions are

$$\begin{aligned}
\Sigma &= \Sigma_0 a^{3\gamma-2} + \Sigma_1 a^{\frac{3}{2}\gamma-3}\,, \\
\mathcal{E} &= -\tfrac{3}{2}\gamma\Sigma_0 a^{3\gamma-2} + \Sigma_1 a^{\frac{3}{2}\gamma-3}\,, \\
\mathcal{H} &= \bar{\Sigma} = \mathcal{H}_0 a^{3\gamma-4} + \mathcal{H}_1 a^{\frac{9}{2}\gamma-3}\,, \\
\bar{\mathcal{E}} &= \mathcal{H}_0 a^{3\gamma-4} - \tfrac{3}{2}\gamma\mathcal{H}_1 a^{\frac{9}{2}\gamma-3}\,,
\end{aligned} \tag{4.53}$$

where Σ_0, Σ_1, $\mathcal{H}_0$, $\mathcal{H}_1$ are four independent constants of integration.

4.5 The Dark Radiation Era

The dark radiation dominated regime is characterized by $\rho^* \gg \rho$, ρ^2/λ or $\Omega_{\rho^*} \gg \Omega_\rho, \Omega_\lambda$, so we now evaluate the perturbation equations in the limit $(\Omega_\rho, \Omega_\lambda, \Omega_{\rho^*}) \to (0, 0, 1)$.

The background solution (R), has the same metric as a flat radiation FLRW model, with $\rho = 0$ and $\rho^* = \rho_0^* a^{-4}$. The scale factor a, Hubble parameter H and deceleration parameter q are given by

$$a(t) = (t/t_0)^{1/2}\,, \quad H = H_0 a^{-2}, \quad q = 1\,. \tag{4.54}$$

As explained in 4.2.2 the perturbation equations are only defined for small but non-zero energy density ρ, or equivalently arbitrarily close but not *on* the exact background (R). We therefore use $\rho = \rho_0 a^{-3\gamma}$ for small but non-zero values of ρ_0.

Scalar Perturbations

In the case of scalar perturbations the propagation equations reduce to:

$$\Sigma' = -\mathcal{E}\,, \tag{4.55}$$
$$\mathcal{E}' = \mathcal{E} - 2\Sigma\,, \tag{4.56}$$

subject to the following constraints:

$$\begin{aligned} W &= \bar{W} = \mathcal{H} = \bar{\mathcal{H}} = \bar{\Sigma} = \bar{\mathcal{E}} = \bar{Q} = 0 \\ 2\Sigma + 3Q_* &= 2Z_*\,, \\ 2\mathcal{E} + 3Q_* - U_* &= 3(\Omega_\rho + 2\Omega_\lambda)\Delta\,, \end{aligned} \tag{4.57}$$

while the equations for the inhomogeneity variables are

$$\begin{aligned} \Delta' &= (3\gamma - 3)\Delta - \gamma Z_*\,, \\ Z_*' &= 6\tfrac{\gamma-1}{\gamma}\Delta - U_*\,, \\ Q_*' &= -Q_* - \tfrac{1}{3}U_* + 4\tfrac{\gamma-1}{\gamma}\Delta\,, \\ U_*' &= -4Z_* + 12\tfrac{\gamma-1}{\gamma}\Delta\,. \end{aligned} \tag{4.58}$$

The following solutions can then be obtained:

$$\begin{aligned} \Sigma &= \Sigma_0 a^2 + \Sigma_1 a^{-1}\,, \\ \mathcal{E} &= -2\Sigma_0 a^2 + \Sigma_1 a^{-1}\,, \end{aligned} \tag{4.59}$$

and

$$\begin{aligned} \Delta &= \Delta_0 + \Delta_1 a^{3\gamma-5} + \tfrac{3\gamma}{3\gamma-7}\Sigma_0 a^2\,, \\ Z_* &= \tfrac{3\gamma-3}{\gamma}\Delta_0 + \tfrac{2}{\gamma}\Delta_1 a^{3\gamma-5} + 3\tfrac{3\gamma-5}{3\gamma-7}\Sigma_0 a^2\,, \\ Q_* &= \tfrac{2\gamma-2}{\gamma}\Delta_0 + \tfrac{4}{3\gamma}\Delta_1 a^{3\gamma-5} + \tfrac{4}{3}\tfrac{3\gamma-4}{3\gamma-7}\Sigma_0 a^2 - \tfrac{2}{3}\Sigma_1 a^{-1}\,, \\ U_* &= \tfrac{6\gamma-6}{\gamma}\Delta_0 + \tfrac{4}{\gamma}\Delta_1 a^{3\gamma-5} + \tfrac{12}{3\gamma-7}\Sigma_0 a^2 \end{aligned} \tag{4.60}$$

for $\gamma \neq \frac{7}{3}$. We do not give the solutions for $\gamma = \frac{7}{3}$, since all values of $\gamma > 2$ are un-physical.

There are four constants of integration Σ_0, Σ_1, Δ_0, Δ_1 corresponding to the four independent modes.

The scalar contributions to the physical quantities defined in (4.10) can then easily be obtained:

$$\begin{aligned} Q^{(\mathrm{DR})} &= \tfrac{2\gamma-2}{3H_0\gamma}\Delta_0 a + \tfrac{4}{9H_0\gamma}\Delta_1 a^{3\gamma-4} + \tfrac{4}{9H_0}\tfrac{3\gamma-4}{3\gamma-7}\Sigma_0 a^3 - \tfrac{2}{9H_0}\Sigma_1 \,, \\ U^{(\mathrm{DR})} &= U_* \,. \end{aligned} \tag{4.61}$$

In particular the density perturbation $\Delta^{(\mathrm{DR})}$ can be written as $\Omega_{\rho^*}\Delta^{(\mathrm{DR})} = \Omega_\rho\Delta$, and using the fact that $\Omega_\rho, \Omega_\lambda \geq 0$, we find that $\Omega_\rho\Delta = \Omega_\lambda\Delta \to 0$ as $\rho \to 0$. Hence the density perturbations are given by

$$\Delta^{(\mathrm{DR})} = \Omega_\rho\Delta_0 + \Omega_\rho\Delta_1 a^{3\gamma-5} + \tfrac{3\gamma}{3\gamma-7}\Omega_\rho\Sigma_0 a^2 \tag{4.62}$$

are suppressed as one approaches the dark radiation solution.

Vector Perturbations

In the case of vector perturbations the propagation equations are

$$W_*' = (3\gamma-4)W_* \,, \tag{4.63}$$
$$\Sigma' = -\mathcal{E} \,, \tag{4.64}$$
$$\mathcal{E}' = \mathcal{E} - 2\Sigma \,, \tag{4.65}$$
$$\mathcal{H}' = \bar{\Sigma}' = \bar{\mathcal{E}}' = 0 \,, \tag{4.66}$$
$$\bar{Q}_*' = 8(3\gamma-4)W_* \,, \tag{4.67}$$

which are subject to the following constraints

$$\begin{aligned} 3\Sigma + 6Q_* &= 4Z_* \,, \\ 3\mathcal{E} + 6Q_* &= 2U_* + 6(\Omega_\rho + 2\Omega_\lambda)\Delta \,, \\ \bar{\Sigma} &= \mathcal{H} \,, \\ \mathcal{H} &= 8W_* - \bar{Q}_* = 0 \,, \\ \bar{\mathcal{H}} &= \bar{W}_* = 0 \,, \\ \bar{\Sigma} &= \bar{\mathcal{E}} = 8W_* - \bar{Q}_* \,. \end{aligned} \tag{4.68}$$

The inhomogeneity variables evolve according to

$$\begin{aligned} \Delta' &= (3\gamma-3)\Delta - \gamma Z_* \,, \\ Z_*' &= 6\tfrac{\gamma-1}{\gamma}\Delta - U_* \,, \\ Q_*' &= -Q_* - \tfrac{1}{3}U_* + 4\tfrac{\gamma-1}{\gamma}\Delta \,, \\ U_*' &= -4Z_* + 12\tfrac{\gamma-1}{\gamma}\Delta \,. \end{aligned} \tag{4.69}$$

The solutions to this system are

$$
\begin{aligned}
\Sigma &= \Sigma_0 a^2 + \Sigma_1 a^{-1}\,,\\
\mathcal{E} &= -2\Sigma_0 a^2 + \Sigma_1 a^{-1}\,,\\
\mathcal{H} &= \bar{\Sigma} = \bar{\mathcal{E}} = \mathcal{H}_0\,,\\
W &= 2W_0^* a^{3\gamma-3}\,,\\
\bar{Q}_* &= 8W_0^* a^{3\gamma-4} + \mathcal{H}_0\,,
\end{aligned}
\tag{4.70}
$$

and

$$
\begin{aligned}
\Delta &= \Delta_0 + \Delta_1 a^{3\gamma-5} + \tfrac{3}{4}\tfrac{3\gamma}{3\gamma-7}\Sigma_0 a^2\,,\\
Z_* &= \tfrac{3\gamma-3}{\gamma}\Delta_0 + \tfrac{2}{\gamma}\Delta_1 a^{3\gamma-5} + \tfrac{9}{4}\tfrac{3\gamma-5}{3\gamma-7}\Sigma_0 a^2\,,\\
Q_* &= \tfrac{2\gamma-2}{\gamma}\Delta_0 + \tfrac{4}{3\gamma}\Delta_1 a^{3\gamma-5} + \tfrac{3\gamma-4}{3\gamma-7}\Sigma_0 a^2 - \tfrac{1}{2}\Sigma_1 a^{-1}\,,\\
U_* &= \tfrac{6\gamma-6}{\gamma}\Delta_0 + \tfrac{4}{\gamma}\Delta_1 a^{3\gamma-5} + \tfrac{9}{3\gamma-7}\Sigma_0 a^2\,,
\end{aligned}
\tag{4.71}
$$

where again $\gamma \neq \frac{7}{3}$.

This time there are six constants of integration: Σ_0, Σ_1, $\mathcal{H}_0$, W_0^*, Q_0^*, Δ_1.

The vector modes of the physical quantities defined in (4.10) can then be found:

$$
\begin{aligned}
\Delta^{(\mathrm{DR})} &= \tfrac{\kappa^2\rho_0}{3H_0^2}\Delta_0 a^{4-3\gamma} + \tfrac{\kappa^2\rho_0}{3H_0^2}\Delta_1 a^{-1} + \tfrac{\kappa^2\rho_0}{4H_0^2}\tfrac{3\gamma}{3\gamma-7}\Sigma_0 a^{6-3\gamma}\,,\\
Q^{(\mathrm{DR})} &= \tfrac{2\gamma-2}{3H_0\gamma}\Delta_0 a + \tfrac{4}{9H_0\gamma}\Delta_1 a^{3\gamma-4} + \tfrac{4}{9H_0}\tfrac{3\gamma-4}{3\gamma-7}\Sigma_0 a^3 - \tfrac{2}{9H_0}\Sigma_1\,,\\
U^{(\mathrm{DR})} &= U_*\,.
\end{aligned}
\tag{4.72}
$$

Tensor Perturbations

The tensor parts of the propagation equations in the long wavelength limit are:

$$
\begin{aligned}
\Sigma' &= -\mathcal{E}\,,\\
\mathcal{E}' &= \mathcal{E} - 2\Sigma\,,\\
\mathcal{H}' &= \mathcal{H} - \bar{\mathcal{E}}\,,\\
\bar{\Sigma}' &= \bar{\Sigma} - \bar{\mathcal{E}}\,,\\
\bar{\mathcal{E}}' &= 2\bar{\mathcal{E}} - 2\bar{\Sigma}\,,
\end{aligned}
\tag{4.73}
$$

subject to the following constraints:

$$
\bar{\mathcal{H}} = 0\,,\quad \bar{\Sigma} = \mathcal{H}\,. \tag{4.74}
$$

The solutions are

$$
\begin{aligned}
\Sigma &= \Sigma_0 a^2 + \Sigma_1 a^{-1} , \\
\mathcal{E} &= -2\Sigma_0 a^2 + \Sigma_1 a^{-1} , \\
\mathcal{H} &= \bar{\Sigma} = \mathcal{H}_0 + \mathcal{H}_1 a^3 , \\
\bar{\mathcal{E}} &= \mathcal{H}_0 - 2\mathcal{H}_1 a^3 ,
\end{aligned} \tag{4.75}
$$

with a constant of integration for each of the independent modes: $\Sigma_0,\ \Sigma_1,\ \mathcal{H}_0,\ \mathcal{H}_1$.

4.6 High Energy Limit: The $\mathcal{F}_b$ Background

The high energy limit is characterized by $\frac{\rho^2}{\lambda} \gg \rho,\ \rho^*$ or $\Omega_\lambda \gg \Omega_\rho, \Omega_{\rho^*}$ so this time we evaluate the perturbation equations in the limit $(\Omega_\rho, \Omega_\lambda, \Omega_{\rho^*}) \to (0,1,0)$.

This model corresponds to a stationary (equilibrium) point $\mathcal{F}_b$ in the phase space of homogeneous Bianchi models [49, 50], as well as in the phase space of the special class of inhomogeneous G_2 cosmological models. In both cases $\mathcal{F}_b$ is found to be the source, or past attractor, for the generic dynamics for $\gamma > 1$ ($\gamma = 1$ is also included in the homogeneous case), consistent with [41, 42, 145, 176]. The stability of this result is now examined through an analysis of the perturbation equations for this case.

The background scale factor a, Hubble function H and deceleration parameter q of these models are given by

$$
a(t) = (t/t_0)^{1/3\gamma} \ , \quad H = H_0 a^{-3\gamma},\ q = 3\gamma - 1 \ , \tag{4.76}
$$

where again we fix an arbitrary initial condition by choosing $a_0 = a(t_0) = 1$. The energy density behaves in the usual way:

$$
\rho = \rho_0 a^{-3\gamma} \ . \tag{4.77}
$$

From the Friedmann equation (4.20) we find that $H_0^2 = \frac{\kappa^2}{6\lambda}\rho_0^2$.

Scalar Perturbations

The scalar propagation equations for this case reduce to

$$
\begin{aligned}
\Sigma' &= (3\gamma - 2)\Sigma - \mathcal{E} , \\
\mathcal{E}' &= (6\gamma - 3)\mathcal{E} - 3\gamma\Sigma ,
\end{aligned} \tag{4.78}
$$

subject to the constraints

$$
\begin{aligned}
W &= \bar{W} = \mathcal{H} = \bar{\mathcal{H}} = \bar{\Sigma} = \bar{\mathcal{E}} = \bar{Q} = 0 , \\
2Z_* &= 2\Sigma + 3Q_* , \\
6\Delta &= 2\mathcal{E} + 3Q_* - U_* .
\end{aligned} \tag{4.79}
$$

The scalar evolution equations for the inhomogeneity variables are

$$\begin{aligned} \Delta' &= (3\gamma-3)\Delta - \gamma Z_* \,, \\ Z_*' &= (3\gamma-2)Z_* - 3(3\gamma+1)\Delta - U_* \,, \\ Q_*' &= (3\gamma-3)Q_* - \tfrac{1}{3}U_* - 6\gamma\Delta \,, \\ U_*' &= (6\gamma-4)U_* \,. \end{aligned} \tag{4.80}$$

Solutions can again be easily obtained by solving the above system of linear equations. They are

$$\begin{aligned} \Sigma &= \Sigma_0 a^{6\gamma-2} + \Sigma_1 a^{3\gamma-3} \,, \\ \mathcal{E} &= -3\gamma\Sigma_0 a^{6\gamma-2} + \Sigma_1 a^{3\gamma-3} \,, \end{aligned} \tag{4.81}$$

and

$$\begin{aligned} \Delta &= \tfrac{1}{2}Q_0^* a^{-3} - \tfrac{\gamma(3\gamma+1)}{6\gamma+1}\Sigma_0 a^{6\gamma-2} - \tfrac{\gamma}{2(6\gamma-1)}U_0^* a^{6\gamma-4} \,, \\ Z_* &= \tfrac{3}{2}Q_0^* a^{-3} + \tfrac{(3\gamma+1)^2}{6\gamma+1}\Sigma_0 a^{6\gamma-2} + \tfrac{3\gamma-1}{2(6\gamma-1)}U_0^* a^{6\gamma-4} \,, \\ Q_* &= Q_0^* a^{-3} + \tfrac{6\gamma^2}{6\gamma+1}\Sigma_0 a^{6\gamma-2} - \tfrac{2}{3}\Sigma_1 a^{3\gamma-3} + \tfrac{3\gamma-1}{3(6\gamma-1)}U_0^* a^{6\gamma-4} \,, \\ U_* &= U_0^* a^{6\gamma-4} \end{aligned} \tag{4.82}$$

for $\gamma \neq \frac{1}{6}$, and

$$\Delta = (\tfrac{1}{2}Q_0^* - \tfrac{1}{6}U_0^*)a^{-3} - \tfrac{1}{8}\Sigma_0 a^{-1} - \tfrac{1}{12}U_0^* \ln a \; a^{-3} \,, \tag{4.83}$$
$$Z_* = \tfrac{3}{2}Q_0^* a^{-3} + \tfrac{9}{8}\Sigma_0 a^{-1} - \tfrac{1}{4}U_0^* \ln a \; a^{-3} \,, \tag{4.84}$$
$$Q_* = Q_0^* a^{-3} + \tfrac{1}{12}\Sigma_0 a^{-1} - \tfrac{2}{3}\Sigma_1 a^{-\frac{5}{2}} - \tfrac{1}{6}U_0^* \ln a \; a^{-3} \,, \tag{4.85}$$
$$U_* = U_0^* a^{-3} \tag{4.86}$$

for $\gamma = \frac{1}{6}$. The scalar parts of the physical quantities $\Delta^{(\mathrm{HE})}$, $Q^{(\mathrm{HE})}$, $U^{(\mathrm{HE})}$ defined by (4.11) are given by

$$\begin{aligned} \Delta^{(\mathrm{HE})} &= \tfrac{\kappa^2\rho_0}{12H_0^2}Q_0^* a^{3\gamma-3} - \tfrac{\gamma(3\gamma+1)}{6\gamma+1}\tfrac{\kappa^2\rho_0}{6H_0^2}\Sigma_0 a^{9\gamma-2} - \tfrac{\gamma}{6\gamma-1}\tfrac{\kappa^2\rho_0}{12H_0^2}U_0^* a^{9\gamma-4} \,, \\ Q^{(\mathrm{HE})} &= \tfrac{1}{6H_0}Q_0^* a^{3\gamma-4} + \tfrac{1}{H_0}\tfrac{\gamma^2}{6\gamma+1}\Sigma_0 a^{9\gamma-3} - \tfrac{1}{9H_0}\Sigma_1 a^{6\gamma-4} + \tfrac{1}{18H_0}\tfrac{3\gamma-1}{6\gamma-1}U_0^* a^{9\gamma-5} \,, \end{aligned} \tag{4.87}$$
$$U^{(\mathrm{HE})} = \tfrac{1}{6}U_0^* a^{6\gamma-4} \tag{4.88}$$

for $\gamma \neq \frac{1}{6}$, and

$$\begin{aligned} \Delta^{(\mathrm{HE})} &= \tfrac{\kappa^2\rho_0}{12H_0^2}(Q_0^* - \tfrac{1}{3}U_0^*)a^{-\frac{5}{2}} - \tfrac{\kappa^2\rho_0}{48H_0^2}\Sigma_0 a^{-\frac{1}{2}} - \tfrac{\kappa^2\rho_0}{72H_0^2}U_0^* \ln a \; a^{-\frac{5}{2}} \,, \\ Q^{(\mathrm{HE})} &= \tfrac{1}{6H_0}Q_0^* a^{-\frac{7}{2}} + \tfrac{1}{6H_0}\tfrac{1}{12}\Sigma_0 a^{-\frac{3}{2}} - \tfrac{1}{9H_0}\Sigma_1 a^{-3} \,, - \tfrac{1}{36H_0}U_0^* \ln a \; a^{-\frac{7}{2}} \end{aligned} \tag{4.89}$$
$$U^{(\mathrm{HE})} = \tfrac{1}{6}U_0^* a^{-3} \tag{4.90}$$

for $\gamma = \frac{1}{6}$.

Σ_0, Σ_1, Q_0^*, U_0^* are arbitrary constants of integration corresponding to the four independent modes.

Vector Perturbations

For vector perturbations the propagation equations are

$$\begin{aligned}
{W_*}' &= (3\gamma-4)W_* \,, \\
\Sigma' &= (3\gamma-2)\Sigma - \mathcal{E} \,, \\
\mathcal{E}' &= (6\gamma-3)\mathcal{E} - 3\gamma\Sigma \,, \\
\mathcal{H}' &= (6\gamma-3)\mathcal{H} - \bar{\mathcal{E}} \,, \\
\bar{\Sigma}' &= (6\gamma-3)\bar{\Sigma} - \bar{\mathcal{E}} \,, \\
\bar{\mathcal{E}}' &= (9\gamma-4)\bar{\mathcal{E}} - 3\gamma\bar{\Sigma} \,, \\
\bar{Q}_*' &= (6\gamma-4)\bar{Q}_* - 36\gamma^2 W_* \,,
\end{aligned} \tag{4.91}$$

subject to the following constraints

$$\begin{aligned}
4Z_* &= 3\Sigma + 6Q_* \,, \\
12\Delta &= 3\mathcal{E} + 6Q_* - 2U_* \,, \\
\mathcal{H} &= \bar{\Sigma} = \bar{\mathcal{E}} = 12\gamma W_* - \bar{Q}_* \,, \\
\bar{\mathcal{H}} &= \bar{W}_* = 0 \,.
\end{aligned} \tag{4.92}$$

The propagation equations characterizing the inhomogeneities are

$$\begin{aligned}
\Delta' &= (3\gamma-3)\Delta - \gamma Z_* \,, \\
{Z_*}' &= (3\gamma-2)Z_* - 3(3\gamma+1)\Delta - U_* \,, \\
{Q_*}' &= (3\gamma-3)Q_* - \tfrac{1}{3}U_* - 6\gamma\Delta \,, \\
{U_*}' &= (6\gamma-4)U_* \,.
\end{aligned} \tag{4.93}$$

Solutions are easily obtained and are given by

$$\begin{aligned}
\Sigma &= \Sigma_0 a^{6\gamma-2} + \Sigma_1 a^{3\gamma-3} \,, \\
\mathcal{E} &= -3\gamma\Sigma_0 a^{6\gamma-2} + \Sigma_1 a^{3\gamma-3} \,, \\
\mathcal{H} &= \bar{\Sigma} = \bar{\mathcal{E}} = \mathcal{H}_0 a^{6\gamma-4} \,, \\
W &= 3\gamma W_0^* a^{6\gamma-5} \,, \\
\bar{Q}_* &= 12\gamma W_0^* a^{3\gamma-4} - \mathcal{H}_0 a^{6\gamma-4} \,,
\end{aligned} \tag{4.94}$$

and

$$\begin{aligned}
\Delta^{(\mathrm{HE})} &= \tfrac{\kappa^2\rho_0}{12H_0^2}Q_0^* a^{3\gamma-3} - \tfrac{\gamma(3\gamma+1)}{6\gamma+1}\tfrac{\kappa^2\rho_0}{9H_0^2}\Sigma_0 a^{9\gamma-2} - \tfrac{\gamma}{6\gamma-1}\tfrac{\kappa^2\rho_0}{12H_0^2}U_0^* a^{9\gamma-4}\,,\\
Z_* &= \tfrac{3}{2}Q_0^* a^{-3} + \tfrac{3}{4}\tfrac{(3\gamma+1)^2}{6\gamma+1}\Sigma_0 a^{6\gamma-2} + \tfrac{3\gamma-1}{2(6\gamma-1)}U_0^* a^{6\gamma-4}\,,\\
Q^{(\mathrm{HE})} &= \tfrac{1}{6H_0}Q_0^* a^{3\gamma-4} + \tfrac{1}{4H_0}\tfrac{3\gamma^2}{6\gamma+1}\Sigma_0 a^{9\gamma-3} - \tfrac{1}{12H_0}\Sigma_1 a^{6\gamma-4} + \tfrac{1}{18H_0}\tfrac{3\gamma-1}{6\gamma-1}U_0^* a^{9\gamma-5}\,,\\
U^{(\mathrm{HE})} &= \tfrac{1}{6}U_0^* a^{6\gamma-4}
\end{aligned} \tag{4.95}$$

for $\gamma \neq \frac{1}{6}$, and

$$\begin{aligned}
\Delta^{(\mathrm{HE})} &= \tfrac{\kappa^2\rho_0}{12H_0^2}(Q_0^* - \tfrac{1}{3}U_0^*)a^{-\frac{5}{2}} - \tfrac{\kappa^2\rho_0}{72H_0^2}\Sigma_0 a^{-\frac{1}{2}} - \tfrac{\kappa^2\rho_0}{72H_0^2}U_0^* \ln a\; a^{-\frac{5}{2}}\,,\\
Z_* &= \tfrac{3}{2}Q_0^* a^{-3} + \tfrac{27}{32}\Sigma_0 a^{-1} - \tfrac{1}{4}U_0^* \ln a\; a^{-3}\,,\\
Q^{(\mathrm{HE})} &= \tfrac{1}{6H_0}Q_0^* a^{-\frac{7}{2}} + \tfrac{1}{6H_0}\tfrac{1}{16}\Sigma_0 a^{-\frac{3}{2}} - \tfrac{1}{12H_0}\Sigma_1 a^{-3} - \tfrac{1}{36H_0}U_0^* \ln a\; a^{-\frac{7}{2}}\,, && (4.96)\\
U^{(\mathrm{HE})} &= \tfrac{1}{6}U_0^* a^{-3} && (4.97)
\end{aligned}$$

for $\gamma = \frac{1}{6}$.

There are six constants of integration: $\Sigma_0,\ \Sigma_1,\ \mathcal{H}_0,\ W_0^*,\ Q_0^*,\ U_0^*$.

Tensor Perturbations

The tensor parts of the propagation equations in the long wavelength limit are:

$$\begin{aligned}
\Sigma' &= (3\gamma-2)\Sigma - \mathcal{E}\,,\\
\mathcal{E}' &= (6\gamma-3)\mathcal{E} - 3\gamma\Sigma\,,\\
\mathcal{H}' &= (6\gamma-3)\mathcal{H} - \bar{\mathcal{E}}\,,\\
\bar{\Sigma}' &= (6\gamma-3)\bar{\Sigma} - \bar{\mathcal{E}}\,,\\
\bar{\mathcal{E}}' &= (9\gamma-4)\bar{\mathcal{E}} - 3\gamma\bar{\Sigma}\,,
\end{aligned} \tag{4.98}$$

subject to the constraints

$$\bar{\mathcal{H}} = 0,\quad \bar{\Sigma} = \mathcal{H}\,. \tag{4.99}$$

The solutions are given by

$$\begin{aligned}
\Sigma &= \Sigma_0 a^{6\gamma-2} + \Sigma_1 a^{3\gamma-3}\,,\\
\mathcal{E} &= -3\gamma\Sigma_0 a^{6\gamma-2} + \Sigma_1 a^{3\gamma-3}\,,\\
\mathcal{H} &= \bar{\Sigma} = \mathcal{H}_0 a^{6\gamma-4} + \mathcal{H}_1 a^{9\gamma-3}\,,\\
\bar{\mathcal{E}} &= \mathcal{H}_0 a^{6\gamma-4} - 3\gamma\mathcal{H}_1 a^{9\gamma-3}\,.
\end{aligned} \tag{4.100}$$

There are four constants of integration: $\Sigma_0,\ \Sigma_1,\ \mathcal{H}_0,\ \mathcal{H}_1$.

4.7 Results and Discussion

In the previous three sections we have developed and solved the perturbation equations for the *low energy*, *dark radiation* and *high energy* backgrounds, respectively.

The main results of our analysis is summarized in table 4.1 in which we present the early time asymptotics $a \to 0$ of the physically relevant quantities for the different energy regimes. These physically relevant quantities are the harmonically decomposed components of the expansion normalized vorticity, shear, and the electric and magnetic parts of the Weyl tensor (4.5,4.6), as well as the appropriate gradients of the energy density ρ, the non-local energy density ρ^* and the non-local flux q^*_a defined in (4.9)-(4.11). The remaining quantities appearing in the previous sections are required to close the system of equations, but are otherwise of no particular physical importance. We can see from table 4.2 that in the low-energy regime the results from General Relativity are recovered. In particular, we find the same decaying mode Σ_1 in both the shear Σ and density gradient Δ, implying that in General Relativity the flat FLRW models are *unstable* with respect to generic linear homogeneous and anisotropic perturbations into the past which was the problem outlined in the introduction and partly the motivation for inflation.

In the dark radiation limit, we find that for any value of γ there is a quantity that diverges as $a \to 0$. The dark radiation background is, however, an unstable equilibrium point in the state space of flat FLRW models [42], and can therefore only be attained for very special initial conditions. For this reason it should be expected that the perturbations around the dark radiation model grow at early times.

The main result of this analysis relates to the evolution of the perturbation quantities in the high energy background $\mathcal{F}_b$. We find that all physically relevant perturbations tend to zero at early times if $\gamma > 4/3$ The same results holds for $\gamma = 4/3$, except that the gradient of the non-local flux tend to a finite constant value. In particular, unlike in GR, both shear and density gradient tend to zero at early times if $\gamma \geq 4/3$. This means that the high-energy models isotropize into the past for realistic equations of state when we include generic linear inhomogeneous and anisotropic perturbations.

4.8 Conclusion

In this chapter we have given a comprehensive large-scale perturbative analysis of flat FLRW braneworld models with vanishing cosmological constant, setting the anisotropic stress on the brane to zero. This extends the work presented in [58] by providing a complete analysis of *scalar*, *vector* and *tensor* perturbations for all the important stages of the braneworld evolution, namely the *low energy GR*, *dark energy* and *high energy* regimes.

To make this precise we defined dimensionless variables that were specially tailored for each regime of interest. In this way we were able to clarify further the past asymptotic

1

Table 4.2: Large-scale behavior of the non-zero physically relevant geometric and kinematic quantities for the different backgrounds in the limit $a \to 0$. The values S, V, T in brackets denote scalar, vector and tensor contributions. One can easily see how the appropriately normalized perturbation quantities defined in (4.9)-(4.10) converge for wider ranges of γ as $a \to 0$. Note that the quantity $\Delta^{(\mathrm{DR})}$ is suppressed by a factor of ρ_0 and therefore not significant when approaching the vacuum model (R).

Quantity	mode	$0<\gamma<\frac{2}{9}$	$\frac{2}{9}<\gamma<\frac{1}{3}$	$\frac{1}{3}<\gamma<\frac{4}{9}$	$\frac{4}{9}<\gamma<\frac{5}{9}$	$\frac{5}{9}<\gamma<\frac{2}{3}$	$\frac{2}{3}<\gamma<\frac{5}{6}$	$\frac{5}{6}<\gamma<1$	$1<\gamma<\frac{10}{9}$	$\frac{10}{9}<\gamma<\frac{4}{3}$	$\frac{4}{3}<\gamma<\frac{5}{3}$	$\frac{5}{3}<\gamma<2$
						Low Energy limit						
$\Sigma,\ \mathcal{E}$	$\Sigma_0(S,V,T)$	∞	∞	∞	∞	∞	0	0	0	0	0	0
	$\Sigma_1(S,V,T)$	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
$\mathcal{H}$	$\mathcal{H}_0(V,T)$	∞	∞	∞	∞	∞	∞	∞	∞	∞	0	0
	$\mathcal{H}_1(T)$	∞	∞	∞	∞	∞	0	0	0	0	0	0
W	$W_0^*(V)$	∞	∞	∞	∞	∞	∞	∞	∞	0	0	0
$\Delta=\Delta^{(\mathrm{LE})}$	$\Sigma_0(S,V)$	∞	∞	∞	∞	∞	0	0	0	0	0	0
	$\Sigma_1, Q_0^*(S,V)$	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
	$U_0^*(S,V)$	∞	∞	∞	∞	∞	∞	∞	∞	∞	0	0
$Q^{(\mathrm{LE})}$	$Q_0^*(S,V)$	∞	∞	∞	∞	∞	∞	∞	∞	∞	0	0
	$U_0^*(S,V)$	∞	∞	∞	∞	∞	∞	∞	∞	0	0	0
$U^{(\mathrm{LE})}$	$U_0^*(S,V)$	∞	∞	∞	∞	∞	∞	∞	∞	∞	0	0
						Dark Radiation limit						
$\Sigma,\ \mathcal{E}$	$\Sigma_0(S,V,T)$	0	0	0	0	0	0	0	0	0	0	0
	$\Sigma_1(S,V,T)$	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
$\mathcal{H}$	$\mathcal{H}_0(V,T)$	const	const	const	const	const	const	const	const	const	const	const
	$\mathcal{H}_1(T)$	0	0	0	0	0	0	0	0	0	0	0
W	$W_0^*(V)$	∞	∞	∞	∞	∞	∞	∞	0	0	0	0
$\Delta^{(\mathrm{DR})}$	$\rho_0\Sigma_0(S,V)$	0	0	0	0	0	0	0	0	0	0	0
	$\rho_0\Delta_0(S,V)$	0	0	0	0	0	0	0	0	0	∞	∞
	$\rho_0\Delta_1(S,V)$	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
$Q^{(\mathrm{DR})}$	$\Sigma_0(S,V)$	0	0	0	0	0	0	0	0	0	0	0
	$\Sigma_1(S,V)$	const	const	const	const	const	const	const	const	const	const	const
	$\Delta_0(S,V)$	0	0	0	0	0	0	0	0	0	0	0
	$\Delta_1(S,V)$	∞	∞	∞	∞	∞	∞	∞	∞	∞	0	0
$U^{(\mathrm{DR})}$	$\Sigma_0(S,V)$	0	0	0	0	0	0	0	0	0	0	0
	$\Delta_0(S,V)$	const	const	const	const	const	const	const	const	const	const	const
	$\Delta_1(S,V)$	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	0
						High Energy limit						
$\Sigma,\ \mathcal{E}$	$\Sigma_0(S,V,T)$	∞	∞	0	0	0	0	0	0	0	0	0
	$\Sigma_1(S,V,T)$	∞	∞	∞	∞	∞	∞	∞	0	0	0	0
$\mathcal{H}$	$\mathcal{H}_0(V,T)$	∞	∞	∞	∞	∞	0	0	0	0	0	0
	$\mathcal{H}_1(T)$	∞	∞	0	0	0	0	0	0	0	0	0
W	$W_0^*(V)$	∞	∞	∞	∞	∞	∞	0	0	0	0	0
$\Delta^{(\mathrm{HE})}$	$\Sigma_0(S,V)$	∞	0	0	0	0	0	0	0	0	0	0
	$Q_0^*(S,V)$	∞	∞	∞	∞	∞	∞	∞	0	0	0	0
	$U_0^*(S,V)$	∞	∞	∞	0	0	0	0	0	0	0	0
$Q^{(\mathrm{HE})}$	$\Sigma_0(S,V)$	∞	∞	0	0	0	0	0	0	0	0	0
	$\Sigma_1(S,V)$	∞	∞	∞	∞	∞	0	0	0	0	0	0
	$Q_0^*(S,V)$	∞	∞	∞	∞	∞	∞	∞	∞	∞	0	0
	$U_0^*(S,V)$	∞	∞	∞	∞	0	0	0	0	0	0	0
$U^{(\mathrm{HE})}$	$U_0^*(S,V)$	∞	∞	∞	∞	∞	0	0	0	0	0	0

behavior of these models and obtain results which exactly match the work of Coley *et. al.* [52] on the spatially inhomogeneous G_2 braneworld models: isotropization towards the $\mathcal{F}_b$ model occurs for an equation of state parameter $\gamma > 4/3$.

This result also agrees with the works in [41, 42, 49, 50, 52, 83, 84, 103, 106, 145, 176], where the issue of isotropization in the context of RS2 braneworld cosmology has been addressed, and where all authors concluded that the past attractor for spatially homogeneous anisotropic models in the brane is the simple high energy FLRW model $\mathcal{F}_b$ [49, 132].

Combining our results with [52], we conclude that the isotropic initial singularity is stable in RS2 cosmology for a range of equations of state, including the physically important cases of radiation. This means that no special initial conditions are required to start inflation, inflation being our best theory producing the temperature fluctuations observed in the CMB.

However, we must point out that our results have been obtained under the simplifying assumption that the non-local anisotropic stress can be neglected. Even though this assumption is valid for the long-wavelength scalar modes [90], it might be flawed when considering vector and tensor modes: Recently a family of anisotropic Bianchi braneworlds has been constructed by solving the five-dimensional field equations in the bulk, and it has been shown that for this particular set of models the non-local anisotropic stress may not be neglected [40]. In another recent work, different ansatze for non-local anisotropic stress have been chosen, and again it has been shown that $\pi^*_{\mu\nu}$ may not be neglected [105]. Both results only apply to very specific cases, but they do indicate that more work on the form of $\pi^*_{\mu\nu}$ needs to be carried out.

Part III

Aspects of Cosmological spacetimes with Horizons

Chapter 5

Why Thermofield dynamics?

In the following chapters we will study some aspects of different spaces with horizons: de Sitter (dS) space, "Rindler space" and Anti-de Sitter (AdS) space. The principle of Horizon Complementarity states that a single causal patch of de Sitter space is described as an isolated finite temperature cavity bounded by a horizon which allows no loss of information. Similarly, the "Rindler wedge" of ordinary Minkowski and AdS space can be described as a finite temperature cavity. This means that the external observer in these spaces, being exposed to a thermal bath of radiation, must describe his physics by a finite (not zero) temperature field theory.

One very powerful tool to study finite temperature field theories is called **thermofield dynamics**. Thermofield theory was invented [10, 123, 155, 178, 148, 198] in the context of many body theory for the purpose of simplifying the calculation of real time correlation functions in finite temperature systems. Its connection with black holes was realized by Israel [115] and elaborated in the Holographic context by Maldacena [152].

5.1 Thermofield Dynamics

We will here introduce the basic concepts of thermofield dynamics. We start with a thermal system characterized by a Hilbert space $\mathcal{H}$, a Hamiltonian H and a temperature $T = 1/\beta$. The thermal ensemble is described by the Boltzmann density matrix

$$\rho = \frac{1}{Z} \exp\left(-\beta H\right). \tag{5.1}$$

Thermal expectation values of real time correlation functions (response functions) contain information not only about equilibrium properties but also non–equilibrium dynamics. A typical response function can be thought of as determining the future behavior of a system that has been kicked out of equilibrium. A typical example has the form

$$\begin{aligned} \langle A(0)B^\dagger(t)\rangle &= \frac{1}{Z} Tr e^{-\beta H} A e^{iHt} B e^{-iHt} \\ &= \sum_{ij} A_{ij} B^\dagger_{ji} e^{i(E_i - E_j)t - \beta E_i} . \end{aligned} \tag{5.2}$$

In fact the full set of thermal real time correlators contains information about physics arbitrarily far from equilibrium.

The thermofield formalism introduces a fictitious system that consists of two copies of the original system. The two copies are labelled 1 and 2. Copy 1 is thought of as the real system, while copy 2 is initially introduced as a trick to simplify calculations [1]. The doubled system is called the **thermofield double**. The **Hilbert space** for the thermofield double is

$$\mathcal{H}_{tf} = \mathcal{H}_1 \bigotimes \mathcal{H}_2 , \tag{5.3}$$

where each factor space is identical to the original Hilbert space. The **Hamiltonian** for the thermofield double is

$$H \equiv H_{tf} = H_1 - H_2 . \tag{5.4}$$

Notice that the energy eigenvalues for subsystem 2 are of opposite sign from those of 1. We now construct the entangled state

$$|\psi\rangle = \frac{1}{Z^{\frac{1}{2}}} \sum_i e^{-\frac{1}{2}\beta E_i} |E_i, E_i\rangle , \tag{5.5}$$

where the state $|E_i, E_j\rangle$ means the eigenvector of H_1 and H_2 with eigenvalues E_i, E_j respectively. The state $|\psi\rangle$ is a particular eigenvector of H_{tf} with eigenvalue zero. Obviously $|\psi\rangle$ has been constructed so that the density matrix for subsystem 1 is just the thermal density matrix ρ at temperature β.

Operators which belong to subsystem 1 have the form $A_1 \otimes I_2$, and will be called A_1. Operators associated with subsystem 2 will be defined in a similar manner except with an additional rule of hermitian conjugation:

$$A_2 \equiv I_1 \bigotimes A_2^\dagger . \tag{5.6}$$

The correlation function in 5.2 may be written as an expectation value:

$$\langle\psi| A_1(0) B_1^\dagger(t) |\psi\rangle . \tag{5.7}$$

[1]When studying cosmological spaces with horizons however, it turns out that copy 2 describes physics behind the horizon as first noticed by Israel in the context of Black Holes [115].

Although no physical significance is ordinarily attached to correlators involving A_2, we may formally define them, for example

$$\langle\psi|A_1(0)B_2(t)|\psi\rangle\,. \tag{5.8}$$

Note that in both 5.7 and 5.8 the time dependence of the operators can be defined via the Heisenberg picture, using the full thermofield Hamiltonian H_{tf}. Let us examine expression 5.8 more carefully. It can be written explicitly in the form

$$\langle\psi|A_1(0)B_2(t)|\psi\rangle = \sum_{ij} e^{-(\frac{\beta}{2}E_i+\frac{\beta}{2}E_j)} e^{i(E_i-E_j)t} A_{ij}B^{\dagger}_{ji}\,. \tag{5.9}$$

This is similar, but not identical, to 5.2. The similarity between the two can be made more apparent by writing 5.9 as

$$\begin{aligned}\langle\psi|A_1(0)B_2(t)|\psi\rangle &= \sum_{ij} e^{-\frac{\beta}{2}E_i} e^{i(E_i-E_j)(t-i\frac{\beta}{2})} A_{ij}B^{\dagger}_{ji} \\ &= \langle A(0)B(t-i\frac{\beta}{2})\rangle\,.\end{aligned} \tag{5.10}$$

This demonstrates that correlators involving both copies can be expressed as analytic continuations of ordinary thermal correlators.

5.2 An example: The eternal Black Hole in AdS

We will now consider the eternal Black Hole in Anti-de Sitter (AdS) space to illustrate how thermofield dynamics can assist our understanding of spacetimes with horizons. We choose this particular example for two reasons. Firstly, the Penrose diagram reflecting the causal structure of this spacetime is identical to the Penrose diagram of de Sitter space. We can thus hope to get some intuition from this example for our study of de Sitter case in chapter 6. Secondly, this spacetime is very useful when studying Holography, since it obeys the AdS/CFT correspondence, which is a concrete implementation of Holography.

We start by presenting the Penrose diagram of AdS - Schwarzschild in figure 5.1. The horizontal lines are space-like singularities, while the vertical lines represent the time-like boundaries of the space. The boost symmetry of this geometry acts as time translation invariance in quadrants I and III and as invariance under space-like translations in quadrants II and IV.

According to the Horizon Complementarity principle, this cosmology can be described by an observer on the boundary in quadrant I as the finite thermal system restricted to the causal patch, surrounded by the horizon represented by the boundaries between quadrant I and quadrants II and IV. We will now consider how the symmetries of the space relate to the causal patch description. Let us first of all consider the Z_2 - symmetry of the

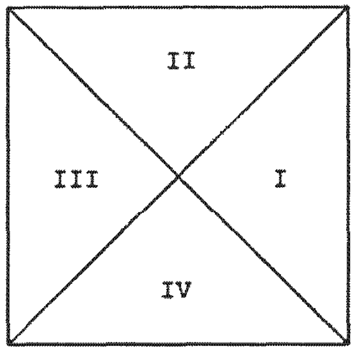

Figure 5.1: The Penrose diagram for AdS-Schwarzschild and de Sitter space.

AdS-Schwarzschild spacetime. This symmetry exchanges quadrants I and III and therefore obviously does not preserve the causal patch. However, we can construct the thermofield double of the thermal system in quadrant I. We can easily see[2] that the second copy must be identified with a thermal system in quadrant III, where time in quadrant III is running in the opposite direction than in quadrant I. It is now clear that the entangled state $|\psi\rangle$ defined in equation 5.5, being the vacuum of the thermofield double, not just the isolated causal patch, is Z_2-invariant.

Another example are time translations and rotations of space. These symmetries preserves the separate boundaries in sectors I and III. However, the thermal state describing quadrant I is not an eigenvector of either the energy or the angular momentum operators of copy 1. In order to implement the symmetries, we again have to construct the thermofield double. We identify the thermofield Hamiltonian H_{tf} (see equation 5.4) and angular momentum operator $J_{tf} \equiv J_1 + J_2$, and construct the Hartle Hawking thermofield double state ψ (see equation 5.5). We can then see that ψ is an eigenvector (with vanishing eigenvalue) of H_{tf} and $J_{tf} \equiv J_1 + J_2$.

The lesson of this section is that the full symmetry group of geometries with horizons acts on the thermofield double Hilbert space, and not on the space of available to an observer one side of the horizon. Moreover the symmetry transformations mix the degrees of freedom of the two copies.

We can now relate this result to the AdS/CFT correspondence [151], according to which the AdS Black Hole can be described by a conformal field theory on the boundary. Since the eternal black hole geometry above has two boundaries, one in quadrant I one in quadrant III, Maldacena argued that the eternal black hole should be described by two copies of the

[2]We will derive this statement more formally when studying 1+1 dimensional Minkowski space in chapter 7.

usual boundary conformal field theory [152]. These two copies are exactly the thermofield double with the state $|\psi\rangle$ being the Hartle Hawking state. Interestingly, it has been noted that this approach allows to probe the physics in sectors II and IV which are behind the horizons of the Black Hole [131, 67, 152].

Chapter 6

The trouble with de Sitter space

6.1 Introduction

Observational evidence indicating that we are living in an asymptotically de Sitter space is accumulating [9]. An exact formulation of quantum theory in an accelerating universe however appears to be very elusive [70].

The causal structure of de Sitter space is the same as the causal structure of the eternal Schwarzschild Anti-de Sitter space geometry mentioned in 5.2. For this reason it was hoped that there exists a dual description of quantum gravity in de Sitter space defined on the asymptotic boundary of de Sitter space. If one could define such a "dS/CFT" correspondence in analogy to the AdS/CFT correspondence [151, 197, 210], we would be equipped with a very powerful tool to study aspects of quantum gravity in de Sitter space.

However, the difference between the geometry of de Sitter space and Anti-de Sitter space poses severe obstacles to constructing the de Sitter dual. Unlike Anti-de Sitter space, de Sitter space has no obvious boundary on which to define the dual theory. It has been suggested [191, 211] to use the infinitely inflated past and/or the infinitely inflated future, which are the two disconnected *asymptotic* space-like boundaries of de Sitter space, since these appear to be the *only* boundaries on which the dual theory can possibly be defined [211]. There remains a severe drawback: no observer will ever be able to measure these quantities, since this would require knowledge of the entire infinitely inflated past and/or future, which is not possible for any real observer living in de Sitter space.

In perturbative String Theory, the well-defined observables are so far S-matrices in asymptotically flat spaces or boundary correlators in asymptotically Anti-de Sitter space. It has been shown that the boundary correlators defined in [191] are not well-defined due to inevitable Poincaré recurrences [61]. Furthermore, Witten has argued that even if one can possibly define a Hilbert space on global de Sitter space by using the asymptotic boundaries, it is impossible to define anything resembling a S-matrix in [211].

In the absence of a convincing dS/CFT correspondence, a number of authors [11, 14, 16, 29, 60, 61, 69, 162, 193] have argued for a de Sitter complementarity principle similar to the black hole version [101, 195]. While differing in the details, all versions agree that physics in a single causal patch of de Sitter space should be described as an isolated quantum system at finite temperature, and that the thermal entropy of the system should be finite. We will here assume that such a dual theory exists and examine the consequences.

From previous experience, especially with Matrix theory and the AdS/CFT duality, we can expect that the holographic dual of de Sitter space will not look much like classical relativity. Most likely its degrees of freedom will be very nonlocally connected to the local semiclassical description of the space. That raises an interesting question: Suppose we were presented with the de Sitter space holographic dual. How would we recognize it as such? The answer for Matrix theory and AdS/CFT was initially through the symmetries, which exactly matched. Supersymmetry, Galilean symmetry, and conformal invariance were especially important. Thus we might try to recognize the de Sitter dual by finding that its symmetries include the $SO(D,1)$ symmetry of D-dimensional de Sitter space. However, most of the group involve transformations which take the causal patch to another patch. The manifest symmetries of the causal patch are the $SO(D-1)$ rotations which preserve the horizon, and the time translations. There are another $D-1$ compact generators and $D-1$ noncompact generators which displace the observer to a new patch. These generators do not act on the Hilbert space of a single observer. As motivated in section 5.2, we must construct the thermofield double in order to implement these symmetries.

6.2 Classical Geometry of de Sitter Space

D-dimensional de Sitter space can be visualized as the hypersurface

$$X_0^2 - X_1^2 - \cdots - X_D^2 = l^2 \tag{6.1}$$

embedded in flat (D+1)-dimensional Minkowski space[1]. This surface is the hyperboloid presented in figure 6.1 We can see that D-dimensional global de Sitter space is a (D-1)-sphere that contracts, reaches a minimum radius and re-expands.

The de Sitter metric $g_{\mu\nu}$ (for $\mu, \nu = 0, \ldots, D-1$) on the hyperboloid is the induced metric from the standard (D+1)-dimensional flat space Minkowski metric. The Einstein tensor of de Sitter space satisfies the Einstein field equations

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \tag{6.2}$$

if we define the cosmological constant Λ by

$$\Lambda = \frac{(D-2)(D-1)}{2l^2}\,. \tag{6.3}$$

[1]In this part of the thesis we choose the signature $(+-\cdots-)$ – which is the opposite of the signature used in the previous part II – in order to be consistent with the relevant field theoretic literature.

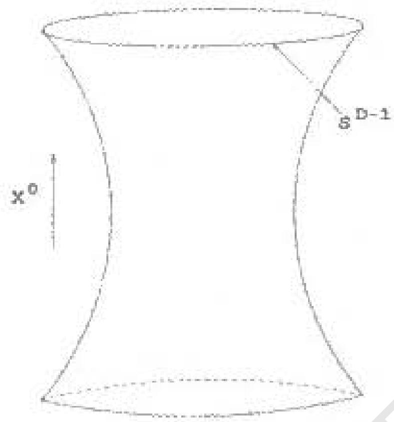

Figure 6.1: 2-dimensional representation of global de Sitter space. Time X^0 runs vertically while slices of constant X^0 are circles or 1-spheres S^1. D-dimensional de Sitter space is obtained by replacing the circles by (D-1)-spheres S^{D-1}.

There exist many different coordinate system in which to describe de Sitter space. A natural choice of coordinates is given by

$$ds^2 = d\tau^2 - l^2 \cosh^2(\tau/l)\; d\Omega^2_{D-1}\,, \tag{6.4}$$

where $d\,\Omega^2_{D-1}$ is the standard metric of a (D-1)-sphere S^{D-1}. From this form of the metric we can clearly see that de Sitter space is indeed an exponentially contracting and re-expanding sphere S^{D-1}. The constant l is the minimal radius of the sphere.

We will now study the de Sitter space Penrose diagram, which captures the causal structure of the spacetime. As mentioned before, the Penrose diagram of de Sitter space is the same as the one of eternal Schwarzschild-AdS space and given by figure 5.1. Due to differences in the geometry, the interpretation of the Penrose diagram in de Sitter space is very different to the interpretation in Anti-de Sitter space. In de Sitter space, the horizontal lines are time-like and correspond to the infinitely inflated future and the infinitely inflated past. The vertical lines are not boundaries of the space, but instead correspond to observers on the opposite poles of the (D-1)-sphere. Every point – except the ones on the vertical lines – represents a (D-2)-sphere.

Let us now consider two observers sitting on opposite poles of the (D-1)-sphere. A signal sent by the observer on the south pole to the observer on the north pole will never get there, as can be seen from the doubled Penrose diagram in figure 6.2.

In fact it can easily be seen that *any two points* on the sphere will eventually fall out of causal contact with each other. Every observer is surrounded by a past and a future horizon

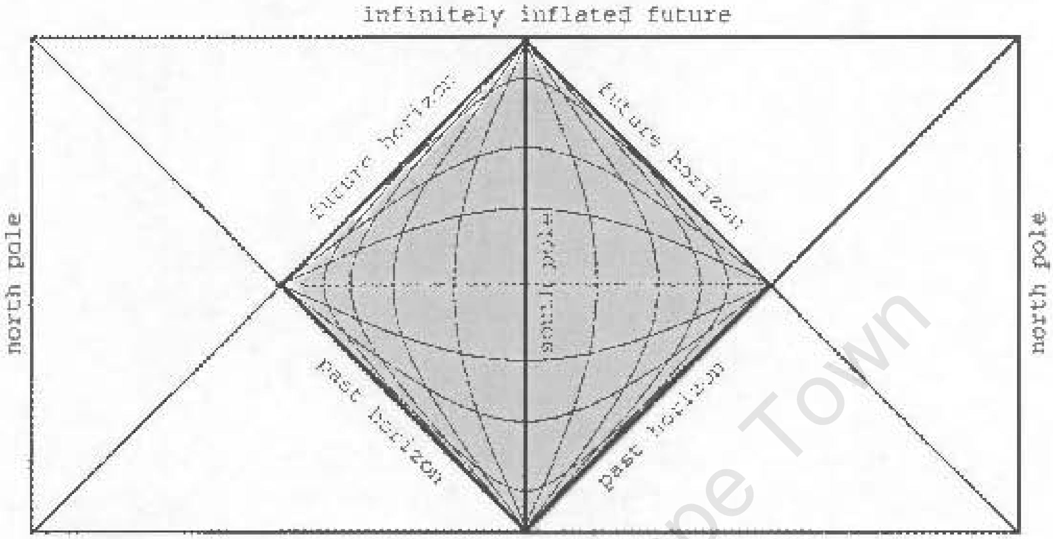

Figure 6.2: The doubled Penrose diagram for de Sitter space. The left and the right vertical borders must be identified. The center area is the causal diamond for the observer on the south pole; static coordinates are indicated.

beyond which he cannot receive or send signals respectively. This motivates the definition of the **causal diamond** or causal patch as the region of spacetime that a given observer is in full causal contact with. The causal diamond for an observer situated at the south pole is the center area in figure 6.2.

It makes sense to define coordinates that are restricted to the causal patch. One very useful parametrization yields the static metric

$$ds^2 = \left(1 - \frac{r^2}{l^2}\right) dt^2 - \left(1 - \frac{r^2}{l^2}\right)^{-1} dr^2 - r^2\, d\Omega_{D-2}^2 \,. \tag{6.5}$$

The observer at the south pole corresponds to $r = 0$. At $r = l$ the metric becomes singular, which is due to the fact that this is the location of the cosmological horizon for this particular observer. The static coordinates are also indicated in the causal diamond of figure 6.2.

6.3 Conditions on the Holographic Dual

As emphasized before, we do not know the details of the holographic dual to quantum gravity in de Sitter space – unlike the AdS case, where the dual theory is a simple supersymmetric conformal field theory defined on the timelike boundary [151]. We will here simply assume that such a dual description exists.

Even without knowing the exact form of the dual theory, we do know that this theory must satisfy certain properties. Firstly, it must preserve the symmetries of the classical space, and secondly it must obey the complementarity principle (see section 1.5.1), according to which every causal patch has its own quantum mechanical description as an isolated system [162].

6.3.1 Symmetries

The symmetry group of classical D dimensional de Sitter space is the group $SO(D,1)$, which is simply the Lorentz group of the $D+1$ dimensional embedding space of the de Sitter hyperboloid. For simplicity we will from now on consider the case of $1+2$ dimensional de Sitter space, but our arguments can easily be extended to de Sitter space of any dimension. In the case of $D=3$, there are six generators: three boosts and three rotations. One of the rotations and one of the boosts preserves the causal patch; we will refer to these as R and H respectively. R generates spatial rotations ($\theta \to \theta + const$), and H generates time translations. More precisely H shifts time forward in quadrant I and backward in quadrant II. The remaining two boosts K_1, K_2 and rotations J_1, J_2 do not preserve the causal patch: they mix the two copies of the thermofield double in a way analogous to the action of Minkowski translations on the Rindler wedges (see appendix D). The generators satisfy the algebra

$$\begin{aligned}
[R, J_i] &= i\epsilon_{ij} J_j \\
[J_i, J_j] &= i\epsilon_{ij} R \\
[H, J_i] &= i\epsilon_{ij} K_j \\
[H, K_i] &= -i\epsilon_{ij} J_j \\
[K_i, K_j] &= -i\epsilon_{ij} R \\
[J_i, K_j] &= i\epsilon_{ij} H \\
[H, R] &= 0\,.
\end{aligned} \tag{6.6}$$

The thermofield formalism requires $H = H_{tf} = H_1 - H_2$, where H_1 acts on $\mathcal{H}_1$ and H_2 acts on $\mathcal{H}_2$ independently. The generators J_i rotate the static patch to a new patch; a rotation by π interchanges the thermofield double copies. Since the J_i mix the two copies, it is evident that they can not be expressed as a sum or difference of operators in $\mathcal{H}_1$ and $\mathcal{H}_2$. The generators K_i are Hamiltonians for static patches which are rotated by a relative angle of $\pi/2$. They too mix the degrees of freedom in the original patch.

The de Sitter group is part of the group of coordinate transformations, which in General Relativity is the gauge group. For this reason physical states should be invariant under the action of any of its generators. Note that this refers to the generators in the thermofield double and not to the individual copies.

According to the definition 5.5, the generator $H_{tf} = H_1 - H_2$ annihilates $|\psi\rangle$.

$$H_{tf}|\psi\rangle = (H_1 - H_2)\sum_i e^{-\frac{1}{2}\beta E_i}|E_i, E_i\rangle = 0\,. \tag{6.7}$$

6.3.2 Finite Entropy

According to the Horizon Complementarity principle, the de Sitter space holographic dual can be described by a theory restricted to the causal patch of a given observer. The properties of the causal patch are described by a thermal density matrix [80] . The additional condition that we must satisfy in order to have a quantum dual of de Sitter space is that the thermal entropy of one causal patch is finite. The value of the entropy is connected to the size of the de Sitter space by the Gibbons Hawking [80] formula for de Sitter entropy:

$$S = \frac{Horizon \;\; Area}{4G}\,. \tag{6.8}$$

For our purposes we can regard 6.8 as defining the size of the de Sitter space through its entropy, which we can compute in terms of the density matrix 5.1:

$$S = -Tr(\rho \log \rho)\,. \tag{6.9}$$

The only thing we will assume is that the entropy is finite. The consequences of that for the spectrum of H_1 are very clear: the eigenvalues E_i must be discrete. More precisely, the number of eigenvalues with energy less than any specified value must be finite. This condition implies that the spectrum of $(H_1 - H_2)$ must be countable and cannot have any accumulation points. A proof of this statement can be found in appendix C.

In the literature much stronger conditions have been assumed for the spectrum of H_1. Banks and Fischler have conjectured that the Hilbert space of states $\mathcal{H}_1$ should be finite dimensional. This may be so, but it does not follow from the finiteness of the entropy. The Entropy is only equal to the dimensionality of the space of states when the temperature is infinite. Entropy can certainly be finite even though the Hilbert space of states is infinite dimensional. We are assuming only the weaker condition that the spectrum is discrete.

6.4 The no-go theorem

We are now ready to prove a no–go theorem: the finiteness of the entropy is incompatible with the existence of the symmetry generators (H, J_i, K_j), and the requirement that H be a hermitian operator. First, define the hermitian operator $L \equiv J_1 + K_2$. It follows from 6.6 that

$$[H, L] = iL\,. \tag{6.10}$$

In general, L is not a good operator on the spectrum of H. If it were, 6.10 would imply that L acts as a raising operator on the spectrum of H but would raise the energy by i, which of course is inconsistent with the hermeticity of H. However, although the generator L itself is not bounded, $\exp(iL)$ is a group element and a good operator. Using 6.10, it is easy to see that

$$e^{iL}(t) \equiv e^{iHt}e^{iL}e^{-iHt} = e^{e^{iHt}iLe^{-iHt}} = e^{iLe^{-t}}\,. \tag{6.11}$$

We will now assume that the spectrum of H is countable, and use the assumption to derive a contradiction. We have

$$\left|\langle\alpha|e^{iL}|\alpha\rangle\right| = 1 - \delta\,. \tag{6.12}$$

Here $|\alpha\rangle$ is some state in the Hilbert space, and $\delta > 0$ because e^{iL} is unitary and L is non-zero. We also have

$$F(t) \equiv \langle\alpha|e^{iL(t)}|\alpha\rangle = \langle\alpha|e^{iHt}e^{iL}e^{-iHt}|\alpha\rangle = \langle\alpha|e^{iLe^{-t}}|\alpha\rangle\,. \tag{6.13}$$

From this, $F(t) \to 1$ as $t \to \infty$, and $F(0) = 1 - \delta < 1$. We will now prove that $F(t)$ is quasiperiodic (see e.g. the appendix of [60]). Any sum of the form

$$\sum_{n=1}^{\infty} f_n e^{i\omega_n t} \tag{6.14}$$

is quasiperiodic if

$$\sum_{n=1}^{\infty} |f_n|^2 < \infty\,. \tag{6.15}$$

Therefore, it suffices to show that $F(t)$ can be written as a sum of this form. But, expanding the state $|\alpha\rangle$ in the energy basis,

$$F(t) = \sum_{n,m} f_n^* f_m \langle n|e^{iL}|m\rangle e^{i(\omega_n - \omega_m)t}\,. \tag{6.16}$$

Consider the sum

$$\sum_{m,n} f_n^* f_m f_m^* f_n \langle n|e^{iL}|m\rangle\langle m|e^{-iL}|n\rangle = \sum_n |f_n|^2 \sum_m |f_m|^2 \,\langle n|e^{iL}|m\rangle\langle m|e^{-iL}|n\rangle\,. \tag{6.17}$$

Considering the inner sum, we have (since $\sum_m \langle n|e^{iL}|m\rangle\langle m|e^{-iL}|n\rangle = 1$, and the terms are real and positive)

$$\sum_m |f_m|^2 \langle n|e^{iL}|m\rangle\langle m|e^{-iL}|n\rangle \leq 1\,, \tag{6.18}$$

and therefore

$$\sum_{m,n} f_n^* f_m f_m^* f_n \langle n|e^{iL}|m\rangle\langle m|e^{-iL}|n\rangle \leq 1\,. \tag{6.19}$$

This shows that $F(t)$ satisfies the criterion 6.15, and hence $F(t)$ is quasiperiodic. Therefore, since $F(0) < 1$, $F(t)$ can not tend to 1 as $t \to \infty$, and we have a contradiction

This proves that H can not have a countable spectrum. and therefore that the E_i cannot be discrete and the entropy cannot be finite.

6.5 Conclusion

Finiteness of entropy appears to be incompatible with de Sitter symmetry. What are we to make of this no–go theorem? One possibility is that de Sitter space has infinite entropy. Perhaps only entropy differences are finite. However, this seems to run against the grain of everything we have learnt over the last decade. The horizon of de Sitter space is locally identical to that of a Schwarzschild black hole. It is hard to see why one would have an infinite entropy and the other not [93].

If the entropy is finite then the symmetry of different causal patches must be broken, the Hamiltonians and energy spectra differing at least slightly. How would these effects manifest themselves? The discreteness of the energy spectrum introduces a new time scale of order $1/\delta$, where δ is of order the typical spacing of levels. For a system with entropy S the level spacing is of order $\exp(-S)$ and the time scale is the Poincaré recurrence [60, 61, 193] time $t_p \sim \exp S$. This is the time scale on which the discreteness of the spectrum causes significant effects. Thus the violations of de Sitter symmetry should become important for times of order t_p and recurrences should not respect the symmetry. We may speculate that no space can behave like idealized de Sitter space for times longer than t_p – eternal de Sitter simply does not exist, because there are always instabilities which cause the space to decay in a time shorter than the recurrence time.

An argument strongly supporting the finite lifetime of de Sitter space in the context of String Theory is based on work by Kachru, Kallosh, Linde and Trivedi [138]. Since String Theory has supersymmetric ground states with vanishing vacuum energy it is possible to tunnel out of any positive local minimum of the scalar potential. Typically there will be a barrier to the tunnelling which may be as high as the string or Planck scale. Kachru, Kallosh, Linde and Trivedi computed the rate for tunnelling over such a barrier and found that it is always less than the recurrence time. This instability has implications for the energy spectrum of the metastable de Sitter space. Since the lifetime is always shorter than the recurrence time, each energy level must develop a width γ at least of order $\exp(-S)$. In a typical situation of the type studied in [138], the width of each level is vastly greater than the level spacing, which is reminiscent of a Schwarzschild black hole in flat space.

An alternative argument has been proposed by Bousso [30], which is simply that because any observer will be destroyed by thermal particles long before t_p, times longer than t_p should be regarded as having no operational meaning. In the same spirit, Banks, Fischler, and Paban [16] speculate that due to the physical constraints on the measuring process in de Sitter space, no experiment can ever last a time comparable to the recurrence time. Hamiltonians that agree for time scales less than $\sim t_p$ should be considered physically equivalent. In terms of the energy levels E_i, this would mean that the level shifts going from one patch to another should be no bigger than the level spacing $\exp(-S)$. If this can be made precise and the physical consequences of different Hamiltonians are identical, then the exact choice of Hamiltonian would become part of the gauge redundancy in a single patch. The transformation from one patch to another would include such a gauge transformation. It is not clear to us whether this would mean that Poincaré recurrences are meaningless or if it means that there is an additional degree of unpredictability in their occurrence.

An interesting viewpoint has been suggested by Shenker [182]. It is fairly clear that in ordinary perturbation theory in de Sitter space the symmetry of the space will never break down. Furthermore, in every order the entanglement entropy of I and III will be infinite. This suggests that the same nonperturbative effects which make the spectrum discrete and the entropy finite, also break the de Sitter symmetry. A tantalizing hint is that the size of such a nonperturbative effect would scale as $\exp(-\text{area}/4G_N) \sim \exp(-S)$, which is the characteristic energy gap δE for a system with entropy S.

Chapter 7

Compactifications of String Theory

7.1 Introduction

String Theory is a theory formulated in 10 spacetime dimensions. In order to achieve a realistic physical model, one has to find solutions where some of the extra dimensions are compactified, rendering them invisible to macroscopic observers such as us. Furthermore, many features of String Theory are complicated even in the four-dimensional theory. It is widely believed that some simplified lower-dimensional toy models still capture the relevant physics, which raises the interest in 1+1 dimensional models in particular.

The question that must be addressed is whether these models are indeed realistic. In this chapter, we will use the principles of Horizon Complementarity and Holography to constrain the possible backgrounds in String Theory, or in fact the possible backgrounds in any consistent quantum gravity theory that obeys these principles.

This chapter is formally very similar to the previous one, where we have argued that the symmetries of de Sitter space are incompatible with the holographic requirement of finite entropy of a causal patch, possibly implying a long time instability of eternally inflating spaces. We will here apply the same argument to 1+1 dimensional Lorentz invariant compactifications of String Theory. We will argue that stable compactifications of String Theory to 1+1 dimensional Minkowski space are in conflict with holography, and that consistency with Holography requires at least three space–time dimensions when considering this class of compactifications.

7.2 Rindler space

7.2.1 The "Rindler wedge" of Minkowski space

We begin by presenting the conformal diagram of Minkowski space in figure 7.1.

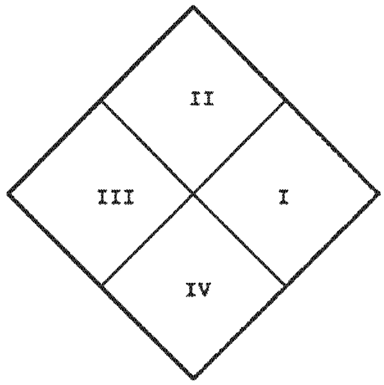

Figure 7.1: The conformal diagram for Minkowski space, showing the Rindler wedges I and III. In Rindler coordinates, continuing from region I to region III involves shifting time by $i\beta/2$, in accord with 5.10.

We can see that 1+1 dimensional Minkowski space can be divided into four quadrants: I, II, III and IV. Quadrants I and III consist of points separated from the origin by a space–like separation, while points in II and IV are displaced by timelike intervals. Quadrant I is the spacetime of an accelerated observer in Minkowski space and called **Rindler space**. It can be described by the metric

$$ds^2 = r^2 dt^2 - dr^2 \,, \tag{7.1}$$

where r is proper distance from the origin, and t is the dimensionless Rindler time. These coordinates are related to the ordinary flat space Minkowski coordinates X^0, X^1 via

$$\begin{aligned} X^0 &= r \sinh t \\ X^1 &= r \cosh t \,. \end{aligned} \tag{7.2}$$

The accelerated Rindler observer sees himself surrounded by a past horizon, the border between quadrants I and IV, and a future horizon, the border between quadrants I and II. In any dimension other than 1+1 the area of the horizon is infinite. However, the horizon of 1+1 dimensional Rindler space $\mathcal{R}_{1,1}$ is a single point.

7.2.2 The thermofield dual

The Rindler quadrant may be described by the Unruh thermal state with temperature

$$T_{rind} = \frac{1}{2\pi} = \frac{1}{\beta_{rind}} \,. \tag{7.3}$$

We can now construct the thermofield double as explained in chapter 5. It turns out that the thermofield double of Rindler space in quadrant I gives us exactly quadrant III.

To see this we first of all note that the Rindler Hamiltonian is the boost generator. Since the Minkowski vacuum is boost invariant, it is an eigenvector of the boost generator with vanishing eigenvalue. Furthermore, the Minkowski vacuum is an entangled state of the degrees of freedom in the two quadrants I and III. Finally, it is well known that when the density matrix for quadrant I is obtained by tracing over III the result is a thermal state at the Rindler temperature.

It is easy to see that correlators between fields in quadrants I and III are related by exactly the same analytic continuations derived from thermofield dynamics: since the inverse Rindler temperature given in equation 7.3 is 2π, the continuation in equation 5.10 is

$$t \to t - i\pi, \tag{7.4}$$

or, from 7.2,

$$X^\mu \to -X^\mu . \tag{7.5}$$

Thus the thermofield continuation takes quadrant I to quadrant III.

7.3 Continuity of the spectrum

7.3.1 Symmetries

The Poincaré group in 1+1 dimensions has three generators, which satisfy the following algebra:

$$\begin{aligned} [P_0, K] &= iP_1 \\ [P_1, K] &= iP_0 \\ [P_0, P_1] &= 0 . \end{aligned} \tag{7.6}$$

The P_i generate translations in space and time, and K is the boost generator. The choice of a Rindler wedge preserves only the generator K. This is obvious if one recalls that the Rindler wedge is the region of space seen by a uniformly accelerating observer. The choice of origin implicit in the identification of Rindler space breaks the symmetries of Minkowski translations. Since the boost generator K acts by generating translations in Rindler time t, it can be identified as the Rindler Hamiltonian H_R.

The generators P_i do not act on the Hilbert space of one wedge; instead they act on the full thermofield double space by mixing degrees of freedom from the two halves. In the case of a free field theory this means that the modes of one Rindler wedge become a linear combination of modes of both wedges under the action of the P_i (see appendix D for more details).

The crucial point (see section 5.2) is that the states of the thermofield double space, namely Minkowski space, form a representation of the full symmetry group.

If we define $P_\pm = P_0 \pm P_1$, and use the Rindler Hamiltonian $H_R = K$ to construct the thermofield Hamiltonian H_{tf} (see equation 5.4), the algebra becomes:

$$\begin{aligned} &[H_{tf}, P_+] = -iP_+ \\ &[H_{tf}, P_-] = iP_- \\ &[P_+, P_-] = 0\,. \end{aligned} \tag{7.7}$$

7.3.2 Finite Entropy

When studying compactifications of String Theory to a 2-dimensional space X_2, we study solutions of String Theory in 10 dimensions of the form $X_2 \times \mathcal{M}_8$. Here $\mathcal{M}_8$ is a compact 8-dimensional manifold and X_2 is taken to be 1+1 dimensional Rindler space $\mathcal{R}_{1,1}$. As argued in section 7.2.1, the horizon of $\mathcal{R}_{1,1}$ is a single point, so the horizon area of $\mathcal{R}_{1,1} \times \mathcal{M}_8$ is simply proportional to the volume[1] of the internal space $\mathcal{M}_8$. This means that the compactification $\mathcal{R}_{1,1} \times \mathcal{M}_8$ has finite horizon area and thus – according to the Bekenstein-Hawking bound – finite entropy. We conclude that a Rindler observer in a compactification of String Theory to 1+1 dimensional Minkowski space sees a horizon of finite entropy unless the compact space $\mathcal{M}_8$ decompactifies.

Similarly to chapter 6, we now make use of the fact that any system with finite entropy must have a discrete spectrum and there must be a finite number of states below any given energy E, so the spectrum can not have discrete accumulation points [2]. As a consequence, the Rindler Hamiltonian H_R must have a discrete spectrum if the entropy of the Rindler wedge is finite. The implications for the thermofield Hamiltonian H_{tf} are weaker. The spectrum of this operator consists of the differences of eigenvalues of H_R, namely $E_i - E_j$. If the spectrum of H_R is discrete, this set of numbers need not be, but is certainly countable. In the next section we will prove that the spectrum of the full thermofield Hamiltonian H_{tf} is not countable, and hence that the entropy of the thermal ensemble defined using H_R is infinite.

7.3.3 The contradiction

Using 7.7, we now proceed to prove that the spectrum of H_{tf} is continuous.

Following chapter 6, consider the operator $e^{iP_-}(t)$:

$$e^{iP_-}(t) \equiv e^{iH_{tf}t} e^{iP_-} e^{-iH_{tf}t} = e^{\left(e^{iH_{tf}t}\, iP_-\, e^{-iH_{tf}t}\right)} = e^{\left(iP_-\, e^{-t}\right)}\,. \tag{7.8}$$

[1] We implicitly assume that basic geometric concepts such as the notion of area and volume hold.

[2] This statement is proven in appendix C.

We will now assume that the spectrum of H is countable, and use the assumption to derive a contradiction. We have

$$\left|\langle\alpha\,|\,e^{iP_-}\,|\,\alpha\rangle\right| = 1-\delta\,. \tag{7.9}$$

Here $|\alpha\rangle$ is some state in the Hilbert space, and $\delta > 0$ because e^{iP_-} is unitary and P_- is non-zero. Define

$$F(t) \equiv \langle\alpha\,|\,e^{iP_-(t)}\,|\,\alpha\rangle = \langle\alpha\,|\,e^{iH_{tf}\,t}e^{iP_-}\,e^{-iH_{tf}\,t}\,|\,\alpha\rangle = \langle\alpha\,|\,e^{iP_-\,e^{-t}}\,|\,\alpha\rangle\,. \tag{7.10}$$

From this, $F(t)\to 1$ as $t\to\infty$, and $F(0)=1-\delta<1$.

We will now prove that $F(t)$ is quasiperiodic as defined in 6.15. We expand the state $|\alpha\rangle$ in the energy basis[3]:

$$F(t) = \sum_{n,m} f_n^* f_m \langle n\,|\,e^{iP_-}\,|\,m\rangle e^{i(\omega_n-\omega_m)t}\,. \tag{7.11}$$

Consider the sum

$$\sum_{m,n} f_n^* f_m f_m^* f_n \langle n|e^{iP_-}|m\rangle\langle m\,|\,e^{-iP_-}\,|\,n\rangle = \sum_n |f_n|^2 \sum_m |f_m|^2 \langle n\,|\,e^{iP_-}\,|\,m\rangle\langle m\,|\,e^{-iP_-}\,|\,n\rangle\,. \tag{7.12}$$

Considering the inner sum, we have (since $\sum_m \langle n\,|\,e^{iP_-}\,|\,m\rangle\langle m\,|\,e^{-iP_-}\,|\,n\rangle = 1$, and the terms are real and positive)

$$\sum_m |f_m|^2 \langle n\,|\,e^{iP_-}\,|\,m\rangle\langle m\,|\,e^{-iP_-}\,|\,n\rangle \le 1\,, \tag{7.13}$$

and therefore

$$\sum_{m,n} f_n^* f_m f_m^* f_n \langle n\,|\,e^{iP_-}\,|\,m\rangle\langle m\,|\,e^{-iP_-}\,|\,n\rangle \le 1\,. \tag{7.14}$$

This shows that $F(t)$ satisfies the criterion 6.15, and hence $F(t)$ is quasiperiodic. Therefore, since $F(0)<1$, $F(t)$ can not tend to 1 as $t\to\infty$, and we have a contradiction.

This proves that H_{tf} can not have a countable spectrum, and therefore that the E_i cannot be discrete and the entropy cannot be finite. However, the area of the Rindler horizon in 1+1 dimensions is finite.

We see that there is a fundamental conflict between the holographic principle and the existence of a stable compactification of String Theory to 1+1 dimensional Minkowski space.

[3]The energies ω_i are the eigenvalues of the thermofield Hamiltonian H_{tf}, and hence are *differences* of pairs of energies of the Rindler Hamiltonian H_R, see the last paragraph in section 7.3.2.

7.4 Conclusion

We have found that maximally symmetric compactifications of String Theory to 1+1 dimensions in which the horizon area is finite are never stable. We used the specific example of 1+1 dimensional Rindler space, but our argument can easily be extended to other maximally symmetric spaces such as AdS space (see [87]). There exists one possible loophole to our argument: if the representation is trivial our argument fails. In that case the generators are zero, and there is no conflict with the algebra. In non-trivial cases however, compactifications to 1+1 dimensional Minkowski space must be unstable and have to decompactify.

We will briefly try to understand this result in the context of String Theory. In general. any compactification will always have massless moduli such as the overall size of the compact space. These massless degrees of freedom can be thought of as $1+1$ dimensional scalar fields. However, it is well known that in $1+1$ dimensions there are logarithmic infrared divergences – the two point function for example has a divergence of the form $\int dk/k$. Unlike the UV divergence which can of course be regulated, the IR divergence represents a true physical effect: the fields fluctuate more and more at longer and longer wavelengths. In particular, if the Hamiltonian includes a mass term $m^2\,\theta(-t)$, where θ is the unit step function, the expectation value $\langle\phi^2(t)\rangle$ will tend to infinity like t for $t > 0$. This means that the field, once released from its confining potential, fluctuates more and more in field space. This effect indicates a decompactification of the manifold: the compactification is unstable. This argument was discussed previously in [17]. It is in some ways more generic than the argument presented in this chapter, since it does not rely on maximal symmetry, but it does on the other hand require that at least some of the compactification moduli are massless.

Finally, we note that there exist $1+1$ dimensional compactifications of String Theory such as pp waves in anti de Sitter space and the linear dilaton vacua in which the dilaton varies linearly, either with respect to time or space. These compactifications violate Lorentz invariance and do not contradict our conclusion.

Part IV

Epilogue

In the final part of this thesis we will discuss the results of the work presented here and point to possible future directions.

In part II of this thesis, we have discussed the stability properties of certain simple classes of Randall-Sundrum braneworlds. One of the main questions we have addressed concerns the nature of the initial singularity in this particular setup. We motivated our work by explaining that the isotropic initial singularity is unstable against inhomogeneous perturbations in General Relativity – a major problem (amongst others) for inflation, since inflation needs a fairly isotropic universe to start at all. In the RS2 scenario on the other hand there exists considerable evidence that the initial singularity is generically isotropic. We have supported this conjecture using two different approaches in chapters 3 and 4.

In chapter 3 we studied the state space of the class of FLRW and the class of Bianchi I inflationary models with a minimally coupled scalar field with exponential potential. We systematically analyzed the state space by identifying and characterizing the equilibrium points of the different sectors of state space. By combining the information obtained in the different sectors, we could identify the attractors and repellers of the full state space. We concluded that the generic repeller of the full state space of FLRW models is the BDL model: an expanding isotropic high energy model [4]. We obtained a similar result when analyzing the anisotropic Bianchi I sector: again the BDL model is a past attractor, but now the slow-rolling anisotropic Kasner model presents another past attractor. We reasoned that the Kasner model, the competing past attractor, corresponds to an anisotropic inflating model for which it has been shown that the initial anisotropy always decays after a finite amount of time, and under certain conditions the initial expansion is quasi-isotropic too [145]. We argued that for this reason the generic past attractor is in fact the isotropic BDL model.

There are two shortcomings to our approach. Firstly, the class of Bianchi I models is highly symmetric. A more general class of models should be studied in order to confirm our results. This has been done for non-inflationary models: in [106] the Bianchi II models were studied, in [49, 103] Bianchi IX models were considered, and in [52] the class of inhomogeneous G_2 models was analyzed. All works agree with the results obtained here.

Secondly, when studying the Bianchi I models we neglected the non-local anisotropic stress $\pi^*_{\mu\nu}$ coupled to the shear. This assumption might be flawed as explained in section 3.5. A direction for future work would be to find the bulk solution containing the *full* class of Bianchi I branes [5] , to solve for $\pi^*_{\mu\nu}$ and insert the solution into our system of equations. In the absence of an exact solution, a viable phenomenological form of the non-local anisotropic stress could be found and used.

In chapter 4 we chose a different approach to analyze the stability of the isotropic initial

[4] In order to agree with the according literature we denote the same model as m in chapter 3 and as $\mathcal{F}_b$ in chapter 4.

[5] In [40] only a very special subset was considered.

braneworld singularity. We focused on flat FLRW backgrounds with three different matter contents corresponding to the physically relevant low energy, high energy and dark radiation dominated regimes. We then allowed for generic anisotropic and inhomogeneous perturbations and found that for the high energy background, all physically relevant perturbations decay when approaching very early times for equations of state with $\gamma > 4/3$. In the physically interesting case of radiation ($\gamma = 4/3$), the perturbations approach a constant value. Both the low energy and dark radiation background suffer from instabilities: at least some of the perturbations here grow when approaching very early times. Our result gives further support to the isotropization conjecture.

The approach in this chapter suffers from the same drawback as the previous one in chapter 3: we had to neglect the contribution of the non-local anisotropic stress since its evolution is not determined by the projected field equations. For scalar perturbations on very large scales this assumption is legitimate [90], but when studying vector and tensor perturbations $\pi^*_{\mu\nu}$ may be important. It is necessary to find and include the exact solution for $\pi^*_{\mu\nu}$ in order to fully understand the vector and tensor modes. It may even be necessary to study the full 5-d perturbation equations – these have been developed, but are highly complicated and can only be solved in special cases [169].

Furthermore, it would be worthwhile to check whether our results remain true when including higher order perturbations along the lines of [46] or [153].

Finally, on a more general note, we need to point out that the Randall-Sundrum scenario presents very special and relatively simple string inspired models. The RS action only includes the term linear in the curvature invariant. For a more general model, in particular when addressing problems at very high energies, it may be necessary to include higher order correction terms. Particulary interesting is the "Gauss-Bonnet" term, which is quadratic in the bulk curvature and leads to ghost-free field equations. The projected field equations for the "Gauss-Bonnet-braneworld" vary from the RS field equations, leading to a different Friedman equation in the high energy limit. It would be interesting to see whether our isotropization results remain true in "Gauss-Bonnet gravity".

In part III of this thesis, we have studied different maximally symmetric cosmological spacetimes with finite temperature horizons: 4-dimensional de Sitter space and 1+1 dimensional Rindler space. We argued that our results remain valid for de Sitter space of any dimension. Two fundamental concepts emerging from semi-classical Black Hole physics, "Horizon Complementarity" and "Holography" were assumed to hold. These concepts imply that there must exist a holographic theory dual to quantum gravity in each of these spacetimes, and this dual theory must be described by a field theory restricted to the causal patch of a given (external) observer. Because of the finite temperature of the horizon, this dual theory must be a finite temperature field theory.

We then showed how to implement the symmetries in de Sitter space (chapter 6) and in 1+1 dimensional Rindler space (chapter 7), and found in each case that these symmetries conflict with the finite temperature of the horizon. Thus one must either give up the finiteness of temperature and entropy or the exact symmetry of the respective classical space.

Our main emphasis is on conventional 4-dimensional de Sitter space in chapter 6, for the simple reason that we in fact seem to be living in an asymptotically de Sitter-like universe [9], and that de Sitter space seems to be a good description for the early (inflationary) universe. In this case we have argued that as a natural consequence the lifetime of any de Sitter phase can not exceed the Poincaré recurrence time introduced in [60]. In particular, we concluded that de Sitter solutions in String Theory are at best metastable, with a finite lifetime followed by a decay to a supersymmetric vacuum.

This troubling result has since been confirmed by various different authors using different approaches, all concluding that de Sitter space in String Theory is at best metastable [12, 15, 60, 61, 77, 81, 118, 173].

If eternal de Sitter space is unstable, it is natural to ask what the possible "decay channels" are. Since de Sitter space has positive vacuum energy, there cannot exist any supersymmetric compactifications of String Theory to de Sitter space. For this reason it is intuitively clear that any de Sitter space compactification may decay or "tunnel" to a de Sitter vacuum with slightly lower vacuum energy, and should eventually tunnel to a supersymmetric vacuum with vanishing vacuum energy (Minkowski space) or one with negative vacuum energy density (Anti-de Sitter space). This statement has been confirmed using certain limits of perturbative String Theory, where de Sitter vacua and the possible decay channels have been found [22, 31, 68, 81, 118, 154, 184]. However, these results are obtained for very specific vacua corresponding to certain "initial conditions" in the state space of all solutions of String Theory. In an attempt to find a more general approach, Susskind has introduced the notion of the "Landscape of String Theory" [194], which by definition extends well-defined supersymmetric moduli-space of String Theory to include vacua with broken super-symmetry. The introduction of a framework in which to study the dynamics within this landscape [75] might well be useful in order to shed more light on the (in-)stability of de Sitter vacua.

In chapter 7 we have concluded that the symmetries of String Theory compactifications to maximally symmetric 1+1 dimensional spacetimes cannot be reconciled with a finite horizon area. One way to avoid this conflict is a very small internal space with vanishing entropy, in which case there is only order one state in the Hilbert space, and the symmetry argument does not apply. Generically however, maximally symmetric 1+1 dimensional compactifications must be unstable, leading to a decompactification of the internal manifold to infinite volume.[6]

This result is particularly interesting since Minkowski space, unlike de Sitter space,

[6]We note that in [23], a supersymmetric compactification of heterotic String Theory to 1+1 dimensions has been presented as a counter-example to our results. We believe however that this model is trivial in the sense that it has a completely trivial S-matrix due to IR divergences, despite the lack of massless propagating degrees of freedom.

has vanishing vacuum energy density and is therefore not inherently prone to tunnel to a (possibly supersymmetric) vacuum with lower vacuum energy density. It would be interesting to see whether String Theory can confirm our result. This could be done by finding 1+1 dimensional Rindler vacua and explicitly calculating the evolution of the moduli, thus obtaining possible decay channels.

In summary, we have found in part III that the delicate symmetries that ensure the equivalence of different observers cannot be implemented for maximally symmetric systems of finite entropy. It seems that for these symmetries to be exact, an infinite horizon area must be available for information to spread out in.

On a final note, we may speculate that this is a general rule: exact observer complementarity is only possible if the horizon is infinite in extent. A finite mass Black Hole could be used to test this hypothesis: the Black Hole entropy is expected to be finite beyond doubt, and the exact symmetry between observers outside the black hole and those which fall through the horizon is expected to be broken. Sufficiently careful measurements of tidal forces should be able to tell a freely falling observer exactly when she crosses the horizon. The horizon according to our hypothesis would only become precisely undetectable in the limit of infinite mass of the Black Hole.

Appendix A

Non-hyperbolic equilibrium points – An explicit example

We will here describe in great detail how to use perturbative methods in order to study the nature of the equilibrium point M^0_ϵ. This point equilibrium point lies in both sectors A and C, and the Jacobian of the dynamical system (3.44), (3.54) is not well-defined in the vicinity of M^0_ϵ.

We will here describe how perturbations around this point evolve in sector A. The same methods must be used to study perturbations in sector C in order to understand the dynamics of the full state space in the vicinity of this equilibrium point.

Defining the small but *positive* perturbations $x(\tau)$, $y(\tau)$, $L(\tau)$, $z(\tau)$ and $r(\tau)$, the perturbed system close to the equilibrium point M^0_ϵ has the coordinates

$$(\gamma, \Omega_k, \Omega_\Lambda, \Omega_\lambda, \Omega_{\rho^*}) = (x, 1-y, L, z, r) \tag{A.1}$$

in state space. Dropping higher order terms, we find that the evolution of the perturbations is governed by the following system:

$$\begin{aligned} x' &= -6\epsilon x - 2\sqrt{3}b\sqrt{x}\sqrt{y-L-z-r}\,, && \text{(A.2)}\\ y' &= 2\epsilon y - 4\epsilon r\,, && \text{(A.3)}\\ L' &= 2\epsilon L\,, && \text{(A.4)}\\ z' &= 2\epsilon z\,, && \text{(A.5)}\\ r' &= -2\epsilon r\,. && \text{(A.6)} \end{aligned}$$

Since we are studying perturbations within the state space, the Friedmann constraint (3.42) must be satisfied, which means that the perturbations must satisfy the constraint

$$0 \le y - L - z - r \le 1\,, \tag{A.7}$$

which means that the last square root in(A.2) is always well-defined.

We aim to find the leading order solutions of these equations, even though it is already clear at this point that both the expanding and the contracting Milne model with $\gamma = 0$ cannot be stable equilibrium points in the full state space, since e.g. L' and r' have opposite signs, meaning that perturbations in Ω_Λ-direction grow whereas perturbations in Ω_{ρ^*}-direction decay or vice versa.

This argument does however break down when restricting to the subspace $\Omega_{\rho^*} = 0$ where $r = 0$. In that case the evolution of y, L and z is of the same nature, and the late time behavior of x is crucial to determine whether the model is a stable or an unstable equilibrium point.

The solutions to (A.4)-(A.6) are easily obtained and given by

$$\begin{aligned} L &= L_0 e^{2\epsilon\tau} , \\ z &= z_0 e^{2\epsilon\tau} , \\ r &= r_0 e^{-2\epsilon\tau} . \end{aligned} \tag{A.8}$$

The constant L_0, z_0, r_0 are arbitrary constant of integration.

Substituting these solutions into (A.3), we find that

$$y = y_0 e^{2\epsilon\tau} + 2 r_0 e^{-2\epsilon\tau} . \tag{A.9}$$

We now turn our attention to (A.2). If $b = 0$, then (A.2) shows that the perturbation x decays for $\epsilon = +1$ and grows for $\epsilon = -1$. If however $b \neq 0$, then there are two terms contributing to the growth of in x. If $\epsilon = -1$, then both these terms are positive, so it is clear that the perturbation grows as $\tau \to \infty$ in that case. The case $\epsilon = +1$ on the other hand is much more complicated, since there are now two conflicting terms, the first one being negative and the second one being positive since $b < 0$. It is not immediately clear which term dominates at late times, i.e. whether the perturbation x grows or decays as $\tau \to \infty$.

In order to study the growth of x for $\epsilon = +1$ and $b \neq 0$, we substitute the solutions (A.8) and (A.9) into (A.2). We do not study the cases $\epsilon = -1$ or $b = 0$ since in that case the dynamics are clear as discussed above. We find that the perturbation x changes according to

$$x' = -6x - 2\sqrt{3} b \sqrt{(y_0 - L_0 - z_0) e^{2\tau} + r_0 e^{-2\tau}} \sqrt{x} . \tag{A.10}$$

We can now define the quantity $s = \sqrt{x}$, which transforms (A.10) into

$$s' = -3s - \sqrt{3} b \sqrt{(y_0 - L_0 - z_0) e^{2\tau} + r_0 e^{-2\tau}} . \tag{A.11}$$

We will first solve this equation in the full state space, and then study the dynamics in specific subsets.

In general, $y_0 - L_0 - z_0$ is non-zero, so that we can drop the second term in the square root. (A.11) then has the simple solution

$$s = s_0 e^{-3\tau} - \tfrac{1}{3}\sqrt{3}b\sqrt{y_0 - L_0 - z_0}e^{\tau}\,, \tag{A.12}$$

where α is a non-zero constant that can be determined from (A.11). Then $x = s^2$ is given by

$$x = s_0^2 e^{-3\tau} + \tfrac{1}{3}b^2(y_0 - L_0 - z_0)^2 e^{2\tau} - \tfrac{2}{3}\sqrt{3}b\sqrt{(y_0 - L_0 - z_0)}s_0 e^{-2\tau}\,. \tag{A.13}$$

This means that the perturbation x grows in general at late times.

If however the initial conditions satisfy $y_0 - L_0 - z_0 = 0$, then (A.11) has the solution

$$s = s_0 e^{-3\tau} - \tfrac{1}{2}\sqrt{3}b\sqrt{r_0}e^{-\tau}\,, \tag{A.14}$$

where the constant β is determined from (A.11), and $x = s^2$ is given by

$$x = s_0^2 e^{-3\tau} + \alpha^2 e^{-2\tau} + 2s_0\alpha e^{-4\tau}\,. \tag{A.15}$$

For these specific initial conditions the perturbation x decay! This means that as we approach the equilibrium point from different "angles", we get repelled unless we hit a point corresponding to the specific initial condition $y_0 - L_0 - z_0 = 0$, in which case we get attracted to the equilibrium point. This odd behavior is due to the non-linearity of the system. It is important to note that the initial condition $y_0 - L_0 - z_0 = 0$ does not correspond to an invariant set of the full state space. If it did we would expect a change of dynamics when restricting to the set. Instead, the set corresponds to equality of fractional energy density and dark energy density contributions $\Omega_\rho = \Omega_{\rho^*}$.

We now want to run through this argument in interesting subspaces of the full state space. As motivated in 3.3.5, there are two subspaces we want to study:

1. The zero cosmological constant subspace with $\Omega_\Lambda = 0$, that means $L = 0$.

2. The general relativistic subspace with vanishing cosmological constant, classified by $\Omega_\Lambda = \Omega_\lambda = \Omega_{\rho^*} = 0$ or $L = z = r = 0$.

In case 1. the propagation equation of s becomes

$$s' = -3s - \sqrt{3}b\sqrt{(y_0 - z_0)e^{2\tau} + r_0 e^{-2\tau}}\,, \tag{A.16}$$

which has the leading order solution

$$s = s_0 e^{-3\tau} - \tfrac{1}{4}\sqrt{3}b\sqrt{(y_0 - z_0)}e^{\tau}\,, \tag{A.17}$$

if $y_0 \neq z_0$ and

$$s = s_0 e^{-3\tau} - \tfrac{1}{2}\sqrt{3}b\sqrt{(r_0)}e^{-\tau}\,, \tag{A.18}$$

if $y_0 = z_0$. This means that $x = s^2$ grows in time if $y_0 \neq z_0$ and decays for the specific initial condition $y_0 = z_0$, which corresponds to $\Omega_\rho = \Omega_{\rho^*}$. Since x grows in a subset of the subspace considered and decays in another subset, the point M_+^0 is a saddle point of that subspace.

For the general relativistic subset equation (A.11) becomes

$$s' = -3s - \sqrt{3}b\sqrt{y_0}e^\tau \ , \tag{A.19}$$

and the solution is

$$s = s_0 e^{-3\tau} + \tfrac{1}{3}b\sqrt{3y_0}e^\tau \ , \tag{A.20}$$

It is immediately clear that $x = s^2$ grows if $y_0 \neq 0$ and decays if $y_0 = 0$. This means that the point is a saddle point even in the subset (γ, Ω_k) of the full state space.

Appendix B

Harmonics

We use harmonics defined through operators constructed with the D_a derivatives, and constant along flow lines (i.e. independent of proper time). Here and elsewhere we shall often denote with Q the whole set of covariant harmonics defined in the next sections: using a standard but misleading terminology: scalars are denoted by Q^S, Q^S_a, Q^S_{ab}, vectors by Q^V_a, Q^V_{ab}, and tensors by Q^T_{ab}. In what follows, we shall omit the type index: for example Q^S_a should denote the vector constructed with the scalar harmonic Q^S, to distinguish it from the solenoidal vector Q^V_a (see below): we use this index only when we have to use the two-vector (or tensor) harmonics together.

In dealing with an almost FLRW model, we can use these harmonics in expanding first-order quantities; for example, for a scalar T we have:

$$T = \sum T_{(k)} Q^{(k)} \; , \tag{B.1}$$

We now look successively at scalar, vector, and tensor harmonics for a flat ($K = 0$) background.

B.1 Scalar harmonics

Definitions

We take the scalar harmonics as being eigenfunctions of the covariantly defined Laplace-Beltrami operator [37]

$$\mathrm{D}^2 Q \equiv \mathrm{D}_a \mathrm{D}^a Q = -\tfrac{k^2}{a^2} Q \; , \tag{B.2}$$

where from now on we drop the index (k) from the Q's. This scalar harmonic can be used to expand scalars; with this we can also define a vector via

$$Q_a = -\tfrac{a}{k} \mathrm{D}_a Q \tag{B.3}$$

and a trace-free symmetric tensor

$$Q_{ab} = \tfrac{k^2}{a^2}\mathrm{D}_b\mathrm{D}_a Q + \tfrac{1}{3}h_{ab}Q\ . \tag{B.4}$$

These harmonics are defined in order to have

$$\dot{Q} = \dot{Q}_a = \dot{Q}_{ab} = 0\ ; \tag{B.5}$$

using this property and applying the commutation relations in the appendix of [64] we get

$$\mathrm{D}_{[a}\mathrm{D}_{b]}Q = -\omega_{ab}\dot{Q} = 0\ , \tag{B.6}$$
$$\mathrm{D}_{[a}\mathrm{D}_{b]}Q_c = 0\ , \tag{B.7}$$
$$\mathrm{D}_{[a}\mathrm{D}_{b]}Q_{cd} = 0\ . \tag{B.8}$$

Properties

Using the derivative D_a we derive various properties satisfied by the above defined scalar harmonics.

$$a\mathrm{D}_a Q^a = kQ \tag{B.9}$$
$$a^2\mathrm{D}^2 Q_a = -k^2 Q_a \tag{B.10}$$
$$a\mathrm{D}_b Q_a = -k(Q_{ab} - \tfrac{1}{3}h_{ab}Q) \tag{B.11}$$
$$Q^a{}_a = 0 \tag{B.12}$$
$$a\mathrm{D}^b Q_{ab} = \tfrac{2}{3}kQ_a \tag{B.13}$$
$$a^2\mathrm{D}^a\mathrm{D}^b Q_{ab} = \tfrac{2}{3}k^2 Q \tag{B.14}$$
$$a^2\mathrm{D}_b\mathrm{D}^c Q_{ac} = -\tfrac{2}{3}k^2(Q_{ab} - \tfrac{1}{3}h_{ab}Q) \tag{B.15}$$
$$a^2\mathrm{D}^2 Q_{ab} = -k^2 Q_{ab} \tag{B.16}$$

B.2 Vector harmonics

Definitions

These are again defined as eigenfunctions of the Helmholtz equation

$$a^2\mathrm{D}^2 Q_a = -k^2 Q_a\ , \tag{B.17}$$

Q_a being a solenoidal vector:

$$\mathrm{D}_a Q^a = 0\ . \tag{B.18}$$

With this we can construct a symmetric trace-free tensor

$$Q_{ab} = -\tfrac{a}{2k}(\mathrm{D}_b Q_a + \mathrm{D}_a Q_b)\ ; \tag{B.19}$$

Both these harmonics are covariantly constant along flow lines:

$$\dot{Q}_a = \dot{Q}_{ab} = 0\,. \tag{B.20}$$

Properties

The following relations hold for the tensor Q_{ab} (B.19) defined from the vector Q_a.

$$Q^a{}_a = 0 \tag{B.21}$$
$$a\mathrm{D}^b Q_{ab} = \tfrac{1}{2} k Q_a \tag{B.22}$$
$$a^2 \mathrm{D}_b \mathrm{D}^c Q_{ac} + a^2 \mathrm{D}_a \mathrm{D}^c Q_{bc} = -k^2 Q_{ab} \tag{B.23}$$
$$a^2 \mathrm{D}^2 Q_{ab} = -k^2 Q_{ab} \tag{B.24}$$

Q_a and Q_{ab} also respectively satisfy the relations (B.7),(B.8).

B.3 Tensor harmonics

Again we have

$$a^2 \mathrm{D}^2 Q_{ab} = -k^2 Q_{ab}\,, \tag{B.25}$$

and

$$\dot{Q}_{ab} = 0\,, \tag{B.26}$$

where this tensor is trace - free and divergence-less

$$Q^a{}_a = 0\,, \tag{B.27}$$
$$\mathrm{D}^b Q_{ab} = 0\,, \tag{B.28}$$

and satisfies (B.8).

Appendix C

Proof: "Finite entropy implies discrete spectrum"

It is a fact that any system with finite entropy must have a discrete spectrum. The statement is actually even stronger: there must be a finite number of states below any given energy E, so the spectrum can not have discrete accumulation points. Because this is central to our analysis in chapters 6 and 6, we present a short proof of this statement below.

The entropy S is defined by $S \equiv -\mathrm{Tr}\,\rho \ln \rho$, where $\rho \equiv e^{-\beta H}/Z$ and $Z \equiv \mathrm{Tr}\, e^{-\beta H}$. Here trace means the sum over all the states in the spectrum of the Hamiltonian, to be replaced by an integral with the proper measure in the case that the spectrum is continuous. Expanding the relation above gives

$$S = (1/Z)\mathrm{Tr}\; e^{-\beta H}\,(\beta H + \ln Z) = \ln Z + (\beta/Z)\mathrm{Tr}\; He^{-\beta H}, \tag{C.1}$$

which is nothing but the ordinary thermodynamic relation $S = -\beta F + \beta E$. As can be easily seen from C.1, S is not affected by the shift $H \to H - E_0$, for any constant E_0. Therefore we can take the ground state energy to be zero (we will assume the energy is bounded from below). Then the second term in C.1 is manifestly positive, and

$$S \geq \ln Z = \ln \sum e^{-\beta H} \geq \ln \left(N e^{-\beta E_N} \right), \tag{C.2}$$

where E_N is the Nth energy level, and we have temporarily assumed that the spectrum is countable. Now the proof is clear: if there exists any energy E_N below which there are an infinite number of states, the entropy will diverge; and hence finite entropy implies a discrete spectrum with no accumulation points. Note that this does *not* imply that the Hilbert space is finite. In general, the level spacing $\delta E \sim T e^{-S}$, at least for $T \gg \delta E$.

Appendix D

Explicit mixing of degrees of freedom in 1+1 Rindler space

In this appendix we illustrate how the group of Minkowski translations acts on the Rindler degrees of freedom. For simplicity we study a 1+1 Minkowski toy model with a free scalar field ζ of mass m. The Minkowski coordinates are denoted by T, X and take values in $[-\infty, \infty]$. The Lagrangian density for a free scalar field of mass m is given by

$$\mathcal{L} = \sqrt{-g}\ [g_{\mu\nu}\partial^{\mu}\zeta\partial^{\nu}\zeta - m^2\zeta], \tag{D.1}$$

Rindler coordinates are defined by

$$r = \sqrt{X^2 - T^2},\ \tau = \frac{1}{2}\ln\frac{X+T}{X-T}. \tag{D.2}$$

They are defined on both the right and the left Rindler wedge, but time and radius run in opposite directions in the two wedges. In these coordinates the wave equation takes the form

$$\ddot{\zeta} - \frac{\partial^2}{\partial u^2}\zeta + m^2 e^{2u}\zeta = 0 \tag{D.3}$$

where $u = \ln r$ and dot indicates derivative with respect to τ. Let us note first that as we discussed earlier, the algebra of the Rindler Hamiltonian and the Minkowski translations can only be consistent if the entropy of Rindler space is infinite. Since the u axis is unbounded at both ends it is not surprising that the entropy of the scalar field at finite temperature is infinite. However the important infinity comes from one end as can be seen from the wave equation (D.3): the mass term acts like a wall preventing the field from penetrating to large positive u. On the other hand the region of large negative u is still massless. In this case the divergent entropy comes from the $u << 0$. This is of course the region very near the horizon.

Any solution of the wave equation can be expanded in terms of the basis functions

$$f_k = \frac{1}{\sqrt{c_k}} e^{iku - i\omega_k \tau} = \frac{1}{\sqrt{c_k}} r^{ik} e^{-i\omega_k \tau} \tag{D.4}$$

where $c_k = 4\pi\omega_k$ are normalization constants and $\omega_k = +\sqrt{k^2 + r^2 m^2}$.

The two thermofield copies of ζ at $\tau = 0$ are just the fields on the two half spaces $X > 0$ and $X < 0$. Each copy can be expanded in creation and annihilation operators for Rindler excitations. Denoting the operators by $a, a^\dagger$ and $b, b^\dagger$ for $X > 0$ and $X < 0$ respectively ζ can be expressed as

$$\zeta(r,\tau) = \int \frac{dk}{\sqrt{c_k}} [r^{ik} e^{-i\omega_k \tau} a(k) + r^{-ik} e^{i\omega_k \tau} a^\dagger(k)] \Theta(X) + \int \frac{dk}{\sqrt{c_k}} [r^{ik} e^{-i\omega_k \tau} b(k) + r^{-ik} e^{i\omega_k \tau} b^\dagger(k)] \Theta(-X). \tag{D.5}$$

The first integral corresponds to the part of the wave equation in the right Rindler wedge while the second integral corresponds to the part in the left Rindler wedge.

To illustrate the action of the Minkowski translations on the Rindler degrees of freedom we consider a translation along the X axis by distance a. The Rindler coordinates transform as

$$r' = \sqrt{(X-a)^2 - T^2} = \sqrt{r^2 + a^2 \mp 2ar\cosh\tau} \tag{D.6}$$

and

$$\tau' = \frac{1}{2} \ln \frac{X - a + T}{X - a - T} = \frac{1}{2} \ln \frac{\pm r\cosh\tau + r\sinh\tau - a}{\pm r\cosh\tau - r\sinh\tau - a}. \tag{D.7}$$

The upper signs apply in the right Rindler wedge ($X > 0$), the lower ones in the left Rindler wedge ($X < 0$).

In the right hand wedge, the shifted fields $\zeta, \dot{\zeta}$ at $\tau = 0$ have the form

$$\begin{aligned}
\zeta(r) &= \int \frac{dk}{\sqrt{c_k}} [(r-a)^{ik} a(k) + (r-a)^{-ik} a^\dagger(k)] \Theta(r-a) \\
&\quad + \int \frac{dk}{\sqrt{c_k}} [(a-r)^{ik} b(k) + (a-r)^{-ik} b^\dagger(k)] \Theta(a-r) \\
\dot{\zeta}(r) &= -i \int \frac{dk}{\sqrt{c_k}} \omega_k r [(r-a)^{ik-1} a(k) - (r-a)^{-ik-1} a^\dagger(k)] \Theta(r-a) \\
&\quad - i \int \frac{dk}{\sqrt{c_k}} \omega_k r [(a-r)^{ik-1} b(k) - (a-r)^{-ik-1} b^\dagger(k)] \Theta(a-r).
\end{aligned} \tag{D.8}$$

The shifted field can also be to expanded in terms of new (shifted) creation and annihilation operators $c^\dagger$ and c. Concentrating on the right hand Rindler wedge:

$$\zeta(r,\tau)\mid_{\tau=0} = \int dk [f_k c(k) + f_k^* c^\dagger(k)] \tag{D.9}$$

where the basis functions f_k are defined as above.

The Bogoliubov transformation giving the operators $c, c^\dagger$ is easily obtained. For example we find

$$c^\dagger(l) = \int dk\ [D(l,k)a^\dagger(k) + E(l,k)a(k) + F(l,k)b^\dagger(k) + G(l,k)b(k)] \tag{D.10}$$

with the Bogoliubov coefficients given by

$$\begin{aligned}
D(l,k) &= \frac{1}{4\pi\sqrt{\omega_k\omega_l}}(\omega_k + \frac{k}{l}\ \omega_l)a^{il-ik}B(-il+ik,-ik) \\
E(l,k) &= -\frac{1}{4\pi\sqrt{\omega_k\omega_l}}(\omega_k + \frac{k}{l}\ \omega_l)a^{il+ik}B(-il-ik,ik) \\
F(l,k) &= \frac{1}{4\pi\sqrt{\omega_k\omega_l}}\frac{l\omega_k - k\omega_l}{l-k}a^{il-ik}B(il,-ik) \\
G(l,k) &= -\frac{1}{4\pi\sqrt{\omega_k\omega_l}}\frac{l\omega_k - k\omega_l}{l+k}a^{il+ik}B(il,ik),
\end{aligned} \tag{D.11}$$

where $B(a,b)$ is the Euler β–function:

$$B(a,b) = \int_0^1 y^{a-1}(1-y)^{b-1}dy \tag{D.12}$$

and we have used the identity $B(a,b+1) = \frac{b}{a+b}B(a,b)$. In the massless limit these coefficients simplify to give

$$\begin{aligned}
D(l,k) &= \frac{1}{2\pi}\sqrt{\frac{k}{l}}a^{il-ik}B(-il+ik,-ik)\ \Theta(kl) \\
E(l,k) &= -\frac{1}{2\pi}\sqrt{\frac{k}{l}}a^{il+ik}B(-il-ik,ik)\ \Theta(kl) \\
F(l,k) &= (sgn(l) - sgn(k))\frac{\sqrt{|kl|}}{4\pi(l-k)}a^{il-ik}B(il,-ik) \\
G(l,k) &= -(sgn(l) - sgn(k))\frac{\sqrt{|kl|}}{4\pi(l+k)}a^{il+ik}B(il,ik).
\end{aligned} \tag{D.13}$$

Similar equations determine the transformed operators for the left wedge of Rindler space. In this way we explicitly see how the symmetries of Minkowski space mix the two copies of the thermofield double.

Bibliography

[1] O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, *Large N Field Theories, String Theory and Gravity*, Phys. Rept. **323** (2000) 183, hep-th/9905111.

[2] S. Alexander, R. Brandenberger and D. Easson, *Brane Gases in the Early Universe*, Phys. Rev. D **62** (2000) 103509, hep-th/0004212.

[3] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. Dvali, *New Dimensions at a Millimeter to a Fermi and Superstrings at a TeV*, Phys. Lett. B **436** (1998) 257, hep-ph/9804398.

[4] N. Arkani-Hamed, S. Dimopoulos and G. Dvali, *The Hierarchy Problem and New Dimensions at a Millimeter*, Phys. Lett. B **429** (1998) 263, hep-ph/9803315.

[5] D. K. Arrowsmith and C. M. Place, *An introduction to dynamical systems*, Cambridge University Press, Cambridge, 1990.

[6] A. Ashtekar, *Gravity and the Quantum*, to appear in the special issue "Space-time Hundred Years Later" of NJP, edited by J. Pullin and R. Price, gr-qc/0410054.

[7] A. Ashtekar and J. Lewandowski, *Background Independent Quantum Gravity: A Status Report*, Class. Quant. Grav. **21** (2004) R53, gr-qc/0404018.

[8] S. Ask, *Search For Extra Dimensions At LEP*, proceedings of the 32nd International Conference on High-Energy Physics (ICHEP04), Beijing, China, 16-22 Aug 2004, hep-ex/0410004.

[9] N. Bahcall, J.P. Ostriker, S. Perlmutter and P. J. Steinhardt, *The Cosmic Triangle: Revealing the State of the Universe*, Science **284** (1999) 1481, astro-ph/9906463.

[10] P. M. Bakshi and K.T. Mahanthappa, *Expectation value formalism in quantum field theory. 1*, JMP **4** (1963) 1;

P. M. Bakshi and K.T. Mahanthappa, *Expectation value formalism in quantum field theory. 2*, JMP **4** (1963) 12.

[11] T. Banks, *Cosmological Breaking of Supersymmetry or Little Lambda Goes Back to the Future II*, hep-th/0007146.

[12] T. Banks, *Heretics of the False Vacuum: Gravitational Effects On and Of Vacuum Decay 2*, hep-th/0211160.

[13] T. Banks and M. Dine, *Couplings and Scales in Strongly Coupled Heterotic String Theory*, Nucl. Phys. B **479** (1996) 173, hep-th/9605136.

[14] T. Banks and W. Fischler, *M-theory Observables for Cosmological Space-times*, hep-th/0102077.

[15] T. Banks and W. Fischler, *An upper bound on the number of e-foldings*, astro-ph/0307459.

[16] T. Banks, W. Fischler and S. Paban, *Recurrent Nightmares?: Measurement Theory in de Sitter Space*, JHEP **0212** (2002) 062, hep-th/0210160.

[17] T. Banks and L. Susskind, *The Number of States of Two Dimensional Critical String Theory*, Phys. Rev. D **54** (1996) 1677, hep-th/9511193.

[18] J. M. Bardeen, *Gauge invariant cosmological perturbations*, Phys. Rev. D **22** (1980) 1882.

[19] J. D. Barrow and R. Maartens, *Kaluza-Klein anisotropy in the CMB*, Phys. Lett. B **532** (2002) 153, gr-qc/0108073.

[20] J. D. Bekenstein, *Entropy Bounds and Black Hole Remnants*, Phys. Rev. D **49** (1994) 1912, gr-qc/9307035.

[21] K. Benakli and S. Davidson, *Baryogenesis in Models with a Low Quantum Gravity Scale*, Phys. Rev. D **60** (1999) 025004, hep-ph/9810280.

[22] P. Berglund, T. Hubsch and D. Minic, *de Sitter Spacetimes from Warped Compactifications of IIB String Theory*, Phys. Lett. B **534** (2002) 147, hep-th/0112079.

[23] A. Bergman, J. Distler and U. Varadarajan, *1+1 dimensional critical string theory and holography*, hep-th/0312115.

[24] D. Bigatti and L. Susskind, *TASI lectures on the Holographic Principle*, hep-th/0002044.

[25] P. Binétruy, C. Deffayet, U. Ellwanger and D. Langlois, *Brane cosmological evolution in a bulk with cosmological constant*, Phys. Lett. B **477** (2000) 285, hep-th/9910219.

[26] P. Binétruy, C. Deffayet and D. Langlois, *Non-conventional cosmology from a brane-universe*, Nucl. Phys. B **565** (2000) 269, hep-th/9905012.

[27] M. Bojowald and H. A. Morales-Tecotl, *Cosmological applications of loop quantum gravity*, Lect. Notes Phys. **646** (2004) 421, gr-qc/0306008.

[28] R. Bousso, *A Covariant Entropy Conjecture*, JHEP **9907** (1999) 004, hep-th/9905177;

R. Bousso, *Holography in General Space-times*, JHEP **9906** (1999) 028, hep-th/9906022;

R. Bousso, *The Holographic Principle for General Backgrounds*, Class. Quant. Grav. **17** (2000) 997, hep-th/9911002.

[29] R. Bousso, *Positive vacuum energy and the N-bound*, JHEP **0011** (2000) 038, hep-th/0010252.

[30] R. Bousso, *The Holographic Principle*, Rev. Mod. Phys. **74** (2002) 825, hep-th/0203101.

[31] R. Bousso and J. Polchinski, *Quantization of four-form fluxes and dynamical neutralization of the cosmological constant*, JHEP **0006** (2000) 006, hep-th/0004134.

[32] P. Bowcock and C. Charmousis, R. Gregory, *General brane cosmologies and their global spacetime structure*, Class. Quant. Grav. **17** (2000) 4745, hep-th/0007177.

[33] R. Brandenberger, *Inflationary Cosmology: Progress and Problems*, proceedings of the "International School on Cosmology", Kish Island, Iran, 22 Jan. - 4 Feb. 1999, hep-ph/9910410.

[34] R. Brandenberger, *Lectures on the Theory of Cosmological Perturbations*, Lect. Notes Phys. **646** (2004) 127, hep-th/0306071.

[35] R. Brandenberger and C. Vafa, *Superstrings in the Early Universe*, Nucl. Phys. B **316** (1989) 391.

[36] H. A. Bridgman, K. A. Malik and D. Wands, *Cosmological perturbations in the bulk and on the brane*, Phys. Rev. D **65** (2002) 043502, astro-ph/0107245.

[37] M. Bruni, P. K. S. Dunsby and G. F. R. Ellis, *Cosmological perturbations and the physical meaning of gauge-invariant variables*, Ap. J. **395** (1992) 34.

[38] M. Bruni, G. F. R. Ellis and P. K. S. Dunsby, *Gauge-invariant perturbations in a scalar field dominated universe*, Class. Quant. Grav. **9** (1992) 921.

[39] A. D. Burd and J. D. Barrow, *Inflationary models with exponential potentials*, Nucl. Phys. B **308** (1988) 929.

[40] A. Campos, R. Maartens, D. Matravers and C. F. Sopuerta, *Braneworld cosmological models with anisotropy*, Phys. Rev. D **68** (2003) 103520, hep-th/0308158.

[41] A. Campos and C. F. Sopuerta, *Evolution of Cosmological Models in the Brane-world Scenario*, Phys. Rev. D **63** (2001) 104012, hep-th/0101060.

[42] A. Campos and C. F. Sopuerta, *Bulk effects in the cosmological dynamics of brane-world scenarios*, Phys. Rev. D **64** (2001) 104011, hep-th/0105100.

[43] S. Carlip, *Quantum Gravity: a Progress Report*, Rept. Prog. Phys. **64** (2001) 885, gr-qc/0108040.

[44] K. Cheung, *Collider Phenomenology for models of extra dimensions*, talk given at the invited session of Beyond the Standard Model in the APS/DPF April meeting, hep-ph/0305003.

[45] D. J. H. Chung, H. Davoudiasl and L. Everett, *Experimental Probes of the Randall-Sundrum Infinite Extra Dimension*, Phys.Rev. D **64** (2001) 065002, hep-ph/0010103.

[46] C. Clarkson, *Cosmological density fluctuations and gravity waves: A covariant approach to gauge-invariant non-linear perturbation theory*, Phys. Rev. D **70** (2004) 103524, astro-ph/0311505.

[47] J. M. Cline, *Cosmological Expansion in the Randall-Sundrum Warped Compactification*, plenary talk given at COSMO-99, 2 October 1999, ICTP, Trieste, Italy, hep-ph/0001285.

[48] J. M. Cline, C. Grojean and G. Servant, *Cosmological Expansion in the Presence of an Extra Dimension*, Phys. Rev. Lett. **83** (1999) 4245, hep-ph/9906523.

[49] A. Coley, *No Chaos in Brane-World Cosmology*, Class. Quant. Grav. **19** (2002) L45, hep-th/0110117.

[50] A. Coley, *Dynamics of Brane-World Cosmological Models*, Phys. Rev. D **66** (2002) 023512, hep-th/0110049.

[51] A. Coley, *Isotropic singularity in brane cosmological models*, gr-qc/0312073.

[52] A. Coley, Y. He and W. C. Lim, *Isotropic singularity in inhomogeneous brane cosmological models*, Class. Quant. Grav. **21** (2004) 1311, gr-qc/0312075.

[53] E. J. Copeland, A. R. Liddle and J. E. Lidsey, *Steep inflation: ending braneworld inflation by gravitational particle production*, Phys. Rev. D **64** (2001) 023509, astro-ph/0006421.

[54] E. J. Copeland, A. R. Liddle and D. Wands, *Exponential potentials and cosmological scaling solutions*, Phys. Rev. D **57** (1998) 4686, gr-qc/9711068.

[55] C. Csaki, Mi. Graesser, C. Kolda and J. Terning, *Cosmology of One Extra Dimension with Localized Gravity*, Phys. Lett. B **462** (1999) 34, hep-ph/9906513.

[56] M. R. Douglas and N. A. Nekrasov, *Noncommutative Field Theory*, Rev. Mod. Phys. **73** (2001) 977, hep-th/0106048.

[57] P. K. S. Dunsby, M. Bruni and G. F. R. Ellis, *Covariant perturbations in a multi-fluid cosmological medium*, Ap. J. **395** (1992) 54.

[58] P. K. S. Dunsby, N. Goheer, M. Bruni and A. Coley, *Are braneworlds born isotropic?*, Phys. Rev. D **69** (2004) 101303, hep-th/0312174.

[59] R. Durrer and P. Kocian, *Testing braneworlds with the binary pulsar*, Class. Quant. Grav. **21** (2004) 2127, hep-th/0305181.

[60] L. Dyson, M. Kleban and L. Susskind, *Disturbing Implications of a Cosmological Constant*, JHEP **0210** (2002) 011, hep-th/0208013.

[61] L. Dyson, J. Lindesay, and L. Susskind, *Is There Really a de Sitter/CFT Duality*, JHEP **0208** (2002) 045, hep-th/0202163.

[62] D. A. Easson, *The Interface of Cosmology with String and M(ILLENNIUM) Theory*, Int. J. Mod. Phys. A **16** (2001) 4803, hep-th/0003086.

[63] G. F. R. Ellis and M. Bruni, *Covariant and gauge-invariant approach to cosmological density fluctuations*, Phys. Rev. D **40** (1989) 1804.

[64] G. F. R. Ellis, M. Bruni and J. Hwang, *Density-gradient-vorticity relation in perfect-fluid Robertson-Walker perturbations*, Phys. Rev. D **42** (1990) 1035.

[65] G. F. R. Ellis and H. van Elst, *Cosmological Models – Cargèse Lectures 1998*, pp 1-116, published in "Theoretical and Observational Cosmology", Kluwer, Dordrecht, 1999, gr-qc/9812046.

[66] U. Ellwanger, *The Cosmological Constant*, lecture given at the XIV Workshop "Beyond the Standard Model", Bad Honnef, 11-14 March 2002, hep-ph/0203252.

[67] L. Fidkowski, V. Hubeny, M. Kleban and S. Shenker, *The Black Hole Singularity in AdS/CFT*, JHEP **0402** (2004) 014, hep-th/0306170.

[68] J. L. Feng, J. March-Russell, S. Sethi and F. Wilczek, *Saltatory Relaxation of the Cosmological Constant*, Nucl. Phys. B **602** (2001) 307, hep-th/0005276.

[69] W. Fischler, *Taking de Sitter Seriously*, talk given at "Role of Scaling Laws in Physics and Biology (Celebrating the 60th Birthday of Geoffrey West)", Santa Fe, Dec. 2000.

[70] W. Fischler, A. Kashani-Poor, R. McNees and S. Paban, *The Acceleration of the Universe, a Challenge for String Theory*, JHEP **0107** (2001) 003, hep-th/0104181.

[71] W. Fischler and L. Susskind, *Holography and Cosmology*, hep-th/9806039.

[72] É. É. Flanagan, S.-H. H. Tye and I. Wasserman, *Cosmological Expansion in the Randall-Sundrum Brane World Scenario*, Phys. Rev. D **62** (2000) 044039, hep-ph/9910498.

[73] É. É. Flanagan, S.-H. H. Tye and I. Wasserman, *Brane World Models With Bulk Scalar Fields*, Phys. Lett. B **522** (2001) 155, hep-th/0110070.

[74] S. Forste, *Strings, Branes and Extra Dimensions*, Fortsch. Phys. 50 (2002) 221, hep-th/0110055.

[75] B. Freivogel and L. Susskind, *A Framework for the Landscape*, hep-th/0408133.

[76] A. R. Frey, *Warped Strings: Self-dual Flux and Contemporary Compactifications*, PhD thesis, hep-th/0308156.

[77] A. R. Frey, M. Lippert and B. Williams, *The fall of stringy de Sitter*, Phys. Rev. D **68** (2003) 046008, hep-th/0305018.

[78] J. Garriga and M. Sasaki, *Brane-world creation and black holes*, Phys. Rev. D **62** (2000) 043523, hep-th/9912118.

[79] C. Germani, M. Bruni and R. Maartens, *Gravitational collapse on the brane: a no-go theorem*, Phys. Rev. Lett. **87** (2001) 231302, gr-qc/0108013.

[80] G. Gibbons and S. Hawking, *Cosmological Event Horizons, Thermodynamics, and Particle Creation*, Phys. Rev. D **15** (1977) 2738.

[81] S. B. Giddings, *The fate of four dimensions*, Phys. Rev. D **68** (2003) 026006, hep-th/0303031.

[82] W. Godlowski and M. Szydlowski, *Brane Universes Tested by Supernovae Ia*, Gen. Rel. Grav. **36** (2004) 767, astro-ph/0404299.

[83] N. Goheer and P. K. S. Dunsby, *Braneworld Dynamics of Inflationary Cosmologies with Exponential Potentials*, Phys. Rev. D **66** (2002) 043527, gr-qc/0204059 .

[84] N. Goheer and P. K. S. Dunsby, *Exponential Potentials on the Brane*, Phys. Rev. D **67** (2003) 103513, gr-qc/0211020.

[85] N. Goheer, P. K. S. Dunsby, A. Coley and M. Bruni, *Large-scale perturbations on the brane and the isotropy of the cosmological singularity*, Phys. Rev. D **70** (2004) 123517, hep-th/0408092.

[86] N. Goheer, M. Kleban and L. Susskind, *The trouble with de Sitter space*, JHEP **0307** (2003) 056 , hep-th/0212209.

[87] N. Goheer, M. Kleban and L. Susskind, *1+1 Dimensional Compactifications of String Theory*, Phys. Rev. Lett. **92** (2004) 191601, hep-th/0310120.

[88] W. D. Goldberger and M. B. Wise, *Modulus Stabilization with Bulk Fields*, Phys. Rev. Lett. **83** (1999) 4922, hep-ph/9907447.

[89] M. Goliath and G. F. R. Ellis, *Homogeneous cosmologies with cosmological constant*, Phys. Rev. D **60** (1999) 023502, gr-qc/9811068.

[90] C. Gordon and R. Maartens *Density perturbations in the brane - world*, Phys. Rev. D **63** (2001) 044022, hep-th/0009010.

[91] S. S. Gubser, *AdS/CFT and gravity*, Phys. Rev. D **63** (2001) 084017, hep-th/9912001.

[92] J. J. Halliwell, *Scalar fields in cosmology with an exponential potential*, Phys. Lett. B **185** (1987) 341.

[93] E. Halyo, *De Sitter Entropy and Strings*, hep-th/0107169.

[94] T. Harko and M. K. Mak, *Anisotropy in Bianchi-type brane cosmologies*, Class. Quant. Grav. **21** (2004) 1489, gr-qc/0401069.

[95] I. P. C. Heard and D. Wands, *Cosmology with positive and negative exponential potentials*, Class. Quant. Grav. **19** (2002) 5435, gr-qc/0206085.

[96] S. Hellerman, private communication.

[97] S. Hellerman, N. Kaloper and L. Susskind, JHEP **0106** (2001) 003, hep-th/0104180.

[98] Y. Himemoto and M. Sasaki, *Brane-world inflation without inflaton on the brane*, Phys. Rev. D **63** (2001) 044015, gr-qc/0010035.

[99] C. J. Hogan, *Brane World Astronomy*, proceedings of the 20th Texas Symposium on Relativistic Astrophysics, astro-ph/0104105.

[100] G. 't Hooft, *Dimensional Reduction in Quantum Gravity*, gr-qc/9310026.

[101] G. 't Hooft, *Quantum Information and Information Loss in General Relativity*, elaborated lecture notes for ISQM-Tokyo'95, gr-qc/9509050.

[102] G. 't Hooft, *Horizons*, lecture presented at the Erice School for Sub-Nuclear Physics, Sept. 2003, gr-qc/0401027.

[103] R. J. van den Hoogen, A. A. Coley and Y. He, *Bianchi IX Brane-world Cosmologies*, Phys. Rev. D **68** (2003) 023502, gr-qc/0212094.

[104] R. J. van den Hoogen, A. A. Coley and D. Wands, *Scaling Solutions in Robertson-Walker Spacetimes*, Class. Quantum Grav. **16** (1999) 1843, gr-qc/9901014.

[105] R. J. van den Hoogen and A. Horne, *Brane-world Cosmologies with non-local bulk effects*, gr-qc/0408014.

[106] *Bianchi II Brane-world Cosmologies ($\mathcal{U} \geq 0$)*, R. J. van den Hoogen and J. Ibanez, Phys. Rev. D **67** (2003) 083510, gr-qc/0212095.

[107] P. Hořava and E. Witten, *Heterotic and Type I String Dynamics from Eleven Dimensions*, Nucl. Phys. B **460** (1996) 506, hep-th/9510209.

[108] P. Hořava and E. Witten, *Eleven-Dimensional Supergravity on a Manifold with Boundary*, Nucl. Phys. B **475** (1996) 94, hep-th/9603142.

[109] G. T. Horowitz, *Quantum Gravity at the Turn of the Millennium*, based on talk presented at the "Ninth Marcel Grossmann Meeting" in Rome, July 2000, gr-qc/0011089.

[110] G. T. Horowitz, *Spacetime in String Theory*, to appear in "Spacetime 100 Years later" edited by J. Pullin and R. Price, 2005, gr-qc/0410049.

[111] C. D. Hoyle, D. J. Kapner, B. R. Heckel, E. G. Adelberger, J. H. Gundlach, U. Schmidt and H. E. Swanson, *Sub-millimeter Tests of the Gravitational Inverse-square Law*, Phys. Rev. D **70** (2004) 042004, hep-ph/0405262.

[112] G. Huey and J. E. Lidsey, *Inflation, braneworlds and quintessence*, Phys. Lett. B **514** (2001) 217, astro-ph/0104006.

[113] D. Ida, *Brane-world cosmology*, JHEP **09** (2000) 014, gr-qc/9912002.

[114] W. Israel, *Singular hypersurfaces and thin shells in general relativity*, Nuovo Cim. B **44** (1966) 1.

[115] W. Israel, *Thermo Field Dynamics Of Black Holes*, Phys. Lett. A **57** (1976) 107.

[116] S. Kachru, *Lectures on Warped Compactifications and Stringy Brane Constructions*, proceedings of TASI 1999 and the 2000 Trieste Spring Workshop on Superstrings, hep-th/0009247.

[117] S. Kachru, R. Kallosh, A. Linde, J. Maldacena, L. McAllister and S. P. Trivedi, *Towards Inflation in String Theory*, JCAP **0310** (2003) 013, hep-th/0308055.

[118] S. Kachru, R. Kallosh, A. Linde and S. P. Trivedi, *De Sitter vacua in string theory*, Phys. Rev. D **68** (2003) 046005, hep-th/0301240.

[119] N. Kaloper, *Bent Domain Walls as Braneworlds*, Phys. Rev. D **60** (1999) 123506, hep-th/9905210.

[120] T. Kaluza, *On the Problem of Unity in Physics*, Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.), **K1** (1921) 966.

[121] G. L. Kane, *Connecting String Theory and Phenomenology*, invited Plenary talk, SUSY02, Hamburg, June 2002, hep-ph/0210352.

[122] A. Kehagias and E. Kiritsis, *Mirage Cosmology*, JHEP **9911** (1999) 022, hep-th/9910174.

[123] L. V. Keldysh, Zh. Eksp. Teor. Fiz. **47** (1964) 1515; Sov. Phys. JETP **20** (1965) 1018.

[124] I. Klebanov and L. Susskind, *Continuum strings from discrete field theories*, Nucl. Phys. B **309** (1988) 175.

[125] O. Klein, *Quantum Theory and Five-Dimensional Theory of Relativity*, Z. Phys. **37** (1926) 89.

[126] S. Kobayashi, K. Koyama and J. Soda, *Quantum Fluctuations of Bulk Inflaton in Inflationary Brane World*, Phys. Lett. B **501** (2001) 157, hep-th/0009160.

[127] H. Kodama and M. Sasaki, *Cosmological Perturbation Theory*, Prog. Theor. Phys. **78** (1984) 1.

[128] E. W. Kolb and M. S. Turner, *The Early Universe*, Addison - Wesley, 1990.

[129] K. Koyama, D. Langlois, R. Maartens and D. Wands, *Scalar perturbations from braneworld inflation*, JCAP **11** (2004) 002, hep-th/0408222.

[130] P. Kraus, *Dynamics of Anti-de Sitter Domain Walls*, JHEP **9912** (1999) 011, hep-th/9910149.

[131] P. Kraus, H. Ooguri and S. Shenker, *Inside the Horizon with AdS/CFT*, Phys. Rev. D **67** (2003) 124022, hep-th/0212277.

[132] D. Langlois, R. Maartens, M. Sasaki and D. Wands, *Large-scale cosmological perturbations on the brane*, Phys. Rev. D **63** (2001) 084009, hep-th/0012044.

[133] D. Langlois, R. Maartens and D. Wands, *Gravitational waves from inflation on the brane*, Phys. Lett. B **489** (2000) 259, hep-th/0006007.

[134] B. Leong, P. K. S. Dunsby, A. Challinor and A. Lasenby, *(1+3) Covariant Dynamics of Scalar Perturbations in Braneworlds*, Phys. Rev. D **65** (2002) 104012, gr-qc/0111033.

[135] B. Leong, A. Challinor, R. Maartens and A. Lasenby, *Braneworld Tensor Anisotropies in the CMB*, Phys. Rev. D **66** (2002) 104010, astro-ph/0208015.

[136] A. R. Liddle and A. J. Smith, *Observational constraints on braneworld chaotic inflation*, Phys. Rev. D **68** (2003) 061301, astro-ph/0307017.

[137] A. R. Liddle and L. A. Urena-Lopez, *Curvaton reheating: an application to braneworld inflation*, Phys. Rev. D **68** (2003) 043517, astro-ph/0302054.

[138] A. Linde, *Inflation and quantum cosmology*, Academic Press (1990).

[139] A. Lukas, B. A. Ovrut, K. S. Stelle and D. Waldram, *The Universe as a Domain Wall*, Phys. Rev. D **59** (1999) 086001, hep-th/9803235;

A. Lukas, B. A. Ovrut and D. Waldram, *Cosmological Solutions of Horava-Witten Theory*, Phys. Rev. D **60** (1999) 086001, hep-th/9806022;

A. Lukas, B. A. Ovrut and D. Waldram, *Cosmology and Heterotic M-Theory in Five-Dimensions*, lectures presented at the Advanced School on Cosmology and Particle Physics, Peniscola, Spain, June 1998, hep-th/9812052;

A. Lukas, B. A. Ovrut and D. Waldram, *Boundary Inflation*, Phys. Rev. D **61** (2000) 023506, hep-th/9902071.

[140] J. Lykken and L. Randall, *The Shape of Gravity*, JHEP **0006** (2000) 014, hep-th/9908076.

[141] R. Maartens, *Cosmological dynamics on the brane*, Phys. Rev. D **62** (2000) 084023, hep-th/0004166.

[142] R. Maartens, *Geometry and Dynamics of the Brane - world*, p.93-119 in "Reference Frames & Gravitomagnetism", edited by J. Pascual - Sanchez et al., World Scientific 2001, gr-qc/0101059.

[143] R. Maartens, *Brane-world cosmological perturbations: a covariant approach*, Prog. Theor. Phys. Suppl. **148** (2003) 213, gr-qc/0304089.

[144] R. Maartens, *Brane-world gravity*, Living Rev. Rel. **7** (2004) 1, gr-qc/0312059.

[145] R. Maartens, V. Sahni and T. D. Saini, *Anisotropy dissipation in brane-world inflation*, Phys. Rev. D **63** (2001) 063509, gr-qc/0011105.

[146] R. Maartens, D. Wands, B. A. Bassett and I. P. C. Heard, *Chaotic inflation on the brane*, Phys. Rev. D **62** (2000) 041301, hep-ph/9912464.

[147] M. S. Madsen, *Scalar fields in curved spacetimes*, Class. Quantum Grav. **5** (1988) 627.

[148] K. T. Mahanthappa, *Multiple Production of Photons in Quantum Electrodynamics*, Phys. Rev. **126** (1962) 329.

[149] M. D. Maia and E. M. Monte, *Covariant Analysis of Experimental Constraints on the Brane-world*, astro-ph/0404370.

[150] A. S. Majumdar, *From brane assisted inflation to quintessence through a single scalar field*, Phys. Rev. D **64** (2001) 083503, astro-ph/0105518.

[151] J. Maldacena, *The large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. **2** (1998) 231; Int. J. Theor. Phys. **38** (1999) 1113, hep-th/9711200.

[152] J. M. Maldacena, *Eternal black holes in Anti-de-Sitter*, JHEP **0304** (2003) 021, hep-th/0106112.

[153] K. A. Malik and D. Wands, *Evolution of second-order cosmological perturbations*, Class. Quant. Grav. **21** (2004) L65, astro-ph/0307055.

[154] A. Maloney, E. Silverstein and A. Strominger, *de Sitter Space in Non-Critical String Theory*, Hawking Festschrift, hep-th/0205316.

[155] P. Martin and J. Schwinger, *Theory of many-particle systems. 1*, Phys. Rev. **115** (1959) 1342.

[156] I. Moss and V. Sahni, *Anisotropy in the chaotic inflationary universe*, Phys. Lett. B **178** (1986) 159.

[157] V. M. Mostepanenko, *Constraints on Non-Newtonian Gravity from Recent Casimir Force Measurements*, proceedings of the "18th Course of the School on Cosmology and Gravitation: The Gravitational Constant. Generalized Gravitational Theories and Experiments", 30 April- 10 May 2003, Erice, gr-qc/0311047.

[158] S. Mukohyama, *Brane-world solutions, standard cosmology, and dark radiation*, Phys. Lett. B **473** (2000) 241, hep-th/9911165.

[159] S. Mukohyama, T. Shiromizu and K. Maeda, *Global structure of exact cosmological solutions in the brane world*, Phys. Rev. D **62** (2000) 024028; Erratum-ibid. D **63** (2001) 029901, hep-th/9912287.

[160] N. J. Nunes and E. J. Copeland, *Tracking quintessential inflation from brane worlds*, Phys. Rev. D **66** (2002) 043524, astro-ph/0204115.

[161] T. Padmanabhan, *Cosmological Constant - the Weight of the Vacuum*, Phys. Rept. **380** (2003) 235, hep-th/0212290.

[162] M. Parikh, I. Savonije and E. Verlinde, *Elliptic de Sitter Space: dS/Z_2*, Phys. Rev. D **67** (2003) 064005, hep-th/0209120.

[163] P. J. E. Peebles, *Principles of Physical Cosmology*, Princeton University Press, Princeton, 1993.

[164] R. Penrose, *Twistor Algebra*, J. Math. Phys. **8** (1967) 345;
R. Penrose, *Nonlinear gravitons and curved twistor theory*, Gen. Rel. Grav. **7** (1976) 31.

[165] R. Penrose, *Singularities and Time-Asymmetry*, in "General Relativity: An Einstein centenary survey", p. 581, edited by S. W. Hawking and W. Israel, Cambridge University Press, 1979.

[166] L. Randall and R. Sundrum, *A Large Mass Hierarchy from a Small Extra Dimension*, Phys. Rev. Lett. **83** (1999) 3370, hep-ph/9905221.

[167] L. Randall and R. Sundrum, *An Alternative to Compactification*, Phys. Rev. Lett. **83** (1999) 4690, hep-th/9906064.

[168] B. Ratra and P. J. E. Peebles, *Cosmological consequences of a rolling homogeneous scalar field*, Phys. Rev. D **37** (1988) 3406.

[169] A. Riazuelo, F. Vernizzi, D. Steer and R. Durrer, *Gauge invariant cosmological perturbation theory for braneworlds*, hep-th/0205220.

[170] A. G. Riess et al., *Observational evidence from supernovae for an accerlerating universe and a cosmological constant*, Astron. J. **116** (1998) 1009.

[171] C. Rovelli, *Loop Quantum Gravity*, Living Rev. Rel. **1** (1998) 1, gr-qc/9710008.

[172] C. Rovelli, *A dialog on quantum gravity*, Int. J. Mod. Phys. D **12** (2003) 1509, hep-th/0310077.

[173] V. A. Rubakov and S. M. Sibiryakov, *False vacuum decay in the de Sitter spacetime*, Theor. Math. Phys. **120** (1999) 1194, gr-qc/9905093.

[174] V. Sahni, *The Cosmological Constant Problem and Quintessence*, Class. Quant. Grav. **19** (2002) 3435, astro-ph/0202076.

[175] V. Sahni, M. Sami and T. Souradeep, *Relic Gravity Waves from Braneworld Inflation*, Phys. Rev. D **65** (2002) 023518, gr-qc/0105121.

[176] M. G. Santos, F. Vernizzi and P. G. Ferreira, *Isotropy and Stability of the Brane*, Phys. Rev. D **64** (2001) 063506, hep-ph/0103112.

[177] M. Sasaki, T. Shiromizu and K. Maeda, *Gravity, Stability and Energy Conservation on the Randall-Sundrum Brane-World*, Phys. Rev. D **62** (2000) 024008, hep-th/9912233.

[178] J. Schwinger, *Brownian motion of a quantum oscillator*, JMP **2** (1961) 407.

[179] N. Seiberg and E. Witten, *String Theory and Noncommutative Geometry*, JHEP **9909** (1999) 032, hep-th/9908142.

[180] Y. Sendouda, K. Kohri, S. Nagataki and K. Sato, *Sub-GeV Galactic cosmic-ray antiprotons from PBHs in the Randall-Sundrum braneworld*, astro-ph/0408369.

[181] T. Shanks, *Problems with the Current Cosmological Paradigm*, proceedings of IAU Symposium 216, Maps of the Cosmos, ASP Conf. Ser., editors M. Colless and L. Staveley-Smith, astro-ph/0401409.

[182] S. Shenker, private communication.

[183] T. Shiromizu, K. I. Maeda and M. Sasaki, *The Einstein Equations on the 3-Brane World*, Phys. Rev. D **62** (2000) 024012, gr-qc/9910076.

[184] E. Silverstein, *(A)dS Backgrounds from Asymmetric Orientifolds*, hep-th/0106209.

[185] P. Singh, R. G. Vishwakarma and N. Dadhich, *A string inspired brane model and supernovae Ia observations*, hep-th/0206193.

[186] L. Smolin, *How far are we from the quantum theory of gravity?*, hep-th/0303185.

[187] L. Smolin, *An invitation to loop quantum gravity*, hep-th/0408048.

[188] J. M. Stewart and M. Walker, *Covariant and gauge-invariant approach to cosmological density fluctuations*, Proc. R. Soc. London A **341** (1974) 49.

[189] D. Stojkovic, *Distinguishing between the small ADD and RS black holes in accelerators*, to be published in Phys. Rev. Lett., hep-ph/0409124.

[190] N. Straumann, *The history of the cosmological constant problem*, invited talk at the "XVIIIth IAP Colloquium: Observational and theoretical results on the accelerating universe", July 1-5 2002, Paris, gr-qc/0208027.

[191] A. Strominger, *The dS/CFT Correspondence*, JHEP **0110** (2001) 034, hep-th/0106113.

[192] L. Susskind, *The World as a hologram*, J. Math. Phys. **36** (1995) 6377, hep-th/9409089.

[193] L. Susskind, *Twenty Years of Debate with Stephen*, p. 330-347 in 'The future of theoretical physics and cosmology - Celebrating Stephen Hawking's 60th Birthday" published in Cambridge 2003, hep-th/0204027.

[194] L. Susskind, *The Anthropic Landscape of String Theory*, hep-th/0302219.

[195] L. Susskind, L. Thorlacius and J. Uglum, *The Stretched Horizon and Black Hole Complementarity*, Phys. Rev. D **48** (1993) 3743, hep-th/9306069.

[196] L. Susskind and J. Uglum, *Black Holes, Interactions, and Strings*, hep-th/9410074.

[197] L. Susskind and E. Witten, *The Holographic Bound in Anti-de Sitter Space*, hep-th/9805114.

[198] Y. Takahashi and H. Umezawa, *Thermo field dynamics*, Collective Phenomena **2** (1975) 55.

[199] T. Thiemann, *Lectures on Loop Quantum Gravity*, Lect. Notes Phys. **631** (2003) 41, gr-qc/0210094.

[200] K. P. Tod, *Isotropic singularities and the gamma = 2 equation of state*, Class. Quant. Grav. **7** (1990) L13.

[201] P. K. Townsend, *Quintessence from M-Theory*, JHEP **0111** (2001) 042, hep-th/0110072.

[202] S. Tsujikawa and A. R. Liddle, *Constraints on braneworld inflation from CMB anisotropies*, JCAP **0403** (2004) 001, astro-ph/0312162.

[203] S. Tsujikawa, P. Singh and R. Maartens, *Loop quantum gravity effects on inflation and the CMB*, Class. Quant. Grav. **21** (2004) 5767, astro-ph/0311015.

[204] A. Vilenkin, *Cosmological constant problems and their solutions*, based on talks given at "The dark Universe" (Space Telescope Institute) and PASCOS-2001 in April 2001, hep-th/0106083.

[205] R. G. Vishwakarma and P. Singh, *Can brane cosmology with a vanishing Λ explain the observations?*, Class. Quant. Grav. **20** (2003) 2033, astro-ph/0211285.

[206] M. Visser, *An exotic class of Kaluza-Klein models*, Phys. Lett. B **159** (1985) 22, hep-th/9910093.

[207] J. Wainwright and G. F. R. Ellis, *Dynamical systems in cosmology*, Cambridge University Press, Cambridge, 1997.

[208] D. Wands, *String-inspired cosmology*, Class. Quant. Grav. **19** (2002) 3403, hep-th/0203107.

[209] E. Witten, *Strong Coupling Expansion Of Calabi-Yau Compactification*, Nucl. Phys. B **471** (1996) 135, hep-th/9602070.

[210] E. Witten, *Anti-de Sitter space and holography*, Adv. Theor. Math. Phys. **2** (1998) 253, hep-th/9802150.

[211] E. Witten, *Quantum Gravity In De Sitter Space*, prepared for "International School of Subnuclear Physics: 39th Course: New Fields and Strings in Subnuclear Physics", Erice, Italy, 29 Aug - 7 Sep 2001, hep-th/0106109.