A STOCHASTIC MODEL OF THE SOUTH AFRICAN GOLD MINES

by

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Abstract

A stochastic model of the South African Gold mines was constructed using Contingent Claims Analysis. This method allows the modelling of the major sources of uncertainty that the gold mines face, namely, uncertainty surrounding the future gold price, the exchange rate, the inflation rate and the interest rate. The trajectories of these variables were modelled by stochastic differential equations. By applying the principles of contingent claims analysis, we could obtain a valuation partial differential equation that described the value of the mine contingent on the current values of the state variables mentioned above. This partial differential equation was solved by the Monte Carlo method and the solution was compared to current estimates of the mines' value.

Dedication

I would like to dedicate this thesis to my parents, Bruce and Lorraine Beelders, and my brother, Reuben; without your support I could not have completed this thesis. Until my return to South Africa:

"The Lord bless thee, and keep thee:

The Lord make his face shine upon thee, and be gracious unto thee:

The Lord lift up His countenance upon thee,

and give thee His peace.

I would also like to thank the staff of the Department of Mathematical Statistics and School of Economics for their support. More specifically, I would like to thank Prof. Barr for his suggestion that I attempt the Masters Programme in Economics and Prof. Abraham for his support as convenor of the course.

Acknowledgments

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I would also like to acknowledge the help of Anton Theunisen who provided me with Ben Bernanke's 1983 paper that evetually led to this thesis.

Chapter 1

Mathematical Preliminaries

1.1 Stochastic Differential equations:

In the model of the extraction of an exhaustible resource I will assume that changes in the state variables can be described by continuous time stochastic processes. By making the following three assumptions, we can derive the continuous stochastic processes from the discrete approximation by letting h, the time increment, tend to zero.

Assumption 1: The state variable is a time-homogeneous Markov process.

Assumption 2: Only local changes in the state variables are allowed.

Assumption 3: At time t, the state variable has an expected rate of return per unit time defined by

$$\mu = E_t[(p(t+h) - p(t))/p(t)]/h \tag{1}$$

and a variance of the rate of return per unit time defined by

$$\sigma^2 = E_t[\{(p(t+h) - p(t))/p(t) - \mu h\}^2]/h \tag{2}$$

that exist, are finite with $\sigma^2 > 0$ and are right continuous functions of h. "E_t" is the conditional expectations operator, conditional on the level of the state variable at time t. In the limit as h tends to 0, μ is called the instantaneous expected return and σ^2 the instantaneous variance of the return.

The term "time homogeneous" in the first assumption merely means that the conditional probabilities of the stochastic process are only dependent on the time interval between two observations. Assumption 2 is the discrete time analog of continuity i.e. with probability 1, $\lim_{h\to 0} [p(t+h) - p(t)] = 0$. This implies that in small time intervals, changes in the state variable are small. Furthermore, this assumption rules out Poisson-type jump processes (Merton (1976); Kushner (1967)). Assumptions 1 and 2 actually imply 3 (Feller (1966) p.321). Assumption 3 ensures that the disturbance does not approach zero or dominate the analysis in small time intervals.

Combining assumptions 1, 2 and 3, as h approaches zero, p(t) is a diffusion process with continuous state space changes and transition probabilities that satisfy the Fokker-Planck or Kolmogorov partial differential equation (Arnold (1974) p.41-44).

In order to gain some insight into the stochastic differential equations consider a discrete form of the dynamics and then let h tend to zero.

$$[p(t+h) - p(t)]/p(t) = \mu h + \sigma y(t)\sqrt{h} \qquad (3)$$

where by construction the $y(t)$ are independently and identically Normally distributed with mean zero and variance 1. If we define the stochastic process,

$$z(t+h) = z(t) + y(t)\sqrt{h} \quad (4)$$

then $z(t)$ has independent increments and since the $y(t)$ are normally distributed, the limit as h tends to zero of $z(t+h) - z(t)$ describes a Wiener process or Brownian motion. In the limit as h tends to zero, we can write (4) as a stochastic differential equation

$$dz = y(t)\ \sqrt{dt}$$

$$E_t(dz) = 0, \quad V(dz) = dt \quad (5)$$

where "d" denotes the total derivative. Note that the presence of the standard Normal random variable causes dz to have a mean of zero and a variance equal to the time increment. We will always refer to dz as a Wiener process or Brownian motion. Therefore, taking the limit of (3) as h tends to zero we obtain the stochastic differential equation

$$dp/p = \mu dt + \sigma dz \quad (6)$$

where we refer to μdt as the drift term and σdz as the disturbance term. We refer to processes of this sort as Itô processes; they are continuous, but not differentiable. In this example, $p(t)$ has a lognormal distribution, however, more will will be said about this process after we have discussed Itô's lemma and differential operator.

One can also write the stochastic differential equation (7) as an integral equation (8) where the limits of integration are $\tau=0$ and $\tau=t$ i.e.

$$dp = \mu dt + \sigma dz \tag{7}$$

$$\int dp = \int \mu dt + \int \sigma dz, \tag{8}$$

which has as its solution

$$p(t) = p(0) + \mu t + \int \sigma dz \tag{9}$$

Since we don't have a concept of a stochastic integral we cannot integrate the last term, this will require the definition of a Itô Integral[1]. However, using a more intuitive approach, since dz is a Wiener process, the increments are independent standard Normal random variables and the integral represents the sum of these normal variates. Therefore, we know that $p(t) - p(0) - \mu t$ is a standard Normal random variable and furthermore, the transition probabilities have a Normal distribution i.e. we can solve the Fokker-Planck equation to obtain a Normal density.[2]

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1. There are other concepts of a stochastic integral eg. Stratinovich, but the Itô integral corresponds to the idea of updating a portfolio: at period t one can choose a portfolio for future periods and is unable to change past events. This interpretation follows from the fact that the Itô integral, as a function of its upper limit, is a Martingale (Arnold (1974) p.61).

2. I could not use (6) to show the integral equation because both the drift and disturbance are functions of p, because p is a function of time, we cannot integrate with respect to time.

1.2 Itô's Lemma and Differential Operator

Itô's lemma is a mathematical result that we use to derive the stochastic differential equation that describes the evolution of a function of a state variable.

<u>Itô's Lemma:</u> For a given state variable s which is described by the stochastic differential equation

$$ds = \mu dt + \sigma dz \tag{1}$$

any function of the state variable $y = F(s,t)$ can be characterised by the stochastic differential equation

$$dF = (\tfrac{1}{2}\sigma^2 F_{ss} + \mu F_s + F_t)dt + \sigma F_s dz. \tag{2}$$

where the subscripts denote partial derivatives and if y is not an explicit function of t, F_t does not appear.

The important result of Itô's Lemma is the fact that the drift of F includes the extra term $\frac{1}{2}\sigma^2 F_{ss}$ that is related to the variance of s; one would expect to have μF_s, which reflects the sensitivity of F to s and the drift in s, and F_t, which relects the movement over time. The presence of the extra term is due to Jensen's inequality which does not fade out due to the nature of the Brownian motion term dz.

Itô's differential operator is defined as $(1/dt)E_t dF(\cdot)$ where $F(\cdot)$ is a function of a state variable. If we apply the operator to (2) above it will simply return the drift term i.e.

$$(1/dt)E_t dF(s,t) = \tfrac{1}{2}\sigma^2 F_{ss} + \mu F_s + F_t \qquad (3)$$

Note that the term involving dz contains a Normal random variable with mean zero, so that when we apply the operator its expected value is zero.

I will now give a heuristic proof of Itô's Lemma using a Taylor expansion, however a more formal proof can be found in Malliaris & Brock (1982) p.80-85 or Arnold (1974) p.96-99.[3] Consider a second-order Taylor expansion of F(s,t) around (s,t) where the discrete increments in s and t are ds and dt, respectively:

$$\begin{aligned} &F(s + \mu dt + \sigma dz,\ t + dt) \\ &= F(s,t) + [\mu dt + \sigma dz]F_s + dtF_t \\ &\quad + \tfrac{1}{2}[\mu^2 + dt^2 + 2\mu\sigma dzdt + \sigma^2 dz^2]F_{ss} \\ &\quad + \tfrac{1}{2}[\mu dt^2 + \sigma dzdt]F_{st} + \tfrac{1}{2}dt^2 F_{tt} \end{aligned} \qquad (4)$$

Subtracting F(s,t) from both sides and taking expectations we obtain

$$E(dF) = [\tfrac{1}{2}\sigma^2 F_{ss} + \mu F_s + F_t]dt, \qquad (5)$$

where the expectation of dz and dz^2 are 0 and $\sigma^2 dt$, respectively. This is just the drift term in (2). In order to obtain the stochastic term in (2), subtract E(dF) from dF and ignore all the higher order terms.

3. Malliaris and Brock (1982) provide some nice examples of Itô's Lemma on p.85-90.

A very popular example of Itô's Lemma is obtained when the state variable is a stock price and it is described by the stochastic differential equation

$$ds = \mu s dt + \sigma s dz \tag{6}$$

where the drift and disturbance are functions of the stock price (Fisher (1974)). The advantage of having the stock price in the variance and drift is that it implies limited liability, but the disadvantage is this implies a lognormal distribution for the stock price returns[4]. If we rewrite (6) as

$$ds/s = \mu dt + \sigma dz \tag{7}$$

we know that $ds/s = d\ln(s)$ i.e. the total derivative of the natural log of s. Thus if we define $y=F(s)=\ln(s)$, by Itô's lemma,

$$dy = (\mu - \tfrac{1}{2}\sigma^2)dt + \sigma dz. \tag{8}$$

Using the integral expression and the fact that $y=\ln(s)$, we obtain the result that the expected return on the share price is given by

$$E[\ln(s(t)) - \ln(s(0))] = \mu - \tfrac{1}{2}\sigma^2. \tag{9}$$

Note that in equation (6) we cannot apply the integral representation to obtain any meaningful results because the drift and disturbance terms are both functions of s which varies over time.

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4. This is too restrictive because the empirical distribution of returns can be modelled by other distributions.

Itô's Lemma can also be applied in the case where there is more than one state variable. In the case of n state variables, each described by the stochastic differential equation

$$ds_i = \mu_i dt + \sigma dz_i, \quad i = 1,2,3...n \tag{10}$$

where each state variable has a different brownian motion and the correlation between the i^{th} and j^{th} brownian motions is given by f_{ij}. Then for a function, $y = F(s_1,s_2,...,s_n)$, by Itô's Lemma

$$dy = (\tfrac{1}{2} \Sigma_i\Sigma_j\ f_{ij}\ \sigma_i\sigma_j\ F_{ii} + \Sigma_i\ \mu_i F_i + F_t)dt$$
$$+ \Sigma_i\ \sigma_i F_i dz_i \tag{11}$$

For examples of models with more than one state variable in the finance literature, refer to Merton's (1973) intertemporal asset pricing model and the stochastic volatility models of Hull and White (1987) or Scott(1987).

1.3 Valuation by arbitrage

Underlying the Contingent Claims Analysis is the assumption that a market exists that values all positive supply assets, and the market is in equilibrium. In order to value the derivative securities we need to obtain a deterministic relationship, in the form of a partial differential equation, between the value of the underlying asset and the derivative security. In order to derive the partial differential equation we need the concept of a no-arbitrage equilibrium condition.[5]

Consider a market with only one traded asset which is described by the stochastic differential equation

$$ds = \mu dt + \sigma dz \tag{1}$$

and one can borrow or lend at a riskless rate r. There are also two derivative securities A and B, based on the asset s. By Itô's lemma we can derive the stochastic differential equations that describe A and B

$$dA = (\tfrac{1}{2}\sigma^2 A_{ss} + \mu A_s + A_t)dt + \sigma A_s dz \tag{2}$$

$$dB = (\tfrac{1}{2}\sigma^2 B_{ss} + \mu B_s + B_t)dt + \sigma B_s dz \tag{3}$$

where we refer to the coefficient of dt as μ_A and μ_B, respectively.

Consider a portfolio which consists of going long in one unit of A, selling short A_s/B_s units of B and borrowing the funds to finance the portfolio. The instantaneous rate of return of the porfolio P will be given by

$$dP = 1.dA - (A_s/B_s)dB - r(A - (A_s/B_s)B)dt \tag{4}$$

where $A - (A_s/B_s)B$ denotes the net borrowing. Substituting in for dB and dA we find that the disturbance terms cancel out and we are left with

$$dP = [\tfrac{1}{2}\sigma^2(A_{ss} - (A_s/B_s)B_{ss}) + \mu(A_s - (A_s/B_s)B_s) + (A_t - (A_s/B_s)B_t) - r(A - (A_s/B_s)B)]dt. \tag{5}$$

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5. Varian (1987) is a very good reference on the arbitrage principle.

If we define the equilibrium set of asset prices such that no arbitrage opportunities may exist, the return on the portfolio must be zero i.e. $dP = 0$ is a requirement for market equilibrium. We can rearrange equation (5) as follows

$$(\mu_A - rA)/A_s = (\mu_B - rB)/B_s \tag{6}$$

$$(\tfrac{1}{2}\sigma^2 A_{ss} + \mu A_s + A_t - rA)/A_s = (\tfrac{1}{2}\sigma^2 B_{ss} + \mu B_s + B_t - rB)/B_s \tag{7}$$

We can see that the numerator of each term is the expected rate of return less the riskless rate of return or the opportunity cost of holding the asset. The denominator reflects the sensitivity of each asset to fluctuations in the state variable.

In a no-arbitrage equilibrium, the excess expected return per unit of security will be equal for all assets that are only functions of the state variable s and time. The common value of the return per unit of sensitivity will be denoted by $\beta(s,t)$, the market price of risk for the state variable s. Thus we can rewrite equation (6) as

$$(\mu_A - rA)/A_s = (\mu_B - rB)/B_s = \beta(s,t) \tag{8}$$

which means that we can obtain the deterministic partial differential equations for assets A and B, and in the case of asset A

$$(\mu_A - rA)/A_s = \beta(s,t)$$

$$\tfrac{1}{2}\sigma^2 A_{ss} + \mu A_s + A_t - rA = \beta(s,t)A_s$$

$$\tfrac{1}{2}\sigma^2 A_{ss} + (\mu - \beta(s,t))A_s + A_t - rA = 0 \tag{9}$$

where μ - $\beta(s,t)$ denotes the risk-adjusted rate of return.[6] One of the most startling results is that all derivative securities of s, must satisfy this partial differential equation e.g. in the case of B, we simply have to replace the capital A's by capital B's in (9) to obtain the partial differential equation for B. The distinguishing factor for the derivative securities will be the boundary conditions: if both A and B were options on s with an exercise price of X except that A was a put and B was a call, then A's boundary condition would be A = Max(0,X-S) while B's boundary condition would be B = Max(0, S-X), but both will still satisfy the same partial differential equation. The other startling result to come out of the ground-breaking work by Black and Scholes (1973), is that this partial differential equation does not involve preferences. The only problem with the above partial differential equation is that the no-arbitrage equilibrium condition does not reveal the functional form of $\beta(s,t)$, however Black and Scholes (1973) do provide a method of obtaining $\beta(s,t)$ when the asset is traded, but more of that later.

A simple rearrangement of the partial differential equation allows an interpretation similar to that of the more familiar Capital Asset Pricing Model (CAPM): take the terms

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6. We usually suppress the arguments of β.

involving β and r to the right hand side and divide through by A i.e.

$$(\tfrac{1}{2}\sigma^2 A_{ss} + \mu A_s + A_t)/A = r + \beta(A_s/A) \tag{11}$$

or $$(\mu_A/A) = r + \beta(A_s/A) \tag{12}$$

On the left hand side of (12) we have the expected return of holding asset A and on the right hand side we have the riskless rate of return plus the price of s-risk multiplied by its proportional sensitivity to s. If s was the "market portfolio" as in the CAPM, then β would be the price of market risk.

We can extend this notion to the case of n state variables where we have an arbitary asset $A = F(s_1,s_2,\ldots,s_n)$. Then A's excess expected return must satisfy

$$\begin{aligned}\mu_A - rA = {} & \beta_1(s_1,t)*(\text{sensitivity of A to } s_1) \\ & + \beta_2(s_2,t)*(\text{sensitivity of A to } s_2) \\ & +\ldots+ \beta_n(s_n,t)*(\text{sensitivity of A to } s_n)\end{aligned} \tag{13}$$

So now we have n prices of risk, one for each state variable. This result is similar to Ross' (1976) Arbitrage Pricing Theory and Merton's (1973) Intertemporal Asset Pricing Model.

An important issue arises in the derivation of the valuation partial differential equation when the state variable is not traded. One either has to try to estimate the drift, disturbance and the price of risk econometrically (Brennan and Schwartz (1979)) or assume that the price of risk is zero (Hull and White (1987)) and use a Method of Moments estimator. However, when the state variable is traded it is easy to proceed as this example from Black and Scholes (1973) will show. If we let $A(s,t) = s$ i.e. the asset is simply the state variable, e.g. a stock that is traded on the NYSE. Then $A_s = 1$, $A_{ss} = 0$ and $A_t = 0$, so that the valuation partial differential equation becomes

$$\tfrac{1}{2}\sigma^2 * 0 + (\mu - \beta) * 1 + 0 - rs = 0 \tag{14}$$

$$\mu - \beta = rs \quad \text{or} \quad \beta = \mu - rs \tag{15}$$

So the risk adjusted rate of return must equal the riskless rate in a no arbitrage equilibrium. Solving for β in (15) and substituing into the valuation partial differential equation for derivative security B we obtain,

$$\tfrac{1}{2}\sigma^2 B_{ss} + rsB_s + B_t - rB = 0 \tag{16}$$

i.e the drift term drops out. So in the final valuation equation, only the variance of the state variable is important in pricing the derivative security and the expected rate of return has no importance.[7]

In the above examples, A and B were derivative securities that paid no dividends. If we are interested in an asset A that pays a continuous stream of dividends, $q_A dt$, per instant in time, we simply have to include $q_A dt$ in the original portfolio, to obtain a return,

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[7]. Constantinides (1978) is a good reference for the application of the risk adjusted processes.

$$dP = dA - (A_s/B_s)dB + q_A dt - (A - (A_s/B_s)r = 0 \qquad (17)$$

and run through the previous steps. In this case the partial differential equation for A will be

$$\tfrac{1}{2}\sigma^2 A_{ss} + (\mu - \beta)A_s + A_t - rA + q_A = 0 \qquad (18)$$

where q_A may be specified to be a function of s and t. We can use the same approach for an asset with n state variables, but we will require a traded asset for each state variable.

1.4 Finite Difference Methods:

Many of the partial differential equations that are derived for derivative securities cannot be solved analytically, so a numerical method is used. The finite difference method is a numerical method that approximates each partial derivative by means of a finite difference between the value of the asset at time t and t-1. For a more in depth explanation consider the partial differential equation

$$aU_{ss} + bU_s + cU - U_t = 0 \qquad (1)$$

which has a boundary condition U(s,T) = f(s). Since the terminal value is known with certainty and can be calculated for all values of s, we work backwards from the terminal time i.e. the terminal time is now 0 and U(s,0) = f(s). Since we are working backwards, we have reversed the sign on the U_t term. The coefficients a, b and c may be functions of the state variable and/or time and we will have to take this into account as we step backwards. However, for this exposition we will regard them as constants, independent of s and t.

In order to use the finite difference method we need to set up a grid with time on the horizontal axis and the state variable s on the vertical axis where the time axis will run from 0 to T and the state variable from Smin to Smax where Smin and Smax are judiciously chosen so as to allow sufficient variation in s. We now need to divide up the time axis into N equally spaced intervals, n= 0,1,2,...,N and the state variable axis into I equally spaced intervals, i=0,1,2,..,I where the length of a time interval is k and the state variable interval is h. We now have a mesh with (N+1)x(I+1) gridpoints.

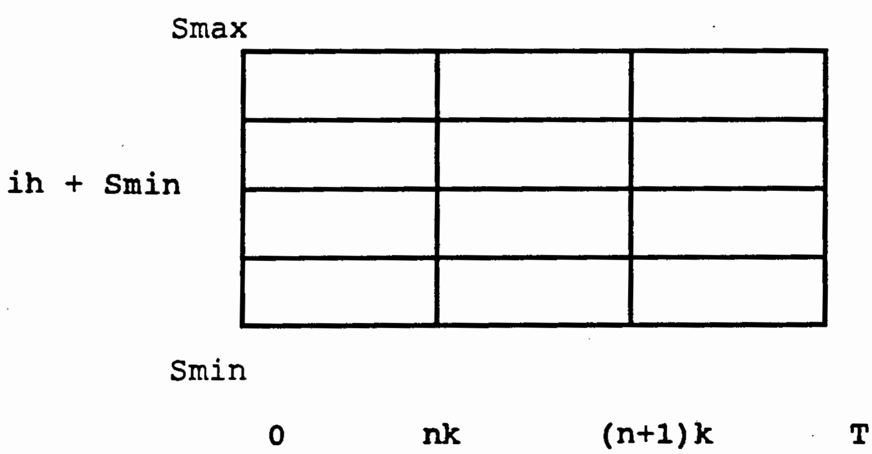

Figure 1. The grid for Finite Difference Approximations

If we added a third axis for V, we would be able to plot a surface above the time and state variable gridpoints that would represent the value of V.

There are two finite difference methods that we can follow, the explicit and the implicit methods, and each uses a particular difference approximation for the partial derivatives. I have given both sets of approximations in the two sections below as well as their solutions.

1.4.1 Explicit Method

Let $U_{i,n}$ denote the finite difference approximation at the point (nk, smin + ih), then we use the following finite differences as approximations for the partial derivatives:

$$U = U_{i,n}$$

$$U_t = (U_{i,n+1} - U_{i,n})/k$$

$$U_s = (U_{i+1,n} - U_{i-1,n})/2h$$

$$U_{ss} = (U_{i+1,n} - 2U_{i,n} + U_{i-1,n})/h^2$$

Substituting into (1) and grouping the $U_{i,n}$ terms we obtain the following solution:

$$\begin{aligned} U_{i,n+1} = {} & [(k/h^2)a + (k/2h)b]U_{i+1,n} \\ & + [1 + kc - (2k/h^2)a]U_{i,n} \\ & + [(k/h^2)a - (k/2h)b]U_{i-1,n}. \end{aligned} \qquad (1)$$

where this solution only applies to the interior gridpoints and not to the boundaries. It is clear from the solution and the diagram below (fig.2) that the value of U at time n+1 is completely determined by known values at time n, so the system is relatively easy to implement. However, this method is only second-order accurate in the state variable direction and first-order accurate in the time direction. Furthermore, the numerical solution method is unstable unless k/h^2 is sufficiently small.

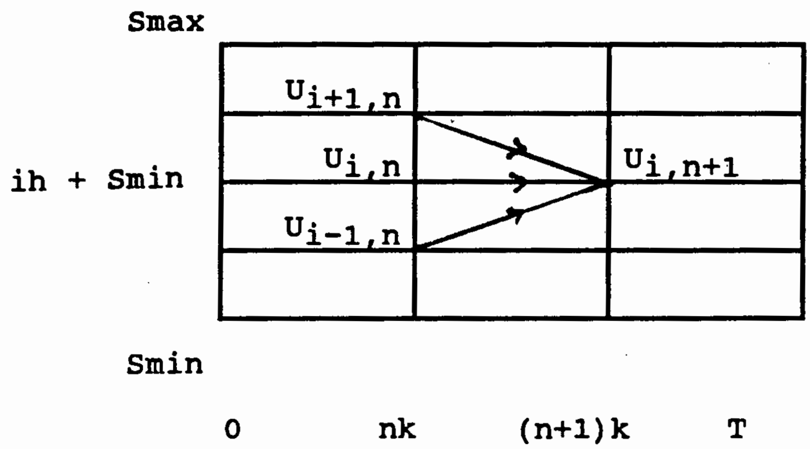

Figure 2. The grid for the Explicit Finite Difference Approximation

By rewriting the solution as follows, the explicit solution also allows a convenient interpretaion:

$$U_{i,n} = \{[(k/h^2)a + (k/2h)b]U_{i+1,n} + [1 - (2k/h^2)a]U_{i,n} + [(k/h^2)a - (k/2h)b]U_{i-1,n}\} + kcU_{i,n} \qquad (2)$$

The coefficients of the terms U_{i+1}, U_i and U_{i-1} can be viewed as probabilities (they sum to 1) and reflect the probability of $U_{i,n}$ moving up or down by an amount h, or remaining the same. The expression in the scrolled brackets can be interpreted as the expected value of $U_{i,n+1}$. the coefficient of $U_{i,n}$ in the last term is made up of k and c where c is usually equal to -r, the negative of the interest rate, in the partial differential equation. This reflects the opportunity cost (interest cost) of holding U for a length of time equal to k. Thus $U_{i,n}$ is just the expected discounted value of the security at the next time point. The concepts of opportunity cost and discounting will be discussed further in the Monte Carlo section. As we step through the grid we are in fact calculating the expected discounted value of U for the boundary condition at the terminal time.

If U was a futures contract, the term $kcU_{i,n}$ would drop out since no money changes hand when a futures contract is struck. So the value of the contract will simply be the expected value of the asset (Samuelson (1965))

1.4.2 Implicit Method:

Using the same notation as above, the finite difference approximations for the partial derivatives are:

$$U = U_{i,n+1}$$

$$U_t = (U_{i,n+1} - U_{i,n})/k$$

$$U_s = (U_{i+1,n+1} - U_{i-1,n+1})/2h$$

$$U_{ss} = (U_{i+1,n+1} + 2U_{i,n+1} - U_{i-1,n+1})/h^2$$

Substituting these approximations into the valuation partial differential equation we obtain the solution:

$$\begin{aligned} U_{i,n} = {} & [(k/h^2)a + (k/2h)b]U_{i+1,n+1} \\ & + [1 + ck + (2k/h^2)a]U_{i,n+1} \\ & + [(k/h^2)a - (k/2h)b]U_{i-1,n+1} \end{aligned} \qquad (3)$$

Now the three unknown values at time n+1 are related to one unknown value at time n (fig.3). This is more accurate than the explicit method and is both first-order accurate in the time and state variable directions. However, one does have the expense of solving a system of equations in order to move forward by one time step.[8]

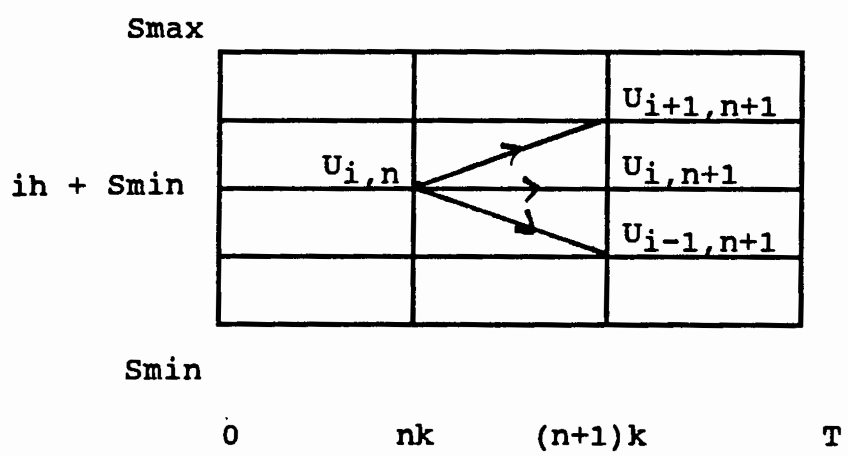

Figure 3. The grid for the Implicit Finite Difference Approximation

1.4.3 Crank-Nicholson Algorithm:

The preferred method to use is the Crank-Nicholson Algorithm which is an average of the explicit and implicit methods centred around time $t+\frac{1}{2}k$ and not at t. It is an easy method to apply because it leads to a tri-diagonal system of I+1 equations which only require 8I floating point operations to

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8. Hull (1989) provides an introduction into the various methods of valuing derivative securities.

determine the next set of $U_{i,n+1}$. The finite difference approximations are given below:

$$U = (U_{i,n} + U_{i,n+1})/2$$

$$U_t = (U_{i,n+1} - U_{i,n})/k$$

$$U_s = (U_{i+1,n} - U_{i-1,n} + U_{i+1,n+1} - U_{i-1,n+1})/4h$$

$$\begin{aligned} U_{ss} = (&U_{i+1,n} - 2U_{i,n} + U_{i-1,n} \\ &+ U_{i+1,n+1} - 2U_{i,n+1} + U_{i-1,n+1})/h^2 \end{aligned}$$

Substituting these finite difference approximations into the valuation partial differential equation we obtain:

$$\begin{aligned} &[4ak + bhk]U_{i+1,n+1} + [4ak - bhk]U_{i-1,n+1} \\ &\qquad + [2ch^2k + 4h^2 - 8ak]U_{i,n+1} \\ &= -[4ak + bhk]U_{i-1,n} - [4ak - bhk]U_{i+1,n} \\ &\qquad - [2ch^2k + 4h^2 - 8ak]U_{i,n} \end{aligned} \tag{4}$$

This is a system of tri-diagonal equations where three values at time n are related to three values at time n+1. Since these equations overlap for different values of i (fig.4), we obtain the desired accuracy.

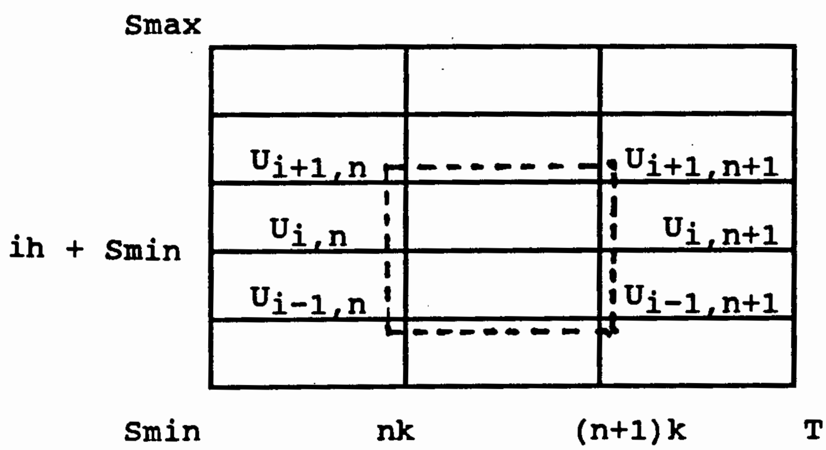

Figure 4. The grid for the Crank-Nicholson Algorithm

1.5 Monte Carlo Methods

The finite difference method becomes intractable beyond the case of the two state variable model and the computing costs become a major factor. A simpler method is the Monte Carlo approach which has the added advantage of being easily adaptable to a model with a number of state variables. I will first derive the Monte Carlo method for a two state variable model where the derivative asset pays a continuous flow of dividends and then extend it to the case of 4 state variables. An important point to note before we begin the Monte Carlo method is that in contrast to the methods mentioned above, the Monte Carlo method works from the present time until the terminal time i.e. it does not use the dynamic programming solution method of starting at the terminal time and work backwards to the present.

Given the partial differential equation

$$a_{11}U_{11} + a_{22}U_{22} + a_{12}U_{12} + b_1U_1 + b_2U_2 + cU + d = 0 \quad (5)$$

where the subscripts on U denote the partial derivative with respect to the i'th state variable and the coefficients a_{ii}, b_i and c may be functions of the state variables and time. We assume that the partial differential equation is parabolic i.e. $a_{11}.a_{22} \geq 0$ and $4a_{11}a_{22} \geq a^2_{12}$. We will now use the explicit finite difference method to find an approximation for each term in the partial differential equation, so we need to discretize the state space by placing a net of mesh size g_i in the ith state direction and k in the time direction. Let V_{tij} denote the solution value at the ij'th gridpoint i.e. $U(ig_1, jg_2, t)$. Using the finite difference methods the object is obtain an expression for V_{tij} in terms of V_{ij}, V_{i+1j} V_{ij+1}, V_{i-1j}, etc. at time t+k. However, we can simplify the problem if we know the sign of a_{12}. When the partial differential equation describes the value of a derivative security, a_{12} will contain the correlation between the Wiener processes of the two state variables. If the state variables are positively correlated, we can simplify the problem by omitting V_{i-1j+1} and V_{i+1j-1} and in the case of negative correlation we can omit V_{i+1j+1} and V_{i-1j-1}.[9]

I will consider the case of negative correlation and use the following short-hand notation for the V_{ij} values: $V_{+-} \equiv V_{i+1j-1}$, $V_{o+} \equiv V_{ij+1}$, etc. where each of theses values is at time t+k. I will include a time subscript to distinguish between period t and t+k where ambiguity may arise. Therefore, we can write the centred difference approximation for the derivatives appearing in the partial differential equation as:

---

9. In the case of V_{i+1j+1}, both state variables have increased so this implies a positive correlation. We can use a similar explanation for V_{i-1j-1}.

$$U_1 = (V_{+o} - V_{-o})/2g_1$$

$$U_{11} = (V_{+o} - 2V_{oo} + V_{-o})/g_1^2$$

$$U = (V_{t+k,oo} + V_{t,o})/2$$

$$U_2 = (V_{o+} - V_{o-})/2g_2$$

$$U_{22} = (V_{o+} - 2V_{oo} + V_{o-})/g_2^2$$

$$U_t = (V_{t+k,oo} - V_{t,oo})/k$$

$$U_{12} = (V_{+o} + V_{o+} + V_{o-} + V_{-o} - V_{+-} - V_{-+} - 2V_{oo})/2g_1g_2$$

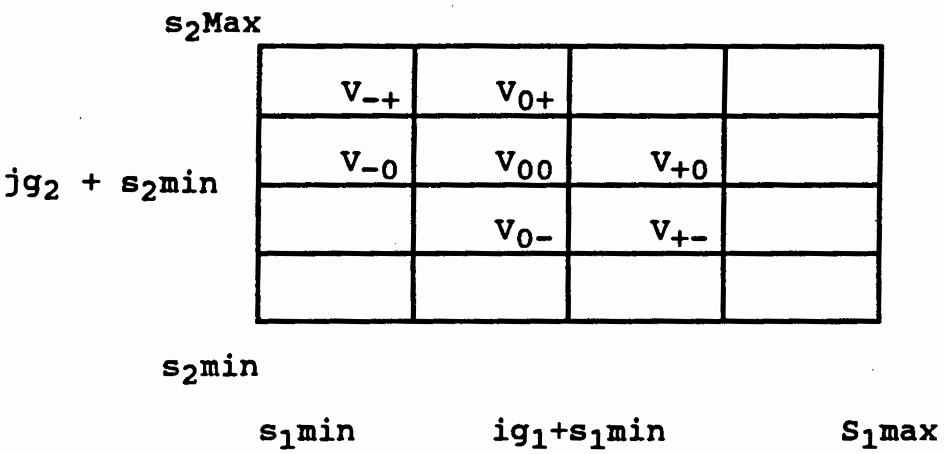

Figure 5. The grid for the Monte Carlo Solution

By substituting them into the partial differential equation we can solve for $V_{t,oo}$ in terms of V at time t+k. After much manipulation we obtain the expression

$$V_{t,oo} = [(1+ck/2)/(1-ck/2)]\{\pi_{oo}V_{oo} + \pi_{+o}V_{+o} + \pi_{o+}V_{o+} + \pi_{-o}V_{-o} + \pi_{o-}V_{o-} + \pi_{-+}V_{-+} + \pi_{+-}V_{+-} + 2dk/(2+kc)\} \quad (6)$$

where $\pi_{oo} = 1 - 4(a_{11}g_2/g_1 + a_{22}g_1/g_2 + a_{12}/2)E$

$$\pi_{+o} = (2a_{11}g_2/g_1 + a_{12} + b_1g_2)E$$

$\pi_{o+} = (2a_{22}g_1/g_2 + a_{12} + b_2g_1)E$

$\pi_{-o} = (2a_{11}g_2/g_1 + a_{12} - b_1g_2)E$

$\pi_{o-} = (2a_{22}g_1/g_2 + a_{12} - b_2g_1)E$

$\pi_{+-} = -a_{12} = \pi_{-+}$

$E = k/[(2+kc)g_1g_2]$

Note that the π's sum to zero by construction and since $4a_{11}a_{22} > a_{12}$, the π's are non-negative for small values of g_1, g_2 and k.[10] If we view the π's as transition probabilities we can view equation (6) as the discounted expected value of V_{t+k} where the discount factor is $(1+kc/2)/(1-kc/2)$ and the probability of being at the various gridpoints at time t+k are conditional on being at i,j at time are given by the π's. However, V_{t+k} is only known at the boundary, so we can step back until the terminal time T where we know the value of V_T. Thus the solution value of V_{tij} is the expected value over all possible paths starting at the point $(ig_1 + s_1min, jg_2 + s_2min, t)$ and ending at time T of the attained maturity values multiplied by the cumulative discount factor. When the derivative security pays a continuous stream of dividends the solution can be written as

$$Vt,ij = \prod^T_{\tau=t} [(1 + c(s_1,s_2)k/2)/(1 - c(s_1,s_2)k/2)].V_T$$
$$+ \sum^T_{\tau=t} [2kd/(2+ck)]*$$

---

10. If the two state variables are positively correlated we simply have to change the sign of a_{12} in the above probabilites.

$$\Pi^{\tau}_{r=t}[(2+c(s_1,s_2)k)/(2-c(s_1,s_2)k) \qquad (7)$$

where $c(\cdot)$ may be a function of s_1 and s_2.

The Monte Carlo method will determine the solution V_{tij} by taking a random walk through the state variable grid starting at ij and using the π's of the current grid location as transition probabilities. The discount factor, which depends on the grid locations through c, is accumulated and at the terminal time T, V_T is multiplied by this factor and then added to the accumulated value of the dividends. We refer to the value in the equation above as the score. After a number of replications, usually 5000 to 10 000, we use the average of the scores as our estimate of V_{tij}.

We can extend this solution to the case of four state variables where all the variables are negatively correlated; this is the maximum that we will require in the empirical application. Using the notation above, the probability of the i^{th} state variable jumping up by a small increment is given by

$$\pi_{oio} = [2a_{ii}g_jg_kg_l/g_i + b_ig_jg_kg_l + a_{ij}g_kg_l + a_{ik}g_jg_l + a_{il}g_jg_k]E \qquad (8)$$

where $i = 1,2,3,4$

$j = 1,2,3,4;\ j \neq i$

$k = 1,2,3,4;\ k \neq i \neq j$

$l = 1,2,3,4;\ l \neq i \neq j \neq k$

$E = k/[(2+ck)g_ig_jg_kg_l]$

In order to obtain the probability of a decrease in the i^{th} state variable we must reverse the sign of the b_i term. The probability of two state variables i and j moving up and down respectively, is given by

$$\pi_{oijo} = a_{ij}g_k g_l \tag{9}$$

where i,j,k and l satisfy the conditions above. Note that the probability of i moving down and j moving up is given by the same expression. Finally, the probability of no move is given by

$$\pi_{oooo} = 1 - 4[\ \Sigma_i a_{ii}\ \pi_j g_j/g^2{}_i + \Sigma_i \Sigma_{j>i} a_{ij} g_k g_l]E \tag{10}$$

where j may not equal i in the first summation in the square brackets and j and l satisfy the conditions given above. If some of the variables were positively correlated we could simply reverse the signs of the appropriate a_{ij} terms and in so doing replace the probabilities of two state variables moving in opposite directions with the probability of the state variables moving in the same direction. If we wanted to reduce this to a three state variable problem, we could simply set a_{kk}, a_{ik} and b_k equal to zero and g_k equal to 1.

There are two points that I would like to elucidate before I end this section: (i) the discount factor and (ii) the Monte Carlo solution (7).

The factor $[(1+ck/2)/(1-ck/2)]$ is a second order approximation to exp(-rt) since "c" is usually equal to "-r" (the negative of the interest rate) in our partial differential equation. If we consider an expansion of this factor, this idea will become transparent

$$[(1+ck/2)/(1-ck/2)]$$

$$= (1+ck/2)(1 + ck/2 + (ck/2)^2 + ...$$

and if we multiply out we obtain

$$= 1 + ck + (ck)^2/2 + (ck)^3/4 + ...$$

which is a second-order approximation to

$$= 1 + ck + (ck)^2/2! + (ck)^3/3! +$$

$$= EXP(ck).$$

Thus the last term in the Monte Carlo solution can also be interpreted as the sum of the discounted value of the dividends.

The Monte Carlo Solution has a very nice form in that it is the discrete analog of the Feynman-Kac Theorem (Karatzas & Shreve (1988) p.366). I will briefly state the Theorem: Given a partial differential equation of the form

$$\tfrac{1}{2}\sigma^2 V_{ss} + \mu V_s - V_t + rV + d = 0$$

subject to the boundary condition $V(T,s) = f(s)$. Then the solution is given by

$$V(t,s) = E_{t,s}\{f(s_T)\exp\{\textstyle\int^T_t r(\theta,s)d\theta \\ + \int^T_t d(\tau,s)\exp[-\int^\tau_t r(\theta,s)d\theta]d\tau\} \qquad (11)$$

where the expectation is conditional on being at time t and on the type of stochastic process that s follows. The first term is simply the discounted value of the boundary value of

V and the second term is the sum (integral) of all the discounted dividend payments. If we are dealing with a futures contract, the term involving r drops out, and the Feynam-Kac solution is simply

$$V(t,s) = E_{t,s}[f(s_T)] \qquad (12)$$

i.e. the expected value of s_T at time t (Samuelson(1965)).

1.6 Alternative Monte Carlo Method

An alternative Monte Carlo method (which was easier to implement) can be derived from the discrete approximation to the stochastic differential equations in section 1.1. Using equation (3) we can generate series of the random variables given the estimated means and variances.

$$p(t+h) = p(t) + \mu h p(t) + \sigma p(t) y(t) \sqrt{h}. \qquad (3)$$

We need to generate standard normal random deviates, y(t) (the Wiener process has independent increments that are Gaussian), and chose a suitable time increment h, in order to calculate p(t+h) given the current value, p(t) and estimates of μ and σ^2.

If we were valuing a European call option exercisable in one year, we could choose h as 1/52 so that the time increments are one week. The value of μ and σ will be estimated from past values of p. 52 standard normal deviates would have to be generated in order to determine the price of the security in one year; this is referred to as a random walk. We could then subtract the future price from the exercise price and discount the difference back one year. The estimate of the value of the European option would be the average of these discounted values for 5000 to 10000 random walks.

When we are dealing with a security that is a function of more than one state variable we will have to simulate all the state variables and model their correlation structure. If μ_i and $\sigma^2{}_i$ are the mean and variance of the i'th random variable and f_{ij} is the correlation coefficient for the i'th and j'th random variables then we can use the following decomposition to calculate the random walks for up to four random variabales:

$$dz_1 = y_1\sqrt{h}$$

$$dz_2 = a_{21}dz_1 + a_{22}y_2\sqrt{h}$$

$$dz_3 = a_{31}dz_1 + a_{32}dz_2 + a_{33}y_3\sqrt{h}$$

$$dz_4 = a_{41}dz_1 + a_{42}dz_2 + a_{43}dz_3 + a_{44}y_4\sqrt{h}$$

where $a_{21} = f_{12}$

$a22 = SQRT(N_1)$

$a31 = (f_{13} - f_{12}f_{23})/N_1$

$a32 = f_{23} - f_{12}a_{31}$

$a33 = SQRT(1 - a^2{}_{31} - a^2{}_{32})$

$a41 = f_{14} - f_{12}a^2{}_{42} - f_{13}a^2{}_{43}$

$a42 = (N_2N_3 - N_4N_5)/DEN$

$a43 = (N_1N_4 - N_5N_3)/DEN$

$a44 = SQRT(1 - a^2{}_{41} - a^2{}_{42} - a^2{}_{43})$

$$N_1 = 1 - f^2_{12}$$

$$N_2 = 1 - f^2_{13}$$

$$N_3 = f_{24} - f_{14}f_{12}$$

$$N_4 = f_{43} - f_{14}f_{13}$$

$$N_5 = f_{23} - f_{13}f_{12}$$

$$DEN = N_1N_2 - N_5N_5$$

This is much easier to implement in a program and is the general method that is used in the finance literature. If one is only dealing with a two state variable model, one can ignore dz_3 and dz_4. In our applications we will consider 1,2,3 and 4 state variable models.

Chapter 2

A Model of Resource Extraction

2.1 Introduction

The standard deterministic model of resource extraction assumes that the firm's objective is one of maximising the discounted profits. Hotelling (1931) popularised this model and showed that efficiency requires that the rate of output be chosen such that the marginal net benefit of extracting the ore today is equated with the discounted marginal net benefit of extracting it in the next time period i.e. the opportunity cost of current extraction. This proposition still holds in the stochastic model of resource extraction that we will be considering, but the added uncertainty leads to different optimal rates of production and prices. In the sections that follow we will set up the basic model, consider the optimal properties and derive the valuation equation that can be used to value the gold mine.

2.2 Setting up the Model

In a simple continuous time stochastic model where the price of the resource and the ore reserves are the only state variables, the rate of extraction is chosen so as to maximise the expected discounted value of profits i.e.

$$\max E_o \int [p(t)d(q) - AC(Q)]qe^{-rt}dt = \max E_o \int \pi(t)dt \quad (1)$$

where q is the rate of extraction at time t;

$E_o(\cdot)$ denotes the expectation conditional on the values of the state variable at time 0;

p(t) is the price of the resource at time t;

d(q) is the demand function for the resource divided by q;

Q is the level of reserves of the exhaustible resource;

AC(Q) is the average cost of mining a unit of the resource and has negative first and second derivatives;

r is the riskless rate of interest;

the limits of integration are t=T and t=0.[1]

We are also maximising with respect to the processes that p(t) and Q are assumed to follow: p(t) is stochastic and is described by the stochastic differential equation

$$dp/p = \mu dt + \sigma dz \quad (2)$$

and Q evolves deterministically but decreases at a rate of q per time period as we extract the resource i.e.

$$dQ = -qdt. \quad (3)$$

We can now go ahead with the maximisation subject to two boundary conditions: $Q \geq 0$ and $\pi(T) = 0$.

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[1]. In this section and the following section, the limits of integration will be t=T and t=0.

2.3 Solving the System

We need to use stochastic dynamic programming to obtain the optimal rate of production and the expected optimal price and production trajectories. Define the optimal value function

$$J = J(p,Q,t) = \max E_t \int\pi(\tau)d\tau. \quad (4)$$

J is a function of the two state variables and time, so we can use Itô's Lemma and differential operator. Therefore, the fundamental equation of optimality is

$$0 = \max[\pi(t) + (1/dt)E_t dJ] \quad (5)$$

From Itô's Lemma we know that the evolution of J is described by

$$dJ = (\tfrac{1}{2}\sigma^2 J_{pp} + \mu p J_p - qJ_Q + J_t)dt + \sigma J_p dz \quad (6)$$

and using Itô's differential operator, we can substitute the drift from equation (6) into (5) to obtain

$$0 = \max[\pi(t) + \tfrac{1}{2}\sigma^2 J_{pp} + \mu p J_p - qJ_Q + J_t]. \quad (7)$$

Maximising with respect to q gives

$$\delta\pi/\delta q = J_Q \quad (8)$$

If the price of the resource is competitively determined, (7) is linear in q^2 and we can derive the equilibrium condition in the market from equation (8):

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[2]. In the case of perfect competition, the market demand will equal the rate of production so that f(q) = 1.

if $\delta\pi/\delta q > J_Q$ - extract at the maximum rate;

if $\delta\pi/\delta q = J_Q$ - market clearing equilibrium;

if $\delta\pi/\delta q < J_Q$ - produce nothing.

Now that we have obtained the conditions for optimality we can follow three routes in order to analyse the problem further: (i) substitute (8) into (7) and solve for J(·) and then use (8) to optain the optimal production trajectory; (ii) use Itô's Lemma in equation (4) to obtain a partial differential equation that describes the value of the mine for given values of the state variable (Brennan and Schwartz(1985)); (iii) Eliminate J from the problem and determine the price dynamics.

It is generally very difficult to solve for J especially in terms of an analytical solution; one often has to use Numerical methods, so the first option is not feasible. The second option is very common in the finance literature and will be done in a later section. However, it is insightful to consider the expected price and production dynamics that are implied by different forms of the model.

2. 4 Expected Price and Production Dyanmics

In order to eliminate J from the problem differentiate equation (7) with respect to Q to obtain

$$\delta\pi/\delta Q + \tfrac{1}{2}\sigma^2 J_{Qpp} + \mu p J_{Qp} - qJ_{QQ} + J_{Qt} = 0 \qquad (9)$$

and using Itô's differential operator we can rewrite (9) as

$$\delta\pi/\delta Q + (1/dt)E_t d(J_Q) = 0. \qquad (10)$$

Apply Itô's differential operator to (8) to obtain

$$(1/dt)E_t d(\delta\pi/\delta q) = (1/dt)E_t d(J_Q) \tag{11}$$

which can be used to eliminate $J(\cdot)$ in (10) i.e.

$$\delta\pi/\delta Q = -\ (1/dt)E_t d(\delta\pi/\delta q). \tag{12}$$

Equation (12) can be more easily interpreted in integral form

$$-\ E_t \int(\delta\pi/\delta Q)dt = \delta\pi/\delta q \tag{13}$$

as the marginal unit from selling a unit of the resource must equal the expected sum of all the future discounted increases in profit that would result from holding that unit in the ground. Since AC(Q) is the only term in $\pi(\cdot)$ that contains Q, we can go even further by assuming that the marginal profit must equal the decrease in production costs from leaving the unit in the ground. i.e. assuming that costs increase as the level of reserves decreases.

We can use (12) to solve for $(1/dt)E_t dp$ for the competitive case:

$$\delta\pi/\delta q = [p - AC(Q)]\exp(-rt) \tag{15}$$

$$\delta\pi/\delta Q = -qAC'(Q)\exp(-rt) \tag{16}$$

where AC'(Q) is the first derivative of the average cost function with respect to cost. Therefore, substituting into (12) and using the Itô differential operator we obtain

$$(1/dt)E_t dp = r[p - AC(Q)] + qAC'(Q) + (1/dt)EtdAC \tag{17}$$

where $(1/dt)E_t dAC = -qAC'(Q)$. Thus the expected price dynamics are characterised by

$$(1/dt)E_t dp = r[p - AC(Q)] \qquad (18)$$

It is interesting to compare this result with Pindyck(1980) where he assumes that the reserves are stochastic:

$$(1/dt)Etdp = r[p - AC(Q)] + \tfrac{1}{2}\sigma^2 AC''(Q) \qquad (19)$$

In Pindyck's model, under the assumption that cost increases at an increasing rate, one expects a higher price in order to compensate for the extra risk embodied in the stochastic reserves.

In the models that we will be considering, the gold price, exchange rate, inflation rate and interest rate will be modelled as stochastic. If we solve for the expected price dynamics in a simple model of the mine where the gold price and the inflation rate are the state variables, and the average cost is a function of the inflation rate, we obtain the following result:

$$(1/dt)E_t dp = r[p - AC(Q,f)] + \tfrac{1}{2}\sigma^2{}_2 AC_{ff} + \mu_2 AC_f \qquad (20)$$

where f is the inflation rate and μ_2 and σ_2 are the instantaneous mean and variance of the inflation rate. Once again, the expected price includes extra positive terms which imply that a higher price is required in order to compensate for the extra risk, assuming that increases in the inflation rate lead to increases in cost at an increasing rate i.e. $AC_f > 0$, $AC_{ff} > 0$.

The production dynamics are somewhat easier to obtain. In a simple model where the gold price is the only state variable, we can use Itô's Lemma to derive the stochastic differential equation for $q = q^*(p)$:

$$dq = q_p dp + \tfrac{1}{2}\sigma^2 q_{pp} \tag{21}$$

Using the differential operator on (21) we obtain:

$$(1/dt)E_t dq = q_p(1/dt)E_t dp + \tfrac{1}{2}\sigma^2 p^2 q_{pp} \tag{22}$$

where we can substitute in from (18) to obtain

$$(1/dt)E_t dq = rq_p[p - AC(Q)] + \tfrac{1}{2}\sigma^2 p^2 q_{pp} \tag{23}$$

So the uncertainty surrounding the gold price raises the expected rate of extraction, assuming that $q_{pp} > 0$. However, in the case of the South African gold mines we find that the rate of extraction of gold decreases when the price of gold increases so as to prolong the life of the mine and lower the taxation. Thus we would expect q_{pp} to be negative.

2.5 Brennan and Schwartz valuation equation

The optimal value function in the stochastic dynamic programming problem ($J(p,Q,t)$) was a function of the gold price, gold reserves and time. Using Itô's Lemma we can derive the stochastic differential equation for J, the value of the mine:[3]

$$dJ = (\tfrac{1}{2}\sigma^2 J_{pp} + \mu p J_p - qJ_Q + J_t)dt + \sigma p J_p dz \tag{1}$$

where $dp/p = \mu dt + \sigma dz$. Since the value of the mine is stochastic, we need to form a portfolio of the mine and some

---

[3]. I will use the principles established in Chapter 1 to obtain the valuation equation.

other asset whose value depends on the gold price. Gold futures are traded on the New York Commodity Exchange (COMEX) so we can specify a stochastic differential equation for the futures price, $F = F(p,t)$:

$$dF = (\tfrac{1}{2}\sigma^2 F_{pp} + \mu p F_p + F_t)dt + \sigma p F_p dz. \qquad (2)$$

where the futures price is assumed to be a function of the gold price and time. If we purchase the mine and sell short (Jp/Fp) units of the futures we can create a hedge portfolio that is riskless and has an instantaneous return of

$$dP = dJ + (R-T)dt - (Jp/Fp)dF \qquad (3)$$

where the term $(R-T)dt$ denotes the revenue less taxation, or instantaneous net cash flow that one receives when purchasing the mine. This cash flow will be a function of the gold price and the return of production and will be expanded upon in a later section.[4]

In a no arbitrage equilibrium we know that this return must equal the riskless rate and invoking a capital market equilibrium condition we can obtain the partial differential equation for the valuation of the mine:

$$\tfrac{1}{2}\sigma^2 J_{pp} + rpJ_p - qJ_Q + J_t + R - T - rJ = 0 \qquad (4)$$

Note that since the gold price is traded, the market price of risk can be determined and drift term does not appear in

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4. One could also include a term reflecting the convenience yield of the resource, but in the case of gold, Fama and French (1988) and Brennan (1958) have found the convenience yield to be zero.

the partial differential equation; thus the coefficient of J_p is the risk adjusted rate of return.

This partial differential equation can easily be extended to the case where there is more than one stochastic state variable, for example when the exchange rate is included as a state variable, the partial differential equation is

$$\tfrac{1}{2}\sigma^2{}_1 J_{pp} + \tfrac{1}{2}\sigma^2{}_2 J_{ee} + f_{12}\sigma_1\sigma_2 J_{ep} + rsJ_p + reJ_e$$
$$- qJ_Q + J_t + R - T - rJ = 0 \qquad (5)$$

where e denotes the exchange rate which has variance $\sigma^2{}_2$ and the correlation between the gold price and interest rate is f_{12}.

The only problem that arises with additional state variables is the problem of non-trading i.e. if the asset is not traded we cannot find the market price of risk and so we need to use different estimation methods to obtain the parameters (method of moments as opposed to maximum likelihood).[5]

2.6 The option to close

One of the major contributions of Brennan and Schwartz(1985) was to value an option of closing the mine when the gold price is low and re-opening the mine once the gold price

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[5]. We will expand on these methods in Chapter 3.

exceeded a certain value. We will now discuss the extra boundary conditions that need to be implemented in valuing the mine and this option.

If the mine owner has an option to close the mine, we can assume that there is a cost of maintaining the mine while it is closed, M. This cost can be included in the valuation partial differential equation:

$$\tfrac{1}{2}\sigma^2 J_{pp} + rpJ_p - qJ_Q + J_t + (R - T)j - M(1-j) - rJ = 0 \qquad (1)$$

where j = 1, if the mine is open and j = 0, if the mine is closed.[6]

Let us distinguish between the value of the mine when open and closed:

$$V(p,Q,t) = \max J(p,Q,t{:}j=1) \qquad (2)$$

$$W(p,Q,t) = \max J(p,Q,t{:}j=0) \qquad (3)$$

Thus the value-maximising output and the value of the mine under the value maximising strategy satisfy:

$$\mathrm{Max}[\tfrac{1}{2}\sigma^2 p^2 V_{pp} + rpV_p - qV_Q + V_t + R-T - rV = 0;$$
$$\tfrac{1}{2}\sigma^2 p^2 W_{pp} + rpW_p - qW_Q + W_t - M - rW = 0] \qquad (4)$$

where the maximisation is conducted subject to q (the optimal rate of production) and the following boundary conditions:

$$W(p_0, Q,t) = 0 \qquad (5)$$

$$V(p_1,Q,t) = \mathrm{Max}[W(p_1,Q,t) - K_1(Q,t);\ 0] \qquad (6)$$

---

6. R - T will drop out of the equation when the mine is closed.

$$W(p_2,Q,t) = V(p_2,Q,t) - K_2(Q,t) \tag{7}$$

$$W(p,0,t) = V(p,0,t) = 0 \tag{8}$$

$$W_p(p_o,Q,t) = 0 \tag{9}$$

$$V_p(p_1,Q,t) = W_p(p_1,Q,t) \text{ if } W(p_1,Q,t) - K_1 \geq 0 \tag{10}$$

$$V_p(p_1,Q,t) = 0 \text{ if } W(p_1,Q,t) - K_1 < 0 \tag{11}$$

$$W_p(p_2,Q,t) = V_p(p_2,Q,t) \tag{12}$$

where K_1 is the cost of closing the mine and K_2 is the cost of re-opening a closed mine. p_o is the gold price at which the closed mine has zero value, so it is abandoned. p_1 is the gold price at which an open mine is closed and p_2 is the gold price at which a closed mine is re-opened. The first boundary condition (5) simply states that when all the reserves have been extracted the mine has no value. Equations (10), (11) and (12) are the Merton-Samuelson high contract conditions (Merton (1973b)) which ensure that there is no discontinuity at p_1 or p_2 otherwise there would be a riskless profit opportunity at p_1 or p_2. Thus it would not be optimal to close the mine at p_1 or open the mine at p_2. By solving for p_1, p_2 and q, one can determine the optimal policy for the mine.

2.7 Infinite Resource Case

When the stock of the resource is infinite, the partial differential equation reduces to

$$\tfrac{1}{2}\sigma^2 J_{pp} + rpJ_p + (R - T)j - M(1-j) - rJ = 0 \tag{1}$$

where the terms involving J_Q and J_t have dropped out because the value of the mine no longer depends on the rate of extraction and time, respectively.[7] This is just a second-order ordinary differential equation that can be solved by conjecturing an initial solution of the form

$$J(p) = p^{\alpha} \tag{2}$$

Substituting into (1) we obtain a quadratic in α

$$\tfrac{1}{2}\sigma^2\alpha(\alpha - 1) + r\alpha - r = 0 \tag{3}$$

where the "R - T - M(1-j)" has been omitted in order to find the complementary solution of the ordinary differential equation. Solving for the roots of α we obtain,

$$\alpha_1 = 1 \tag{4a}$$

$$\alpha_2 = -2r/\sigma^2 \tag{4b}$$

As a second conjectured solution consider a linear form, i.e

$$J(p) = a + bp \tag{5}$$

where we need to solve for the constants a and b. Substituting into (1) and solving for "a" we obtain[8]

$$a = ((R - T)j - M(1-j))/r. \tag{6}$$

Note that we can express the revenue less taxation ,R-T, as the rate of production multiplied by the price less average cost multiplied by 1 minus the tax rate, $q(p-c)(1-\tau)$, so that (6) can be interpreted as the discounted value of all

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7. There is some optimal rate of production in this case, however we will not concern ourselves with the choice of q.

8. b will be represented by ϕ_i.

future income streams assuming the mine is kept open forever (j=1) or closed indefinitely (j=0), i.e.

$$((R-T)j-M(1-j))/r = E[\int((R-T)j-M(1-j))\exp(-rt)dt] \quad (7)$$

where the cash flows are discounted at the riskless rate r.[9] Therefore, if we distinguish between the mine when it is open and closed, we obtain the following solutions:

$$W(p) = \phi_1 p + \phi_2 p^{(-2r/\sigma^2)} - M/r \quad (8)$$

$$V(p) = \phi_3 p + \phi_4 p^{(-2r/\sigma^2)} - (q(p-c)(1-\tau))/r \quad (9)$$

Since α_2 is negative, ϕ_2 must be zero because w(p) must remain finite as p approaches zero. Similarly, since $\alpha_2 >$ 0, $\phi_3 = 0$, so that (8) and (9) reduce to

$$W(p) = \phi_1 p - M/r \quad (10)$$

$$V(p) = \phi_4 p^{(-2r/\sigma^2)} + (q(p\cancel{+}c)(1-\tau)/r \quad (11)$$

We can interpret equation (10) as being made up of the value of an option to re-open a closed (but not abandoned) mine and the present value of a stream of payments M/r i.e. the cost of maintaining a closed mine forever. Similarly, equation (11) is made up of an option to close an open mine plus the present value of the stream of revenue less taxes. We must still solve for the parameters ϕ_1 and ϕ_4 subject to the equations (6), (7),(10), (11) and (12) in section 2.6. We will also solve for p_1 and p_2 simultaneously with ϕ_1 and ϕ_4.

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9. The level of taxation will also depend on p, but we will abstract from this for the current purpose of considering how the option values affect the value of the mine.

For the case, $W(p_1) - k_1 < 0$, we can solve for p_1 as

$$p_1 = c/(1 + (\sigma^2/2r)) \tag{12}$$

and when $W(p_1) - k_1 < 0$, we can solve for the ratio of p_1 to p_2 as

$$p_1/p_2 = [(qc(1-\tau)-M)/r - k_1]/[(qc(1-\tau)-M)/r + k_2] \tag{13}$$

Equation (12) has a nice interpretation since the numerator is the net present value of the after-tax cost of operating the mine less the cost of maintaining the closed mine, less the cost of closing the mine. The denominator has a similar interpretation although we now add the cost of re-opening the mine to the net present value of the after-tax cost of operating the mine less the maintenance cost. Thus combining (12) and (13) one can compute p_1 and p_2. Furthermore, equations (12) and (13) indicate the importance of the tax rate, the cost of maintaining a closed mine, and the cost of opening and closing the mine in computing the critical prices p_1 and p_2.

Using the boundary conditions we can also solve for ϕ_1 and ϕ_4:

$$\phi_1 = q(1-\tau)/r - [((qc(1-\tau)-M)/r - k_1)]/[1+(\sigma^2/2r)p_1] \tag{14}$$

$$\phi_4 = \{[((qc(1-\tau)-M)/r - k_1)]/[1+(2r/\sigma^2)]\}\cdot p_1^{(2r/\sigma^2)} \tag{15}$$

Therefore, we can solve for W(p) and V(p):

$$W(p) = (qp(1-\tau) - M)/r$$

$$- \{[((qc(1-\tau)-M)/r - k_1)]/[1+(\sigma^2/2r)]\}(p/p_1) \quad (16)$$

$$V(p) = \{[(qc(1-\tau)-M)/r - k_1)]/[1+(2r/\sigma^2)]\}\cdot$$

$$\cdot(p_1/p)^{(2r/\sigma^2)} + (q(p-c)(1-\tau) - T)/r \quad (17)$$

where we would impose the neccessary conditions on W(p) and V(p) so as to keep them nonnegative. In the case of W(p), the value of the option to open the mine will increase as the cost of maintaining the closed mine increases and as the gold price increases. The option value will also decrease as the cost of re-opening the mine increases. The value of the option to close increases as p_1 increases and decreases as the price of gold increases. Furthermore, the value of the option decrease as the cost of closing the mine increases.

2.8 Conclusion

Using the techniques of Contingent Claims Analysis introduced in Chapter 1 we constructed a model of a firm that extracts a natural resource. We obtained the valuation partial differential equation that describes the value of the firm. Furthermore, we considered the nature of the solution and its implications for the optimal rate of extraction and the required price in a perfectly competitive output market. The solution to the valuation (ordinary) differential equation was obtained in the infinite resource and it was determined that the value of the firm was the sum of an option and a discounted cash flow. In Chapter 3, we will value a sample of the South African Gold mines.

Chapter 3:

Empirical Application

3.1 Estimating the parameters of the Stochastic Differential Equations:

3.1.1 The Method

For stochastic processes such as

$$dp = \mu dt + \sigma dz$$

where dz is a Wiener process, the Fokker-Plank or forward equation which describes the transition probabilities has a Normal distribution. Furthermore, based on the Markov assumption, the joint density of a sample of n observations can be written as

$$f(p_1, \ldots p_n | p_o) = f_1(p_1 | p_o) \cdot \cdot \cdot f_n(p_n | p_{n-1}) \qquad (1)$$

In our model the parameters μ and σ are independent of time, so the conditional probability at time t is

$$f(p_t | p_{t-1}) = (2\pi\sigma^2\tau)^{-\frac{1}{2}} EXP\{(-1/(2\sigma^2\tau))[p_t - p_{t-1} - \mu\tau]\}$$

where $\sigma^2\tau$ and $\mu\tau$ are the integrals of σ^2 and μ with respect to time over the time interval $[t;t+\tau]$, where τ is the length of the time period (Lo(1986)). When the observations are made at equally spaced intervals of time (eg.monthly) then we can let $\tau=1$ and simply treat μ and σ^2 as the monthly mean and variance. If the observations are not made at equally spaced intervals of time, each observation will have a unique τ.[1]

Given a sample of n equally spaced observations, the Maximum Likelihood estmates of μ and σ^2 in equation (1) are the sample mean and variance. In the case of the lognormal distribution,

$$dp/p = \mu dt + \sigma dz \qquad (2)$$

where we can find the stochastic differential equation for $x = \ln(p)$,

$$dx = (\mu - \tfrac{1}{2}\sigma^2)dt + \sigma dz \qquad (3)$$

the transition probabilities of x are Normally distributed i.e.

$$f(x_t | x_{t-1}) = (2\pi\sigma^2\tau)^{-\frac{1}{2}}EXP\{(-1/2\sigma^2\tau)[x_t - x_{t-1} - \mu + \tfrac{1}{2}\sigma^2]\}$$

Using the change of variable formula one can obtain the Maximum likelihood estimates of μ and σ:

$$\sigma^2 = T^{-1}\Sigma_t[\ln(p_t/p_{t-1}) - T^{-1}\Sigma_t\ln(p_t/p_{t-1})]^2 \qquad (4)$$

$$\mu = T^{-1}\Sigma_t\ln(p_t/p_{t-1}) + \tfrac{1}{2}\sigma^2 \qquad (5)$$

where the summation limits are t=1 and t=T, assuming that the first observation is at t=0.

The parametric method discussed above can be used whenever the state variable is traded. In the case of the inflation rate and the interest rate, the method of moments approach

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1. The presence of the τ-factor relates back to the construction of the Wiener process where $E(dz^2) = \sigma^2 dt$.

has to be used. Following the approach by Ball and Torous (1983), one has to calculate the cummulants of the data series and use the relationship between the moments and cumulants to calculate the implied mean and variance. If the data is Normally distributed, this method reduces to calculating the sample mean and variance of the first differences of the series. More sophisticated methods are being developed as this method of modelling grows in popularity e.g. Duffie and Singleton (1989)

3.1.2 The Data

Quarterly data was used to coincide with the publication of the Chamber of Mines Quarterly Report. The quarter-ending New York gold price (in US$) and the three-month South African BA-rate were used to estimate the parameters for the gold price and interest rate, respectively, and the South African Consumer Price Index was used to estimate the rate of inflation The gold price and exchange rate were assumed to have have a lognormal distribution and the inflation rate and interest rate were assumed to have a Normal distribution. The The data were obtained from the IMF database on International Trade for the time period, January 1977 to March 1988. Unfortunately, more recent data was not available on the Simon Fraser University Database. The corresponding means and variances are contained in table 1 and table 2 contains the lower triangle of the correlation matrix.

Table 1: The Means and Variances of the State Variables

Variable	Mean	Variance
Gold Price	0.035 875	0.013 626
Exchange Rate	0.024 303	0.007 617
Inflation Rate	-0.000 020	0.000 356
Interest Rate	0.000 270	0.000 228

Table 2: The Correlation Matrix of the State Variables

Variable	Gold	Exchange	Inflation	Interest
Gold	1.000			
Exchange	-0.3559	1.000		
Inflation	0.2153	-0.1270	1.000	
Interest	-0.3837	0.1833	-0.06748	1.000

3.2 Tax and Lease Payments

Each mine is required to pay a leasing fee and government taxation. The lease payment is calculated and deducted from the working profit, while the tax is calculated and deducted from the working profit after payment of the lease. The structure of both formulas is of the the following form

$$y = a - b/x \qquad (1)$$

where y is the percentage of profit that will be paid as the lease or tax;

a and b are chosen by the government and are specific to each mine. The value of a and b differ for the lease and tax formula;

x is the ratio of profit to total revenue.

Using the following income statement we will explain the tax calculation:

Total Revenue

<u>less</u> Working COsts

Working Profit(1)

<u>less</u> Lease Payment

Working Profit(2)

<u>less</u> Taxation

Net Profit

In order to calculate the lease payment we need to express the ratio of working profit(1) less a capital allowance, to total revenue as a percentage and then substitute into equation (1). This will provide us with the percentage of working profit(1) that must be paid as a lease payment. Thus by multiplying working profit(1) by y, we obtain the lease payment.

In order to obtain the tax payment, we need to obtain the ratio of working profit(2) less a capital allowance, to total revenue as a percentage and substitute it into equation 1. This will provide the rate of taxation of working profit(2). Simply multiplying y by working profit(2) will provide the total taxation. The capital redemption allowance is based on the amount of capital expenditure that the mine incurs during the previous quarter of operations and is usually augmented by a 5% or 6% allowance. In section 3.3 we have listed the lease and taxation formulas of the mines used in this study. The parameters a and b are the values of a and b in equation (1) for the lease formula and c and d are the corresponding parameters for the tax rates.

3.3 Choosing the Mines

We used cost (excluding capital expenditure) and market capitalisation as the two criteria for choosing the mines for this study. Using the June 1991 review of the gold mine quarterlies by Simpson McKie Inc., we chose the following 6 gold mines: Vaal Reefs, Western Deep Levels, Kinross, Western Areas, Loraine and Venters. In table 1 we have listed ranking of each mine used in the study based on Market Capitalisation and Cost - the complete set of rankings is found in the report by Simpson McKie Inc. The final column indicates whether the mine was profitable in the second quarter of 1991. In table 2, we have listed the relevant information for the mines that was required for the Monte Carlo programs: the average cost in Rands of producing one troy ounce of gold, the quarterly rate of production in millions of troy ounces, the tax and lease parameters, and the prospective life of the mine in years. The information in table 2 was obtained from the quarterly review of the gold mines by Ed Hern, Rudolph, INc.

Table 1. The Ranking of the Mines used in this study

Mine	Market Cap	Cost	Profit/Loss
V. Reefs	2	13	Profit
W. Deep	4	6	Profit
Kinross	10	7	Profit
W. Areas	20	39	Profit
Loraine	26	43	Loss
Venters	30	41	Loss

Table 2. Mine Characteristics

Mine	Cost	Prod'n	a	b	c	d	life
V.Reefs	772	0.6021	12.5	75	61	305	30
W.Deep	700	0.3284	15.0	90	61	305	45
Kinross	707	0.0967	12.5	75	61	305	27
W. Areas	1013	0.0956	12.5	75	61	305	25
Loraine	1132	0.0561	15.0	90	61	305	15
Venters	1032	0.0385	10.0	80	61	305	23

Table 3. Contains the capital expenditure in rands per troy ounce and the Lease and Tax allowances that are used the calculation of the capital redemption allowance. The value of capital expenditure per troy ounce was estimated from the previous two years of data and the projected expenditure in the Quarterly Gold Share Review and Forecast of Ed Hern, Rudoplh Inc.

Table 3. Capital Expenditure Allowance Information

Mine	Capex	Lease	Tax
V.Reefs	150	5%	0%
W.Deep	220	5%	0%
Kinross	50	5%	0%
W.AReas	25	5%	0%
Loraine	20	5%	0%
Venters	50	6%	0%

3.4 The Models and Results

We will consider four different models:

1. A one state variable model where the gold price is the only state variable.
2. A two state variable model where the gold price and one of other economic variables will be the second state variable. This will provide us with three two state variable models.

3. A three state variable model where the gold price and two of the other economic variables will be the second and third state variables. This will provide us with three three state variable models.

4. A four state variable model will include all the economic variables available.

The one and two state variable models will be solved by means of a finite difference method while the three and four state variable models will be solved by Monte Carlo Methods.

Model 1: Gold price

Model 2A: Gold Price, Exchange Rate.

Model 2B: Gold Price, Inflation Rate

Model 2C: Gold Price, Interest Rate

Model 3A: Gold Price, Exchange Rate, Inflation Rate

Model 3B: Gold Price, Exchange Rate, Interest Rate

Model 3C: Gold Price, Inflation Rate, Interest Rate

Model 4: Gold Price, Exchange Rate, Inflation Rate, Interest Rate.

The finite difference method was applied but did not produce results consistent with those of the Monte Carlo program, so we resorted to valuing all the mines using the Monte Carlo program. After some of the mines were valued using the models that included the inflation rate as a state variable, it was found that allowing the inflation rate to be unbounded produced extremely large negative present values. Thus we felt it necessary to consider two scenarios, one where the inflation rate remained at 14.7% for the rest of the life of the mine and secondly, where the inflation rate was stochastic, but was bounded between 0% and 20%. These two variants are represented by the suffix "1" for the constant inflation case and the suffix "2" for the bounded inflation case in the tables below.

Table 1: The Monte Carlo value of each Mine

(Millions of Rands)

Model	V. Reefs	W. Deep	Kinross
M.Cap	4 357.0000	3 630.0000	999.0000
1.	1 255.6055	885.3650	295.6929
2A.	1 318.447	902.5907	302.4684
2B.1	-26 071.7201	-18 307.0708	-30136.6448
2B.2	-13 118.8146	-7 465.7229	-1 548.6293
2C.	2 957.2067	2 190.7136	566.7136
3A.1	-26 323.6177	-18 461.6627	-3 181.7101
3A.2	-13 419.5327	-7 811.6532	-1 597.7222
3B.	2 295.3464	1 621.1275	476.0257
3C.1	-114 783.8730	-334 067.6330	-11 245.7928
3C.2	-49 824.6261	-92 578.8263	-5 095.4470
4.2	-56 310.1793	-101 548.7420	-5 933.2444

Table 1 (continued):

Model	W. Areas	Loraine	Venters
M.Cap	276.0000	106.0000	71.0000
1.	-99.5811	-207.0307	-148.5506
2A.	-74.8412	-197.2240	-141.3956
2B.1	-4 932.4873	-2 057.1878	-2 234.2455
2B.2	-2 908.4895	-1 469.3251	-1 399.1994
2C.	77.3151	-152.9169	-101.0322
3A.1	-4 968.6626	-2078.8029	-2 254.3858
3A.2	-2 930.2413	-1 485.0894	-1 424.2197
3B.	-34.8473	-198.0753	-154.6701
3C.1	-13 873.7983	-2 971.8371	-5 340.5243
3C.2	-7 164.7775	-2 045.9653	-3 033.8693
4.2	-8 219.8947	-2 370.7351	-3 416.4986

Interpreting the Results:

The initial values of the four state variables were chosen to be: gold $360, Rand/U.S. Dollar exchange rate R2.85, Inflation Rate 14.7%, Interest Rate 17%. We assumed that the term structure was flat based on the yield curve in the beginning of September 1991.

1. The most favourable valuation model, 2C, still undervalues the mines by more than 50% in comparison to their market capitalisation. Two possible explanations are, firstly, that the model is not an adequate representation of

the gold mines and the uncertainty they face, and secondly, the market participants have discounted a much brighter future for the gold mines. By definition, the first objection can be levelled at any model, but based on the results in the table above we feel that the model has captured the salient aspects of the gold mines. Regarding the second criticism, the initial values of the state variables may not be those that market participants expect to prevail in the future.

2. Due to the negative correlation between the changes in the gold price and the changes in the exchange rate, the addition of the exchange rate to model 1 does not add much to the value of the mines in model 2A. Similar results are evident in the comparison of model 2B and 3A.

3. The addition of the interest rate produces a large increase in the value of most mines due to its negative correlation with the gold price. This correlation stems from the use of the interest rate as a policy instrument in South Africa and from the fact that business cycles can be associated with movements in the gold price. Comparing model 1 and 2C, the value of Vaal Reefs doubles when the interest rate is added because large positive cash flows are discounted at a lower interest rate and negative cash flows are discounted at a higher interest rate. The effect is even greater in the case of Western Deep Levels because the

prospective life of the mine is 45 years as opposed to only 30 years for Vaal Reefs. Western Areas which was negatively valued by model 1 is now positively valued and the two other marginal mines, Loraine and Venters, have increased slightly in value.

4. The effect of inflation on the cost of the extraction of gold is the most serious problem facing the gold mines. Comparing models 2B, 3A and 3C, we see that if inflation continues at a rate of 14.7% per annum for the rest of the life of the mines, none of them is positively valued. In model 3C Vaal Reefs, Western Deep Levels and Western Areas have "abnormally" large negative values. It is not clear why these three mines have such similar valuations since the only similarity is their lease rates. The results of models 2B.2 and 3A.2 are consistent with the observation that the exchange rate has very little effect on the value of the mines.

5. The results of model four are seriously affected by the presence of the interest and inflation rates, so much so that results of model 4.1 were not reported - the values could not fit in the table. The values of the shorter-lived mines like Loraine and Venters are consistent with previous models. The results for the low-cost and long-lived mines, Vaal Reefs and Western Deep Levels cause concern. It appears that the presence of inflation and the interest rate along

with the life of the mine have produced extremely negative results. It suggests that we should test the sensitivity of the Monte Carlo model to these two variables.One of the examiners has questioned the large negative valuations of some of the mines: having reconsidered the computer program, and finding no error, we conjecture that the problem lies with the modelling of the inflation rate. As mentioned in Chapter 1, the stochastic differential equation is the continuous time analogue of the Random Walk, so with an initially high rate of inflation of 14.7% it is possible for the growth in the inflation rate to outstrip that of the gold price. This result is evident when one compares models 2B.1 to 2B.2, 3B.1 to 3B.2, etc.

6. In a recent paper by Cloete (1985) and Cloete and van Rensburg (1984), it was suggested that subsidising the maginal gold mines might be defended on the grounds of net social benefit. That is, we can compare the cost of subsidising the mine to the social benefit of creating employment for an unskilled labour force. The Monte Carlo results in Table 1 provide a guide as to the extent of the social cost - between R2 billion and R3 billion (in 1991 rands) for Venters and Loraine over the next 23 and 15 years, respectively.

Cloete (1985) also refers to the trade-off between re-distribution and efficiency, and the extraction of low grade (un-economic) ores in the marginal gold mines. This model is not rich enough to consider the value of different ore bodies, but with the relevant information the model could easily be extended to cover differences in ore grade. This model cannot answer all the questions related to subsidising marginal gold mines and this was not the intention from the outset. However, it does show that this framework can be useful for analysing this question from the point of view of social cost and it can be augmented to provide better results.[2]

3.5 Conclusion

We have set out to use current financial theory - contingent claims analysis - to create a model of the South African gold mines and determine its performance in valuing these mines. The model had to be "solved" by means of the Monte Carlo method. When the gold price was the only state variable, the mines were undervalued by as much as 50% in comparison to their market capitalisations. The importance of the inflation rate and its serious affect on the cost of producing gold was underscored by the model valuations: none

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[2]. I wrote this paper while enrolled at the Simon Fraser University in Canada, so the information relating to the South African gold mines was limited. I have detailed some of the improvements that can be made to the model in the section, Avenues for Further Research, which follows below.

of the mines were positively valued when we took inflation into account. This method of modelling proved to be very tractable and captured the uncertainty that the mines face in both the input and output markets.

3.6 Avenues For Further Research

1. The first improvement that could be made would be to obtain better data and thus obtain better estimates of the parameters. Furthermore, the Normality assumption for the distribution of returns may not be correct.

2. The model could take into account the difference in the grades of the ore bodies and the mining of these ore bodies.

3. A related issue is the response of the mines to a higher gold price and the mining of lower grade ores. Cloete (1985) feels that this may not be efficient in the case of marginal mines which are being subsidised.

5. The implied extraction rate that is obtained from the model in Chapter 2 can be compared to current extraction rates in terms of optimal extraction rates.

6. Are there any other state variables that need to be modelled?

7. Lastly, but perhaps the most important of all the points, we should get some input from the mining community.

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