Isometries on symmetric spaces associated with semi-finite von Neumann algebras

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Abstract

Isometries on Banach spaces of measurable functions can typically be characterized as weighted composition operators. In the non-commutative setting, isometries between symmetric spaces (of trace-measurable operators) can often be described in terms of a Jordan $*$-homomorphism (which may be considered a non-commutative composition operator) weighted by a partial isometry and/or a positive operator. In this thesis we describe the structures of isometries on various (non-commutative) symmetric spaces associated with semi-finite von Neumann algebras. This is achieved by extending certain results from the finite setting to the semi-finite setting, exploring the applicability of disjointness-preserving techniques in generalizations of L_p-spaces, and developing characterizations of extreme points in a certain class of Lorentz spaces and in various types of Orlicz spaces.

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> *The steadfast love of the Lord never ceases; His mercies never come to an end; they are new every morning; great is Your faithfulness.* -Lamentations 3:22-23.

Contents

Introduction

Every unital commutative C^*-algebra can be represented as a space of continuous functions on a compact Hausdorff space and every commutative von Neumann algebra as an L_∞-space over a localizable measure space. These representation theorems have led to the interpretation of general (not necessarily commutative) C^*- and von Neumann algebras as non-commutative function spaces, and this has inspired research into the extent to which results for commutative spaces have non-commutative counterparts. In the setting of von Neumann algebras this research area is called non-commutative integration theory and focuses on the study of non-commutative analogues of the classical Banach spaces of (equivalence classes of) measurable functions such as the Lebesgue spaces, Orlicz spaces and Lorentz spaces. Symmetric spaces (both in the commutative and non-commutative settings) form a common generalization of these spaces and provide a framework for a unified study of these spaces. In the context of semi-finite von Neumann algebras, the existence of a semi-finite normal faithful trace makes it possible to represent these non-commutative spaces as spaces of unbounded, closed and densely defined operators affiliated with the von Neumann algebra. The construction of non-commutative spaces associated with more general von Neumann algebras has been achieved for L_p-spaces ([19]) and, more recently, Orlicz spaces ([32]). These constructions are a great deal more technically involved and so we will restrict ourselves to non-commutative spaces associated with semi-finite von Neumann algebras. The specific concepts and notation to be used throughout this text will be introduced in Chapter 1.

It follows from the spectral theorem that every von Neumann algebra is generated by its lattice of projections. It is therefore unsurprising that any isometric isomorphism between von Neumann algebras has to be implemented by a map that preserves this lattice structure, namely a Jordan $*$-isomorphism, possibly multiplied by a unitary operator ([22]). Furthermore, one would anticipate that there would a relationship between the isometries of symmetric spaces associated with semi-finite von Neumann algebras and the isometries of the underlying von Neumann algebras. In describing the structure of an isometry between symmetric spaces it is therefore natural to use the isometry to initially define a map on projections. In order to ensure that this map preserves the projection lattice structure and can be extended in a well-defined and linear manner, this map should preserve orthogonality of projections. In the setting of commutative and non-commutative L_p-spaces, for example, this can be achieved by showing that the isometry is disjointness-preserving ([33] and [49]). Since surjective isometries preserve extreme points, a description of the extreme points of the unit ball of a symmetric space can provide a further tool in characterizing surjective isometries. In the setting of unital C^*-algebras and Lorentz spaces, for example, extreme points of the of the unit balls of these spaces can be characterized in terms of partial isometries, and facilitate the description of the structure of surjective isometries on such spaces ([22], [2], [4]). A third method that can be employed in describing the structure of isometries on symmetric spaces is the use of characterizations of Hermitian operators on such spaces ([50],[35] and [42]). In Chapter 2 we describe how these approaches have been used in the commutative and non-commutative settings and outline some of the results contained in the literature.

Several results in the non-commutative setting have only been obtained for spaces associated with von Neumann algebras equipped with finite traces. Our first aim is therefore to extend these results to spaces associated with semi-finite von Neumann algebras. We see in the proofs of Lamperti's, Zaidenberg's and Yeadon's results examples of how this may be achieved. Motivated by these and our earlier discussion, we investigate in Chapter 3 the possibility of defining a map on projections with finite trace and then extending it to a Jordan $*$-homomorphism on the whole von Neumann algebra. The first application of these extension procedures will be in describing the structure of a positive surjective isometry between a symmetric space and a fully symmetric space. This characterization has been obtained in the finite setting by Chilin et al. ([4, Theorem 3.1]). An important component in the proof of this result is being able to show that the inverse of a surjective positive isometry is itself positive under certain conditions. This will also form a significant component of other proofs and so in Chapter 4 we provide sufficient conditions for this to be achieved in the semi-finite setting. In Chapter 5 we describe the structure of a positive surjective isometry between a strongly symmetric space and a fully symmetric space, both with absolutely continuous norm, and hence provide a partial generalization of [4, Theorem 3.1] to the semi-finite setting. Another aim of this thesis is to consider the applicability of disjointness-preserving techniques in spaces more general than L_p-spaces. Indeed, in Chapter 6 we show that the structure of a surjective isometry between a strongly symmetric space and a fully symmetric space (both with absolutely continuous norm), which is not necessarily positive, can be described if we know, in addition, that the isometry is disjointness-preserving. In Chapter 7 we characterize surjective isometries between Lorentz spaces associated with semi-finite von Neumann algebras by showing that such isometries are disjointness-preserving and in the process we provide a generalization of the corresponding characterization in the finite setting ([4, Theorem 5.1]). The techniques employed in Chapter 7 also illustrate how the extreme points of the unit balls of such spaces can be used in determining the structure of isometries. In essence, the characterization of these extreme points allows one to show that isometries between Lorentz spaces map partial isometries onto partial isometries, which plays a critical role in showing that these isometries are disjointness-preserving, and can therefore be described using the results obtained in Chapter 6. The final aim of this thesis is to consider further applicability of extreme point methods. In Chapter 8 we show that characterizations of the extreme points in the unit balls of various types of Orlicz spaces can be used to describe the structure of positive surjective isometries on these Orlicz spaces, including the Orlicz space formed by the intersection of a von Neumann algebra with the L_1-space associated with it.

As mentioned earlier, a framework for describing Orlicz spaces associated with general von Neumann algebras has recently been developed. The possibility of developing similar frameworks for other symmetric spaces and describing the structures of isometries on such spaces provide interesting possibilities for further research. Furthermore, there are several results in the semi-finite setting that could potentially be refined further.

CHAPTER 1

Preliminaries

We introduce the concepts and notation to be used throughout the text. To facilitate readability a number of results to be used in the sequel will be provided in Appendix B. A basic understanding of measure theory, functional analysis and operator algebras as presented in [37], [6] and [23] will be assumed.

1.1. Measure theory and integration

Let (Ω, Σ, μ) be a measure space, where Ω denotes a set, Σ denotes a σ-algebra of subsets of Ω and μ denotes a positive measure on Σ. (Ω, Σ, μ) is *semi-finite (or has the finite subset property)* if for every $A \in \Sigma$ with $\mu(A) > 0$ there exists a $B \in \Sigma$ with $0 < \mu(B) < \infty$ such that $B \subseteq A$. $A, B \in \Sigma$ are μ-almost equal if $\mu(A \triangle B) = 0$. The *measure algebra* of (Ω, Σ, μ) is obtained from Σ by identifying sets which are μ-almost equal. If the measure algebra obtained from (Ω, Σ, μ) is a complete Boolean algebra and (Ω, Σ, μ) is semi-finite, then (Ω, Σ, μ) is called *localizable*. Any σ-finite measure space is localizable. Unless stated otherwise, we assume that the measure spaces under consideration in this thesis are localizable. We will use $L_0(\mu)$ to denote the complexification of the Riesz space of all equivalence classes of almost-everywhere finite real-valued measurable functions on Ω, where functions which are equal μ-a.e. have been identified. For $f \in L_0(\mu)$, the distribution function d_f of f is defined as

$$d_f(\lambda) = \mu(\{t \in \Omega : |f(t)| > \lambda\}) \qquad \lambda \geq 0$$

and the decreasing rearrangement f^* of f as

$$f^*(t) = \inf\{\lambda \geq 0 : d_f(\lambda) \leq t\} \qquad t \geq 0.$$

An important subspace of $L_0(\mu)$ is the set of all (equivalence classes of) functions in $L_0(\mu)$ that are bounded, except possibly on a set of finite measure. We will denote this space $L_{00}(\mu)$. It can be shown that $L_{00}(\mu)$ is also given by

$$\{f \in L_0(\mu) : f^*(t) < \infty \ \forall t > 0\}. \tag{1.1.1}$$

Furthermore, if $L_{00}(\mu)$ is equipped with the topology of convergence in measure, then it is a complete metrizable topological vector space. We present a few results that will be used in the sequel.

THEOREM 1.1.1. [20, p.94] *For every integrable function f,*

$$\frac{d}{dx}\int_a^x f(t)dt = f(x) \qquad a.e.$$

THEOREM 1.1.2. [20, p.58] *If the function f is integrable on the set E, then for every $\epsilon > 0$ there exists a $\delta > 0$ such that*

$$\int_A |f(x)|dx < \epsilon$$

whenever $A \subseteq E$ is a measurable set with measure less than δ.

THEOREM 1.1.3. [37, p.15] *Suppose (Ω, Σ, μ) is a measure space. If $f : \Omega \to [0, \infty)$ is measurable, then there exists a sequence $(f_n)_{n=1}^{\infty}$ of simple functions such that*

(1) $0 \leq f_1 \leq f_2 \leq ... \leq f$ *and*
(2) $f_n \to f$ *pointwise* μ-*a.e.*

Proposition 1.1.4. *If* $f \in L_0(\Omega, \Sigma, \mu)$ *such that* $f \neq 0$*, then there exists an* $\epsilon > 0$ *and a* $B \in \Sigma$ *with* $\mu(B) = \delta > 0$ *such that* $|f(x)| \geq \epsilon$ *for all* x *in* B*.*

Proof. Let $A = \{x \in \Omega : |f(x)| > 0\}$. $f \neq 0$ implies that $\mu(A) > 0$. For each $n \in \mathbb{N}^+$, define $A_n = \{x \in \Omega : |f(x)| > \frac{1}{n}\}$. Then $A = \bigcup_{n=1}^{\infty} A_n$ and so

$$\mu(A) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n)$$

A simple proof by contradiction shows that this implies that $\mu(A_k) > 0$ for some $k \in \mathbb{N}^+$. Let $\epsilon = \frac{1}{k}$ and $B = A_k$. ☐

1.2. von Neumann Algebras

Unless indicated otherwise, we will use H, K to denote Hilbert spaces, ξ, η to denote elements of Hilbert spaces, x, y, z to denote (bounded or unbounded) operators on Hilbert spaces, and $\mathcal{A}, \mathcal{B}$ to denote von Neumann algebras. If H is a Hilbert space, then $\mathcal{B}(H)$ will be used to denote the algebra of all bounded linear operators on H. A von Neumann algebra $\mathcal{A}$ is a unital, weak operator closed, self-adjoint subalgebra of $\mathcal{B}(H)$. We will use $\mathbf{1}$ or $\mathbf{1}_{\mathcal{A}}$ to denote the identity element of $\mathcal{A}$, $\mathcal{A}^{sa}$ to denote the set of all self-adjoint operators in $\mathcal{A}$ and $\mathcal{A}^+$ to denote the set of all positive operators. Using the double commutant theorem, one can also characterize a von Neumann algebra as a unital, self-adjoint subalgebra $\mathcal{A} \subseteq \mathcal{B}(H)$ with the additional property that $\mathcal{A} = \mathcal{A}''$, where $\mathcal{A}'$ denotes the commutant of $\mathcal{A}$. For any $\mathcal{A} \subseteq \mathcal{B}(H)$, $\mathcal{A}'$ is a von Neumann algebra and it is easily shown that $\mathcal{A}'' = (\mathcal{A}')'$ is the smallest von Neumann algebra containing $\mathcal{A}$. We will therefore refer to $\mathcal{A}''$ as the von Neumann algebra generated by $\mathcal{A}$. We will use $\mathcal{P}(\mathcal{A})$ to denote the set of all projections in $\mathcal{A}$, $\mathcal{V}(\mathcal{A})$ to denote the set of all partial isometries in $\mathcal{A}$ and $U(\mathcal{A})$ to denote the set of all unitaries in $\mathcal{A}$. The *center* of $\mathcal{A}$ is denoted $Z(\mathcal{A})$. If $Z(\mathcal{A}) = \{\alpha\mathbf{1} : \alpha \in \mathbb{C}\}$, then $\mathcal{A}$ is called a *factor*.

Example 1.2.1. [23]

(1) The simplest example of a von Neumann algebra is $\mathcal{B}(H)$.
(2) Suppose (Ω, Σ, μ) is some localizable measure space and let $H = L_2(\mu)$. For $f \in L_\infty(\mu)$, we will use M_f to denote the *multiplication operator* induced by f, i.e. $M_f(g) = f.g$ for all $g \in L_2(\mu)$. Let $\mathcal{A} = \{M_f : f \in L_\infty(\mu)\}$. It is easily shown that $\mathcal{A}$ is a unital, self-adjoint subalgebra of $\mathcal{B}(H)$. It is in fact weak operator closed and therefore a von Neumann algebra. Furthermore, the mapping $f \to M_f$ is an isometric $*$-isomorphism of $L_\infty(\mu)$ onto $\mathcal{A}$. The weak operator topology on $\mathcal{A}$ coincides with the weak$*$ topology on $L_\infty(\mu)$ when M_f and f are identified. Furthermore, a von Neumann algebra $\mathcal{A}$ is commutative if and only if $\mathcal{A}$ is isometrically $*$-isomorphic to $L_\infty(\Omega, \Sigma, \mu)$ for some localizable measure space (Ω, Σ, μ).

We will work exclusively with semi-finite von Neumann algebras. These can be defined in terms of the type decomposition of von Neumann algebras.

Theorem 1.2.2. [44, p.296] *Every von Neumann algebra is uniquely decomposable into the direct sum of those of type* I*, type* II_1*, type* II_∞ *and type* III*, i.e. there exist mutually orthogonal projections* $p_I, p_{II_I}, p_{II_\infty}, p_{III}$ *such that*

$$\mathbf{1} = p_I + p_{II_I} + p_{II_\infty} + p_{III}$$

and the range of each of these projections is a von Neumann algebra of the corresponding type.

If $p_{III} = 0$ in the decomposition above, then the von Neumann algebra is called *semi-finite*. A detailed discussion of the types of von Neumann algebras can be found in [24]. We will however be more interested in the characterization of semi-finite von Neumann algebras in terms of faithful normal semi-finite traces. Let $\mathcal{A}$ be a von Neumann algebra. We will write $x_\lambda \uparrow x$ if $\{x_\lambda\}_{\lambda\in\Lambda} \subseteq \mathcal{A}^{sa}$ is an increasing net with supremum $x \in \mathcal{A}^{sa}$. A map $\tau : \mathcal{A}^+ \to [0, \infty]$ is called a *weight* if $\tau(x + y) = \tau(x) + \tau(y)$ and $\tau(\lambda x) = \lambda\tau(x)$ for all x and y in $\mathcal{A}^+$ and all $\lambda \geq 0$ (where the convention $0.\infty = 0$ has been used). If in addition $\tau(u^*xu) = \tau(x)$ for all $x \in \mathcal{A}^+$ and all $u \in U(\mathcal{A})$, then τ is called a *trace*. A trace $\tau : \mathcal{A}^+ \to [0, \infty]$ is called

(1) finite if $\tau(\mathbf{1}) < \infty$;
(2) semi-finite if $\tau(x) = \sup\{\tau(y) : 0 \leq y \leq x; \tau(y) < \infty\}$ for all $x \in \mathcal{A}^+$
(3) faithful if $\tau(x) = 0$ implies that $x = 0$;
(4) normal if $0 \leq x_\lambda \uparrow x$ implies that $\tau(x_\lambda) \uparrow \tau(x)$

PROPOSITION 1.2.3. *Let τ be a weight on a von Neumann algebra $\mathcal{A}$.*

(1) *If $0 \leq x \leq y$, then $\tau(x) \leq \tau(y)$*
(2) *τ is a trace if and only if $\tau(x^*x) = \tau(xx^*)$ for all x in $\mathcal{A}$;*
(3) *If τ is a normal trace, then τ is semi-finite if and only if for every non-zero x in $\mathcal{A}^+$ there exists an y in $\mathcal{A}^+$ with $0 < y \leq x$ and $\tau(y) < \infty$*

Semi-finite von Neumann algebras can be characterized in terms of their traces as follows. A von Neumann algebra $\mathcal{A}$ is semi-finite if and only if there exists a faithful, normal, semi-finite trace on $\mathcal{A}^+$. If $\mathcal{A}$ is a semi-finite von Neumann algebra, we will usually speak of the semi-finite von Neumann algebra $(\mathcal{A}, \tau)$ and mean that $\mathcal{A}$ is a semi-finite von Neumann algebra equipped with a (distinguished) faithful normal semi-finite trace τ. If $\mathcal{A}$ is a von Neumann algebra, equipped with a trace, then we will use $\mathcal{P}(\mathcal{A})^f$ (or $\mathcal{D}$ when dealing with subscripts) to denote the set of all projections in $\mathcal{A}$ with finite trace and $\mathcal{V}(\mathcal{A})^f$ to denote the set of all partial isometries in $\mathcal{A}$ whose support projections have finite trace.

EXAMPLE 1.2.4. [15]

(1) Let H be a Hilbert space with orthonormal basis $\{\eta_i\}_{i\in I}$. Let $\tau : \mathcal{B}(H) \to [0, \infty]$ be defined by
$$\tau(x) = \sum_{i\in I} \langle x(\eta_i), \eta_i\rangle .$$
Then τ defines a faithful, semi-finite normal trace on $\mathcal{B}(H)$.
(2) Let (Ω, Σ, μ) be a measure space, $H = L_2(\mu)$ and $\mathcal{A} = \{M_f : f \in L_\infty(\mu)\}$. Define $\tau : \mathcal{A}^+ \to [0, \infty]$ by $\tau(M_f) = \int_\Omega f\, d\mu$. Then τ is a faithful, semi-finite normal trace. In the special case where the underlying measure space is the set of natural numbers, equipped with counting measure, we have that $\tau\left((x_n)_{n=1}^\infty\right) = \sum_{n=1}^{\infty} x_n$.

We finish this section by mentioning some of the terminology to be used with regards to projections. Let $p, q \in \mathcal{P}(\mathcal{A})$. We will say that p is equivalent to q, relative to the von Neumann algebra $\mathcal{A}$, if there exists a partial isometry $v \in \mathcal{A}$ such that $p = v^*v$ and $q = vv^*$. If this is the case, we will write $p \sim q$. If there exists a $q_1 \in \mathcal{P}(\mathcal{A})$ such that $q_1 \leq q$ and $q_1 \sim p$, then we will write $p \precsim q$. A non-zero projection $p \in \mathcal{P}(\mathcal{A})$ is called *minimal* if $q \in \mathcal{P}(\mathcal{A})$, $q \leq p$ implies that $q = 0$ or $q = p$. This is the abstract analogue of a projection onto a one-dimensional subspace. If a von Neumann algebra does not contain minimal projections, then it is called *non-atomic*.

Further results regarding von Neumann algebras can be found in Section B.1. In particular, results to be used in the sequel regarding the weak and strong operator topologies, spectral theory, order structure of von Neumann algebras, projections and partial isometries will be described there.

1.3. Trace-measurable operators

Trace-measurable operators are the non-commutative analogues of measurable functions. We will introduce some of the salient features relating to such operators. Suppose x is a closed, densely defined operators on a Hilbert space and let $\mathcal{D}(x)$ denote its domain. The projection $k(x)$ onto the kernel of x is called the *null projection* of x; the projection $r(x)$ onto the closure of the range of x is called the *range projection* of x; and the projection $s(x)$ onto $\overline{\mathrm{ran}(x^*)}$ is called the *support projection* of x. The *central support projection* $z(x)$ of x is defined as the projection $\mathbf{1}-p$, where p is the supremum of all central projections $q \in \mathcal{A}$ such that $qx = 0$. If $\mathcal{A} \subseteq \mathcal{B}(H)$ is a von Neumann algebra, equipped with a faithful normal semi-finite trace τ, then we will often be interested in the set $\mathcal{F}(\tau)$ defined by

$$\mathcal{F}(\tau) := \{x \in \mathcal{A} : \tau(r(x)) < \infty\}.$$

The projections mentioned above have the following properties.

Proposition 1.3.1. [25, p.77] *Let x be a closed densely defined operator on a Hilbert space H. Then*

(1) *$s(x)$ is the smallest projection $p \in \mathcal{P}(\mathcal{B}(H))$ such that $x = xp$*
(2) *$r(x)$ is the smallest projection $p \in \mathcal{P}(\mathcal{B}(H))$ such that $x = px$*
(3) *$k(x) = (k(x^*))^\perp$ and $k(x) = (r(x^*))^\perp = (s(x))^\perp$*
(4) *$r(x^*x) = r(x^*)$ and $k(x^*x) = k(x)$.*

Let $\mathcal{A} \subseteq \mathcal{B}(H)$ be a von Neumann algebra. A closed, densely defined operator x is said to be *affiliated* with $\mathcal{A}$ if $u^*xu = x$ for all unitary operators $u \in \mathcal{A}'$. If this is the case we will write $x\eta\mathcal{A}$. If $\mathcal{A} = \mathcal{B}(H)$, then $\mathcal{A}' = \mathbb{C}\mathbf{1} := \{\alpha\mathbf{1} : \alpha \in \mathbb{C}\}$ and so all closed, densely defined operators are affiliated with $\mathcal{A}$. If $H = L_2(\mu)$ and $\mathcal{A} = \{M_f : f \in L_\infty(\mu)\}$, then a closed, densely defined operator on $L_2(\mu)$ is affiliated with $\mathcal{A}$ if and only if it is of the form M_f for some $f \in L_0(\mu)$ (where the domain of the multiplication operator M_f is given by the set of all $g \in L_2(\mu)$ such that $f.g \in L_2(\mu)$).

A subspace $D \subseteq H$ is called *τ-dense* if there exists a sequence of projections $(p_n)_{n=1}^\infty \subseteq \mathcal{P}(\mathcal{A})$ such that $p_n \uparrow \mathbf{1}$ and $p_n(H) \subseteq E$, $\tau(p_n^\perp) < \infty$ for all $n \in \mathbb{N}^+$. A closed, densely defined operator x, affiliated with $\mathcal{A}$, is called a *τ-measurable operator* if its domain $\mathcal{D}(x)$ is τ-dense. It can be shown ([45, p.4]) that if x and y are τ-measurable, then $x+y$ and xy are pre-closed operators whose closures (the strong sum of x and y, and the strong product of x and y, respectively) are τ-measurable. Furthermore, x^* is τ-measurable for any τ-measurable operator x, and the set of all τ-measurable operators (denoted $S(\mathcal{A},\tau)$), equipped with strong-sum, strong-product and adjoint is a $*$-algebra. In this context the usual notation for sums and products will be used for strong sums and strong products. The set of self-adjoint elements of $S(\mathcal{A},\tau)$ will be denoted $S(\mathcal{A},\tau)^{sa}$. If $x \in S(\mathcal{A},\tau)$, then defining

$$\begin{aligned}\mathrm{Re}\,(x) &= \frac{1}{2}(x + x^*)\\ \mathrm{Im}\,(x) &= \frac{1}{2i}(x - x^*)\end{aligned}$$

yields $\mathrm{Re}\,(x), \mathrm{Im}\,(x) \in S(\mathcal{A},\tau)^{sa}$ and $x = \mathrm{Re}\,(x) + i\,\mathrm{Im}\,(x)$. It follows that $S(\mathcal{A},\tau) = S(\mathcal{A},\tau)^{sa} + iS(\mathcal{A},\tau)^{sa}$. An operator $x \in S(\mathcal{A},\tau)^{sa}$ is called positive, denoted $x \geq 0$, if $\langle x\xi, \xi\rangle \geq 0$ for all $\xi \in \mathcal{D}(x)$. The set $S(\mathcal{A},\tau)^+$ of all positive elements in $S(\mathcal{A},\tau)^{sa}$ is a proper cone and therefore a partial order may be introduced on $S(\mathcal{A},\tau)^{sa}$ by defining $x \leq y$ if $y - x \in S(\mathcal{A},\tau)^+$. With respect to this ordering $S(\mathcal{A},\tau)^{sa}$ is a partially ordered vector space.

Recall that if $x : \mathcal{D}(x) \to H$ is a closed, densely defined normal operator, then there exists a uniquely determined spectral measure e^x such that $x = \int_{\mathbb{C}} \lambda de^x$. For $x \in S(\mathcal{A},\tau)$, we will use $\mathcal{B}(\sigma(x))$ to denote the set of all Borel measurable functions on the spectrum of x, $\mathcal{B}_b(\sigma(x))$ to denote the set of all bounded Borel measurable functions on the spectrum of x, and $\mathcal{B}_{bc}(\sigma(x))$ to denote the set of all Borel measurable functions, which are bounded on compact subsets of the spectrum of x. Let $x \in S(\mathcal{A},\tau)^{sa}$ and $f \in \mathcal{B}(\sigma(x))$. Define

$$f(x) := \int_{\sigma(x)} f(\lambda)\, de^x(\lambda).$$

The map $f \mapsto f(x)$ is called the *functional calculus* for x. If G is a subset of $\mathbb{C}$, then we will sometimes use $\mathbb{B}(G)$ to denote the set of all Borel subsets of G.

REMARK 1.3.2. We will sometimes prefer to work with a spectral family rather than with a spectral measure. A family of projections $\{e(\lambda)\}_{\lambda\in\mathbb{R}}$ satisfying

(1) $\underset{\lambda\in\mathbb{R}}{\wedge} e(\lambda) = 0$ and $\underset{\lambda\in\mathbb{R}}{\vee} e(\lambda) = \mathbf{1}$;
(2) $e(\lambda) \leq e(\lambda')$ if $\lambda \leq \lambda'$;
(3) $e(\lambda) = \underset{\lambda'>\lambda}{\wedge} e(\lambda')$

is called a *resolution of the identity* or *spectral family*. If, in addition, there exists a constant $\alpha \geq 0$ such that $e(\lambda) = 0$ for $\lambda < -\alpha$ and $e(\lambda) = \mathbf{1}$ for $\lambda > \alpha$, then $\{e(\lambda)\}_{\lambda\in\mathbb{R}}$ is called a *bounded resolution of the identity*. If $x \in S(\mathcal{A},\tau)$ is a self-adjoint operator with spectral measure e^x, then letting $e(\lambda) = e^x(-\infty,\lambda)$ for $\lambda \in \mathbb{R}$ yields a spectral family. Furthermore, we note that

$$e^{\perp}(\lambda) := \Big(e(\lambda)\Big)^{\perp} = \Big(\underset{\lambda'>\lambda}{\wedge} e(\lambda')\Big)^{\perp} = \underset{\lambda'>\lambda}{\vee} e^{\perp}(\lambda').$$

For $x \in S(\mathcal{A},\tau)^{sa}$, let

$$\begin{aligned} x^+ &= xe^x[0,\infty) \\ x^- &= -xe^x(-\infty,0] \end{aligned}$$

Then $x^+, x^- \in S(\mathcal{A},\tau)^+$ and $x = x^+ - x^-$. The positive cone $S(\mathcal{A},\tau)^+$ is therefore generating in $S(\mathcal{A},\tau)^{sa}$. For $x \in S(\mathcal{A},\tau)$, we define $|x| := (x^*x)^{1/2}$, using the functional calculus. If x is self-adjoint, then $|x| = x^+ + x^-$. It is easily checked that $\mathcal{A}$ is an *absolutely solid subspace* of $S(\mathcal{A},\tau)$, i.e. if $x \in S(\mathcal{A},\tau)$ and $y \in \mathcal{A}$ is such that $|x| \leq |y|$, then $x \in \mathcal{A}$. Furthermore, if $(\mathcal{A},\tau)$ is an abelian von Neumann algebra, then $S(\mathcal{A},\tau)$ is also abelian.

The functional calculus allows us to provide the following characterizations of trace-measurability, which will provide a clearer understanding of this notion.

PROPOSITION 1.3.3. [45, p.16] *Let x be a closed, densely-defined operator affiliated with $\mathcal{A}$ and suppose x has polar decomposition $x = v|x|$. The following are equivalent:*

(1) *x is τ-measurable;*
(2) *$|x|$ is τ-measurable;*
(3) *there exists a projection $p \in \mathcal{P}(\mathcal{A})$ such that $p(H) \subseteq \mathcal{D}(x)$ and $\tau(p^{\perp}) < \infty$;*
(4) *$\forall\delta > 0$ there exists an $\epsilon > 0$ such that $\tau\left(e^{|x|}(\epsilon,\infty)\right) \leq \delta$, where $e^{|x|}$ is spectral measure obtained in the spectral decomposition of $|x|$;*
(5) *There exists a $\lambda > 0$ such that $\tau(e^{|x|}(\lambda,\infty)) < \infty$;*
(6) *$\tau\left(e^{|x|}(\lambda,\infty)\right) \to 0$ as $\lambda \to \infty$.*

It is worth highlighting that a closed, densely defined operator x, affiliated with $\mathcal{A}$, is τ-measurable if and only if there exists $\lambda > 0$ such that $\tau(e^{|x|}(\lambda,\infty)) < \infty$. In general, if $x \in S(\mathcal{A},\tau)$, then there may be certain values of $\lambda > 0$ such that $\tau(e^{|x|}(\lambda,\infty)) = \infty$. The set of all $x \in S(\mathcal{A},\tau)$ such that $\tau(e^{|x|}(\lambda,\infty)) < \infty$ for all $\lambda > 0$ will be denoted $S_c(\mathcal{A},\tau)$ and is called the set of all *compact τ-measurable operators*.

Next, we define a topology on $S(\mathcal{A},\tau)$. For $\epsilon,\delta > 0$, define

$$\mathcal{N}(\epsilon,\delta) := \left\{x \in S(\mathcal{A},\tau) : \exists p \in \mathcal{P}(\mathcal{A}) \text{ such that } p(H) \subseteq \mathcal{D}(x),\ \big\|xp\big\| \leq \epsilon,\ \tau(p^{\perp}) \leq \delta\right\}$$

It can be shown ([45, p.18]) that $\{\mathcal{N}(\epsilon,\delta) : \epsilon,\delta > 0\}$ is a neighbourhood basis at 0 for a metrizable vector topology on $S(\mathcal{A},\tau)$, called the *topology of convergence in measure* or the *measure topology*. Moreover, $S(\mathcal{A},\tau)$ is a topological $*$-algebra with respect to this topology. We will denote this topology by $\mathcal{T}_m$. In the following example we demonstrate what trace-measurable operators look like in two important contexts and also describe the measure topology in these settings.

EXAMPLE 1.3.4. [45, p.22]

(1) If $\mathcal{A} = \mathcal{B}(H)$ and τ is the usual trace on $\mathcal{B}(H)$, then $S(\mathcal{A},\tau) = \mathcal{A}$ and

$$\mathcal{N}(\epsilon,\delta) = \{T \in \mathcal{A} : \|T\| \leq \epsilon\}$$

for any $\epsilon > 0$ and $0 < \delta < 1$, since $\tau(p^{\perp}) < 1$ implies that $p = \mathbf{1}$. It follows that the measure topology coincides with the norm topology. Furthermore, $S_c(\mathcal{A},\tau)$ is the set of all compact operators on H and $\mathcal{F}(\tau)$ is the set of all finite rank operators.

(2) Suppose (Ω,Σ,μ) is a measure space and $H = L_2(\mu)$. If $\mathcal{A} = \{M_f : f \in L_\infty(\mu)\}$ and $\tau(M_f) := \int_\Omega f d\mu$, then it can be shown that $S(\mathcal{A},\tau) = \{M_f : f \in L_{00}(\mu)\}$ and the map $f \mapsto M_f$ is a $*$-isomorphism from $L_{00}(\mu)$ onto $S(\mathcal{A},\tau)$. Furthermore, $M_{f_n} \xrightarrow{\mathcal{T}_m} 0$ if and only if f_n converges to zero in measure, i.e. the measure topology on $S(\mathcal{A},\tau)$ corresponds to the usual topology of convergence in measure, via the $*$-isomorphism $f \mapsto M_f$.

If $(\mathcal{A},\tau)$ is a semi-finite von Neumann algebra such that $\tau(\mathbf{1}) < \infty$, then $(\mathcal{A},\tau)$ will be called a *trace-finite von Neumann algebra*. We list three easily-checked properties of trace-measurable operators associated with trace-finite von Neumann algebras.

PROPOSITION 1.3.5. *Suppose $(\mathcal{A},\tau)$ is a trace-finite von Neumann algebra equipped with a finite normal faithful trace. Then*

(1) *All closed, densely defined operators affiliated with $\mathcal{A}$ are τ-measurable and $S_c(\mathcal{A},\tau) = S(\mathcal{A},\tau)$*
(2) *The closure of $\mathcal{F}(\tau)$ in $S(\mathcal{A},\tau)$ with respect to the measure topology is $S(\mathcal{A},\tau)$*
(3) *If $x \in S(\mathcal{A},\tau)$ and $s(x) = \mathbf{1}$, then x is invertible in $S(\mathcal{A},\tau)$*

Further properties of τ-measurable operators can be found in Section B.2.

1.4. The generalized singular value function

The (generalized) singular value function of a trace-measurable operator is the non-commutative analogue of the decreasing rearrangement of a measurable function. Throughout this section we will use $(\mathcal{A},\tau)$ to denote a semi-finite von Neumann algebra. Let $x \in S(\mathcal{A},\tau)$ and let $|x| = \int_0^\infty \lambda\, de^{|x|}(\lambda)$ be the spectral decomposition of $|x|$. We define the *distribution function* of $|x|$ as

$$d\left(|x|\right)(s) := \tau\left(e^{|x|}(s,\infty)\right) \qquad s \geq 0.$$

Note that this is in fact a generalization of the notion of a distribution as defined in the commutative setting, since if $H = L_2(\Omega,\Sigma,\mu)$ and $\mathcal{A} = \{M_f : f \in L_\infty(\mu)\}$, then $d\left(M_f\right)(s) = \mu\{t \in \Omega : |f(t)| > s\} = d_f(s)$. This follows from the fact that $e^{M_f}(s,\infty) = \chi_{\{t\in\Omega:|f(t)|>s\}}$. For $x \in S(\mathcal{A},\tau)$, the *singular value function* of x is denoted μ_x and is defined to be the right continuous inverse of the spectral distribution function of $|x|$, i.e.

$$\mu_x(t) = \inf\left\{s \geq 0 : d\left(x\right)(s) \leq t\right\} \qquad t \geq 0.$$

This is a generalization of the concept of a decreasing rearrangement of a measurable function, since if $H = L_2(\mu)$, $\mathcal{A} = \{M_f : f \in L_\infty(\mu)\}$ and $g \in L_{00}(\mu)$, then

$$\mu_{M_g}(t) = \inf\left\{s \geq 0 : d\left(M_g\right)(s) \leq t\right\} = \inf\left\{s \geq 0 : d_g(s) \leq t\right\} = g^*(t)$$

The following are some examples of singular functions that can be calculated explicitly.

EXAMPLE 1.4.1. [15]

(1) If p is a projection in $\mathcal{A}$, then $\mu_p = \chi_{[0,\tau(p))}$.

(2) Suppose x is a simple operator of the form $x = \sum_{j=1}^{m} \alpha_j p_j$, where $p_1, ..., p_m \in \mathcal{P}(\mathcal{A})^f$ with $p_j p_k = 0$ for $j \neq k$ and $\alpha_1 > \alpha_2 > ... > \alpha_m > 0$. Let $p_{m+1} = \mathbf{1} - \sum_{j=1}^{m} p_j$, $\alpha_{m+1} = 0$, $\gamma_j = \sum_{i=1}^{j} \tau(p_i)$ and $\gamma_0 = 0$. Then

$$\mu_x = \sum_{j=1}^{m+1} a_j \chi_{[\gamma_{j-1}, \gamma_j)}.$$

(3) The simple operator x, described above, can also be written in the form $x = \sum_{j=1}^{m+1} \beta_j q_j$, where $q_j = \sum_{i=1}^{j} p_i$, $\beta_j = \alpha_j - \alpha_{j+1}$ and $\beta_{m+1} = 0$. Note that $q_1 \leq q_2 \leq ... \leq q_m \leq q_{m+1} = \mathbf{1}$. One can show that

$$\mu_x = \sum_{j=1}^{k} \beta_j \chi_{[0,\tau(q_j))} = \sum_{j=1}^{k} \beta_j \mu_{q_j}.$$

The following describes some of the most important properties of singular value functions to be used in the sequel.

PROPOSITION 1.4.2. [15] *For any $x \in S(\mathcal{A}, \tau)$, the singular value function of x is a positive decreasing right-continuous function. Furthermore, for any $x, y \in S(\mathcal{A}, \tau)$*

(1) $\mu_{\lambda x} = |\lambda| \mu_x$ *for all* $\lambda \in \mathbb{C}$
(2) $\mu_{x^*} = \mu_x$ *and* $\mu_{|x|} = \mu_x$
(3) *if* $|x| \leq |y|$, *then* $\mu_x \leq \mu_y$
(4) $\mu_x = 0$ *if and only* $x = 0$.
(5) $\mu_x(0) < \infty$ *if and only if* $x \in \mathcal{A}$, *in which case* $\mu_x(0) = \|x\|_{\mathcal{A}}$
(6) $\lim_{t \to \infty} \mu_x(t) = 0$ *if and only if* $x \in S_c(\mathcal{A}, \tau)$
(7) $\sigma(x) = \overline{\{\mu_x(t) : t \in [0, \tau(\mathbf{1}))\}}$ *if* $x \in S_c(\mathcal{A}, \tau)^+$

If $x, y \in S(\mathcal{A}, \tau)$, then we will say that x is *submajorized* by y and write $x \ll y$ if

$$\int_0^t \mu_x(s) ds \leq \int_0^t \mu_y(s) ds \qquad \text{for all } t > 0.$$

Note that in the commutative setting $f \ll g$ if and only if

$$\int_0^t f^*(s) ds \leq \int_0^t g^*(s) ds \qquad \forall t > 0.$$

THEOREM 1.4.3. [9, p.217] *If $x, y \in S(\mathcal{A}, \tau)$, then*

(1) $\mu_{x+y} \ll \mu_x + \mu_y$
(2) $\mu_x - \mu_y \ll \mu_{x-y}$

We finish this section by listing two more important properties of singular value functions.

PROPOSITION 1.4.4. [15] *Let $x \in S(\mathcal{A}, \tau)^+$ and let $\phi : [0, \infty) \to [0, \infty)$ be an increasing function which is left-continuous on $(0, \infty)$.*

(1) *If $\tau(\mathbf{1}) = \infty$, then*

$$\mu_{\phi(x)} = \phi \circ \mu_x$$

(2) *If* $\tau(\mathbf{1}) < \infty$, *then*

$$\mu_{\phi(x)} = \phi \circ \mu_x \chi_{[0,\tau(\mathbf{1}))}$$

If in addition $\phi(0) = 0$, *then*

$$\mu_{\phi(x)} = \phi \circ \mu_x$$

PROPOSITION 1.4.5. *Suppose* $x \in S(\mathcal{A}, \tau)$ *and* $\beta > 0$. *Then* $\mu_x = \beta\chi_{[0,\alpha)}$ *if and only if* $|x| = \beta p$ *for some* $p \in \mathcal{P}(\mathcal{A})$ *with* $\tau(p) = \alpha$

PROOF. If $|x| = \beta p$ for some $p \in \mathcal{P}(\mathcal{A})$, then using the properties of the singular value function and Example 1.4.1, we have that

$$\mu_x = \mu_{|x|} = \mu_{\beta p} = \beta\mu_p = \beta\chi_{[0,\tau(p))}.$$

Conversely, if $\mu_x = \beta\chi_{[0,\alpha)}$, then

$$\inf\{s \geq 0 : \tau(e^{|x|}(s,\infty)) \leq t\} = \mu_x(t) = \begin{cases} \beta & \text{if } 0 \leq t < \alpha \\ 0 & \text{if } t > \alpha \end{cases} \tag{1.4.1}$$

Let $0 \leq t < \alpha$. Then $\tau(e^{|x|}(s,\infty)) > t$ for all $s < \beta$ by (1.4.1). It follows that if we fix $s < \beta$, then $\tau(e^{|x|}(s,\infty)) > \alpha - \epsilon$ for all $\epsilon > 0$ and therefore

$$\tau(e^{|x|}(s,\infty)) \quad \geq \quad \alpha \tag{1.4.2}$$

Note further, that for $t > \alpha$, $\inf\{s \geq 0 : \tau(e^{|x|}(s,\infty)) \leq t\} = 0$ by (1.4.1) and so $\tau(e^{|x|}(s,\infty)) \leq t$ for all $s \geq 0$. Since this holds for all $t > \alpha$, we have that $\tau(e^{|x|}(s,\infty)) \leq \alpha$ for all $s \geq 0$. Combining this with (1.4.2), we obtain

$$\tau(e^{|x|}(s,\infty)) \quad = \quad \alpha \qquad \forall s < \beta \tag{1.4.3}$$

It follows that for $s < \beta$, we have

$$\tau(e^{|x|}(0,s]) = \tau(e^{|x|}(0,\infty) - e^{|x|}(s,\infty)) = \tau(e^{|x|}(0,\infty)) - \tau(e^{|x|}(s,\infty)) = \alpha - \alpha = 0,$$

by (1.4.3). Since τ is faithful, this implies that $e^{|x|}(0,s] = 0$ for all $s < \beta$. Furthermore, it is clear that $x \in \mathcal{A}$ and $\|x\|_{\mathcal{A}} = \beta$. Therefore, $e^{|x|}(s,\infty) = 0$ for all $s > \beta$. It follows that $|x|$ has two eigenvalues, namely 0 and β, from which it follows that $|x| = \beta p$ for some $p \in \mathcal{P}(\mathcal{A})$. Therefore $\mu_x = \beta\chi_{[0,\tau(p))}$ and hence $\tau(p) = \alpha$. ◻

1.5. Banach function spaces

In this section we consider spaces of (equivalence classes of) measurable functions. We introduce Banach function spaces and a few related spaces before considering in more detail Lorentz spaces, Orlicz spaces and Orlicz-Lorentz spaces, as examples of such spaces. Throughout this section (Ω, Σ, μ) will denote a localizable measure space. A mapping $\rho : L_0(\mu)^+ \to [0,\infty]$ is called a *function norm* if it satisfies the following properties, for all f, $g \in L_0(\mu)^+$ and for all $\alpha \in \mathbb{F}$,

(1) $\rho(f) = 0$ if and only if $f = 0$
(2) $\rho(|\alpha|f) = |\alpha|\rho(f)$
(3) $\rho(f+g) \leq \rho(f) + \rho(g)$
(4) $g \leq f$ μ-a.e. implies that $\rho(g) \leq \rho(f)$

Let ρ be a function norm on $L_0(\mu)^+$. The *Köthe function space* $E = E_\rho$ is defined to be the set of all $f \in L_0(\mu)$ such that $\|f\|_E := \rho(|f|) < \infty$. It is clear from the definition of a function norm that a Köthe function space is a normed space. Furthermore, if $f \in E$ and $g \in L_0(\mu)$ with $|g| \leq |f|$, then $g \in E$ and $\|g\|_E = \|f\|_E$. A Köthe function space E which is complete with respect to the norm induced by ρ will be called a *Banach function space*. It is worth noting that some authors (eg. [1] and [16]) include the Fatou property in the definition of a (Banach) function norm, i.e. $\rho(f_n) \uparrow \rho(f)$, whenever $(f_n)_{n=1}^{\infty} \cup \{f\} \subseteq L_+^0(\mu)$ is such that $f_n \uparrow f$ pointwise μ-a.e. A Banach function space consisting of (equivalence classes of) measurable functions on the positive real line, equipped with Lebesgue measure, will often be denoted $E(0,\infty)$. We present one important result regarding Banach function spaces.

THEOREM 1.5.1. [1, p.3] *Suppose $E \subseteq L_0(\mu)$ is a Banach function space. If $f_n \xrightarrow{E} f$, then $f_n \to f$ in measure on sets of finite measure (i.e. E is continuously embedded into $L_0(\mu)$, equipped with the topology of local convergence in measure), and hence some subsequence converges pointwise μ-a.e. to f.*

If $E \subseteq L_0(\mu)$ is a Banach function space, then E is called *rearrangement invariant* if$f \in E$, $g \in L_0(\mu)$ and $d_f = d_g$ implies that $g \in E$ and $\|g\|_E = \|f\|_E$. The spaces $L_1 \cap L_\infty(\mu)$ and $L_1 + L_\infty(\mu)$ are important examples of rearrangement invariant Banach function spaces and are defined as the subspaces of $L_0(\mu)$ generated by the following function norms

$$\begin{aligned} \|f\|_{L_1 \cap L_\infty} &:= \max\{\|f\|_1, \|f\|_\infty\} \\ &= \sup_{t>0} \frac{1}{\min\{t,1\}} \int_0^t f^*(s)ds \qquad (1.5.1) \\ \|f\|_{L_1 + L_\infty} &:= \inf\{\|g\|_1 + \|h\|_\infty : f = g+h, g \in L_1(\mu), h \in L_\infty(\mu)\} \\ &= \int_0^1 f^*(s)ds. \qquad (1.5.2) \end{aligned}$$

Since the decreasing rearrangement f^* of a function f is finite-valued almost everywhere if and only if $f \in L_{00}(\mu)$ (see (1.1.1)), it is clear that $L_1 \cap L_\infty(\mu)$ and $L_1 + L_\infty(\mu)$ are contained in $L_{00}(\mu)$ (they are in fact continuously embedded when $L_{00}(\mu)$ is equipped with the topology of convergence in measure). Let $E \subseteq L_{00}(\mu)$ be a Banach function space. If

(1) E is rearrangement invariant,
(2) $L_1 \cap L_\infty(\mu) \subseteq E \subseteq L_1 + L_\infty(\mu)$ with continuous embeddings, and
(3) $f, g \in E$ and $f \ll g$ implies $\|f\|_E \leq \|g\|_E$,

then E is called a *strongly symmetric Banach function space* (also sometimes referred to in the literature as a symmetric space). If, in addition, $f \in E$, whenever $f \in L_{00}(\mu)$ and $f \ll g$ for some $g \in E$, then E is called a *fully symmetric Banach function space*.

The first family of examples of fully symmetric Banach function spaces we will consider is Lorentz spaces. There are numerous types of Lorentz spaces found in the literature (see for example [1] and [3]). We will however focus on the $L_{w,1}$-spaces. Let $w : (0,\infty) \to [0,\infty)$ denote a locally integrable decreasing function satisfying $\lim_{t\to 0} w(t) = \infty$, $\lim_{t\to\infty} w(t) = 0$, $\int_0^\infty w(t)dt = \infty$ and $\int_0^1 w(t)dt = 1$. A function w with these properties will be called a *weight function*. The Lorentz space $L_{w,1}(\mu)$ is given by

$$L_{w,1}(\mu) = \{f \in L_{00}(\mu) : \int_0^{\mu(\Omega)} f^*(t)w(t)dt < \infty\}.$$

Equipped with the norm

$$\|f\|_{w,1} = \int_0^{\mu(\Omega)} f^*(t)w(t)dt$$

$L_{w,1}(\mu)$ is a fully symmetric Banach function space. A closely related family of Lorentz spaces is the Λ_ψ-spaces. Let $\psi : [0, \tau(\mathbf{1})) \to [0, \infty)$ denote an increasing concave function with $\psi(0) = 0$. The space $\Lambda_\psi(\mu)$ and its norm is given by

$$\Lambda_\psi(\mu) = \{f \in L_{00}(\mu) : \int_0^{\mu(\Omega)} f^*(t)d\psi(t) < \infty\}, \qquad \|f\|_{\Lambda_\psi} = \int_0^{\mu(\Omega)} f^*(t)d\psi(t).$$

If w is a weight function and we let $\psi(t) := \int_0^t w(s)ds$, then $\Lambda_\psi(\mu) = L_{w,1}(\mu)$, with equality of norms.

The next family of fully symmetric Banach function spaces we will consider is Orlicz spaces. A function $\phi : [0, \infty) \to [0, \infty]$ is called an *Orlicz (Young) function* if ϕ is convex, $\phi(0) = 0$ and $\lim_{t\to\infty} \phi(t) = \infty$. We assume further that ϕ is neither identically zero nor identically infinite on $(0, \infty)$ and that ϕ is left continuous. Let $a_\phi := \inf\{t > 0 : \phi(t) > 0\}$ and $b_\phi := \sup\{t > 0 : \phi(t) < \infty\}$. We can use an Orlicz function to define a modular I_ϕ in the following way

$$I_\phi(f) := \int_\Omega \phi(|f(t)|)d\mu.$$

If ϕ is an Orlicz function, then the collection of all $f \in L_0(\mu)$ such that $I_\phi(\lambda f) < \infty$ for some $\lambda > 0$ is called an *Orlicz space* and is denoted by $L_\phi(\mu)$. Restricted to $L_\phi(\mu)$, the functional $\|\cdot\|_\phi : L_0(\mu) \to [0, \infty)$ defined by

$$\|f\|_\phi = \inf\{\lambda^{-1} : I_\phi(\lambda f) \le 1\}$$

is a norm, called the *Luxemburg norm*. Next, we consider an important growth parameter on an Orlicz function ϕ. If there exists a $t_0 > 0$ and a $C > 0$ such that $\phi(2t) \le C\phi(t) < \infty$ for all t such that $t_0 \le t < \infty$, then ϕ is said to satisfy the Δ_2-*condition* for large t. If $t_0 = 0$, then ϕ is said to satisfy the Δ_2-condition globally. We will sometimes write $\phi \in \Delta_2$ for large t (respectively globally) if this is the case. We present two important consequences of an Orlicz function satisfying the Δ_2-condition.

PROPOSITION 1.5.2. [1] *Let ϕ be an Orlicz function satisfying the Δ_2-condition globally, then*

(1) $\forall k \ge 1$, $\exists M_k > 0$ *such that* $\phi(kt) \le M_k\phi(t)$ *for all* $t \ge 0$
(2) ϕ *is invertible*

As examples of Orlicz spaces, we note that if $\phi(t) = t^p$ for $t \ge 0$, then ϕ is an Orlicz function, satisfying the Δ_2-condition globally, and $L_\phi(\mu) = L_p(\mu)$, with equality of norms. Furthermore, the Orlicz functions

$$\phi(t) = \begin{cases} t & if\ 0 \le t \le 1 \\ \infty & if\ t > 1 \end{cases} \qquad \text{and} \qquad \phi(t) = \begin{cases} 0 & if\ 0 \le t \le 1 \\ t-1 & if\ t > 1 \end{cases}$$

generate $L_1 \cap L_\infty(\mu)$ and $L_1 + L_\infty(\mu)$, respectively.

We finish this section by mentioning Orlicz-Lorentz spaces, which are a common generalization of Orlicz spaces and Lorentz spaces. The process of defining an Orlicz-Lorentz space is similar to the one employed for Orlicz spaces, except that decreasing rearrangements are used and a weight function is introduced. To define Orlicz-Lorentz spaces, let $\phi : [0, \infty) \to [0, \infty]$ be an Orlicz function, and let $w : [0, \mu(\Omega)) \to (0, \infty)$ be a weight function. Consider the following modular on $L_0(\mu)$:

$$I_{\phi,w}(f) := \int_0^{\mu(\Omega)} \phi(f^*(t))w(t)dt$$

The Orlicz-Lorentz space $L_{\phi,w}(\mu)$ is defined as the set of all $f \in L_0(\mu)$ such that $I_{\phi,w}(\lambda f) < \infty$ for some $\lambda > 0$. It can be shown ([29, p.80]) that the Orlicz-Lorentz space $L_{\phi,w}(\mu)$ is a Banach space when equipped with the norm

$$\|f\|_{\phi,w} := \inf\{\lambda > 0 : I_{\phi,w}(f/\lambda) \leq 1\}$$

The following illustrates how Orlicz spaces and Lorentz spaces may be obtained from Orlicz-Lorentz spaces.

EXAMPLE 1.5.3. [29]

(1) If ϕ is an Orlicz function and $w \equiv 1$, then $L_{\phi,w}(\mu) = L_\phi(\mu)$ with equality of norms.
(2) If $\phi(t) = t$ and w is a weight function, then $L_{\phi,w}(\mu) = L_{w,1}(\mu)$
(3) If $\phi = \begin{cases} 0 & 0 \leq t \leq 1 \\ \infty & t > 1 \end{cases}$ and $w \equiv 1$, then $L_{\phi,w}(\mu) = L_\infty(\mu)$ with equality of norms.

1.6. Symmetric spaces

Symmetric spaces and the related spaces to be introduced in this section are the non-commutative analogues of Banach function spaces. In this section we provide precise definitions for these spaces and mention some general properties. We will also discuss Köthe duality, absolute continuity of the norm, and reduced spaces. Throughout this section, unless stated otherwise, $\mathcal{A} \subseteq \mathcal{B}(H)$ will be a semi-finite von Neumann algebra, equipped with a faithful normal semi-finite trace τ. A linear subspace $E \subseteq S(\mathcal{A},\tau)$ is called an *$\mathcal{A}$-bimodule of τ-measurable operators* if $uxv \in E$ whenever $x \in E$ and $u, v \in \mathcal{A}$. If an $\mathcal{A}$-bimodule is equipped with a norm $\|\cdot\|_E$, satisfying

$$\|uxv\|_E \leq \|u\|_{\mathcal{A}} \|v\|_{\mathcal{A}} \|x\|_E \qquad x \in E, u, v \in \mathcal{A},$$

then E is called a *normed $\mathcal{A}$-bimodule*. If, in addition, E is complete, then E is called a *Banach $\mathcal{A}$-bimodule*. A linear subspace $E \subseteq S(\mathcal{A},\tau)$, equipped with a norm $\|\cdot\|_E$, is called *symmetrically normed* if $x \in S(\mathcal{A},\tau)$, $y \in E$ and $\mu_x \leq \mu_y$ implies that $x \in E$ and $\|x\|_E \leq \|y\|_E$. If, in addition E is a Banach space, then E is termed a *symmetric space* (of τ-measurable operators). It is easily shown that E is a normed $\mathcal{A}$-bimodule if E is a symmetrically normed space. If $E \subseteq S(\mathcal{A},\tau)$ is a symmetrically normed space and $\|x\|_E < \|y\|_E$ whenever $x, y \in E$ is such that $|x| < |y|$, then E is said to have *strictly monotone norm*. A symmetrically normed space $E \subseteq S(\mathcal{A},\tau)$ is called *strongly symmetrically normed* if its norm $\|\cdot\|_E$ has the additional property that $\|x\|_E \leq \|y\|_E$ whenever $x, y \in E$ satisfy $x \ll y$. E is called a *strongly symmetric space* if, in addition, E is a Banach space. A linear subspace $E \subseteq S(\mathcal{A},\tau)$, equipped with a norm $\|\cdot\|_E$, is called *fully symmetrically normed* if it follows from $x \in S(\mathcal{A},\tau)$, $y \in E$ and $x \ll y$ that $x \in E$ and $\|x\|_E \leq \|y\|_E$. If, in addition, E is a Banach space, then E is called a *fully symmetric space*. The next result demonstrates that all the fully symmetric Banach function spaces mentioned in the previous section can be used to construct natural non-commutative analogues.

THEOREM 1.6.1. [9, p.218,219] *Suppose $E(0,\infty)$ is a fully symmetric Banach function space and $(\mathcal{A},\tau)$ is a semi-finite von Neumann algebra. Let*

$$\begin{aligned} E(\tau) &:= \{x \in S(\mathcal{A},\tau) : \mu_x \in E(0,\infty)\} \\ \|x\|_{E(\tau)} &:= \|\mu_x\|_{E(0,\infty)} \end{aligned}$$

Then $(E(\tau), \|\cdot\|_{E(\tau)})$ is a fully symmetric space.

The converse, that every fully symmetric space is derived from a fully symmetric Banach function space on $(0,\infty)$ is in fact also true (see [12, p.951]). It can also be shown (see [27, Theorems 8.7 and 8.11] and [14, Theorem 57]) that if $E(0,\infty)$ is a symmetric Banach function space, then $x \mapsto \|\mu_x\|_E$ is a norm and $E(\tau)$ is a Banach space. It is not yet known if every symmetric space is derived from a symmetric Banach function space on $(0,\infty)$.

If $E \subseteq S(\mathcal{A},\tau)$ is a symmetric space, then E^{sa} will be used to denote the set of all self-adjoint elements of E. E^{sa} is a real linear subspace of E and E is the complexification of E^{sa}, i.e. $E = E^{sa} + iE^{sa}$. E^+ will be used to denote the set of all positive self-adjoint elements of E. It can be shown that E^+ is a proper cone which is closed and generating in E^{sa}. Therefore $(E^{sa}, \|\cdot\|_E)$ is an ordered Banach space.

REMARK 1.6.2. It will often be convenient to restrict attention to fully symmetric spaces associated with non-atomic von Neumann algebras. The following technique, described in [7] and [15], allows one to embed a fully symmetric space associated with a semi-finite von Neumann algebra into a fully symmetric space associated with a non-atomic semi-finite von Neumann algebra. Let $E(0,\infty)$ be a fully symmetric space, let $H = L_2(0,1)$ and let $\mathcal{B} := \{M_f : f \in L_\infty(0,1)\}$ denote the von Neumann subalgebra of $\mathcal{B}(H)$ consisting of multiplication operators on H, equipped with the trace $\nu(M_f) = \int_0^1 f dm$, where m is the Lebesgue measure on $(0,1)$. Forming the tensor product $\mathcal{B} \otimes \mathcal{A}$, equipped with the trace $\tau_\otimes = \nu \otimes \tau$ (i.e. $\tau_\otimes(M_f \otimes x) = \nu(M_f)\tau(x)$ for all simple tensors) yields a non-atomic von Neumann algebra. Let $\mathbb{C}\mathbf{1} = \{\lambda\mathbf{1} : \lambda \in \mathbb{C}\}$, where $\mathbf{1}$ denotes the identity operator on $L_2[0,1]$. Note that $\mathbb{C}\mathbf{1} \otimes \mathcal{A}$ is a von Neumann subalgebra of $\mathcal{B} \otimes \mathcal{A}$. The map

$$\pi : x \mapsto \mathbf{1} \otimes x \qquad x \in \mathcal{A}$$

is a unital trace-preserving $*$-isomorphism from $\mathcal{A}$ onto $\mathbb{C}\mathbf{1} \otimes \mathcal{A}$ and extends uniquely to a $*$-isomorphism $\tilde{\pi}$ from $S(\mathcal{A},\tau)$ onto $S(\mathbb{C}\mathbf{1} \otimes \mathcal{A}, \tau_\otimes)$. Furthermore, if $\tilde{\mu}_{\tilde{\pi}(x)}$ denotes the singular value function of $\tilde{\pi}(x)$ computed with respect to $\tau_\otimes$, then $\tilde{\mu}_{\tilde{\pi}(x)} = \mu_x$ and therefore it can be shown that

$$E(\tau_\otimes) := \{y \in S(\mathbb{C}\mathbf{1} \otimes \mathcal{A}, \tau_\otimes) : \tilde{\mu}_y \in E(0,\infty)\} = \{\tilde{\pi}(x) : x \in S(\mathcal{A},\tau) \text{ and } \mu_x \in E(0,\infty)\}$$

and

$$\|\tilde{\pi}(x)\|_{E(\tau_\otimes)} = \|x\|_{E(\tau)} \qquad \forall x \in E(\tau).$$

The restriction of $\tilde{\pi}$ to $E(\tau)$ is therefore an isometric $*$-preserving linear map from $E(\tau)$ onto $E(\tau_\otimes)$ and is multiplicative in the sense that $\tilde{\pi}(xy) = \tilde{\pi}(x)\tilde{\pi}(y)$, whenever $x, y, xy \in E(\tau)$.

Let $\mathcal{P}(E) := \{p \in \mathcal{P}(\mathcal{A}) : p \in E\}$. If $E \subseteq S(\mathcal{A},\tau)$ is an $\mathcal{A}$-bimodule, then we define the *carrier projection* c_E by $c_E := \vee\{p : p \in \mathcal{P}(E)\}$. It can be shown that c_E is a central projection in $\mathcal{A}$ and

$$x = xc_E = c_E x = c_E x c_E \qquad \forall x \in E. \tag{1.6.1}$$

When the carrier projection is equal to the identity, we have the following important inclusions and embeddings.

PROPOSITION 1.6.3. [15][9] *Suppose $E \subseteq S(\mathcal{A},\tau)$ is a symmetrically normed space with $c_E = \mathbf{1}$. Then*

1. *the embedding of E in $S(\mathcal{A},\tau)$ is continuous with respect to the norm topology in E and the measure topology in $S(\mathcal{A},\tau)$*
2. *$\mathcal{P}(\mathcal{A})^f \subseteq \mathcal{P}(E)$ and hence, $\mathcal{F}(\tau) \subseteq E$*
3. *$L_1 \cap L_\infty(\tau) \subseteq E(\tau) \subseteq L_1 + L_\infty(\tau)$ and these embeddings are continuous*
4. *$\mathcal{A} \subseteq E(\tau) \subseteq L_1(\tau)$, if $(\mathcal{A},\tau)$ is a trace-finite von Neumann algebra*

We will assume throughout the text that $c_E = \mathbf{1}$. As will be shown later (see Remark 1.6.10), this assumption is not too restrictive.

Next, we discuss Köthe duality. Suppose $E \subseteq S(\mathcal{A},\tau)$ is a normed $\mathcal{A}$-bimodule. Let

$$E^\times := \{x \in S(\mathcal{A},\tau) : yx \in L_1(\tau)\, \forall y \in E(\tau)\}$$

It is easily checked that $E^\times$ is a linear subspace of $S(\mathcal{A},\tau)$ and that $E^\times$ has the following properties.

PROPOSITION 1.6.4. [13][9]

(1) *Let $x \in S(\mathcal{A},\tau)$. Then $x \in E^\times \iff |x| \in E^\times \iff x^* \in E^\times$*
(2) *If $x \in E^\times$ and $y \in \mathcal{A}$, then $xy, yx \in E^\times$*
(3) *Let $x \in S(\mathcal{A},\tau)$. Then $x \in E^\times$ if and only if $xy \in L_1(\tau)$ for all $y \in E$*
(4) *If $x \in E^\times$ and $y \in E$, then $\tau(yx) = \tau(xy)$. If in addition $x, y \geq 0$, then $\tau(yx) \geq 0$.*

We can define a norm on $E^\times$ by letting

$$\left\|x\right\|_{E^\times} := \sup\{\tau(|xy|) : y \in E, \left\|y\right\|_E \leq 1\}.$$

It can be shown that if $E(0,\infty)$ is a strongly symmetric Banach function space, then $E(\tau)^\times$ is a fully symmetric space. Furthermore, $E(\tau)^\times = E^\times(\tau)$ in this case (see [9, Theorem 7.4]), where

$$E^\times(0,\infty) := \{f \in L_0(\mu) : \int_0^\infty |fg| dm < \infty \,\forall g \in E(0,\infty)\}$$

is equipped with the norm $\left\|f\right\|_{E^\times(0,\infty)} := \sup\{\int_0^\infty |fg|dm : g \in E(0,\infty), \left\|g\right\|_{E(0,\infty)} \leq 1\}$. We mention briefly the relationship between Köthe duality and Banach duality. Suppose $E \subseteq S(\mathcal{A},\tau)$ is a normed $\mathcal{A}$-bimodule. For $x \in E^\times$, define $\varphi_x^E : E \to \mathbb{C}$ by

$$\varphi_x^E(y) = \tau(yx) \qquad y \in E.$$

It can be shown that $\varphi_x^E \in E^*$, where E^* denotes the Banach dual of E. If it is clear from the context which space we are working in, we will use φ_x to denote φ_x^E. If $x \in E^\times$ and $x \geq 0$, then φ_x is a positive functional (i.e. $\varphi_x(y) \geq 0$ for all $y \in E^+$). Furthermore, the map $\Phi : E^\times \to E^*$, defined by $\Phi(x) = \varphi_x$, is linear and isometric.

In the sequel we will often be interested in the conditions under which the set of all finite linear combinations of projections with finite trace is dense in a particular space. We will see that the desired condition is order continuity or absolute continuity of the norm. We devote the next part of this section to these concepts. If $E \subseteq S(\mathcal{A},\tau)$ is a normed $\mathcal{A}$-bimodule, then the norm $\left\|\cdot\right\|_E$ is called *order continuous* if $\left\|x_\alpha\right\|_E \downarrow 0$ whenever $\{x_\alpha\}$ is a downwards directed net in E^+ satisfying $x_\alpha \downarrow 0$. The set E^{on} is defined by setting

$$E^{on} = \{x \in E : |x| \geq x_\alpha \downarrow 0 \Rightarrow \left\|x_\alpha\right\| \downarrow 0\}$$

It is evident that the norm in E is order continuous if and only if $E = E^{on}$. An element $x \in E$ is said to have *absolutely continuous norm* if $\left\|p_n x p_n\right\|_E \to 0$ for every sequence $(p_n)_{n=1}^\infty$ in $P(\mathcal{A})$ satisfying $p_n \downarrow 0$. The set of all elements of absolutely continuous norm is denoted by E^{an}. It can be shown ([10, Proposition 6.12]) that if E is a strongly symmetrically normed space, then $E^{an} = E^{on}$. Closely related concepts are the Fatou property and the notion of a Fatou norm. The norm $\left\|\cdot\right\|_E$ in a normed $\mathcal{A}$-bimodule $E \subseteq S(\mathcal{A},\tau)$ is called a *Fatou norm* if $\left\|x_\lambda\right\|_E \uparrow \left\|x\right\|_E$, whenever $\{x_\lambda\}_{\lambda\in\Lambda} \cup \{x\} \subseteq E^+$ is such that $x_\lambda \uparrow x$; a normed $\mathcal{A}$-bimodule E is said to have the *Fatou property* if for every upwards directed net $\{x_\lambda\}_{\lambda\in\Lambda} \subseteq E^+$, satisfying $\sup_\lambda \left\|x_\lambda\right\|_E < \infty$, there exists an $x \in E^+$ such that $x_\lambda \uparrow x$ and $\left\|x\right\|_E = \sup_\lambda \left\|x_\lambda\right\|_E$.

REMARK 1.6.5. If $E \subseteq S(\mathcal{A},\tau)$ is a symmetrically normed space with order continuous norm and $\{x\} \cup \{x_\lambda\}_{\lambda\in\Lambda} \subseteq E^+$ is such that $x_\lambda \uparrow x$, then $\{\mu_{x_\lambda}\}_\lambda$ is an increasing net, bounded above by μ_x, by Proposition

1.4.2(3). Since E is symmetrically normed, this implies that $\{\|x_\lambda\|_E\}_\lambda$ is an increasing net, bounded above by $\|x\|_E$. It follows that

$$\|x\|_E \geq \|x_\lambda\|_E = \|x - (x - x_\lambda)\|_E \geq \|x\|_E - \|x - x_\lambda\|_E \to \|x\|_E,$$

since $0 \leq x - x_\lambda \downarrow 0$ and E has order continuous norm. It follows that $\|x_\lambda\|_E \uparrow \|x\|_E$ and therefore any order continuous norm is a Fatou norm.

Note that if $E(0,\infty)$ is a strongly symmetric Banach function space with order continuous norm, then $E(\tau)$ is a strongly symmetric space with order continuous (equivalently absolutely continuous) norm (see [13, Proposition 3.6]). The following result therefore follows from the corresponding result in the commutative setting.

PROPOSITION 1.6.6. [34][1]

(1) *$L_p(\tau)$ has absolutely continuous norm for any $1 \leq p < \infty$.*
(2) *$L_{w,1}(\tau)$ has absolutely continuous norm for any weight function $w : [0,\infty) \to [0,\infty)$.*
(3) *$L_\phi(\tau)$ has absolutely continuous norm if the Orlicz function satisfies the Δ_2-condition globally.*

THEOREM 1.6.7. [15] *Let $E \subseteq S(\mathcal{A},\tau)$ be a strongly symmetrically normed space. If E has order continuous norm (or equivalently absolutely continuous norm), then $\mathcal{F}(\tau)$ is dense in E (and $\mathcal{F}(\tau)^+$ is dense in E^+) . In this case $E \subseteq S_c(\mathcal{A},\tau)$ and so $\lim_{t\to\infty} \mu_x(t) = 0$ for all $x \in E$.*

COROLLARY 1.6.8. *Suppose $E \subseteq S(\mathcal{A},\tau)$ is a strongly symmetric space and let $\mathcal{G}_f$ denote the set of all finite linear combinations of mutually orthogonal projections, each with finite trace. If E has order continuous norm (or equivalently absolutely continuous norm), then*

(1) *$\mathcal{G}_f^+$, is dense in $E(\tau)^+$.*
(2) *$\mathcal{G}_f$ is dense in $E(\tau)$*
(3) *the set of projections in E corresponds to the set of projections in $\mathcal{A}$ with finite trace.*

PROOF. 1) We show that $\mathcal{F}(\tau)^+$ is contained in the closure of $\mathcal{G}_f^+$ in E and therefore that

$$E^+ = \overline{\mathcal{F}(\tau)^+} \subseteq \overline{\overline{\mathcal{G}_f^+}} = \overline{\mathcal{G}_f^+},$$

using Theorem 1.6.7. If $x \in \mathcal{F}(\tau)^+$, then by Remark B.1.12, there exists $(x_n)_{n=1}^\infty \subseteq \mathcal{G}_f^+$ such that $x_n \xrightarrow{\mathcal{A}} x$ and $s(x_n) \leq s(x)$ for all n. By Proposition B.3.3, $x_n \xrightarrow{E} x$ and therefore $x \in \overline{\mathcal{G}_f^+}$. Since elements in $E(\tau)$ and $\mathcal{G}_f$ can be written as linear combinations of positive elements, (2) follows immediately.
3) If $p \in \mathcal{P}(\mathcal{A})^f$, then $p \in E$, by Proposition 1.6.3(2). If $p \in \mathcal{P}(\mathcal{A})$ with $\tau(p) = \infty$, then $\mu_p = \chi_{[0,\infty)}$. It follows that $\mu_p(t) \nrightarrow 0$ as $t \to 0$ and therefore $p \notin E$, by Theorem 1.6.7. □

We finish this section by considering reduced von Neumann algebras and associated symmetric spaces. Let $p \in P(\mathcal{B}(H))$ be a projection onto $K = p(H)$. For $x \in \mathcal{B}(H)$, define $x_p \in \mathcal{B}(K)$ by setting

$$x_p(\eta) = px(\eta) \qquad \forall \eta \in K,$$

i.e. $x_p = (px) \restriction K$. For any non-empty subset $\mathcal{G} \subseteq \mathcal{B}(H)$, denote

$$\mathcal{G}_p := \{x_p : x \in \mathcal{G}\} \qquad \text{and} \qquad p\mathcal{G}p := \{pxp : x \in \mathcal{G}\}.$$

If $\mathcal{A} \subseteq \mathcal{B}(H)$ is a von Neumann algebra and $p \in \mathcal{P}(\mathcal{A})$, then $p\mathcal{A}p$ is a $*$-subalgebra of $\mathcal{A}$ with unit element p. Note that an element $x \in \mathcal{A}$ belongs to $p\mathcal{A}p$ if and only if $x(K) \subseteq K$ and $x(K^\perp) = \{0\}$. Furthermore, if $x \in p\mathcal{A}p$, then $x_p = x \restriction K$. In this situation $\mathcal{A}_p$ is a von Neumann algebra contained in $\mathcal{B}(K)$ and the mapping $\phi_p : p\mathcal{A}p \to \mathcal{A}_p$

defined by $\phi_p(x) = x_p$ is a surjective unital $*$-isomorphism. $\mathcal{A}_p$ is called the reduced von Neumann algebra of $\mathcal{A}$ with respect to $p \in P(\mathcal{A})$. For the remainder of this section we will use $(\mathcal{A}, \tau)$ to denote a semi-finite von Neumann algebra on H. Suppose $p \in \mathcal{P}(\mathcal{A})$. It will sometimes be convenient to identify $\mathcal{A}_p$ with $p\mathcal{A}p$ via the map ϕ_p. With this identification, it is clear that

$$\mathcal{P}(\mathcal{A}_p) = \{q \in \mathcal{P}(\mathcal{A}) : q \leq p\}.$$

Defining $\tau_p : \mathcal{A}_p^+ \to [0, \infty]$ by setting

$$\tau_p(a_p) = \tau(pap) \qquad a \in \mathcal{A}^+,$$

it is easy to verify that τ_p is a semi-finite normal faithful trace. Furthermore, τ_p is a finite trace if and only if $\tau(p) < \infty$.

If $x \in S(\mathcal{A}, \tau)$, then let $x_p := (pxp) \restriction K$, i.e. $\mathcal{D}(x_p) = \mathcal{D}(pxp) \cap K$ and $x_p(\eta) = (px)(\eta)$ for $\eta \in \mathcal{D}(x_p)$. It can be shown that the map $x \mapsto x_p$ is a unital $*$-isomorphism from $pS(\mathcal{A}, \tau)p$ onto $S(\mathcal{A}_p, \tau_p)$ which is a homeomorphism for the measure topology and extends the map $x \mapsto x_p : p\mathcal{A}p \to \mathcal{A}_p$. Moreover, if $x \in S(\mathcal{A}, \tau)^{sa}$, then $e^{x_p}(s, \infty) = e^{pxp}(s, \infty) \restriction_{p(H)}$ for all $s \geq 0$ and therefore $\mu_{x_p} = \mu_{pxp}$, where μ_{x_p} is computed with respect to the reduced von Neumann algebra $\mathcal{A}_p$ and the trace τ_p. It follows that the extended traces τ and τ_p, on $S(\mathcal{A}, \tau)^+$ and $S(\mathcal{A}_p, \tau_p)^+$, respectively, satisfy

$$\tau_p(x_p) = \tau(pxp) \qquad x \in S(\mathcal{A}, \tau)^+.$$

REMARK 1.6.9. If $x \in pS(\mathcal{A}, \tau)^{sa}p$ and $f \in \mathcal{B}_{bc}(\mathbb{R})$, then one can check that $f(x) \in pS(\mathcal{A}, \tau)p$. For $f \in \mathcal{B}_{bc}(\mathbb{R})$, let Γ be the map which takes f to $\phi(f(x))$, where ϕ denotes the $*$-isomorphism from $pS(\mathcal{A}, \tau)p$ onto $S(\mathcal{A}_p, \tau_p)$. It is easily checked that Γ satisfies the conditions of Proposition B.2.6 (with respect to the element $\phi(x) \in S(\mathcal{A}_p, \tau_p)$) and so $\Gamma(f) = f(\phi(x))$ for all $f \in \mathcal{B}_{bc}(\mathbb{R})$, i.e.

$$f(\phi(x)) = \phi(f(x)) \qquad \forall f \in \mathcal{B}_{bc}(\mathbb{R}).$$

Now suppose, in addition, that $E \subseteq S(\mathcal{A}, \tau)$ is an $\mathcal{A}$-bimodule. Define

$$pEp = \{pxp : x \in E\} \qquad E_p = \{x_p : x \in E\}.$$

The map $x \mapsto x_p$ is a $*$-preserving linear isomorphism from pEp onto E_p. This map is also an order isomorphism. Moreover, E_p is an $\mathcal{A}_p$-bimodule of τ_p-measurable operators. If, in addition, E is a normed $\mathcal{A}$-bimodule, and for $x_p \in E_p$ its norm is defined by

$$\|x_p\|_{E_p} = \|pxp\|_E,$$

then E_p is a normed $\mathcal{A}_p$-bimodule and the map $x \mapsto x_p$ is an isometry.

REMARK 1.6.10. Let $E \subseteq S(\mathcal{A}, \tau)$ be a normed $\mathcal{A}$-bimodule. Recall that c_E denotes the central carrier of E. Note that $c_E \in \mathcal{P}(\mathcal{A})$ and so $c_E x c_E \in E$ for every $x \in E$, since E is an $\mathcal{A}$-bimodule. It follows that $c_E E c_E \subseteq E$. If $x \in E$, then $c_E x c_E = x$ by (1.6.1) and so $x \in c_E E c_E$. Therefore $E = c_E E c_E$. It follows that E is isometrically isomorphic to the reduced space E_{c_E}, whose carrier projection is equal to the identity of $\mathcal{A}_{c_E}$. This motivates our decision to assume throughout that $c_E = \mathbf{1}$.

It is easily checked that numerous properties of a normed $\mathcal{A}$-bimodule are transferred to the corresponding reduced space. We highlight some of these.

PROPOSITION 1.6.11. *Let $E \subseteq S(\mathcal{A}, \tau)$ be a normed $\mathcal{A}$-bimodule and $p \in P(\mathcal{A})$.*

(1) *If E is symmetrically normed, strongly symmetrically normed or fully symmetrically normed, then so is E_p*

(2) *If E is complete, then so is E_p*
(3) *If E is a symmetric space, strongly symmetric space or fully symmetric space, then so is E_p*
(4) *If E has order continuous norm or strictly monotone norm, then so does E_p*

Further properties of symmetric spaces to be used in the sequel can be found in Appendix B.3.

1.7. Regular set isomorphisms and σ-homomorphisms

We will see later that characterizations of isometries in the commutative setting typically involve maps induced by regular set isomorphisms. Furthermore, regular set isomorphisms are often induced by measurable transformations, in which case, the map induced by the regular set isomorphism is in fact a composition operator. It is therefore natural to consider the relationships between these types of maps. We will see that it will also be necessary to consider maps induced by σ-homomorphisms. In the present section we will detail some of the properties of regular set isomorphisms, σ-homomorphisms and the maps induced by these.

Since regular set isomorphisms and σ-homomorphisms are defined on equivalence classes of measurable sets, we mention a few details regarding the properties of these equivalence classes. Let (Ω, Σ, μ) be a measure space. Let Σ_0 denote the family of sets of zero measure. For $[A], [B] \in \Sigma/\Sigma_0$, let

$$\begin{aligned} [A] \cup [B] &:= [A \cup B] \\ [A] \cap [B] &:= [A \cap B] \\ [A]^c &:= [A^c] \end{aligned}$$

Note that since Σ is a σ-algebra and Σ_0 is a σ-ideal, $[\Sigma] := \Sigma/\Sigma_0$, equipped with the operations defined above, is a Boolean algebra. We will call a sequence $([A_n])_{n=1}^{\infty} \subseteq [\Sigma]$ pairwise disjoint if $[A_n] \cap [A_m] = [\emptyset]$ whenever $n \neq m$. If $[A] \in [\Sigma]$ and $B, C \in [A]$, then $\mu(B \triangle C) = 0$ and so $\mu(C) = \mu(B)$. Note that $[A] = [B]$ if and only if $\mu(A \triangle B) = 0$ and so if we define $[A] \subseteq [B]$ if and only if $[A] \cap [B] = [A]$, then

$$\begin{aligned} [A] \subseteq [B] &\iff [A \cap B] = [A] \\ &\iff \mu((A \cap B) \triangle A) = 0 \\ &\iff \mu(A \backslash B) = 0 \end{aligned}$$

Furthermore if we define $[A] \backslash [B] := [A] \cap [B]^c$ then $[A] \backslash [B] = [A] \cap [B^c] = [A \cap B^c] = [A \backslash B]$. Regular set isomorphisms are defined on equivalence classes of measurable sets, but to simplify notation, we will generally identify A and its equivalence class $[A]$ and Σ with $[\Sigma]$. Under such circumstances it will be understood that all equalities, relations and statements hold modulo sets of measure zero.

Definition 1.7.1. [16, p.52] Suppose $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ are measure spaces. A map $\eta : \Sigma_1 \to \Sigma_2$ is called a *regular set isomorphism* if

(1) $\eta\left(\bigcup\limits_{n=1}^{\infty} A_n\right) = \bigcup\limits_{n=1}^{\infty} \eta(A_n)$ for any pairwise disjoint sequence $(A_n)_{n=1}^{\infty} \subseteq \Sigma_1$
(2) $\eta(\Omega_1 \backslash A) = \eta(\Omega_1) \backslash \eta(A)$ for all $A \in \Sigma_1$ and
(3) $\mu_2(\eta(A)) = 0$ if and only if $\mu_1(A) = 0$.

The following properties follow from the definition of a regular set isomorphism.

Proposition 1.7.2. [16, p.52] *Let $A, B \in \Sigma_1$ and $\eta : \Sigma_1 \to \Sigma_2$ be a regular set isomorphism. Then*

(1) *If $A \subset B$, then $\eta(A) \subseteq \eta(B)$*
(2) $\eta\left(\bigcup\limits_{n=1}^{\infty} A_n\right) = \bigcup\limits_{n=1}^{\infty} \eta(A_n)$ *for any sequence $(A_n)_{n=1}^{\infty} \subseteq \Sigma_1$*

(3) $\eta\left(\bigcap_{n=1}^{\infty} A_n\right) = \bigcap_{n=1}^{\infty} \eta(A_n)$ *for any sequence* $(A_n)_{n=1}^{\infty} \subseteq \Sigma_1$
(4) $\eta(A) \cap \eta(B) = \emptyset$ *if and only if* $A \cap B = \emptyset$

We wish to show that a regular set isomorphism induces a linear map from $L_0(\mu_1)$ into $L_0(\mu_2)$. In defining this map we will be careful to distinguish between a measurable function f and the equivalence class $[f]$ containing it and between a measurable set A and the equivalence class $[A] \in [\Sigma]$ containing it. Suppose $\eta : [\Sigma_1] \to [\Sigma_2]$ is a regular set isomorphism. For $[f] \in L_0(\mu_1)$, let

$$T_0(f)(t) = s, \qquad \text{if } t \in \cap_{r>s} B_r \setminus \cup_{r<s} B_r,$$

where the r's represent rational numbers and B_r is a representative from $\eta([f^{-1}(-\infty, r]])$. If we let $T_\eta([f]) := [T_0(f)]$, then T_η is a well-defined linear map from $L_0(\mu_1)$ into $L_0(\mu_2)$.

Remark 1.7.3. T_η can also be characterized as the unique linear map from $L_0(\mu_1)$ into $L_0(\mu_2)$ such that $T_\eta([\chi_A]) = [\chi_{\eta([A])}]$ for every $[A] \in [\Sigma_1]$.

In listing the properties of the map induced by a regular set isomorphism we will return to our convention of identifying functions and sets with the equivalence classes containing them.

Proposition 1.7.4. [16, p.52] *For any* $f, g \in L_0(\mu_1)$,

(1) $T_\eta(f_n) \to T_\eta(f)$ *pointwise* μ_2*-a.e. if* $(f_n)_{n=1}^{\infty} \subseteq L_0(\Omega_1, \Sigma_1, \mu_1)$ *is such that* $f_n \to f$ *pointwise* μ_1*-a.e.*
(2) $T_\eta(f.g) = (T_\eta(f)).(T_\eta(g))$ *and* $T_\eta(\overline{f}) = \overline{T_\eta(f)}$
(3) $T_\eta(|f|) = |T_\eta(f)|$
(4) T_η *is positive*
(5) *If* $f.g = 0$*, then* $T_\eta(f)T_\eta(g) = 0$
(6) $T_\eta(f)\chi_B = 0$ *if* $B \subseteq \eta(\Omega_1) \backslash \eta(supp\,(f))$
(7) T_η *is injective*

Proof. Properties (1) - (3) are given in [16]. Properties (4)-(6) are easily checked.
7) If $f = \chi_A$ for some $A \in \Sigma_1$ and $T_\eta(\chi_A) = 0$, then $\chi_{\eta(A)} = T_\eta(\chi_A) = 0$. It follows that $\mu_2(\eta(A)) = 0$ and hence $\mu_1(A) = 0$, since η is a regular set isomorphism. Therefore $\chi_A = 0$. Next, suppose $f = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$ with $A_i \cap A_j = \emptyset$ for $i \neq j$, $\alpha_i \neq 0$ for each i and $T_\eta(f) = 0$. Since $\chi_{A_j}\chi_{A_i} = 0$ if $i \neq j$ and applying (2), we have that

$$\alpha_j T_\eta(\chi_{A_j}) = \sum_{i=1}^{n} \alpha_i T_\eta(\chi_{A_i}\chi_{A_j}) = \sum_{i=1}^{n} \alpha_i (T_\eta(\chi_{A_i}) T_\eta(\chi_{A_j})) = (\sum_{i=1}^{n} \alpha_i T_\eta(\chi_{A_i})) T_\eta(\chi_{A_j}) = T_\eta(f).T_\eta(\chi_{A_j}) = 0.$$

It follows $T_\eta(\chi_{A_j}) = 0$ for each j and hence $\chi_{A_j} = 0$ for each j, by what has been shown already. Therefore $f = 0$. Next, suppose $f \in L_0(\mu_1)^+$ and $T_\eta(f) = 0$. By Theorem 1.1.3, there exists a sequence $(f_n)_{n=1}^{\infty}$ of simple functions such that $0 \leq f_1 \leq f_2 \leq ... \leq f$ and $f_n \to f$ pointwise μ_1-a.e.. Using (4), we have that

$$0 \leq T_\eta(f_n) \leq T_\eta(f) = 0$$

for all $n \in \mathbb{N}^+$ and hence $T_\eta(f_n) = 0$ for each $n \in \mathbb{N}^+$. By what has been shown already (and noting that each f_n could be written in the form $f = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$ with $A_i \cap A_j = \emptyset$ for $i \neq j$), we have that $f_n = 0$ for each $n \in \mathbb{N}^+$. Since $f_n \to f$ pointwise and $f_n = 0$ for each $n \in \mathbb{N}^+$, we have that $f = 0$. Next, suppose $f \in L_0(\mu_1)^{sa}$ and $T_\eta(f) = 0$. f can be written in the form $f = f_+ - f_-$ with $f_+, f_- \geq 0$ and $f_+.f_- = 0$. Using (4), (6) and the linearity of T_η, we have that $T_\eta(f_+), T_\eta(f_-) \geq 0$, $T_\eta(f_+).T_\eta(f_-) = 0$ and $0 = T_\eta(f) = T_\eta(f_+) - T_\eta(f_-)$. It follows that $T_\eta(f_+), T_\eta(f_-) = 0$ and hence by what has been shown before, we have that

$$0 - 0 = f_+ - f_- = f.$$

Finally, if $f \in L_0(\mu_1)$ and $T_\eta(f) = 0$, then using (2), the fact that $f = f_1 + if_2$, with $f_1, f_2 \in L_0(\mu_1)^{sa}$, and a similar argument to the one used for positive functions, we obtain $f = 0$. $\square$

REMARK 1.7.5. It is often possible to show that continuity of the map induced by a regular set isomorphism follows from the positivity of this map. In particular, suppose $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ are measure spaces. If $\eta : \Sigma_1 \to \Sigma_2$ is a regular set isomorphism, then $T_\eta \restriction L_\infty(\mu_1)$ is a continuous map (with respect to the norm topologies) from $L_\infty(\mu_1)$ into $L_\infty(\mu_2)$.

Next, we consider σ-homomorphisms and the maps induced by them. As in the case of regular set isomorphisms, these are in fact mappings of equivalence classes of measurable sets. Following our convention we identify a set and its equivalence class in what follows.

DEFINITION 1.7.6. Suppose $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ are measure spaces. A map $\eta : \Sigma_1 \to \Sigma_2$ is called a *(Boolean) homomorphism* if

(1) $\eta(A \cup B) = \eta(A) \cup \eta(B)$ for every $A, B \in \Sigma_1$
(2) $\eta(A \cap B) = \eta(A) \cap \eta(B)$ for every $A, B \in \Sigma_1$
(3) $\eta(A^c) = \eta(A)^c$ for every $A \in \Sigma_1$

If, in addition, η preserves countable unions, then η is called a *σ-homomorphism.*

We list some properties of (Boolean) homomorphisms.

PROPOSITION 1.7.7. [40, p.15] *Suppose $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ are measure spaces. If $\eta : \Sigma_1 \to \Sigma_2$ is a (Boolean) homomorphism, then*

(1) *$\eta(\emptyset) = \emptyset$ and $\eta(\Omega_1) = \Omega_2$*
(2) *$A \cap B = \emptyset$ implies $\eta(A \cap B) = \emptyset$*
(3) *$\eta(B \backslash A) = \eta(B) \backslash \eta(A)$ for every $A, B \in \Sigma_1$*

The following remark details the relationship between regular set isomorphisms and σ-homomorphisms.

REMARK 1.7.8. Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces. It follows from Proposition 1.7.7 and the definition of a σ-homomorphism, that if $\eta : \Sigma_1 \to \Sigma_2$ is a σ-homomorphism with the property that $\eta(A) = \emptyset$ implies $A = \emptyset$, then η is a regular set isomorphism. Conversely, suppose $\eta : \Sigma_1 \to \Sigma_2$ is a regular set isomorphism. Let $\Omega_3 = \eta(\Omega_1)$, let Σ_3 denote the Σ_2-measurable subsets of Ω_3 and let $\mu_3 = \mu_2 \restriction \Sigma_3$. Note that since $\eta(A) \subseteq \eta(\Omega_1) = \Omega_3$ and $\eta(A) \in \Sigma_2$ for all $A \in \Sigma_2$, we have that $\eta : \Sigma_1 \to \Sigma_3$. By Properties (2) and (3) of Proposition 1.7.2, we have that η preserves countable union and intersection. Furthermore, by definition of a regular set isomorphism, we have that for any $A \in \Sigma_1$,

$$\eta(\Omega_1 \backslash A) = \eta(\Omega_1) \backslash \eta(A) = \Omega_3 \backslash A$$

and hence $\eta : \Sigma_1 \to \Sigma_3$ is a σ-homomorphism.

If (Ω, Σ, μ) is a measure space and $E \subseteq L_0(\mu)$ is a Banach function space, then we will define

$$\mathcal{K}(E) := \{\chi_A : A \in \Sigma,\, \chi_A \in E\}.$$

In the following result we detail some of the properties of a continuous linear map induced by a σ-homomorphism.

PROPOSITION 1.7.9. *For $i = 1, 2$, suppose $(\Omega_i, \Sigma_i, \mu_i)$ is a measure space, $L_{00}(\mu_i)$ has been equipped with the topology of convergence in measure and $E_i \in \{L_p(\mu_i) : p \in [1, \infty]\} \cup \{L_{00}(\mu_i)\}$. If $U : E_1 \to E_2$ is a continuous linear operator and there exist an $\Omega_3 \in \Sigma_2$ and a σ-homomorphism $\eta : \Sigma_1 \to \Sigma_3$ (where Σ_3 is the family of Σ_2-measurable subsets of Ω_3) such that $U(\chi_A) = \chi_{\eta(A)}$ for all $\chi_A \in \mathcal{K}(E_1)$, then*

(1) *U is induced by η;*
(2) *U is positve;*
(3) *$U(|f|) = |U(f)|$ for every $f \in E_1$;*
(4) *$U(f.g) = U(f).U(g)$ whenever f, g and $f.g$ belong to E_1; and*
(5) *$U(\mathcal{K}(E_1)) \subseteq \mathcal{K}(E_2)$.*

PROOF. Under the stated conditions $\mathcal{S}_1 := \text{span}\,(\mathcal{K}(E_1))$ is dense in E_1. In this case we could use η to define a continuous linear map $T_\eta : E_1 \to E_2$ in the following way: for $\chi_A \in \mathcal{K}(E_1)$, let $T_\eta(\chi_A) := \chi_{\eta(A)}$. We could then extend T_η to $\mathcal{S}_1$ by setting

$$T_\eta(\sum_{i=1}^{n} \alpha_1 \chi_{A_i}) = \sum_{i=1}^{n} \alpha_1 \chi_{\eta(A_i)}, \qquad \sum_{i=1}^{n} \alpha_1 \chi_{A_i} \in \mathcal{S}_1.$$

In general one would have to check that T_η is well-defined and linear, but in this case it follows automatically from the fact that T_η defined in this way agrees with U on $\mathcal{S}_1$. For $f \in E_1$, we can find a sequence $(f_n)_{n=1}^{\infty} \subseteq \mathcal{S}_1$ such that $f_n \xrightarrow{E_1} f$. We could then define

$$T_\eta(f) = \lim_{n \to \infty} T_\eta(f_n).$$

In general one would have to check that this limit exists, that T_η is well-defined and that T_η is continuous, but once again these follow from the fact that T_η defined in this way agrees with U on E_1. The properties of U are easily verified. $\square$

For $i = 1, 2$, suppose $(\Omega_i, \Sigma_i, \mu_i)$ is a σ-finite measure space and $E_i \subseteq L_0(\mu_i)$ is a Banach function space. Let $A \in \Sigma_2$. A map $\sigma : A \to \Omega_1$ is called a *measurable transformation* if $\sigma^{-1}(B) \in \Sigma_2$ for all $B \in \Sigma_1$. A continuous linear operator $T : E_1 \to E_2$ is called a *(generalized) composition operator* if T is of the form

$$T(f)(t) = \begin{cases} f \circ \sigma(t) & if\ t \in A \\ 0 & if\ t \in \Omega_2 \backslash A \end{cases} \qquad f \in E_1,$$

for some $A \in \Sigma_2$ and some measurable transformation $\sigma : A \to \Omega_1$. If this is the case we will often use C_σ to denote the composition operator T. We wish to describe the relationship between (generalized) composition operators and maps induced by σ-homomorphisms. The key component in this relationship is being able to determine when a σ-homomorphism is induced by a point mapping and is described in Sikorski's Theorem, which holds for absolute Borel spaces. A topological space is said to be an *absolute (standard) Borel space* if it is homeomorphic to a Borel subset of the Hilbert cube. The positive real line and the unit interval are examples. For the statement of this theorem we will distinguish between a set and the equivalence class containing it.

THEOREM 1.7.10 (Sikorski). [40, p.110] *Let Ω_1 be an absolute Borel space, Σ_1 the σ-algebra of all Borel subsets of Ω_1 and Σ_1' a σ-ideal of Σ_1. If Σ_2 is a σ-algebra of subsets of a set Ω_2 and Σ_2' a σ-ideal of Σ_2 then every σ-homomorphism $\eta : \Sigma_1/\Sigma_1' \to \Sigma_2/\Sigma_2'$ is induced by a point mapping, i.e. there exists a map $\sigma : \Omega_2 \to \Omega_1$ such that*

$$\eta([A]) = [\sigma^{-1}(A)] \qquad \forall [A] \in \Sigma_1/\Sigma_1'$$

We finish this section with a characterization of maps induced by σ-homomorphisms.

PROPOSITION 1.7.11. [30, p.436] *For $i = 1, 2$, suppose $(\Omega_i, \Sigma_i, \mu_i)$ is a measure space, $L_{00}(\mu_i)$ has been equipped with the topology of convergence in measure and $E_i \in \{L_p(\mu_i) : p \in [1, \infty]\} \cup \{L_{00}(\mu_i)\}$. If $U : E_1 \to E_2$ is a continuous linear operator, then the following are equivalent*

(1) *U is induced by a σ-homomorphism $\eta : \Sigma_1 \to \Sigma_3$ (where $\Omega_3 \in \Sigma_2$ and Σ_3 is the family of Σ_2-measurable subsets of Ω_3);*

(2) *$U(f.g) = U(f).U(g)$ whenever f, g and $f.g$ belong to $L_{p_1}(\mu_1)$, and*
(3) *$U(\mathcal{K}_1) \subseteq \mathcal{K}_2$.*

If, in addition $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ are standard Borel spaces, then these statements are equivalent to U being a generalized composition operator.

Remark 1.7.12. The previous result is given in [30] for spaces associated with standard Borel spaces, but appropriate modifications to the proof yields the desired result.

1.8. Jordan ∗-homomorphisms

Descriptions of the structures of isometries in the non-commutative setting typically involve Jordan ∗-homomorphisms. In this section we will define Jordan ∗-homomorphisms, describe some of their properties and present a characterization of Jordan ∗-homomorphisms in terms of their action on projections. We will then consider Jordan ∗-isomorphisms, present some of the additional properties they possess and provide a characterization in terms of the order structure of von Neumann algebras. We will also describe the relationship between Jordan ∗-homomorphisms and the linear maps induced by σ-homomorphisms. Finally, some motivations will be presented for viewing Jordan ∗-homomorphisms as non-commutative composition operators and a definition of a non-commutative composition operator will be provided.

Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. A linear mapping $\Phi : \mathcal{A} \to \mathcal{B}$ is called a *Jordan homomorphism* if

$$\Phi(yx + xy) = \Phi(y)\Phi(x) + \Phi(x)\Phi(y) \qquad \text{for all } x, y \in \mathcal{A}.$$

If, in addition, $\Phi(x^*) = \Phi(x)^*$ for all $x \in \mathcal{A}$, then Φ is called a *Jordan ∗-homomorphism*.

Remark 1.8.1. Some authors define a Jordan ∗-homomorphism to be ∗-preserving linear map such that $\Phi(x^2) = \Phi(x)^2$ for all $x \in \mathcal{A}^{sa}$. Noting that $(x + y)^2 = x^2 + xy + yx + y^2$ for all $x, y \in \mathcal{A}$ and recalling that any element $x \in \mathcal{A}$ can be written in the form $x = x_1 + ix_2$, where $x_1, x_2 \in \mathcal{A}^{sa}$, it is easily checked that these definitions are equivalent.

A bijective Jordan (∗-)homomorphism will be called a *Jordan (∗-)isomorphism*. A Jordan ∗-isomorphism is sometimes also called a C^*-isomorphism. In the next proposition we collect together some of the important algebraic properties of Jordan ∗-homomorphisms.

Proposition 1.8.2. [24] [26] *Suppose $\Phi : \mathcal{A} \to \mathcal{B}$ is a Jordan ∗-homomorphism and let $\mathcal{B}_0$ denote the smallest norm-closed subalgebra of $\mathcal{B}$ that contains $\Phi(\mathcal{A})$. The following holds:*

(1) *$\Phi(x^n) = \Phi(x)^n$ for all $x \in \mathcal{A}$ and for all $n \in \mathbb{N}^+$*
(2) *$\Phi(xyx) = \Phi(x)\Phi(y)\Phi(x)$ for all $x, y \in \mathcal{A}$*
(3) *if $x \in \mathcal{A}^+$, then $\Phi(x) \geq 0$, i.e. Φ is positive*
(4) *if $x, y \in \mathcal{A}$ and $xy = yx$, then $\Phi(x)\Phi(y) = \Phi(y)\Phi(x)$ and $\Phi(xy) = \Phi(x)\Phi(y)$*
(5) *if p is a projection in $\mathcal{A}$, then $\Phi(p)$ is a projection in $\mathcal{B}_0$*
(6) *if p and q are mutually orthogonal projections in $\mathcal{A}$, then $\Phi(p)$ and $\Phi(q)$ are mutually orthogonal projections in $\mathcal{B}_0$*
(7) *$\Phi(\mathbf{1})$ is the unit element of $\mathcal{B}_0$ and so $\Phi(\mathbf{1})\Phi(x) = \Phi(x) = \Phi(x)\Phi(\mathbf{1})$ for all $x \in \mathcal{A}$*

We will also be interested in maps with similar properties to Jordan ∗-homomorphisms, but which are not necessarily defined on C^*-algebras.

PROPOSITION 1.8.3. *Suppose $(\mathcal{A},\tau)$ and $(\mathcal{B},\nu)$ are semi-finite von Neumann algebras. If $\Phi:\mathcal{F}(\tau)\to\mathcal{B}$ is a positive linear map such that*

$$\Phi(x^2)=\Phi(x)^2 \qquad \forall x\in\mathcal{F}(\tau)^{sa},$$

then, for all $x,y\in\mathcal{F}(\tau)$,

(1) $\Phi(x^*)=\Phi(x)^*$*;*
(2) $\Phi(xy+yx)=\Phi(x)\Phi(y)+\Phi(y)\Phi(x)$
(3) $\Phi(x^n)=\Phi(x)^n$ *for all* $n\in\mathbb{N}^+$*;*
(4) $\Phi(xyx)=\Phi(x)\Phi(y)\Phi(x)$*;*
(5) $\Phi(p)$ *is a projection for every* $p\in\mathcal{P}(\mathcal{A})$
(6) $\Phi(p)\Phi(q)=0$ *whenever* $p,q\in\mathcal{P}(\mathcal{A})$ *with* $pq=0$.
(7) *if* $x,y\in\mathcal{F}(\tau)$ *and* $xy=yx$, *then* $\Phi(x)\Phi(y)=\Phi(y)\Phi(x)$ *and* $\Phi(xy)=\Phi(x)\Phi(y)$
(8) $\|\Phi(x)\|_{\mathcal{B}}\le\|x\|_{\mathcal{A}}$ *for every* $x\in\mathcal{F}(\tau)^{sa}$

PROOF. It is easily checked that (1) follows from the fact that Φ is linear and positive. A similar argument to the one used in Remark 1.8.1 shows that (2) holds. Properties (3) to (7) are algebraic in nature and can therefore be shown using the same techniques as in the proofs of the corresponding properties in Proposition 1.8.2 (see [26]). To prove (8), let $x\in\mathcal{F}(\tau)^{sa}$. It follows by Proposition B.1.6 that $-\|x\|\mathbf{1}\le x\le\|x\|\mathbf{1}$. Since x is self-adjoint $s(x)xs(x)=x$ and therefore

$$-\|x\|s(x)\le x\le\|x\|s(x),$$

by Proposition B.2.2(4). Since $x\in\mathcal{F}(\tau)^{sa}$, $\tau(s(x))<\infty$ and therefore $s(x)\in\mathcal{F}(\tau)$. Since Φ is positive and linear we therefore have

$$-\|x\|\Phi(s(x))\le\Phi(x)\le\|x\|\Phi(s(x)).$$

By Proposition B.1.7, this implies that $\|\Phi(x)\|\le\|x\|\|\Phi(s(x))\|$. This completes the proof, since $\Phi(s(x))$ is a projection (using (5)) and therefore $\|\Phi(s(x))\|=1$. $\square$

We have the following characterization of Jordan $*$-homomorphisms.

THEOREM 1.8.4. [30, p.442] *Let $\mathcal{A}$ and $\mathcal{B}$ be von Neumann algebras and suppose $\Phi:\mathcal{A}\to\mathcal{B}$ is a continuous linear operator. Then Φ is a Jordan $*$-homomorphism if and only if Φ maps projections onto projections.*

A useful tool for dealing with algebraic calculations involving Jordan $*$-homomorphisms is the fact that any such map is a sum of a homomorphism and an anti-homomorphism.

PROPOSITION 1.8.5. [41, p.444] *If $\Phi:\mathcal{A}\to\mathcal{B}$ is a Jordan $*$-homomorphism, then there exists a central projection $p\in\mathcal{B}$ such that the map Φ_1 defined by $\Phi_1(x)=\Phi(x)p$, for $x\in\mathcal{A}$, is a $*$-homomorphism; the map Φ_2 defined by $\Phi_2(x)=\Phi(x)p^\perp$, for $x\in\mathcal{A}$, is a $*$-anti-homomorphism; and $\Phi=\Phi_1+\Phi_2$.*

Based on the properties of Jordan $*$-homomorphisms described thus far we present some supplementary results that will be used in the sequel.

PROPOSITION 1.8.6. *If $\Phi:\mathcal{A}\to\mathcal{B}$ is a Jordan $*$-homomorphism, then*

(1) $\Phi(|x|)=|\Phi(x)|$ *whenever* $x\in\mathcal{A}$ *is normal;*
(2) $\Phi(v)$ *is a partial isometry whenever* $v\in\mathcal{A}$ *is a partial isometry;*
(3) $\Phi(p)\sim\Phi(q)$ *whenever* $p,q\in\mathcal{P}(\mathcal{A})$ *are such that* $p\sim q$.

PROOF. 1) If $x \in \mathcal{A}^{sa}$, then, using Proposition 1.8.2(1) and the fact that Φ is $*$-preserving, we have

$$|\Phi(x)|^2 = \Phi(x)^*\Phi(x) = \Phi(x^*)\Phi(x) = \Phi(x)^2 = \Phi(x^2) = \Phi(|x|^2) = \Phi(|x|)^2.$$

Furthermore, $\Phi(|x|) \geq 0$, since Φ is positive. Positive square roots are unique; therefore $|\Phi(x)| = \Phi(|x|)$. Next, suppose $x \in \mathcal{A}$ is normal. Then $x = x_1 + ix_2$, where $x_1, x_2 \in \mathcal{A}^{sa}$. Since x is normal, it is easily checked that $x_1x_2 = x_2x_1$. It is easily checked that $|x|^2 = x_1^2 + x_2^2$ and $\Phi(x_1)\Phi(x_2) = \Phi(x_2)\Phi(x_1)$, by Proposition 1.8.2(4). Therefore

$$\Phi(|x|)^2 = \Phi(|x|^2) = \Phi(x_1^2+x_2^2) = \Phi(x_1)^2+\Phi(x_2)^2 = \Big(\Phi(x_1)-i\Phi(x_2)\Big)\Big(\Phi(x_1)+i\Phi(x_2)\Big) = \Phi(x)^*\Phi(x) = |\Phi(x)|^2.$$

Since positive square roots are unique, $\Phi(|x|) = |\Phi(x)|$.
2) Suppose v is a partial isometry. By using the properties of a Jordan $*$-homomorphism it is easily checked that $\Phi(v)^*\Phi(v)$ is self-adjoint. Furthermore, $v^*v = s(v) = r(v^*)$ and so $v^*vv^* = v^*$. Using the properties of a Jordan $*$-homomorphism, we therefore have that

$$\Big(\Phi(v)^*\Phi(v)\Big)^2 = \Big(\Phi(v^*)\Phi(v)\Phi(v^*)\Big)\Phi(v) = \Phi(v^*vv^*)\Phi(v) = \Phi(v^*)\Phi(v) = \Phi(v)^*\Phi(v).$$

It follows that $\Phi(v)^*\Phi(v)$ is a projection and hence $\Phi(v)$ is a partial isometry, by Proposition B.1.28.
3) By Proposition 1.8.5, there exists a central projection $e \in \mathcal{B}$ such that the map Φ_1 defined by $\Phi_1(x) = \Phi(x)e$, for $x \in \mathcal{A}$ is a $*$-homomorphism; the map Φ_2 defined by $\Phi_2(x) = \Phi(x)e^{\perp}$, for $x \in \mathcal{A}$, is a $*$-anti-homomorphism; and $\Phi = \Phi_1 + \Phi_2$. Since $p \sim q$, there exists a partial isometry v such that $v^*v = p$ and $vv^* = q$. Let $w = \Phi(v)e + \Phi(v)^*e^{\perp}$. It is easily checked that $w^*w = \Phi(v^*v) = \Phi(p)$ and $ww^* = \Phi(vv^*) = \Phi(q)$. Since $\Phi(p)$ is a projection, by Proposition 1.8.2(5), it follows that w is a partial isometry, by Proposition B.1.28, and $\Phi(p) \sim \Phi(q)$. □

Next, we consider Jordan $*$-isomorphisms. We start by presenting some of the additional properties that Jordan $*$-isomorphisms possess. One of these properties is normality. A map $\Phi : \mathcal{A}_0 \to \mathcal{B}_0$ between subspaces of C^*-algebras is called *normal* if $\Phi(x_\lambda) \uparrow \Phi(x)$ whenever $\{x_\lambda\}_{\lambda\in\Lambda} \cup \{x\} \subseteq \mathcal{A}_0^{sa}$ is such that $x_\lambda \uparrow x$.

REMARK 1.8.7. If $\Phi : \mathcal{A}_0 \to \mathcal{B}_0$ is a linear map such that $\Phi(x_\lambda) \uparrow \Phi(x)$ whenever $\{x_\lambda\}_{\lambda\in\Lambda} \cup \{x\} \subseteq \mathcal{A}_0^+$ is such that $x_\lambda \uparrow x$, then Φ is normal.

PROPOSITION 1.8.8. [**24**, p.778][**26**, p.596] *Suppose $\mathcal{A}$ and $\mathcal{B}$ are von Neumann algebras and Φ is a Jordan $*$-isomorphism from $\mathcal{A}$ onto $\mathcal{B}$. Then*

(1) *Φ is an order isomorphism from $\mathcal{A}$ onto $\mathcal{B}$ (i.e. Φ and Φ^{-1} are positive);*
(2) *Φ is an isometry;*
(3) *Φ is unital;*
(4) *Φ is normal;*
(5) *Φ^{-1} is a Jordan $*$-isomorphism.*

PROOF. The proofs of (1) and (2) are given in [**26**, p.596]. (3) follows by Proposition 1.8.2(7), since $\Phi(\mathcal{A}) = \mathcal{B}$. (4) follows from (1) (see Proposition 4.1.2) and (5) is easily checked. □

By considering the following proposition in the light of Proposition 1.8.8(1 and 3), we obtain a useful characterization of Jordan $*$-isomorphisms.

PROPOSITION 1.8.9. [**26**, p.597] *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. If $\Phi : \mathcal{A} \to \mathcal{B}$ is a surjective linear order isomorphism such that $\Phi(\mathbf{1}) = \mathbf{1}$, then Φ is a Jordan $*$-isomorphism.*

REMARK 1.8.10. It is worth noting that if $\Phi : \mathcal{A} \to \mathcal{B}$ is a linear order isomorphism such that $\Phi(\mathbf{1}) = \mathbf{1}$, but which is not surjective, then Φ need not be Jordan $*$-homomorphism (see [26, p.598]).

In the next part of this section we describe the relationship between Jordan $*$-homomorphisms, maps induced by σ-homomorphisms and generalized composition operators. The following non-commutative analogue of Proposition 1.7.11 plays a significant role in analyzing these relationships.

THEOREM 1.8.11. [30, p.441] *Let* $(\mathcal{A}, \tau)$ *and* $(\mathcal{B}, \nu)$ *be semi-finite von Neumann algebras. Suppose* $E \in \{S(\mathcal{A}, \tau)\} \cup \{L_p(\tau) : 1 \leq p \leq \infty\}$ *and* $F \in \{S(\mathcal{B}, \nu)\} \cup \{L_p(\nu) : 1 \leq p \leq \infty\}$. *If* $\Phi : E \to F$ *is a continuous linear operator, then the following are equivalent:*

(1) Φ *maps projections in* E *onto projections in* F*;*

(2) Φ *is adjoint-preserving and*

$$\Phi(xy + yx) = \Phi(x)\Phi(y) + \Phi(y)\Phi(x)$$

whenever $x, y \in E$ *is such that all second order combinations of* x, y, x^* *and* y^* *also belong to* E*;*

(3) $\Phi(x) \geq 0$ *and* $\Phi(x^2) = \Phi(x)^2$ *for any* $x \in E^+$ *with* $x^2 \in E^+$.

We demonstrate that in the commutative setting a Jordan $*$-homomorphism between von Neumann algebras corresponds to a map induced by a σ-homomorphism and therefore in the special case where the underlying measure spaces are standard Borel spaces, a Jordan $*$-homomorphism is a generalized composition operator.

COROLLARY 1.8.12. *Suppose* $(\Omega_i, \Sigma_i, \mu_i)$ *is a measure space and* $\mathcal{K}_i = \{\chi_A : A \in \Sigma_i\}$ *for* $i = 1, 2$. *Suppose* $\Phi : L_\infty(\mu_1) \to L_\infty(\mu_2)$ *is a continuous linear operator. The following are equivalent*

(1) Φ *is a Jordan* $*$*-homomorphism*

(2) $\Phi(\mathcal{K}_1) \subseteq \mathcal{K}_2$

(3) $\Phi(f.g) = \Phi(f).\Phi(g)$ *for every* $f, g \in L_\infty(\mu_1)$

(4) Φ *is induced by a* σ*-homomorphism* $\eta : [\Sigma_1] \to [\Sigma_3]$ *(where* $\Omega_3 \in \Sigma_2$ *and* Σ_3 *is the family of* Σ_2*-measurable subsets of* Ω_3*)*

If, in addition, $(\Omega_1, \Sigma_1, \mu_1)$ *and* $(\Omega_2, \Sigma_2, \mu_2)$ *are standard Borel spaces, then these statements are equivalent to saying that* Φ *is a (generalized) composition operator.*

PROOF. The equivalence of (1) and (2) follows from Theorem 1.8.11. The equivalence of conditions (2), (3) and (4), and the final claim follows from Proposition 1.7.11. $\square$

Motivated by the fact that in the commutative setting a Jordan $*$-homomorphism is a generalized composition operator (provided the underlying measure space is a standard Borel space), we will sometimes refer to a Jordan $*$-homomorphism as non-commutative composition operator and rather imprecisely to a Jordan $*$-homomorphism multiplied on the left by a unitary operator and/or a positive trace-measurable operator as a weighted non-commutative composition operator. We note that in [31], the notion of a non-commutative composition operator is defined precisely. Before supplying this definition, we first extend the definition of a Jordan $*$-homomorphism and present a result that will play a significant role in the formulation of the definition of a non-commutative composition operator. A linear $*$-preserving map Ψ from $S(\mathcal{A}, \tau)$ into $S(\mathcal{B}, \nu)$ will be called a Jordan $*$-homomorphism if $\Psi(xy + yx) = \Psi(x)\Psi(y) + \Psi(y)\Psi(x)$ for all $x, y \in S(\mathcal{A}, \tau)$.

PROPOSITION 1.8.13. [31, p.1068] *Suppose* $(\mathcal{A}, \tau)$ *and* $(\mathcal{B}, \nu)$ *are semi-finite von Neumann algebras and* $\Phi : \mathcal{A} \to \mathcal{B}$ *is a Jordan* $*$*-homomorphism. Then* Φ *extends uniquely to a continuous (with respect to the measure topologies) Jordan* $*$*-homomorphism* $\Phi' : S(\mathcal{A}, \tau) \to S(\mathcal{B}, \nu)$ *if and only if* $\nu \circ \Phi$ *is* $\epsilon - \delta$ *absolutely continuous* *with respect to* τ *on* $\mathcal{P}(\mathcal{A})$ *(i.e. for any* $\epsilon > 0$ *there exists a* $\delta > 0$ *such that* $\nu(\Phi(p)) < \epsilon$ *whenever* $p \in \mathcal{P}(\mathcal{A})$ *is such that* $\tau(p) < \delta$*).*

Suppose $(\mathcal{A}, \tau)$ and $(\mathcal{B}, \nu)$ are semi-finite von Neumann algebras and $E(\tau) \subseteq S(\mathcal{A}, \tau)$ and $F(\nu) \subseteq S(\mathcal{B}, \nu)$ are symmetric spaces. Let $\Phi : \mathcal{A} \to \mathcal{B}$ be a normal Jordan $*$-homomorphism such that $\nu \circ \Phi$ is $\epsilon - \delta$ absolutely continuous with respect to τ on $\mathcal{P}(\mathcal{A})$. If the unique continuous extension $\Phi' : S(\mathcal{A}, \tau) \to S(\mathcal{B}, \nu)$ maps $E(\tau)$ into $F(\nu)$, then we call the induced linear map $E(\tau) \to F(\nu)$ a *non-commutative composition operator*.

CHAPTER 2

Literature review

We will present some of the results contained in the literature and provide outlines for some of the proofs to illustrate techniques employed in different settings and to show how techniques in the commutative setting have been adapted to develop non-commutative analogues of earlier results. We will see that, broadly speaking, these can be divided into methods utilizing the disjointness-preserving property of isometries; methods involving the characterization of extreme points and, finally, methods involving the characterization of Hermitian operators. If $(\mathcal{A},\tau)$ and $(\mathcal{B},\nu)$ are semi-finite von Neumann algebras and $E \subseteq S(\mathcal{A},\tau)$ and $F \subseteq S(\mathcal{B},\nu)$ are symmetric spaces, then a map $U : E \to F$ is called *disjointness-preserving* if

$$U(p)^*U(q) = 0 = U(p)U(q)^*,$$

whenever $p, q \in \mathcal{P}(E) = \mathcal{P}(\mathcal{A}) \cap E$ are such that $pq = 0$. Other aims of this chapter include illustrating how results obtained in the finite setting can be extended to the σ-finite or semi-finite setting; demonstrating some of the similarities in the methods employed and the results obtained by different authors; and describing the relationship between results and techniques presented in this thesis and those contained in the literature.

2.1. Isometries on function spaces: the commutative setting

Due to its historical significance, we start by presenting the Banach-Stone Theorem for spaces of continuous functions.

THEOREM 2.1.1 (Banach-Stone). [6, p.172] *Suppose X and Y are compact Hausdorff spaces and $C(X)$ and $C(Y)$ denote the sets of continuous, scalar-valued functions on X and Y, respectively. A linear map $T : C(X) \to C(Y)$ is a surjective isometry if and only if there exists a homeomorphism $\sigma : Y \to X$ and a unimodular function $h \in C(Y)$ such that*

$$Tf = h.C_\sigma f \qquad \forall f \in C(X),$$

where C_σ is the composition operator induced by σ (i.e. $C_\sigma(f) = \sigma \circ f$, for all $f \in C(X)$).

For the remainder of this section we will be interested in spaces of (equivalence classes of) measurable functions. Isometries on spaces of measurable functions can typically be characterized in terms of linear maps induced by regular set isomorphisms. Strictly speaking these are defined on equivalence classes of measurable sets. We will, however, identify a measurable set $A \in \Sigma$ with the equivalence class $[A] \in [\Sigma]$ containing it and a function f with the equivalence class $[f]$ containing it. All equalities and statements regarding functions and measurable sets in these instances will therefore be understood to hold modulo sets of zero measure.

The next result we will consider is a characterization of isometries on L_p-spaces. The outline of the proof that will be provided illustrates how being able to show that an isometry is disjointness-preserving can facilitate a description of its structure. The most significant structural feature of L_p-spaces utilized in showing that an L_p-isometry is disjointness-preserving, is the conditions under which equality holds in Clarkson's inequality, as described in the following lemma.

LEMMA 2.1.2. [33, p.461] *If f and g belong to $L_p(\mu)$, then*

$$\begin{aligned} \|f+g\|_p^p + \|f-g\|_p^p &\leq 2\|f\|_p^p + 2\|g\|_p^p \qquad 0 < p \leq 2 \\ \|f+g\|_p^p + \|f-g\|_p^p &\geq 2\|f\|_p^p + 2\|g\|_p^p \qquad p \geq 2 \end{aligned}$$

For $p \neq 2$, equality holds if and only if $f.g = 0$.

REMARK 2.1.3. If $p = 2$ in the above Corollary, then $L_p(\mu)$ is a Hilbert space and so by the parallelogram law ([6, p.8])

$$\|f+g\|_2^2 + \|f-g\|_2^2 = 2\|f\|_2^2 + 2\|g\|_2^2$$

for any $f, g \in L_2(\mu)$, not just for f and g such that $f.g = 0$. Furthermore, if H and K are Hilbert spaces, then a linear map $U : H \to K$ is an isometry if and only if

$$\langle U\xi, U\eta\rangle = \langle \xi, \eta\rangle \qquad \forall \xi, \eta \in H$$

and two Hilbert spaces are isometrically isomorphic if and only if they have the same dimension ([6, p.19,20]).

We now present Lamperti's characterization of isometries between L_p-spaces.

THEOREM 2.1.4 (Lamperti). [33, p.461] [16, p.53] *Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be σ-finite measure spaces and let $U : L_p(\Omega_1, \Sigma_1, \mu_1) \to L_p(\Omega_2, \Sigma_2, \mu_2)$, $(1 \leq p < \infty,\ p \neq 2)$ be a linear isometry. Then there exists a regular set isomorphism $\eta : \Sigma_1 \to \Sigma_2$ and a function $h : \Omega_2 \to \mathbb{F}$ such that for all $f \in L_p(\mu_1)$*

$$U(f) = h.T_\eta(f) \tag{2.1.1}$$

where T_η is the transformation induced by η (see Proposition 1.7.4). Furthermore,

$$\int_{\eta(A)} |h|^p \, d\mu_2 = \int_{\eta(A)} \frac{d(\mu_1 \circ \eta^{-1})}{d\mu_2} \, d\mu_2 = \mu_1(A) \qquad \forall A \in \Sigma_1. \tag{2.1.2}$$

Conversely, if η is a regular set isomorphism and $h : \Omega_2 \to \mathbb{F}$ is such that (2.1.2) holds, then the operator U defined by (2.1.1) is a linear isometry.

PROOF. We start with an outline of the proof of the necessary condition. The result is initially shown under the additional assumption that $\mu_1(\Omega_1) < \infty$. To define the map $\eta : \Sigma_1 \to \Sigma_2$, which one will show is a regular set isomorphism, one lets

$$\eta(A) := \operatorname{supp}(U(\chi_A)).$$

The way in which one shows that U is disjointness-preserving and the role this plays in the process of describing its structure is as follows. Suppose $A, B \in \Sigma_1$ such that $A \cap B = \emptyset$. It follows by Lemma 2.1.2 that

$$\|\chi_A + \chi_B\|^p + \|\chi_A - \chi_B\|^p = 2\|\chi_A\|^p + 2\|\chi_B\|^p$$

and hence

$$\|U(\chi_A) + U(\chi_B)\|^p + \|U(\chi_A) - U(\chi_B)\|^p = 2\|U(\chi_A)\|^p + 2\|U(\chi_B)\|^p$$

since U is linear and an isometry. It follows from Lemma 2.1.2 that

$$(U(\chi_A)).(U(\chi_B)) = 0,$$

i.e. U is disjointness-preserving, and hence

$$\eta(A \cup B) = \eta(A) \cup \eta(B).$$

It is then straightforward to check that η is a regular set isomorphism. The disjointness-preserving property of U therefore plays a crucial role in showing that η is a regular set isomorphism. Having assumed that $\mu(\Omega_1) < \infty$, one can let $h = U(\chi_{\Omega_1}) = U(\mathbf{1})$. Recalling that η induces a linear map T_η with the property that $T_\eta(\chi_A) = \chi_{\eta(A)}$, it can then be shown that for any $A \in \Sigma_1$,

$$U(\chi_A) = h.\chi_{\eta(A)}.$$

Noting that the map $f \mapsto h.T_\eta(f)$ is an isometry, agreeing with U on characteristic functions and hence, by linearity on simple functions, one obtains the desired representation, by using the continuity of U and the fact that the set of simple functions forms a dense subspace of $L_p(\mu_1)$.

The extension to σ-finite measure spaces is achieved as follows. If Ω_1 is σ-finite, then Ω_1 can be written in the form $\Omega_1 = \bigcup_{n=1}^{\infty} \Omega_n'$, where $\mu_1(\Omega_n') < \infty$ for each n and $\Omega_n' \cap \Omega_m' = \emptyset$ if $n \neq m$. Let $\Sigma_n' := \{A \cap \Omega_n' : A \in \Sigma_1\}$. For each n, U induces an isometry from $L_p^{(n)}(\mu_1) := L_p(\Omega_n', \Sigma_n', \mu_1 \restriction \Omega_n')$ into $L_p(\mu_2)$ in the following way: Let $f \in L_p^{(n)}(\mu_1)$ and let $\tilde{f}(t) = \begin{cases} f(t) & t \in \Omega_n' \\ 0 & t \notin \Omega_n' \end{cases}$. Then $(\tilde{f}) \in L_p(\mu_1)$. Define $U_n : L_p^{(n)}(\mu_1) \to L_p(\mu_2)$ by

$$U_n(f) := U(\tilde{f}) \tag{2.1.3}$$

From what has been shown in the case where $\mu(\Omega_1) < \infty$, we therefore have for each $n \in \mathbb{N}^+$, a regular set isomorphism $\eta_n : \Sigma_n' \to \Sigma_2$ and a function h_n defined on Ω_2 such that

$$U_n f = h_n.T_{\eta_n}(f) \qquad f \in L_p^{(n)}(\mu_1) \tag{2.1.4}$$

Define $\eta : \Sigma_1 \to \Sigma_2$ by

$$\eta(A) := \bigcup_{n=1}^{\infty} \eta_n(A \cap \Omega_n') \tag{2.1.5}$$

It is easily checked that η is a regular set isomorphism. Let

$$h := \sum_{n=1}^{\infty} h_n$$

One can show that h is well-defined and that

$$Uf = h.T_\eta f \qquad \text{for any } f \in L_p(\mu_1) \tag{2.1.6}$$

For the converse, we note that if $\eta : \Sigma_1 \to \Sigma_2$ is a regular set isomorphism and $h : \Omega_2 \to \mathbb{F}$ is a function such that (2.1.2) holds, then using (2.1.1) to define a map U, immediately yields U isometric on characteristic functions of sets with finite measure, since in this case

$$\int_{\eta(A)} |h|^p \, d\mu_2 = \int_{\Omega_2} |h.T_\eta(\chi_A)|^p \, d\mu_2 = \|U(\chi_A)\|^p$$

and

$$\mu_1(A) = \|\chi_A\|^p.$$

Noting that $|\sum_{i=1}^{n} \alpha_i \chi_{A_i}|^p = \sum_{i=1}^{n} |\alpha_i|^p \chi_{A_i}$ (provided $A_i \cap A_j = \emptyset$ when $i \neq j$), we can show that U is isometric on the set of simple functions supported on sets of finite measure. If $f \in L_p(\mu_1)$, we can use the Monotone Convergence Theorem and the fact that there exists an increasing sequence $(f_n)_{n=1}^{\infty}$ of positive simple functions (supported on sets of finite measure) such that $f_n \uparrow |f|$ to show that the image of $|f|$ has the same norm as $|f|$ (and hence the same holds for f). $\square$

Next, we consider isometries of Lorentz spaces as an illustration of how extreme point methods may be used in the description of isometries. Recall that if w is a weight function, then we define $\psi(t) = \int_0^t w(s)ds$.

PROPOSITION 2.1.5. [2, p.22] *Let w be a strictly decreasing weight function and let E be the Lorentz spaces $L_{w,1}(0,\infty)$. f is an extreme point of the unit ball B_E of E if and only if $|f| = \frac{1}{\psi(m(A))}\chi_A$ for some $A \subset (0,\infty)$ with $0 < m(A) < \infty$.*

THEOREM 2.1.6 (Carothers). [2, p.22] *If U is a surjective isometry of the space $L_{w,1}[0,1]$, then there exist a ± 1-valued Borel measurable function h and a trace-preserving Borel measurable map $\sigma : [0,1] \to [0,1]$ such that*

$$(Uf)(t) = h(t)(C_\sigma(f))(t) \qquad 0 \leq t \leq 1,$$

where C_σ is the composition operator induced by σ, i.e., $(C_\sigma(f))(t) = f(\sigma(t))$ for all $0 \leq t \leq 1$.

PROOF. Suppose A is a Borel measurable subset of $[0,1]$ and let $\alpha_A := \frac{1}{\psi(m(a))}$. Using the characterization of the extreme points and the fact that surjective isometries preserve extreme points it follows that $|U(\alpha_A \chi_A)| = \alpha_{A'}\chi_{A'}$ for some Borel measurable set $A' \subseteq [0,1]$. Let $\eta(A) := A'$. If A and B are disjoint Borel subsets of $[0,1]$, then a direct calculation using the equation

$$|U(\chi_A) + U(\chi_B)| = |U(\chi_{A\cup B}| = \frac{\psi(A \cup B)}{\psi(\eta(A \cup B))}\chi_{\eta(A\cup B)}$$

and the characterization of the extreme points can be used to show that U is disjointness-preserving. One can then check that η is a regular set isomorphism and that letting $h := U(\chi_{[0,1]}) = U(\mathbf{1})$ yields

$$U(f) = h.T_\eta(f) \qquad \forall f \in L_{w,1}[0,1].$$

Since $[0,1]$ is an absolute Borel measure space, it follows that there exists a Borel point mapping $\sigma : [0,1] \to [0,1]$ such that $\eta(A) = \sigma^{-1}(A)$ for every Borel set $A \subseteq [0,1]$ (see Theorem 1.7.10). Therefore,

$$U(f) = h.C_\sigma(f) \qquad \forall f \in L_{w,1}[0,1].$$

□

REMARK 2.1.7. We note that there are more general results available for isometries on Lorentz spaces (see for example [3]), but we felt that the previous result most clearly illustrates how extreme point methods can be used in the commutative setting and most closely resembles the techniques we will be employing in the sequel.

We finish this section by considering isometries on (arbitrary) rearrangement invariant Banach function spaces as an example of how Hermitian operators can be used in the description of isometries. Suppose X is a Banach space and $[\cdot,\cdot]$ is a semi-inner product on X which is compatible with its norm, i.e. $\|x\|_X = [x,x]^{1/2}$ for every $x \in X$. A bounded operator H on X is called *Hermitian* if $[Hx, x]$ is real for every $x \in X$. Note that there always is a semi-inner product compatible with the norm on any Banach space, but that it might not be unique. One can show the definition does not depend on the choice of such a semi-inner product. The primary value of Hermitian operators in the characterization of isometries is the relationship between Hermitian operators and multiplication operators.

THEOREM 2.1.8. [16, p.121] *Suppose (Ω, Σ, μ) is a purely non-atomic σ-finite measure space and let E be a rearrangement invariant space such that the norm on E is not proportional to the norm on $L_2(\mu)$. H is a bounded Hermitian operator on E, if and only if there exists an $h \in L_\infty(\mu)$ such that $H(f) = h.f = M_h(f)$ for all $f \in E$. In this case, $\|H\| = \|h\|_\infty$.*

THEOREM 2.1.9 (Zaidenberg). [16, p.126] *Suppose $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ are purely non-atomic σ-finite measure spaces and suppose E_1 and E_2 are rearrangement invariant spaces of functions on $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$, respectively. Assume that the norm on E_1 is not proportional to the norm of the space $L_2(\mu_1)$. If U*

is a surjective isometry from E_1 onto E_2, then there exist a measurable function h and a regular set isomorphism η from Σ_1 onto Σ_2 such that

$$U(f) = h.T_\eta(f) \qquad \forall f \in E_1.$$

PROOF. One can show that for any Hermitian operator H on E_1, UHU^{-1} is a Hermitian operator on E_2 and that the association $H \mapsto UHU^{-1}$ is an algebraic isomorphism between the set of Hermitian operators on E_1 and the set of Hermitian operators on E_2. Application of these properties and the characterization given in Theorem 2.1.8 can then be used to show that if $A \in \Sigma_1$, then there exists an $A' \in \Sigma_2$ such that

$$UM_{\chi_A}U^{-1} = M_{\chi_{A'}}.$$

Letting $\eta(A) = A'$ yields the desired regular set isomorphism. In the case where $\mu_1(\Omega_1) < \infty$ one can let $h := U(\chi_{\Omega_1}) = U(\mathbf{1})$ and show that $U(f) = h.T_\eta(f)$ for all $f \in E_1$. In the σ-finite case one can use the fact that there exists an increasing sequence $\{\Omega_{1,n}\}_{n=1}^{\infty}$ of measurable sets, each of finite measure, such that $\Omega = \bigcup_{n=1}^{\infty} \Omega_{1,n}$. Letting $h_n := U(\chi_{\Omega_{1,n}})$, one obtains a pointwise convergent sequence. h can then be defined as this pointwise limit. The definitions of η and T_η are obtained in the same as in the finite case.

$\square$

2.2. Isometries on spaces of measurable operators: the non-commutative setting

In this section we will show that many of the techniques employed in the commutative setting have natural non-commutative analogues. In particular, results concerning isometries on non-commutative Lebesgue, Lorentz and symmetric spaces will be described and brief outlines of their proofs provided in order to illustrate how approaches involving the disjointness-preserving property of an isometry, the characterization of extreme points and the description of Hermitian operators, respectively may be adapted to the non-commutative setting. We start, however, by stating a result that is of historical interest and will be referenced when considering isometries on $L_1 \cap L_\infty$.

THEOREM 2.2.1 (Kadison). [**22**, p.329,330] *Suppose $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras with identity and suppose $U : \mathcal{A} \to \mathcal{B}$ is a linear isomorphism. U is isometric if and only if there exist a Jordan $*$-isomorphism $\Phi : \mathcal{A} \to \mathcal{B}$ and a unitary operator $u = \Phi(\mathbf{1})$ such that*

$$U(x) = u\Phi(x) \qquad \forall x \in \mathcal{A}.$$

REMARK 2.2.2. The previous theorem is originally stated in terms of a C^*-isomorphism. One can however show (see Section 1.8) that the definition of a C^*-isomorphism is equivalent to the definition of a Jordan $*$-isomorphism. Furthermore, Paterson and Sinclair ([**36**]) showed that a similar result holds even if the C^*-algebras are not necessarily unital.

To demonstrate how the disjointness-preserving property of an isometry can be utilized in the non-commutative setting, we consider Yeadon's characterization of isometries on L_p-spaces. The outline of the proof will also demonstrate how an analogous technique to the one employed by Lamperti can be developed in the non-commutative setting. The following non-commutative analogue of the conditions under which equality holds in Clarkson's inequality is the main structural feature of L_p-spaces that will be utilized.

THEOREM 2.2.3. [**17**, p.175] *Suppose $(\mathcal{A}, \tau)$ is a semi-finite von Neumann algebra and $1 \le p < \infty$, $p \neq 2$. If $x, y \in L_p(\tau)$, then equality*

$$\|x + y\|_p^p + \|x - y\|_p^p = 2\|x\|_p^p + 2\|y\|_p^p$$

holds in Clarkson's inequality

$$\begin{aligned} \|x+y\|_p^p + \|x-y\|_p^p &\leq 2\|x\|_p^p + 2\|y\|_p^p \qquad (1 \leq p < 2) \\ \|x+y\|_p^p + \|x-y\|_p^p &\geq 2\|x\|_p^p + 2\|y\|_p^p \qquad (2 < p < \infty) \end{aligned}$$

if and only if $xy^* = 0 = x^*y$.

THEOREM 2.2.4 (Yeadon). [49, p.45] *Suppose* $(\mathcal{A}_i, \tau_i)$ *is a semi-finite von Neumann algebra for* $i = 1, 2$ *and* $1 \leq r < \infty$, $r \neq 2$. *A continuous linear operator* $U : L_r(\tau_1) \to L_r(\tau_2)$ *is an isometry if and only if there exist, uniquely, a partial isometry* $w \in \mathcal{A}_2$, *a positive operator* b *affiliated with* A_2, *and a Jordan* $*$-*isomorphism* Φ *of* $\mathcal{A}_1$ *onto a WOT-closed* $*$-*subalgebra of* $\mathcal{A}_2$ *such that*

(1) $w^*w = \Phi(\mathbf{1}) = s(b)$;
(2) *every spectral projection of* b *commutes with* $\Phi(x)$ *for all* $x \in \mathcal{A}_1$;
(3) $\tau_1(x) = \tau_2(b^r\Phi(x))$ *for all* $x \in \mathcal{A}_1^+$; *and*
(4) $U(x) = wb\Phi(x)$ *for all* $x \in L_r(\tau_1) \cap \mathcal{A}_1$.

PROOF. Suppose $U : L_r(\tau_1) \to L_r(\tau_2)$ is an isometry. One starts by defining a map Φ on projections of finite trace. For $p \in \mathcal{P}(\mathcal{A}_1)^f$, let

$$\Phi(p) = {w_{(p)}}^* w_{(p)},$$

where $U(p) = w_{(p)}b_{(p)}$ and $w_{(p)}$ is the partial isometry and $b_{(p)}$ is the positive operator in the polar decomposition of $U(p)$. It follows that $\Phi(p) = s(U(p))$. As in the commutative setting, the conditions under which equality holds in Clarkson's inequality can be used to show that U is disjointness-preserving. This is the crucial step in showing that the map Φ is additive on orthogonal projections with finite trace. This enables extension of Φ to

$$\mathcal{G}_f^{sa} := \{x = \sum_{i=1}^{n} \lambda_i p_i \in \mathcal{A}_1 : \lambda_i \in \mathbb{R},\, p_i \in \mathcal{P}(\mathcal{A}_1)^f, p_i p_j = 0 \text{ for } i \neq j,\, n \in \mathbb{N}^+\}$$

by letting

$$\Phi(\sum_{i=1}^{n} \lambda_i p_i) := \sum_{i=1}^{n} \lambda_i \Phi(p_i), \qquad \sum_{i=1}^{n} \lambda_i p_i \in \mathcal{G}_f^{sa}.$$

The additivity of Φ ensures that Φ is well-defined and real linear on commuting elements from $\mathcal{G}_f^{sa}$. Furthermore, the additivity of Φ is also significant in showing that Φ is square-preserving and isometric. Since any $x \in \mathcal{F}(\tau_1)^{sa}$ can be written as the limit of a sequence $(x_n)_{n=1}^{\infty}$ of commuting elements from $\mathcal{G}_f^{sa}$ and Φ is isometric, we can define

$$\Phi(x) = \lim_{n \to \infty} \Phi(x_n).$$

It is worth noting that in order to show that this map is real linear one shows that if $x \in \mathcal{F}(\tau_1)^{sa}$ and $p \in \mathcal{P}(\mathcal{A}_1)^f$ with $p \geq s(x)$, then

$$U(x) = U(p)\Phi(x) = w_{(p)}b_{(p)}\Phi(x) \tag{2.2.1}$$

The linearity of U can then be employed to show that Φ is real linear. Φ can then be extended to a complex linear map on $\mathcal{F}(\tau_1)$ in the natural way. In the case where $\tau(\mathbf{1}) < \infty$, $\mathcal{A} = \mathcal{F}(\tau_1)$ and so we can complete the proof by letting $w = w_{\mathbf{1}}$ and $b = b_{\mathbf{1}}$. It follows from (2.2.1) that

$$U(x) = wb\Phi(x)$$

for all $x \in \mathcal{A}$ and one can check that Φ is a Jordan $*$-isomorphism of $\mathcal{A}_1$ onto a weakly closed $*$-subalgebra of $\mathcal{A}_2$ with the desired properties. In the case where $\tau_1(\mathbf{1}) = \infty$, one shows that for any $x \in \mathcal{A}_1$, $\{\Phi(pxp)\}_{p \in \mathcal{D}_1}$

($\mathcal{D}_1 := \mathcal{P}(\mathcal{A}_1)^f$) is strong-operator topology convergent and defines $\Phi(x)$ to be this limit. One can also show that $\{w_{(p)}\}_{p\in\mathcal{D}_1}$ and $\{e_{(p)}(\lambda,\infty)\}_{p\in\mathcal{D}_1}$ ($\lambda \geq 0$) are strong operator topology convergent, where

$$b_{(p)} = \int_0^\infty \lambda de_{(p)}(\lambda)$$

is the spectral decomposition of $b_{(p)}$. We define

$$\begin{aligned} w &= \mathrm{SOT}\lim w_{(p)} \\ e(\lambda,\infty) &= \mathrm{SOT}\lim e_{(p)}(\lambda,\infty) \qquad \lambda \geq 0 \\ b &= \int_0^\infty \lambda de(\lambda) \end{aligned}$$

One can then check that these maps have the desired properties and that

$$U(x) = wb\Phi(x) \qquad \text{for all } x \in L_p(\tau_1) \cap \mathcal{A}_1.$$

For the converse, suppose U is a continuous linear map and Φ, w and b are as in the statement of the theorem. One can use the fact that there exists a central projection $e \in \mathcal{P}(\mathcal{A}_1)$ such that Φ restricted to $\mathcal{A}_1 e$ is a $*$-homomorphism and Φ restricted to $\mathcal{A}_1 e^\perp$ is a $*$-anti-homomorphism to show (using condition (4)) that $|U(x)|^r = b^r\Phi(e|x|^r + e^\perp|x^*|^r)$ and hence (using condition (3)) that

$$\left\|U(x)\right\|_r^r = \tau_2(b^r\Phi(e|x|^r + e^\perp|x^*|^r)) = \tau_1(e|x|^r + e^\perp|x^*|^r) = \tau_1(|x|^r) = \left\|x\right\|_r^r.$$

□

REMARK 2.2.5. The sufficiency part of Yeadon's Theorem shows that if we have a continuous linear map which happens to have the prescribed structure, then it is an isometry. The method employed in the proof of Yeadon's Theorem can be adapted to show that if Φ is a Jordan $*$-isomorphism of $\mathcal{A}_1$ onto a WOT-closed $*$-subalgebra of $\mathcal{A}_2$, w is a partial isometry in $\mathcal{A}_2$, and b is a positive operator affiliated with A_2, such that

(1) $w^*w = s(b)$;
(2) every spectral projection of b commutes with $\Phi(x)$ for all $x \in \mathcal{A}_1$ and
(3) $\tau_1(x) = \tau_2(b^p\Phi(x))$ for all $x \in \mathcal{A}_1^+$,

then defining a map

$$U(x) = wb\Phi(x) \qquad \text{for all } x \in L_p(\tau_1) \cap \mathcal{A}_1 \tag{2.2.2}$$

yields a linear isometry from $L_p(\tau_1) \cap \mathcal{A}_1$ into $L_p(\tau_2)$. Since $L_p(\tau_1) \cap \mathcal{A}_1$ is dense in $L_p(\tau_1)$, U has a unique linear extension to an isometry from $L_p(\tau_1)$ into $L_p(\tau_2)$. In other words, we do not need the prior existence of a continuous linear map $U : L_p(\tau_1) \to L_p(\tau_2)$, nor do we require $w^*w = \Phi(\mathbf{1})$ in order to show that (2.2.2) yields an isometry. In particular if Φ is a trace-preserving Jordan $*$-isomorphism from $\mathcal{A}$ onto $\mathcal{B}$, then using $w = \mathbf{1} = b$, we obtain

$$\left\|\Phi(x)\right\|_{L_p(\tau_2)} = \left\|x\right\|_{L_p(\tau_1)} \qquad \forall x \in \mathcal{A}_1 \cap L_p(\tau_1).$$

REMARK 2.2.6. As an example of how a result in the commutative setting may follow from the corresponding result in the non-commutative setting, we show that Lamperti's Theorem follows from Yeadon's Theorem. Suppose $(\Omega_1,\Sigma_1,\mu_1)$ and $(\Omega_2,\Sigma_2,\mu_2)$ are σ-finite measure spaces, $1 \leq p < \infty$, $p \neq 2$ and $U : L_p(\mu_1) \to L_p(\mu_2)$ is a continuous linear mapping. If U is an isometry, then by Yeadon's Theorem there exist uniquely a Jordan $*$-isomorphism Φ from $L_\infty(\mu_1)$ onto a weakly closed $*$-subalgebra of $L_\infty(\mu_2)$, a partial isometry $w \in L_\infty(\mu_2)$ and a positive function $b \in L_{00}(\mu_2)$ such that

(1) $\bar{w}w = \Phi(\mathbf{1}) = s(b)$ (i.e. $|w(t)| = 1$ for μ_2-a.e $t \in \mathrm{supp}\,(b)$);
(2) $\int_{\Omega_1} f d\mu_1 = \int_{\Omega_2} b^p\Phi(f)d\mu_2$ for all $f \in L_\infty(\mu_1)^+$; and

(3) $U(f) = wb\Phi(f)$ for all $f \in L_p(\mu_1) \cap L_\infty(\mu_1)$.

Φ is a Jordan $*$-isomorphism and so by Corollary 1.8.12, Φ is induced by a σ-homomorphism $\eta : \Sigma_1 \to \Sigma_3$ (where $\Omega_3 \in \Sigma_2$ and Σ_3 is the family of all Σ_2 measurable subsets of Ω_3). Furthermore, if $\mu_2(\eta(A)) = 0$, then $0 = \chi_{\eta(A)} = \Phi(\chi_A)$. Since Φ is injective, it follows that $\chi_A = 0$ and hence $\mu_1(A) = 0$. $\eta : \Sigma_1 \to \Sigma_3 \subseteq \Sigma_2$ is a regular set isomorphism, by Remark 1.7.8 and therefore induces a linear map $T_\eta : L_{00}(\mu_1) \to L_{00}(\mu_2)$, by Proposition 1.7.4. Let $\mathcal{S}$ denote the family of all simple functions supported on sets of finite measure. Since $\Phi : L_\infty(\mu_1) \to L_\infty(\mu_2)$ is induced by η, we have that $\Phi(f) = T_\eta(f)$ for all $f \in \mathcal{S}$ and hence, for any $f \in \mathcal{S}$,

$$\begin{aligned} U(f) &= w.b.\Phi(f) \\ &= w.b.T_\eta(f) \end{aligned}$$

Let $h = w.b$ and suppose $g \in L_p(\mu_1)$. $\mathcal{S}$ is dense in $L_p(\mu_1)$ and so there exists a sequence $(g_n)_{n=1}^\infty \subseteq \mathcal{S}$ such that $g_n \overset{L_p(\mu_1)}{\to} g$. Therefore $U(g_n) \overset{L_p(\mu_2)}{\to} U(g)$, since U is continuous. By repeated application of Theorem 1.5.1, we can find a subsequence $(g_{n_k})_{k=1}^\infty$ such that $g_{n_k} \to g$ pointwise μ_1-a.e. and $U(g_{n_k}) \to U(g)$ pointwise μ_2-a.e. By Proposition 1.7.4, $T_\eta(g_{n_k}) \to T_\eta(g)$ pointwise μ_2-a.e. and therefore

$$U(g_{n_k}) = h.T_\eta(g_{n_k}) \to h.T_\eta(g) \qquad \text{pointwise } \mu_2\text{-a.e.}$$

But $U(g_{n_k}) \to U(g)$ pointwise μ_2-a.e. and hence

$$U(g) = h.T_\eta(g) \qquad \mu_2\text{-a.e.}$$

Furthermore, for $A \in \Sigma_1$,

$$\begin{aligned} \int_{\eta(A)} |h|^p d\mu_2 &= \int_{\eta(A)} |w.b|^p d\mu_2 \\ &= \int_{\eta(A)} b^p d\mu_2 \qquad \text{since } b \geq 0 \text{ and } |w(t)| = 1 \ \mu_2\text{-a.e } t \in \operatorname{supp}(b) \\ &= \int_{\Omega_2} b^p \Phi(\chi_A) d\mu_2 \qquad \text{since } \Phi(\chi_A) = \chi_{\eta(A)} \\ &= \int_{\Omega_1} \chi_A d\mu_1 \qquad \text{since } \chi_A \in L_\infty(\mu_1)^+ \\ &= \mu_1(A) \end{aligned}$$

Conversely, suppose $\eta : \Sigma_1 \to \Sigma_2$ is a regular set isomorphism and h is a function on Ω_2 such that

$$\int_{\eta(A)} |h|^p d\mu_2 = \mu_1(A) \qquad \forall A \in \Sigma_1. \tag{2.2.3}$$

By Proposition 1.7.4, η defines a linear map $T_\eta : L_{00}(\mu_1) \to L_{00}(\mu_2)$. Let $\Phi = T_\eta \restriction L_\infty(\mu_1)$. By Remark 1.7.5, Φ is a continuous map from $L_\infty(\mu_1)$ into $L_\infty(\mu_2)$ and hence a Jordan $*$-homomorphism, by Remark 1.7.8 and Corollary 1.8.12. T_η, and hence Φ, is injective by Proposition 1.7.4 and so Φ is a Jordan $*$-isomorphism from $L_\infty(\mu_1)$ onto a WOT-closed subalgebra of $L_\infty(\mu_2)$. Let $b = |h|$ and $w(t) := \begin{cases} \frac{h(t)}{|h(t)|} & \text{if } h(t) \neq 0 \\ 0 & \text{if } h(t) = 0 \end{cases}$.

Then $b \in L_{00}(\mu)^+$ and $\overline{w(t)}w(t) := \begin{cases} 1 & \text{if } t \in \text{supp}\,(b) \\ 0 & \text{otherwise} \end{cases}$. Furthermore,

$$\begin{aligned} \int_{\Omega_1} \chi_A d\mu_1 &= \mu_1(A) \\ &= \int_{\eta(A)} |h|^p d\mu_2 \qquad \text{by (2.2.3)} \\ &= \int_{\Omega_2} b^p \Phi(\chi_A) d\mu_2 \qquad \text{since } b = |h| \text{ and } \Phi(\chi_A) = T_\eta(\chi_A) = \chi_{\eta(A)} \end{aligned} \tag{2.2.4}$$

If $f = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$, with $\alpha_i > 0$ and $A_i \in \Sigma_i$ for $1 \leq i \leq n$, then

$$\begin{aligned} \int_{\Omega_1} f d\mu_1 &= \sum_{i=1}^{n} \int_{\Omega_1} \chi_{A_i} d\mu_1 \\ &= \sum_{i=1}^{n} \alpha_i \int_{\Omega_2} b^p \Phi(\chi_{A_i}) d\mu_2 \qquad \text{by (2.2.4)} \\ &= \int_{\Omega_2} b^p \Phi(f) d\mu_2 \end{aligned} \tag{2.2.5}$$

Finally, if $f \in L_\infty(\mu_1)^+$, then, by Theorem 1.1.3, there exists an increasing sequence $(f_n)_{n=1}^{\infty}$ of positive simple functions such that $f_n \uparrow f$ pointwise μ_1-a.e. By Proposition 1.7.4, $\Phi(f_n) = T_\eta(f_n) \to T_\eta(f) = \Phi(f)$ pointwise μ_2-a.e. and $(\Phi(f_n))_{n=1}^{\infty}$ is increasing and positive. Since $b^p \geq 0$, we have that $b^p\Phi(f_n) \uparrow b^p\Phi(f)$. Using the Monotone Convergence Theorem, we have that

$$\int_{\Omega_1} f_n d\mu_1 \uparrow \int_{\Omega_1} f d\mu_1 \qquad \text{and}$$
$$\int_{\Omega_2} b^p \Phi(f_n) d\mu_2 \uparrow \int_{\Omega_2} b^p \Phi(f) d\mu_2$$

However, $\int_{\Omega_1} f_n d\mu_1 = \int_{\Omega_2} b^p \Phi(f_n) d\mu_2$ for every $n \in \mathbb{N}^+$ and so

$$\int_{\Omega_1} f d\mu_1 = \int_{\Omega_2} b^p \Phi(f) d\mu_2.$$

For $f \in L_p(\mu_1) \cap L_\infty(\mu_1)$, let

$$T(f) := w.b.\Phi(f)$$

By Yeadon's Theorem (see Remark 2.2.5), T has a unique extension to an isometry (which we will also denote T) from $L_p(\mu_1)$ into $L_p(\mu_2)$. For $f \in L_p(\mu_1)$, let

$$U(f) := h.T_\eta(f).$$

It follows from the definitions of Φ, w and b that U and T agree on $L_p(\mu_1) \cap L_\infty(\mu_1)$. Suppose $f \in L_p(\mu_1)$. Since $L_p(\mu_1) \cap L_\infty(\mu_1)$ is dense in $L_p(\mu_1)$, there exists a sequence $(f_n)_{n=1}^{\infty} \subseteq L_p(\mu_1) \cap L_\infty(\mu_1)$ such that $f_n \overset{L_p(\mu_1)}{\to} f$. T is continuous and so $T(f_n) \overset{L_p(\mu_2)}{\to} T(f)$. By repeated application of Theorem 1.5.1, we can find a subsequence $(f_{n_k})_{k=1}^{\infty}$ of $(f_n)_{n=1}^{\infty}$ such that

$$\begin{aligned} f_{n_k} &\to f \qquad \text{pointwise } \mu_1\text{-a.e. and} \\ T(f_{n_k}) &\to T(f) \qquad \text{pointwise } \mu_2\text{-a.e.} \end{aligned}$$

By Proposition 1.7.4, the former implies that $T_\eta(f_{n_k}) \to T_\eta(f)$ pointwise μ_2-a.e. and hence $h.T_\eta(f_{n_k}) \to h.T_\eta(f) = U(f)$ pointwise μ_2-a.e. However, $h.T_\eta(f_{n_k}) = U(f_{n_k}) = T(f_{n_k})$ for each $k \in \mathbb{N}^+$ and $T(f_{n_k}) \to T(f)$ pointwise μ_2-a.e.Therefore $U(f) = T(f)$ for each $f \in L_p(\mu_1)$ and hence U is an isometry.

To illustrate how extreme point methods can be used in the non-commutative setting we will consider a result characterizing the structure of isometries on a particular type of Lorentz space. The proof of this result uses a description of the structure of a positive surjective isometry between a symmetric space and a fully symmetric space, both associated with finite von Neumann algebras. In the sequel we will provide a partial generalization of this result to the semi-finite setting; we therefore include this result and an outline of its proof.

THEOREM 2.2.7. [4, p.534] *Let $(\mathcal{A}, \tau)$ and $(\mathcal{B}, \nu)$ be trace-finite von Neumann algebras and suppose $E \subseteq S(\mathcal{A}, \tau)$ is a symmetric space and $F \subseteq S(\mathcal{B}, \nu)$ is a fully symmetric space. If $U : E \to F$ is a positive linear bijection from E onto F such that $\|U(x)\|_F = \|x\|_E$ for every self-adjoint $x \in E$, then there exist uniquely a positive operator $a \in S(Z(B), \nu)$ and a Jordan $*$-isomorphism Φ of $\mathcal{A}$ onto $\mathcal{B}$ such that $s(a) = \mathbf{1}$ and*

$$U(x) = a\Phi(x) \qquad \forall x \in \mathcal{A}.$$

PROOF. Let $a = U(\mathbf{1}_{\mathcal{A}})$. By showing that U^{-1} is positive, one can show that $s(a) = \mathbf{1}_{\mathcal{B}}$, which implies that a has a closed, densely defined inverse. Since one is working in the finite setting, this inverse is a ν-measurable operator. One can then define the map

$$\Phi(x) := a^{-1/2}U(x)a^{-1/2} \qquad x \in \mathcal{A}.$$

Using the properties of U and U^{-1} one can show that Φ is a unital linear order isomorphism from $\mathcal{A}$ onto $\mathcal{B}$ and hence a Jordan $*$-isomorphism, by Proposition 1.8.9. The desired representation can then be obtained by showing that a is affiliated with the center of $\mathcal{B}$ and hence that $a^{-1/2}$ commutes with $\Phi(x)$ for every $x \in \mathcal{A}$. $\square$

REMARK 2.2.8. Theorem 2.2.7 is implicitly stated with the added assumption that the traces on the respective von Neumann algebras are normalized. It is easily checked however that the result does not depend on this added assumption.

The other important component in the characterization of isometries on Lorentz spaces (Theorem 2.2.10) is a description of the extreme points of the unit balls of these Lorentz spaces.

THEOREM 2.2.9. [4, p.536] *Let $(\mathcal{A}, \tau)$ be a finite von Neumann algebras with $\tau(\mathbf{1}) = 1$ and suppose $\psi : [0, 1] \to [0, \infty)$ is a strictly concave continuous increasing function with $\psi(0) = 0$. An element $x \in \Lambda_\psi(\tau)$ is an extreme point of the unit ball of $\Lambda_\psi(\tau)$ if and only if $x = \frac{1}{\psi(\tau(|v|))}v$ for some partial isometry $v \in \mathcal{A}$.*

THEOREM 2.2.10 (Chilin et al.). [4, p.534] *Let $(\mathcal{A}, \tau)$ and $(\mathcal{B}, \nu)$ be finite von Neumann algebras with $\tau(\mathbf{1}) = 1 = \nu(\mathbf{1})$ and suppose $\psi : [0, 1] \to [0, \infty)$ is a strictly concave continuous increasing function with $\psi(0) = 0$. A continuous surjective linear mapping $U : \Lambda_\psi(\tau) \to \Lambda_\psi(\nu)$ is an isometry if and only if there exist uniquely a unitary operator $u \in \mathcal{B}$ and a Jordan $*$-isomorphism Φ of $\mathcal{A}$ onto $\mathcal{B}$ such that*

$$\tau(x) = \nu(\Phi(x)) \qquad \forall x \in \mathcal{A}$$

and

$$U(x) = a\Phi(x) \qquad \forall x \in \mathcal{A}.$$

PROOF. The idea of the proof is that one wishes to use U to construct a positive surjective isometry from $\Lambda_\psi(\tau)$ onto $\Lambda_\psi(\nu)$. Theorem 2.2.7 can then be applied to obtain the desired Jordan $*$-isomorphism. The characterization of the extreme points of the unit ball of a Lorentz space plays an important role in constructing the aforementioned positive surjective isometry. Since one can assume, without loss of generality that $\psi(1) = 1$, $\mathbf{1} = \frac{1}{\psi(\tau(\mathbf{1}))}\mathbf{1}$ is an extreme point of the unit ball of $\Lambda_\psi(\tau)$, by Theorem 2.2.9. U is a surjective isometry and therefore preserves extreme points. It follows, using Theorem 2.2.9, that $U(\mathbf{1}) = \frac{1}{\psi(\nu(|a|))}a$ for some partial isometry $a \in \mathcal{B}$. Since $\mathcal{B}$ is a finite von Neumann algebra, there exists a unitary operator $u \in \mathcal{B}$ such that $a = u|a|$. Let

$$T(x) := u^* U(x) \qquad x \in \Lambda_\psi(\tau).$$

It is easily checked that this map is a surjective linear isometry from $\Lambda_\psi(\tau)$ onto $\Lambda_\psi(\nu)$. It can also be checked (although it involves considerably more work) that T is positive and $|a| = \mathbf{1}$. Application of Theorem 2.2.7 yields a Jordan $*$-isomorphism Φ from $\mathcal{A}$ onto $\mathcal{B}$ and a positive operator $b \in S(Z(\mathcal{B}), \nu)$ such that

$$T(x) = b\Phi(x) \qquad \forall x \in \mathcal{A}.$$

In particular,

$$\mathbf{1} = |a| = \frac{1}{\psi(\nu(|a|))}|a| = u^* \left(\frac{1}{\psi(\nu(|a|))} a \right) = u^* U(\mathbf{1}) = T(\mathbf{1}) = b\Phi(\mathbf{1}) = b.$$

Therefore $T(x) = \Phi(x)$ for all $x \in \mathcal{A}$ and hence,

$$U(x) = uT(x) = u\Phi(x) \qquad \forall x \in \mathcal{A}.$$

Furthermore, since $\Phi(p) = T(p)$ for any $p \in \mathcal{P}(\mathcal{A})$ and T is a Lorentz space isometry,

$$\psi(\nu(\Phi(p))) = \left\|\Phi(p)\right\|_{\Lambda_\psi(\nu)} = \left\|p\right\|_{\Lambda_\psi(\tau)} = \psi(\tau(p))$$

and hence Φ is trace-preserving on projections, using the fact that ψ is strictly increasing. This can be used to show that Φ is trace-preserving on all of $\mathcal{A}$.

For the converse, it is shown (using non-commutative interpolation-type techniques) that if Φ is a trace-preserving Jordan $*$-isomorphism, then

$$\left\|\Phi(x)\right\|_{\Lambda_\psi(\nu)} = \left\|x\right\|_{\Lambda_\psi(\tau)} \qquad \forall x \in \mathcal{A}.$$

Since $\left\|v\Phi(x)\right\|_{\Lambda_\psi(\nu)} = \left\|\Phi(x)\right\|_{\Lambda_\psi(\nu)}$, whenever v is a unitary operator, it follows that $U = u\Phi$ is isometric (with respect to the Lorentz space norm) on $\mathcal{A}$. Since $\mathcal{A}$ is dense in $\Lambda_\psi(\tau)$, U is an isometry. $\square$

REMARK 2.2.11. It is worth noting that extreme points appear only to play a role in describing the structure of $U(\mathbf{1})$ and indirectly in showing that $T = u^*U$ is positive. In the sequel (see Theorem 7.2.9) we will prove a semi-finite generalization of this result and, interestingly, extreme points will play a crucial role in showing that the isometry under consideration is disjointness-preserving.

Next, we provide a brief outline of the proof of Sukochev's characterization of surjective isometries of a separable symmetric space as an illustration of how the characterization of Hermitian operators may be used in the description of isometries in the non-commutative setting. The proof will further demonstrate how techniques employed in the commutative setting may be adapted to obtain results in the non-commutative setting.

THEOREM 2.2.12. [42, p.116] *Let H be a Hermitian operator on the separable symmetric space $E(\mathcal{A}, \tau)$. If $E(\mathcal{A}, \tau) \neq L_2(\mathcal{A}, \tau)$, then H^2 is Hermitian if and only if H can be represented as either a left multiplication or a right multiplication by a self-adjoint operator in $\mathcal{A}$.*

REMARK 2.2.13. [42, p.116] If a and b are self-adjoint operators in $\mathcal{A}$ and

$$ax = bx \qquad \forall x \in E(\tau),$$

then $a = b$. In particular, if representation of Hermitian operators as operators of left (right) multiplication is possible, then such a representation is uniquely determined.

THEOREM 2.2.14 (Sukochev). [42, p.116] *Suppose* $(\mathcal{A}, \tau)$ *is an AFD factor of type* II_1 *or* II_∞ *and suppose* $E(0, \infty)$ *is a separable symmetric Banach function space such that the norms on* $E(\tau)$ *and* $L_2(\tau)$ *are not proportional. Then a continuous linear mapping* U *of* $E(\tau)$ *onto itself is an isometry if and only if there exist a unitary operator* $u \in \mathcal{A}$ *and a Jordan* $*$*-automorphism* Φ *of* $\mathcal{A}$ *such that*

$$U(x) = u\Phi(x) \qquad \forall x \in \mathcal{A} \cap E(\tau).$$

PROOF. Suppose U is a surjective isometry on $E(\tau)$. For $a \in \mathcal{A}^{sa}$ and $x \in E(\tau)$, let

$$l_a(x) := ax \qquad \text{and} \qquad r_a(x) := xa.$$

One can show that Ul_aU^{-1} and $(Ul_aU^{-1})^2$ are Hermitian operators for any $a \in \mathcal{A}^{sa}$. It follows, using the characterization of Hermitian operators (Theorem 2.2.12), that Ul_aU^{-1} is a left or a right multiplication by a self-adjoint operator. It can be shown that Ul_aU^{-1} must be either a left multiplication for each $a \in \mathcal{A}^{sa}$ or a right multiplication for each $a \in \mathcal{A}^{sa}$, which allows one to associate with each $a \in \mathcal{A}^{sa}$ a self-adjoint operator $\Phi(a) \in \mathcal{A}$, such that

$$\begin{aligned} Ul_aU^{-1} &= l_{\Phi(a)} \\ &\text{or} \\ Ul_aU^{-1} &= r_{\Phi(a)} \end{aligned}$$

It can be shown that this map $\Phi : \mathcal{A}^{sa} \to \mathcal{A}^{sa}$ is real linear, square-preserving, surjective and $\Phi(\mathbf{1}) = \mathbf{1}$. It follows that the natural complex linear extension of Φ onto $\mathcal{A}$ is a Jordan $*$-automorphism of $\mathcal{A}$, which will also be denoted using Φ. To define the unitary operator that will be used in the representation of U, one starts by showing that if $p \in \mathcal{P}(\mathcal{A})^f$, then $U(p)$ is a partial isometry $v_{(p)}$ with $s(U(p)) = \Phi(p)$. Moreover, if $p, q \in \mathcal{P}(\mathcal{A})^f$ with $pq = 0$, then $\Phi(p)\Phi(q) = 0$ and hence one can show that $v_{(p+q)} = v_{(p)} + v_{(q)}$. Therefore, if $p, q \in \mathcal{P}(\mathcal{A})^f$ with $p \leq q$, then $v_{(p)} = v_{(q)}\Phi(p)$. This enables one to define a unitary operator u as the SOT-limit of the net $\{v_{(p)}\}_{p \in \mathcal{P}(\mathcal{A})^f}$ and to show that

$$U(p) = u\Phi(p) \qquad \forall p \in \mathcal{P}(\mathcal{A})^f.$$

Using the linearity of Φ and U and the density in $E(\tau) \cap \mathcal{A}$ of the set of finite linear combinations of projections with finite trace, one obtains the desired representation.

Conversely, suppose $U(x) = u\Phi(x)$ for all x in $E(\tau) \cap \mathcal{A}$, for some unitary u and Jordan $*$-automorphism Φ. Note that since Φ is a Jordan $*$-automorphism of $\mathcal{A}$, Φ is trace-preserving. A similar interpolation-type argument to the one employed in showing that a trace-preserving Jordan $*$-isomorphism is isometric (on suitable elements) with respect to the Lorentz space norm, can then be used to show that for all x in $E(\tau) \cap \mathcal{A}$,

$$\left\|U(x)\right\|_E = \left\|u\Phi(x)\right\|_E = \left\|\Phi(x)\right\|_E = \left\|x\right\|_E.$$

Since $\mathcal{A} \cap E(\tau)$ is dense in $E(\tau)$, U is an isometry.

□

REMARK 2.2.15. We note that all the symmetric spaces we have considered are associated with trace-finite or semi-finite von Neumann algebras. L_p-spaces and Orlicz spaces associated with general von Neumann algebras have been defined (see [19] and [32]) and in the case of L_p-spaces associated with general von Neumann algebras, their isometries have been characterized (see [21], [38] and [39]). Isometries on such spaces, however, are not considered in this thesis.

We finish this chapter by making a few comments about conditions that are either explicitly or implicitly required in proofs in both the commutative and non-commutative settings. The first comment is regarding surjectivity. We note that surjectivity is not assumed in Lamperti and Yeadon's results whereas it is assumed in all other results mentioned here. The reason for this is that in order to ensure that isometries map extreme points to extreme points (as used in the proofs of Carothers, Kadison and Chilin et al.); or to ensure that UHU^{-1} exists and is a Hermitian operator whenever U is an isometry and H is a Hermitian operator (as in the proofs of Zaidenberg and Sukochev), surjectivity of the isometry is required; whereas showing that an isometry on an L_p-space is disjointness-preserving (as shown by Lamperti and Yeadon) involves only the structure of the L_p-space and the linear and isometric properties of the isometry. The second comment is regarding absolute continuity of the norm. Although not mentioned explicitly, all of the spaces considered in this chapter, except the ones mentioned in Theorem 2.1.9 and Theorem 2.2.7, have absolutely continuous norm. If $(\mathcal{A}, \tau)$ is a semi-finite von Neumann algebra and $E \subseteq S(\mathcal{A}, \tau)$ is a strongly symmetric space with absolutely continuous norm, then the set of finite linear combinations of projections, each with finite trace, is dense in E (in the commutative setting this yields density of the set of simple functions, supported on sets of finite measure). One implication of this is that having a structural description for an isometry U on E, which holds for all elements in $\mathcal{A} \cap E$, completely describes the isometry. Another practical advantage of working with spaces with absolutely continuous norm is that in the process of describing the structure of an isometry on such a space, it often suffices to show that a particular structural representation holds for projections of finite trace (characteristic functions of sets of finite measure in the commutative setting); the remainder will then follow by linearity and density.

CHAPTER 3

Extension procedures

As we saw in the literature review, the procedure involved in using an L_p-isometry to define a Jordan $*$-homomorphism involved defining a map on projections with finite trace and then extending this map in a number of successive steps to the whole von Neumann algebra. Our descriptions of isometries will employ similar methods. In order to facilitate this process in different settings and to avoid repetition as much as possible we develop some of these extension procedures in a general context (i.e. independent of an isometry on a particular type of symmetric space).

Suppose $\mathcal{A}$ and $\mathcal{B}$ are von Neumann algebras and Φ is a map from $\mathcal{P}(\mathcal{A})^f$ into $\mathcal{P}(\mathcal{B})$. Ideally, one would like to know the conditions on Φ which would ensure that it is uniquely extensible to a Jordan $*$-homomorphism from $\mathcal{A}$ into $\mathcal{B}$ (or preferably, in certain instances, from $\mathcal{A}$ onto $\mathcal{B}$). There appears to be several difficulties. The first is in ensuring that the extension to linear combinations of projections is linear (and not just linear on commuting elements). This can be overcome if there exists a linear map $U : \mathcal{A} \to S(\mathcal{B}, \nu)$ with the property that $\Phi(p) = s(U(p))$ for all $p \in \mathcal{P}(\mathcal{A})^f$. The second difficulty is in extending Φ from $\mathcal{F}(\tau)$ to $\mathcal{A}$, which appears to require normality of the map Φ on $\mathcal{F}(\tau)$ to ensure that the extension to $\mathcal{A}$ is linear. We have therefore divided the extension process into several steps. These steps will be described in the first three sections of this chapter. The result of these extension procedures is a Jordan $*$-homomorphism; in the final section we consider sufficient conditions to ensure the surjectivity of such a Jordan $*$-homomorphism.

3.1. Extension from a set of projections to finite linear combinations of these projections

There will be two scenarios where we will extend a map from a set of projections to the set of all self-adjoint finite linear combinations of these projections. The one will be for a map defined on the set of all projections of finite trace and the other will be for a map defined on the set of all projections. The same technique can be employed in both cases and will be demonstrated in the following lemma.

LEMMA 3.1.1. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are von Neumann algebras, $\mathcal{E} \subseteq \mathcal{P}(\mathcal{A})$ is a lattice of projections and $\mathcal{H}$ is the set of all finite linear combinations of mutually orthogonal projections from $\mathcal{E}$. If $\Phi : \mathcal{E} \to \mathcal{P}(\mathcal{B})$ is a map such that $\Phi(p) = 0$ if and only if $p = 0$, and $\Phi(p+q) = \Phi(p) + \Phi(q)$, whenever $p, q \in \mathcal{E}$ with $pq = 0$, then defining*

$$\Phi_1\Big(\sum_{i=1}^{n}\alpha_i p_i\Big) = \sum_{i=1}^{n}\alpha_i \Phi(p_i) \qquad \alpha_i \in \mathbb{R},\ p_i \in \mathcal{E}$$

yields a well-defined map $\Phi_1 : \mathcal{H}^{sa} \to \mathcal{B}$, which extends Φ and is real linear on commuting elements from $\mathcal{H}^{sa}$. Furthermore,

$$\Phi_1(x^2) = \Phi_1(x)^2 \qquad \textit{and} \qquad \left\|\Phi_1(x)\right\|_{\mathcal{B}} = \left\|x\right\|_{\mathcal{A}}$$

for all $x \in \mathcal{H}^{sa}$.

PROOF. We show that Φ_1 is well-defined on $\mathcal{H}^{sa}$. Suppose $x = \sum_{i=1}^{n}\alpha_i p_i$, with $p_i p_j = 0$ if $i \neq j$ and $\alpha_i \neq 0$ for all i. Suppose $x = \sum_{j=1}^{m}\beta_j q_j$ is another representation of x, with $q_i q_j = 0$ if $i \neq j$ and $\beta_j \neq 0$ for all i. Note that if $p, q \in \mathcal{E}$ with $pq = 0$, then $\Phi(p)$ and $\Phi(q)$ are orthogonal, i.e. $\Phi(p)\Phi(q) = 0$, by Proposition B.1.16, since $\Phi(p) + \Phi(q) = \Phi(p+q) \in \mathcal{P}(\mathcal{B})$. Note that $\sum_{i=1}^{k} p_i = \bigvee_{i=1}^{k} p_i$, since this is a sum of mutually orthogonal projections. $\mathcal{E}$ is a lattice and therefore it contains this supremum. Similarly, $\sum_{j=1}^{m} q_j = \bigvee_{j=1}^{m} q_j \in \mathcal{E}$. Furthermore,

$$
\begin{aligned}
\sum_{i=1}^{k} p_i &= \sum_{j=1}^{m} q_j \qquad \text{since } \alpha_i \neq 0 \text{ for all } i \text{ and } \beta_j \neq 0 \text{ for all } j \text{ and therefore} \\
& \qquad \text{these are representations of the support projection of } x \\
\implies \Phi(\sum_{i=1}^{k} p_i) &= \Phi(\sum_{j=1}^{m} q_j) \\
\implies \sum_{i=1}^{k} \Phi(p_i) &= \sum_{j=1}^{m} \Phi(q_j) \qquad \text{since } \Phi \text{ is additive on orthogonal projections} \\
\implies \forall i,\ \Phi(p_i) &= (\sum_{j=1}^{m} \Phi(q_j))\Phi(p_i) \qquad \text{since } p_j p_i = 0 \text{ for } i \neq j \text{ implies } \Phi(p_j)\Phi(p_i) = 0 \text{ for } i \neq j \\
&= \sum_{j=1}^{m} (\Phi(q_j)\Phi(p_i)) \qquad (3.1.1)
\end{aligned}
$$

We can similarly show that for every j,

$$\Phi(q_j) = \Phi(q_j)(\sum_{i=1}^{k} \Phi(p_i)) = \sum_{i=1}^{k} (\Phi(q_j)\Phi(p_i)) \qquad (3.1.2)$$

Furthermore, if $1 \leq i' \leq k$ and $1 \leq j' \leq m$, then $p_i p_{i'} = 0$ if $i \neq i'$ and $q_{j'} q_j = 0$ if $j \neq j'$. It follows that

$$\alpha_{i'} q_{j'} p_{i'} = q_{j'}(\sum_{i=1}^{k} \alpha_i p_i) p_{i'} = q_{j'}(\sum_{j=1}^{m} \beta_j q_j) p_{i'} = \beta_{j'} q_{j'} p_{i'}$$

This implies that $\alpha_{i'} = \beta_{j'}$ or $q_{j'} p_{i'} = 0$. Since this holds for every $1 \leq i' \leq k$, $1 \leq j' \leq m$ and Φ maps orthogonal projections onto orthogonal projections, we have that

$$\text{for every } i, j, \text{ either } \alpha_i = \beta_j \text{ or } \Phi(q_j)\Phi(p_i) = 0 \qquad (3.1.3)$$

Therefore,

$$
\begin{aligned}
\sum_{i=1}^{n} \alpha_i \Phi(p_i) &= \sum_{i=1}^{n} \alpha_i (\sum_{j=1}^{m} \Phi(q_j)\Phi(p_i)) \qquad \text{using (3.1.1)} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \Phi(q_j)\Phi(p_i) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} \beta_j \Phi(q_j)\Phi(p_i) \qquad \text{using (3.1.3)} \\
&= \sum_{j=1}^{m} \sum_{i=1}^{n} \beta_j \Phi(q_j)\Phi(p_i) \\
&= \sum_{j=1}^{m} \beta_j \sum_{i=1}^{n} \Phi(q_j)\Phi(p_i) \\
&= \sum_{j=1}^{m} \beta_j \Phi(q_j) \qquad \text{using (3.1.2)}
\end{aligned}
$$

It follows that Φ_1 is well-defined. Next, we show that Φ_1 is real linear on commuting operators in $\mathcal{H}^{sa}$. Suppose $x, y \in \mathcal{H}^{sa}$ are commuting operators and $\delta, \gamma \in \mathbb{R}$. We can write x in the form $x = \sum_{i=1}^{k} \alpha_i p_i$, where $\alpha_i \neq 0$ for all i and $\alpha_i \neq \alpha_j$ if $i \neq j$. Similarly, y can be written in the form $y = \sum_{j=1}^{m} \beta_j q_j$, where where $\beta_i \neq 0$ for all i and $\beta_i \neq \beta_j$ if $i \neq j$. Put $p = \sum_{i=1}^{k} p_i$ and $q = \sum_{j=1}^{m} q_j$ and define

$$\begin{aligned} r_{i,j} &:= p_i q_j \\ s_i &:= p_i - p_i q \\ t_j &:= q_j - q_j p \\ \lambda_{i,j} &:= \delta\alpha_i + \gamma\beta_j \end{aligned}$$

Note that since $\alpha_i \neq \alpha_j$ if $i \neq j$, the p_i's are spectral projections of x. Similarly, the q_i's are spectral projections of y. Since $xy = yx$, we have that $p_i y = y p_i$ for all i, by Corollary B.2.8. It therefore follows by the same proposition that $p_i q_j = q_j p_i$ for all i, j and so $r_{i,j}, s_i$ and t_j are projections for all i, j. Furthermore, these are mutually orthogonal, since $(p_i)_{i=1}^{k}$ and $(q_j)_{j=1}^{m}$ are families consisting of mutually orthogonal projections. It is easily verified that

$$\delta x + \gamma y = \sum_{i=1}^{k}\sum_{j=1}^{m} \lambda_{i,j} r_{i,j} + \sum_{i=1}^{k} \delta\alpha_i s_i + \sum_{j=1}^{m} \gamma\beta_j t_j \in \mathcal{H}^{sa} \tag{3.1.4}$$

Furthermore,

$$\begin{aligned} p_i &= s_i + \sum_{j=1}^{m} r_{i,j} \\ \Longrightarrow \Phi(p_i) &= \Phi(s_i) + \sum_{j=1}^{m} \Phi(r_{i,j}) \qquad \text{since } \Phi \text{ is additive on orthogonal projections} \end{aligned} \tag{3.1.5}$$

and similarly,

$$\Phi(q_j) = \Phi(t_j) + \sum_{i=1}^{k} \Phi(r_{i,j}) \tag{3.1.6}$$

Therefore,

$$\begin{aligned} \delta\Phi_1(x) + \gamma\Phi_1(y) &= \delta\sum_{i=1}^{k} \alpha_i \Phi(p_i) + \gamma\sum_{j=1}^{m} \beta_j \Phi(q_j) \\ &= \sum_{i=1}^{k} \delta\alpha_i [\Phi(s_i) + \sum_{j=1}^{m} \Phi(r_{i,j})] + \sum_{j=1}^{m} \gamma\beta_j [\Phi(t_j) + \sum_{i=1}^{k} \Phi(r_{i,j})] \\ &\quad \text{using (3.1.5) and (3.1.6)} \\ &= \sum_{i=1}^{k}\sum_{j=1}^{m} [\delta\alpha_i + \gamma\beta_j]\Phi(r_{i,j}) + \sum_{i=1}^{k} \delta\alpha_i \Phi(s_i) + \sum_{j=1}^{m} \gamma\beta_j \Phi(t_j) \\ &\quad \text{by grouping appropriate terms} \\ &= \sum_{i=1}^{k}\sum_{j=1}^{m} \lambda_{i,j} \Phi(r_{i,j}) + \sum_{i=1}^{k} \delta\alpha_i \Phi(s_i) + \sum_{j=1}^{m} \gamma\beta_j \Phi(t_j) \qquad \text{by definition of } \lambda_{i,j} \\ &= \Phi_1(\delta x + \gamma y) \qquad \text{using (3.1.4)} \end{aligned}$$

We show that Φ_1 is square-preserving. Suppose $x = \sum_{i=1}^{n} \alpha_i p_i \in \mathcal{H}^{sa}$. Then

$$\begin{aligned} x^2 &= (\sum_{i=1}^{n} \alpha_i p_i)^2 \\ &= \sum_{i=1}^{n} \alpha_i^2 p_i \qquad \text{since } p_i p_j = 0 \text{ for } i \neq j \end{aligned}$$

Therefore $\Phi_1(x^2)$ is defined and

$$\begin{aligned} \Phi_1(x^2) &= \sum_{i=1}^{n} \alpha_i^2 \Phi(p_i) \\ &= (\sum_{i=1}^{n} \alpha_i \Phi(p_i))^2 \qquad \text{since } \Phi(p_i)\Phi(p_j) = 0 \text{ for } i \neq j \\ &= \Phi_1(x)^2 \end{aligned}$$

Finally, we show that Φ_1 is isometric. Suppose $x = \sum_{i=1}^{n} \alpha_i p_i \in \mathcal{H}^{sa}$. Then

$$\begin{aligned} \|x\|_{\mathcal{A}} &= \|\sum_{i=1}^{n} \alpha_i p_i\|_{\mathcal{A}} \\ &= \max\{|\alpha_i| : i = 1, 2, ..., n\} \qquad \text{since the } p_i\text{'s are mutually orthogonal projections} \\ &= \|\sum_{i=1}^{n} \alpha_i \Phi(p_i)\|_{\mathcal{B}} \qquad \text{since the } \Phi(p_i)\text{'s are mutually orthogonal projections and} \\ &\quad \Phi(p_i) = 0 \text{ iff } p_i = 0 \\ &= \|\Phi_1(x)\|_{\mathcal{B}} \end{aligned}$$

□

3.2. Extension from $\mathcal{P}(\mathcal{A})^f$ to $\mathcal{F}(\tau)$

In this section, we will prove the following result.

PROPOSITION 3.2.1. *Suppose $\Phi_0 : \mathcal{P}(\mathcal{A})^f \to \mathcal{P}(\mathcal{B})$ is a map such that $\Phi_0(p) = 0$ if and only if $p = 0$; and $\Phi_0(p+q) = \Phi_0(p) + \Phi_0(q)$ whenever $p, q \in \mathcal{P}(\mathcal{A})^f$ with $pq = 0$. If there exists a linear map U from $\mathcal{F}(\tau)$ into $S(\mathcal{B}, \nu)$ such that $\Phi_0(p) = s(U(p))$ for all $p \in \mathcal{P}(\mathcal{A})^f$, and which has the property that $U(x_n) \overset{\mathcal{T}_m}{\to} U(x)$ whenever $(x_n)_{n=1}^{\infty} \cup \{x\} \subseteq \mathcal{F}(\tau)$ is such that $x_n \overset{\mathcal{A}}{\to} x$ and $s(x_n) \leq s(x)$ for all $n \in \mathbb{N}^+$, then Φ_0 can be extended to a positive linear map $\Phi_3 : \mathcal{F}(\tau) \to \mathcal{B}$ such that*

$$\|\Phi_3(x)\|_{\mathcal{B}} = \|x\|_{\mathcal{A}} \qquad \text{and} \qquad \Phi_3(x^2) = \Phi_3(x)^2$$

for all $x \in \mathcal{F}(\tau)^{sa}$.

The proof is fairly lengthy and so we have divided it into a number of lemmas. Φ_0 satisfies the conditions of Lemma 3.1.1 and can therefore be extended to self-adjoint finite linear combinations of projections with finite trace. Since Lemma 3.1.1 only guarantees the linearity of this extension on commuting elements, we will then prove a sequence of lemmas that will enable us to show the linearity on all elements in its domain. This will enable us to further extend this map to a complex linear map on all of $\mathcal{F}(\tau)$. We will finish this section by showing that this extension has the desired properties.

Let $\mathcal{G}_f$ denote the set of all finite linear combinations of mutually orthogonal projections from $\mathcal{P}(\mathcal{A})^f$. Since $\mathcal{P}(\mathcal{A})^f$ is a lattice, it follows by Lemma 3.1.1 that defining

$$\Phi_1(\sum_{i=1}^n \alpha_i p_i) = \sum_{i=1}^n \alpha_i \Phi_0(p_i) \qquad \sum_{i=1}^n \alpha_i p_i \in \mathcal{G}_f^{sa}$$

yields a well-defined map $\Phi_1 : \mathcal{G}_f^{sa} \to \mathcal{B}$, which extends Φ_0 and is real linear on commuting elements from $\mathcal{G}_f^{sa}$. Furthermore, Φ_1 is square-preserving and isometric. It follows by Proposition B.1.14 that Φ_1 can be extended to $\mathcal{F}(\tau)^{sa}$ in an isometric fashion. We will denote this extension Φ_2. Throughout this section we will use the fact that if $p, q \in \mathcal{P}(\mathcal{A})^f$ with $pq = 0$, then $\Phi_0(p)\Phi_0(q) = 0$ by Proposition B.1.16, since $\Phi_0(p+q) = \Phi_0(p) + \Phi_0(q)$. Furthermore, if $p, q \in \mathcal{P}(\mathcal{A})^f$ with $p \geq q$, then $\Phi_0(p) = \Phi_0(p-q) + \Phi_0(q) \geq \Phi_0(q)$. The following two lemmas will be used to show that Φ_2 is real linear.

Lemma 3.2.2. *If $x \in \mathcal{F}(\tau)^{sa}$, then*

$$U(x) = U(p)\Phi_2(x)$$

for any $p \in \mathcal{P}(\mathcal{A})^f$ with $p \geq s(x)$.

Proof. Suppose $x = q$ for some $q \in \mathcal{P}(\mathcal{A})^f$. If $p \in \mathcal{P}(\mathcal{A})^f$ with $p \geq s(q) = q$, then $p - q \in \mathcal{P}(\mathcal{A})^f$ and

$$\begin{aligned} s(U(p-q)) &= \Phi_0(p-q) \\ \Longrightarrow U(p-q)\Phi_0(q) &= 0 \qquad \text{since } q(p-q) = 0 \text{ implies } \Phi_0(q)\Phi_0(p-q) = 0 \end{aligned} \tag{3.2.1}$$

Furthermore,

$$\begin{aligned} U(p)\Phi_1(q) &= (U(p-q) + U(q))\Phi_0(q) \qquad \text{since } \Phi_1 \text{ extends } \Phi_0 \\ &= U(q)\Phi_0(q) \qquad \text{using (3.2.1)} \\ &= U(q) \qquad \text{since } \Phi_0(q) = s(U(q)) \end{aligned} \tag{3.2.2}$$

Next suppose $x = \sum_{i=1}^n \alpha_i p_i \in \mathcal{G}^{sa}$ and $p \geq s(x) = \sum_{i=1}^n p_i$. Then

$$\begin{aligned} U(p)\Phi_1(x) &= U(p)\sum_{i=1}^n \alpha_i \Phi_1(p_i) \qquad \text{since } \Phi_1 \text{ is real linear on commuting elements in } \mathcal{G}^{sa} \\ &= \sum_{i=1}^n \alpha_i U(p)\Phi_1(p_i) \\ &= \sum_{i=1}^n \alpha_i U(p_i) \qquad \text{using (3.2.2)} \\ &= U(x) \end{aligned} \tag{3.2.3}$$

Finally, suppose x is an arbitrary element in $\mathcal{F}(\tau)^{sa}$ and suppose $p \in \mathcal{P}(\mathcal{A})^f$ with $p \geq s(x)$. By Remark B.1.12, there exists a sequence $(x_n)_{n=1}^\infty \subseteq \mathcal{G}^{sa}$ such that $x_n \overset{\mathcal{A}}{\to} x$, $s(x_n) \leq s(x) \leq p$ for every $n \in \mathbb{N}^+$, and the x_n's commute with each other and with x. By the assumption on U we have that

$$U(x_n) \overset{\mathcal{T}_m}{\to} U(x). \tag{3.2.4}$$

Φ_2 is real linear on commuting elements (see Proposition B.1.14) and is an isometric extension of Φ_1. It follows that $\Phi_1(x_n) \overset{\mathcal{B}}{\to} \Phi_2(x)$. Therefore $U(p)\Phi_1(x_n) \overset{\mathcal{B}}{\to} U(p)\Phi_2(x)$ and hence $U(p)\Phi_1(x_n) \overset{\mathcal{T}_m}{\to} U(p)\Phi_2(x)$, by Proposition 1.6.3(1). However, $U(p)\Phi_1(x_n) = U(x_n)$ for all $n \in \mathbb{N}^+$, using (3.2.3). It follows that

$$U(x_n) \overset{\mathcal{T}_m}{\to} U(p)\Phi_2(x). \tag{3.2.5}$$

Combining (3.2.4) and (3.2.5), we obtain $U(x) = U(p)\Phi_2(x)$, since the measure topology is Hausdorff. $\square$

LEMMA 3.2.3. *If $x \in \mathcal{F}(\tau)^{sa}$, then*

$$r(\Phi_2(x)) \leq s(U(p)) = \Phi_0(p)$$

for any $p \in \mathcal{P}(\mathcal{A})^f$ with $p \geq s(x)$.

PROOF. Suppose $x = q \in \mathcal{P}(\mathcal{A})^f$ and $p \in \mathcal{P}(\mathcal{A})^f$ with $p \geq s(x) = q$. As shown in the introduction this implies that $s(U(p)) = \Phi_0(p) \geq \Phi_0(q) = r(\Phi_2(q))$. If $x = \sum_{i=1}^{n} \alpha_i p_i \in \mathcal{G}^{sa}$ and $p \geq s(x) = \sum_{i=1}^{n} p_i$, then

$$\begin{aligned}
r(\Phi_2(x)) &= r(\sum_{i=1}^{n} \alpha_i \Phi_0(p_i)) \\
&= \sum_{i=1}^{n} \Phi_0(p_i) \qquad \text{since the } \Phi_0(p_i)\text{'s are orthogonal projections} \\
&= \Phi_0(\sum_{i=1}^{n} p_i) \qquad \text{since } \Phi_0 \text{ is additive on orthogonal projections} \\
&\leq \Phi_0(p) \qquad \text{since } p \geq \sum_{i=1}^{n} p_i \\
&= s(U(p)) \qquad (3.2.6)
\end{aligned}$$

Suppose $x \in \mathcal{F}(\tau)^{sa}$ and $p \in \mathcal{P}(\mathcal{A})^f$ with $p \geq s(x)$. By Remark B.1.12, there exists a sequence $(x_n)_{n=1}^{\infty} \subseteq \mathcal{G}^{sa}$ such that $x_n \stackrel{\mathcal{A}}{\to} x$, $s(x_n) \leq s(x) \leq p$ for all $n \in \mathbb{N}^+$, and the x_n's commute with each other and with x. It follows by (3.2.6) that $r(\Phi_2(x_n)) \leq s(U(p))$ for all $n \in \mathbb{N}^+$. Furthermore, $\Phi_2(x_n) \stackrel{\mathcal{B}}{\to} \Phi_2(x)$, since Φ_2 is real linear on commuting elements and isometric. Recall that for $y \in S(\mathcal{A}, \tau)$, we will use $k(y)$ to denote the projection onto the kernel of y. Suppose $\eta \in \ker(U(p))$. Note that for any $n \in \mathbb{N}^+$,

$$\begin{aligned}
k(U(p)) &= [s(U(p))]^{\perp} \qquad \text{by Proposition 1.3.1(3)} \\
&\leq [r(\Phi_2(x_n))]^{\perp} \qquad \text{since } r(\Phi_2(x_n)) \leq s(U(p)), \text{ by (3.2.6)} \\
&= [r(\Phi_2(x_n))^*)]^{\perp} \qquad \text{since } \Phi_2(x_n) \text{ is a real linear combination of projections} \\
&= k(\Phi_2(x_n)) \qquad \text{by Proposition 1.3.1(3)} \qquad (3.2.7)
\end{aligned}$$

Furthermore, $\Phi_2(x_n) \stackrel{\mathcal{B}}{\to} \Phi_2(x)$ implies that $\Phi_2(x_n) \stackrel{SOT}{\to} \Phi_2(x)$ and so

$$\begin{aligned}
\|\Phi_2(x)(\eta)\| &= \lim_{n\to\infty} \|\Phi_1(x_n)(\eta)\| \\
&= 0 \qquad \text{since } \eta \in \ker(\Phi_2(x_n)) \text{ for every } n \in \mathbb{N}^+, \text{ using (3.2.7)}
\end{aligned}$$

It follows that $\eta \in k(\Phi_2(x))(H_2) = [r(\Phi_2(x)^*)]^{\perp}(H_2) = [r(\Phi_2(x))]^{\perp}(H_2)$ using Proposition 1.3.1 and Remark B.1.15. Therefore $s(U(p))^{\perp} = k(s(U(p))) \leq r(\Phi_2(x))^{\perp}$ and hence $r(\Phi_2(x)) \leq s(U(p))$. $\square$

We are now in a position to show that Φ_2 is real linear on $\mathcal{F}(\tau)^{sa}$. Suppose $x, y \in \mathcal{F}(\tau)^{sa}$ and $\alpha, \beta \in \mathbb{R}$. Let $p = s(x) \vee s(y)$. Then

$$\begin{aligned}
U(p)[\Phi_2(\alpha x + \beta y) - \alpha\Phi_2(x) - \beta\Phi_2(y)] &= U(\alpha x + \beta y) - \alpha U(x) - \beta U(y) \qquad \text{using Lemma 3.2.2} \\
&= 0 \qquad \text{since } U \text{ is real linear} \qquad (3.2.8)
\end{aligned}$$

Furthermore,

$$\begin{aligned}
r([\Phi_2(\alpha x + \beta y) - \alpha\Phi_2(x) - \beta\Phi_2(y)]) &\leq r(\Phi_2(\alpha x + \beta y)) \vee r(\Phi_2(x)) \vee r(\Phi_2(y)) \\
&\leq s(U(p)) \qquad \text{by Lemma 3.2.3, since } s(x), s(y), s(\alpha x + \beta y) \leq p \qquad (3.2.9)
\end{aligned}$$

Applying Proposition B.2.12(1) to (3.2.8) and (3.2.9), we obtain $[\Phi_2(\alpha x + \beta y) - \alpha\Phi_2(x) - \beta\Phi_2(y)] = 0$. Since $\alpha, \beta \in \mathbb{R}$ and $x, y \in \mathcal{F}(\tau)^{sa}$ were arbitrary, it follows that Φ_2 is real linear on $\mathcal{F}(\tau)^{sa}$.

Before extending Φ_2 to all of $\mathcal{F}(\tau)$, we show that Φ_2 is positive and square-preserving.

LEMMA 3.2.4. *Φ_2 is positive and if $x \in \mathcal{F}(\tau)^{sa}$, then*

$$\Phi_2(x^2) = \Phi_2(x)^2.$$

PROOF. To show that Φ_2 is positive we start by considering its action on projections. Suppose $p \in \mathcal{P}(\mathcal{A})^f$. Then $\Phi_2(p) = \Phi_0(p) \in \mathcal{P}(\mathcal{B})$ and so $\Phi_2(p) \geq 0$. If $x \in \mathcal{G}_f^+$, i.e. $x = \sum_{i=1}^{n} \alpha_i p_i$ with $\alpha_i > 0$ for all i, $p_i \in \mathcal{P}(\mathcal{A})^f$ for all i and $p_i p_j = 0$ if $i \neq j$, then $\Phi_2(x) = \sum_{i=1}^{n} \alpha_i \Phi_2(p_i)$. Therefore $\Phi_2(x) \geq 0$, since $\Phi_2(p_i) \geq 0$ and $\alpha_i > 0$ for every i. Finally, suppose $x \in \mathcal{F}(\tau)^+$. By Remark B.1.12 there exists a sequence $(x_n)_{n=1}^{\infty} \subseteq \mathcal{G}_f^+$ such that $x_n \xrightarrow{\mathcal{A}} x$. Since $x_n \in \mathcal{G}_f^+$ for each $n \in \mathbb{N}^+$, $\Phi_2(x_n(x)) \geq 0$, for every $n \in \mathbb{N}^+$. It follows that $\Phi_2(x) \geq 0$, since $\Phi_2(x_n) \xrightarrow{\mathcal{B}} \Phi_2(x)$ and $\mathcal{B}^+$ is closed, by Theorem B.1.5.

Next, we show that Φ_2 is square-preserving. Suppose $x \in \mathcal{F}(\tau)^{sa}$. By Remark B.1.12 there exists a sequence $(x_n)_{n=1}^{\infty} \subseteq \mathcal{G}_f^{sa}$ such that $x_n \xrightarrow{\mathcal{A}} x$. Since Φ_2 is real linear and isometric on $\mathcal{F}(\tau)^{sa}$, this implies that $\Phi_2(x_n) \xrightarrow{\mathcal{B}} \Phi_2(x)$. Using the joint continuity of multiplication in $\mathcal{A}$ and $\mathcal{B}$, equipped with their respective norm topologies, it follows that $x_n^2 \xrightarrow{\mathcal{A}} x^2$ and

$$\Phi_2(x_n)^2 \xrightarrow{\mathcal{B}} \Phi_2(x)^2 \tag{3.2.10}$$

Again using the isometric nature of Φ_2, but this time applied to the sequence $\{x_n^2\}_{n=1}^{\infty}$, we obtain

$$\Phi_2(x_n^2) \xrightarrow{\mathcal{B}} \Phi_2(x^2) \tag{3.2.11}$$

However, since $x_n \in \mathcal{G}_f^{sa}$ for each $n \in \mathbb{N}^+$ and Φ_2 extends Φ_1, we have that $\Phi_2(x_n^2) = (\Phi_2(x_n))^2$ for each $n \in \mathbb{N}^+$, by Lemma 3.1.1. Combining this with (3.2.10) and (3.2.11), we obtain

$$\Phi_2(x^2) = \Phi_2(x)^2.$$

□

Finally, we extend Φ_2 to all of $\mathcal{F}(\tau)$. To do so, note that if $x \in \mathcal{F}(\tau)$, then $x = x_1 + ix_2$, where $x_1, x_2 \in \mathcal{F}(\tau)^{sa}$. Let

$$\Phi_3(x) := \Phi_2(x_1) + i\Phi(x_2).$$

Since this decomposition of x is unique, Φ_3 is well-defined and extends Φ_2. It is easily checked that Φ_3 is complex linear and, since Φ_2 is positive, isometric and square-preserving on $\mathcal{F}(\tau)^{sa}$, so is Φ_3.

3.3. Extension from $\mathcal{F}(\tau)$ to $\mathcal{A}$

Suppose $(\mathcal{A}, \tau)$ and $(\mathcal{B}, \nu)$ are semi-finite von Neumann algebras on the Hilbert spaces H and K, respectively and recall that we will sometimes use $\mathcal{D}$ to denote $\mathcal{P}(\mathcal{A})^f$. We show that if $\Phi_3 : \mathcal{F}(\tau) \to \mathcal{B}$ is linear, positive, normal,

$$\Phi_3(p) > 0 \qquad \text{if } 0 \neq p \in \mathcal{P}(\mathcal{A})^f$$

and

$$\Phi_3(x^2) = \Phi_3(x)^2 \qquad \forall x \in \mathcal{F}(\tau)^{sa},$$

then Φ_3 can be extended uniquely to a normal Jordan $*$-homomorphism Φ from $\mathcal{A}$ into $\mathcal{B}$. Furthermore, in this case,

$$\Phi(x) = \mathrm{SOT}\lim_{p\in\mathcal{D}} \Phi_3(pxp) \qquad \forall x \in \mathcal{A}$$

and

$$\left\|\Phi(x)\right\|_{\mathcal{B}} = \left\|x\right\|_{\mathcal{A}} \qquad \forall x \in \mathcal{A}^{sa}.$$

We start by defining Φ on projections. Since $\mathcal{A}$ is semi-finite, $\mathcal{D}_{(p)} := \{q \in \mathcal{P}(\mathcal{A})^f : q \leq p\}$ is an increasing net whose supremum is p. For the sake of convenience we will use an indexing set and let $\mathcal{D}_{(p)} = \{p_\alpha : \alpha \in I_p\}$. Note that Φ_3 is positive and therefore $\{\Phi_3(p_\alpha)\}_{\alpha\in I_p}$ is increasing. By Proposition 1.8.3(5), $\Phi_3(p_\alpha)$ is a projection for every $\alpha \in I_p$ and so $\mathrm{SOT}\lim_{\alpha\in I_p} \Phi_3(p_\alpha)$ exists and is a projection. Unless stated otherwise, we will assume throughout this section that convergence and limits are with respect to the strong operator topology. Let

$$\Phi(p) := \lim_{\alpha\in I_p} \Phi_3(p_\alpha) \in \mathcal{P}(\mathcal{B}) \tag{3.3.1}$$

Note that $\Phi(p) = \Phi_3(p)$ for $p \in \mathcal{P}(\mathcal{A})^f$. In order to apply Lemma 3.1.1 to extend Φ to $\mathcal{G}^{sa}$, the set of all self-adjoint finite linear combinations of mutually orthogonal projections in $\mathcal{A}$, we need to show that Φ maps non-zero projections onto non-zero projections and that Φ is additive on orthogonal projections. If $p = 0$, then $\mathcal{D}_{(p)} = \{0\}$ and so $\Phi(p) = \Phi_3(0) = 0$, since Φ_3 is linear. If $p > 0$, then there exists a $\beta \in I_p$ such that $0 \neq p_\beta \in \mathcal{D}_{(p)}$, since $\mathcal{A}$ is semi-finite. $\{\Phi_3(p_\alpha)\}_{\alpha\in I_p}$ is an increasing net and so $\Phi(p) \geq \Phi_3(p_\beta) > 0$. It follows that

$$\Phi(p) = 0 \iff p = 0.$$

Next, suppose p and q are orthogonal projections in $\mathcal{P}(\mathcal{A})$. Note that if $p_\alpha \in \mathcal{D}_{(p)}$ and $q_\beta \in \mathcal{D}_{(q)}$, then $p_\alpha + q_\beta \in \mathcal{D}_{(p+q)}$, since p_α and q_β are disjoint projections, each with finite trace. Furthermore $p_\alpha + q_\beta \uparrow p + q$ (see Remark B.1.4) and so if $r \in \mathcal{D}_{(p+q)}$, then there exists a $(\alpha, \beta) \in I_p \times I_q$ such that $p_\alpha + q_\beta \geq r$. It follows, using the positivity of Φ_3, that $\{\Phi_3(p_\alpha + q_\beta) : \alpha \in I_p, \beta \in I_q\}$ is a subnet of $\{\Phi_3(r) : r \in \mathcal{D}_{(p+q)}\}$ and hence $\Phi_3(p_\alpha) + \Phi_3(q_\beta) = \Phi_3(p_\alpha + q_\beta) \to \Phi(p + q)$, since $\Phi(p + q) := \lim_{r\in\mathcal{D}_{(p+q)}} \Phi_3(r)$. However, we also have that $\Phi_3(p_\alpha) + \Phi_3(q_\beta) \to \Phi(p) + \Phi(q)$, by definition of $\Phi(p)$ and $\Phi(q)$ and the SOT-continuity of addition. Therefore

$$\Phi(p + q) = \Phi(p) + \Phi(q).$$

By Proposition B.1.16 this further implies that $\Phi(p)$ and $\Phi(q)$ are orthogonal, i.e. $\Phi(p)\Phi(q) = 0$ if $pq = 0$.

REMARK 3.3.1. An orthoisomorphism between two von Neumann algebras $\mathcal{A}$ and $\mathcal{B}$ is defined to be an injective mapping θ of $\mathcal{P}(\mathcal{A})$ onto $\mathcal{P}(\mathcal{B})$ such that

$$\theta(p)\theta(q) = 0 \iff pq = 0.$$

In 1955, Dye ([8, p.83]) showed that in the case where $\mathcal{A}$ does not have summands of type I_2, such an orthoisomorphism has to be the restriction of a Jordan $*$-isomorphism from $\mathcal{A}$ onto $\mathcal{B}$. Since we may not be able to show that Φ, as defined above and considered as a mapping from $\mathcal{P}(\mathcal{A})$ into $\mathcal{P}(\mathcal{B})$, is surjective nor that $pq = 0$ whenever $\Phi(p)\Phi(q) = 0$, we will proceed with the process of extending Φ more directly.

For $x = \sum_{i=1}^{n} \alpha_i p_i \in \mathcal{G}^{sa}$, define

$$\Phi(x) = \sum_{i=1}^{n} \alpha_i \Phi(p_i).$$

It follows by Lemma 3.1.1 that Φ is well-defined and real linear on commuting operators in $\mathcal{G}^{sa}$. Furthermore, Φ is square-preserving and isometric on $\mathcal{G}^{sa}$. It follows, by Proposition B.1.13, that Φ can be extended, in an isometric fashion, to the set of all self-adjoint elements in $\mathcal{A}$. This extension is real linear on commuting elements, but need not be linear on all elements of $\mathcal{A}^{sa}$. Recall that in section 3.2, we showed that the existence of a linear map $U : \mathcal{F}(\tau) \to S(\mathcal{B}, \nu)$ such that $\Phi_0(p) = s(U(p))$ for all $p \in \mathcal{P}(\mathcal{A})^f$, could, under certain circumstances, be used to show that the extension of Φ_0 is linear. A similar idea is not applicable in this context, since the map U we will be interested in in the sequel will typically not be defined on projections with infinite trace. We can however use the normality of Φ_3 to prove several lemmas that will enable us to view $\Phi(y)$ as a SOT-limit for any $y \in \mathcal{A}^{sa}$ and hence demonstrate the real linearity of Φ on $\mathcal{A}^{sa}$.

LEMMA 3.3.2. *If $p \in \mathcal{P}(\mathcal{A})$, then*

$$\Phi_3(qpq) = \Phi_3(q)\Phi(p)\Phi_3(q)$$

for all $q \in \mathcal{P}(\mathcal{A})^f$.

PROOF. Note that

$$\begin{aligned} p_\alpha \uparrow_{\alpha \in I_p} p \quad &\Longrightarrow \quad qp_\alpha q \uparrow_{\alpha\in I_p} qpq \qquad \text{by Proposition B.2.2(7)} \\ &\Longrightarrow \quad \Phi_3(qp_\alpha q) \uparrow_{\alpha \in I_p} \Phi_3(qpq) \qquad \text{since } \Phi_3 \text{ is normal} \\ &\Longrightarrow \quad \Phi_3(qp_\alpha q) \underset{\alpha\in I_p}{\overset{SOT}{\to}} \Phi_3(qpq) \qquad \text{by Corollary B.1.9} \end{aligned} \tag{3.3.2}$$

Furthermore,

$$\begin{aligned} \lim_{\alpha \in I_p} \Phi_3(qp_\alpha q) &= \lim_{\alpha\in I_p} \Phi_3(q)\Phi_3(p_\alpha)\Phi_3(q)) \qquad \text{by Proposition 1.8.3(4)} \\ &= \Phi_3(q)(\lim_{\alpha\in I_p} \Phi_3(p_\alpha))\Phi_3(q) \qquad \text{by Proposition B.1.1} \\ &= \Phi_3(q)\Phi(p)\Phi_3(q) \qquad \text{by definition of } \Phi \end{aligned} \tag{3.3.3}$$

Combining (3.3.2) and (3.3.3), we obtain

$$\Phi_3(qpq) = \Phi_3(q)\Phi(p)\Phi_3(q),$$

since SOT-limits are unique. □

LEMMA 3.3.3. *If $p \in \mathcal{P}(\mathcal{A})$, then*

$$\Phi(\mathbf{1})\Phi(p)\Phi(\mathbf{1}) = \Phi(p).$$

PROOF. If $\alpha \in I_p$, then $p_\alpha \in \mathcal{D}_{(\mathbf{1})} = \mathcal{D}$ and so $\Phi_3(p_\alpha) \le \bigvee_{q\in\mathcal{D}} \Phi_3(q) = \Phi(\mathbf{1})$. It follows that

$$\begin{aligned} \Phi_3(p_\alpha) \quad &= \quad \Phi(\mathbf{1})\Phi_3(p_\alpha)\Phi(\mathbf{1}) \qquad \text{by Proposition B.1.16(3)} \\ &\underset{\alpha\in I_p}{\overset{SOT}{\to}} \quad \Phi(\mathbf{1})\Phi(p)\Phi(\mathbf{1}) \qquad \text{by Proposition B.2.2(7) and Corollary B.1.9} \end{aligned}$$

However, $\Phi_3(p_\alpha) \underset{\alpha\in I_p}{\overset{SOT}{\to}} \Phi(p)$ and so

$$\Phi(\mathbf{1})\Phi(p)\Phi(\mathbf{1}) = \Phi(p).$$

□

These two lemmas enable us to show that for any self-adjoint element x, $\Phi(x)$ can be viewed as a SOT-limit.

LEMMA 3.3.4. *If $x \in \mathcal{A}^{sa}$, then $pxp \in \mathcal{F}(\tau)$ for every $p \in \mathcal{P}(\mathcal{A})^f$ and*

$$\Phi(x) = \lim_{p\in\mathcal{D}} \Phi_3(pxp).$$

PROOF. Suppose $x = q \in \mathcal{P}(\mathcal{A})$. Then for $p \in \mathcal{P}(\mathcal{A})^f$,

$$\begin{aligned}
\Phi_3(pqp) &= \Phi_3(p)\Phi(q)\Phi_3(p) && \text{by Lemma 3.3.2} \\
&\overset{SOT}{\to} \Phi(\mathbf{1})\Phi(q)\Phi(\mathbf{1}) && \text{using } \Phi_3(p) \overset{SOT}{\to} \Phi(\mathbf{1}) \text{ and repeated application of Remark B.1.4} \\
&= \Phi(q) && \text{by Lemma 3.3.3} \qquad (3.3.4)
\end{aligned}$$

If $x = \sum_{i=1}^{n} \alpha_i q_i \in \mathcal{G}^{sa}$, then for $p \in \mathcal{P}(\mathcal{A})^f$

$$\begin{aligned}
\Phi_3(pxp) &= \sum_{i=1}^{n} \alpha_i \Phi_3(pq_ip) && \text{since } \Phi_3 \text{ is linear} \\
&\overset{SOT}{\to} \sum_{i=1}^{n} \alpha_i \Phi(q_i) && \text{using (3.3.4) and the fact that the SOT is a vector topology} \\
&= \Phi(x) && (3.3.5)
\end{aligned}$$

Next, suppose that $x \in \mathcal{A}^{sa}$. Recall that Φ has been defined using Proposition B.1.13. Let $(x_n)_{n=1}^{\infty} \subseteq \mathcal{G}^{sa}$ be the sequence described in Proposition B.1.13 such that $x_n \overset{\mathcal{A}}{\to} x$ and $\Phi(x_n) \overset{\mathcal{B}}{\to} \Phi(x)$. Let $\epsilon > 0$ and fix $\eta \in K$, with $\|\eta\|_K = 1$. It follows that there exist $n_0, n_1 \in \mathbb{N}^+$, such that

$$\|x_n - x\|_{\mathcal{A}} < \frac{\epsilon}{3} \quad \text{for all } n \geq n_0 \text{ and} \qquad (3.3.6)$$

$$\|\Phi(x_n) - \Phi(x)\|_{\mathcal{B}} < \frac{\epsilon}{3} \quad \text{for all } n \geq n_1 \qquad (3.3.7)$$

Furthermore, for $p \in \mathcal{P}(\mathcal{A})^f$ and $n \geq \max\{n_0, n_1\}$, we have

$$\begin{aligned}
\|(\Phi_3(p(x - x_n)p))\eta\|_K &\leq \|(\Phi_3(p(x - x_n)p))\|_{\mathcal{B}} \|\eta\|_K \\
&\leq \|p(x - x_n)p\|_{\mathcal{A}} && \text{by Proposition 1.8.3(8)} \\
&\leq \|p\|_{\mathcal{A}}^2 \|x - x_n\|_{\mathcal{A}} \\
&< \frac{\epsilon}{3} && \text{by (3.3.6)} \qquad (3.3.8)
\end{aligned}$$

Fix $n \geq \max\{n_0, n_1\}$. By (3.3.5), $\Phi(x_n) = \lim_{p \in \mathcal{D}} \Phi_3(px_np)$. Let $q \in \mathcal{P}(\mathcal{A})^f$ be such that $p \in \mathcal{P}(\mathcal{A})^f$, $p \geq q$ implies

$$\|(\Phi_3(px_np) - \Phi(x_n))\eta\|_K < \frac{\epsilon}{3} \qquad (3.3.9)$$

Then for $p \in \mathcal{P}(\mathcal{A})^f$ with $p \geq q$, we have

$$\begin{aligned}
\|(\Phi_3(pxp) - \Phi(x))\eta\|_K &\leq \|(\Phi_3(pxp) - \Phi_3(px_np))\eta\|_K + \|(\Phi_3(px_np) - \Phi(x_n))\eta\|_K + \\
&\quad \|(\Phi_3(x_n) - \Phi((x))\eta\|_K \\
&< \epsilon && \text{by (3.3.8), (3.3.9) and (3.3.7)}
\end{aligned}$$

It follows that $\Phi_3(pxp)\eta \to \Phi(x)\eta$ for any $\eta \in K$ with $\|\eta\|_K = 1$ and therefore also for any $0 \neq \eta \in K$ by considering $\tilde{\eta} = \frac{\eta}{\|\eta\|}$. Therefore $\Phi(x) = \lim_{p \in \mathcal{D}} \Phi_3(pxp)$. $\square$

Lemma 3.3.4 can be used to show that Φ is real linear on $\mathcal{A}^{sa}$ in the following way. If $x, y \in \mathcal{A}^{sa}$ and $\alpha, \beta \in \mathbb{R}$, then

$$\begin{aligned}
\alpha\Phi(x) + \beta\Phi(y) &= \alpha \lim_{p \in \mathcal{D}} \Phi_3(pxp) + \beta \lim_{p \in \mathcal{D}} \Phi_3(pyp) && \text{by Lemma 3.3.4} \\
&= \lim_{p \in \mathcal{D}} (\alpha\Phi_3(pxp) + \beta\Phi_3(pyp)) && \text{since the SOT is a vector topology} \\
&= \lim_{p \in \mathcal{D}} \Phi_3(p(\alpha x + \beta y)p) && \text{since } \Phi_3 \text{ is linear} \\
&= \Phi(\alpha x + \beta y) && \text{by Lemma 3.3.4}
\end{aligned}$$

Next, we extend Φ to $\mathcal{A}$. Suppose $y \in \mathcal{A}$. Then $y = y_1 + iy_2$, where $y_1, y_2 \in \mathcal{A}^{sa}$. Let

$$\Phi(y) := \Phi(y_1) + i\Phi(y_2).$$

Φ is well-defined, since the decomposition for y is unique. It is easily checked that Φ is a complex linear map on $\mathcal{A}$ and

$$\Phi(x) = \lim_{p\in\mathcal{D}} \Phi_3(pxp) \qquad \forall x \in \mathcal{A} \tag{3.3.10}$$

In particular, if $x \in \mathcal{F}(\tau)$, then $x \in q\mathcal{A}q$ for some $q \in \mathcal{P}(\mathcal{A})^f$. Furthermore,

$$\begin{aligned} \Phi(x) &= \lim_{p\in\mathcal{D}} \Phi_3(pxp) \qquad \text{using (3.3.10)} \\ &= \lim_{p\in\mathcal{D}:p\geq q} \Phi_3(pxp) \\ &= \Phi_3(x) \qquad \text{since } pxp = x \text{ for all } p \geq q \end{aligned}$$

and so Φ is an extension of Φ_3.

Lemma 3.3.5. *Φ is a normal Jordan $*$-homomorphism from $\mathcal{A}$ into $\mathcal{B}$.*

Proof. We know that Φ maps projections onto projections (see (3.3.1)) and so if we can show that Φ is continuous, then it will follow by Theorem 1.8.4 that Φ is a Jordan $*$-homomorphism. If $x \in \mathcal{A}$, then $x = x_1 + ix_2$ for some $x_1, x_2 \in \mathcal{A}^{sa}$ and so

$$\begin{aligned} \|\Phi(x)\|_{\mathcal{B}} &\leq \|\Phi(x_1)\|_{\mathcal{B}} + \|\Phi(x_2)\|_{\mathcal{B}} \\ &= \|x_1\|_{\mathcal{A}} + \|x_2\|_{\mathcal{A}} \qquad \text{since } \Phi \text{ is isometric on self-adjoint elements} \\ &\leq 2\|x\|_{\mathcal{A}} \end{aligned}$$

It follows that Φ is continuous. Next, we show that Φ is normal. By Remark 1.8.7, it suffices to consider $\{x_\gamma\}_\gamma \cup \{x\} \subseteq \mathcal{A}^+$ such that $x_\gamma \uparrow x$. In this case, we have that for any $p \in \mathcal{P}(\mathcal{A})^f$,

$$\begin{aligned} px_\gamma p &\uparrow pxp \qquad \text{by Proposition B.2.2(7)} \\ \Longrightarrow \Phi_3(px_\gamma p) &\uparrow \Phi_3(pxp) \qquad \text{since } \Phi_3 \text{ is normal} \\ \Longrightarrow \Phi(px_\gamma p) &\uparrow \Phi(pxp) \qquad \text{since } \Phi_3 \text{ and } \Phi \text{ agree on } \mathcal{F}(\tau) \\ \Longrightarrow \Phi(pxp) &= \lim_\gamma \Phi(px_\gamma p) \end{aligned} \tag{3.3.11}$$

Since Φ is positive, we have that $\{\Phi(x_\gamma)\}_\gamma$ is increasing and $0 \leq \Phi(x_\gamma) \leq \Phi(x) \leq \|\Phi(x)\|_{\mathcal{B}}\mathbf{1}$ for all γ, by Proposition B.1.6. Therefore $\lim_\gamma \Phi(x_\gamma)$ exists, by Lemma B.1.8. Furthermore, for any $p \in \mathcal{P}(\mathcal{A})^f$,

$$\begin{aligned} \Phi(p)\Phi(x)\Phi(p) &= \Phi(pxp) \qquad \text{by Proposition 1.8.2(2)} \\ &= \lim_\gamma \Phi(px_\gamma p) \qquad \text{by (3.3.11)} \\ &= \lim_\gamma [\Phi(p)\Phi(x_\gamma)\Phi(p)] \qquad \text{by Proposition 1.8.2(2)} \\ &= \Phi(p)[\lim_\gamma \Phi(x_\gamma)]\Phi(p) \qquad \text{by Remark B.1.4} \end{aligned} \tag{3.3.12}$$

Furthermore, $\Phi(p) = \Phi_3(p) \overset{SOT}{\rightarrow} \Phi(\mathbf{1})$ and so

$$\begin{aligned} \Phi(p)\Phi(x)\Phi(p) &\overset{SOT}{\rightarrow} \Phi(\mathbf{1})\Phi(x)\Phi(\mathbf{1}) \qquad \text{by Remark B.1.4} \\ &= \Phi(x) \qquad \text{by Proposition 1.8.2(7)} \end{aligned} \tag{3.3.13}$$

Similarly,

$$\Phi(p)[\lim_{\gamma}\Phi(x_\gamma)]\Phi(p) \underset{p\in\mathcal{D}}{\overset{SOT}{\to}} \lim_{\gamma}\Phi(x_\gamma) \tag{3.3.14}$$

By combining (3.3.12), (3.3.13) and (3.3.14), we obtain $\Phi(x) = \lim_{\gamma}\Phi(x_\gamma)$. Since, $\lim_{\gamma}\Phi(x_\gamma) = \bigvee_{\gamma}\Phi(x_\gamma)$, by Lemma B.1.8, we therefore have that $\Phi(x_\gamma) \uparrow \Phi(x)$. □

We have shown that Φ_3 can be extended to a normal Jordan $*$-homomorphism from $\mathcal{A}$ into $\mathcal{B}$. We conclude by showing that this extension is unique. Suppose $\Psi : \mathcal{A} \to \mathcal{B}$ is another normal Jordan $*$-homomorphism extending Φ_3 and let $x \in \mathcal{A}$. Since Ψ is normal, we have that $\Psi(p) \uparrow_{p\in\mathcal{D}} \Psi(\mathbf{1})$ and hence $\Psi(p) \underset{p\in\mathcal{D}}{\overset{SOT}{\to}} \Psi(\mathbf{1})$, by Corollary B.1.9. Therefore,

$$\begin{aligned}
\Psi(pxp) &= \Psi(p)\Psi(x)\Psi(p) \qquad \text{by Proposition 1.8.2(2)}\\
&\overset{SOT}{\to} \Psi(\mathbf{1})\Psi(x)\Psi(\mathbf{1}) \qquad \text{by Remark B.1.4}\\
&= \Psi(x) \qquad \text{by Proposition 1.8.2(7)}
\end{aligned}$$

Similarly, $\Phi(pxp) \overset{SOT}{\to} \Phi(x)$, but $\Phi(pxp) = \Phi_3(pxp) = \Psi(pxp)$ for every $p \in \mathcal{P}(\mathcal{A})^f$ and so $\Phi(x) = \Psi(x)$. Since this holds for every $x \in \mathcal{A}$, we have that $\Psi = \Phi$. We summarize the results of this section in the following proposition.

Proposition 3.3.6. *Suppose $(\mathcal{A},\tau)$ and $(\mathcal{B},\nu)$ are semi-finite von Neumann algebras. If $\Phi_3 : \mathcal{F}(\tau) \to \mathcal{B}$ is linear, positive, normal, square-preserving on self-adjoint elements, and has the property that for $p \in \mathcal{P}(\mathcal{A})^f$*

$$\Phi_3(p) = 0 \iff p = 0,$$

then Φ_3 can be extended uniquely to a normal Jordan $*$-homomorphism Φ from $\mathcal{A}$ into $\mathcal{B}$. Furthermore, in this case,

$$\Phi(x) = SOT\lim_{p\in\mathcal{D}}\Phi_3(pxp) \qquad \forall x \in \mathcal{A}$$

and

$$\|\Phi(x)\|_{\mathcal{B}} = \|x\|_{\mathcal{A}} \qquad \forall x \in \mathcal{A}^{sa}.$$

3.4. Surjectivity

We have seen that the result of the extension procedures we have detailed is a normal Jordan $*$-homomorphism. In this section we describe sufficient conditions to ensure that such a map is surjective. We need a preliminary result. To prove this result we will need some information regarding the ultra weak operator topology (UWOT). The ultra-weak operator topology is the locally convex Hausdorff topology on $\mathcal{B}(H)$ generated by the semi-norms $\rho_{(\xi),(\eta)}$, where

$$\rho_{(\xi),(\eta)}(x) := |\sum_{n=1}^{\infty}\langle x\xi_n, \eta_n\rangle|$$

and $(\xi) = (\xi_n)_{n=1}^{\infty}$ and $(\eta) = (\eta_n)_{n=1}^{\infty}$ are sequences in H satisfying $\sum_{n=1}^{\infty}\|\xi_n\|^2 < \infty$ and $\sum_{n=1}^{\infty}\|\eta_n\|^2 < \infty$. Three facts that we require regarding the ultra-weak operator topology are that the ultra-weak operator topology is stronger than the weak operator topology; the weak operator topology and ultra-weak operator topology coincide on norm bounded sets; and a positive linear mapping between von Neumann algebras is normal if and only if it is continuous with respect to the ultra-weak operator topologies ([15]).

Lemma 3.4.1. *Suppose $(\mathcal{A},\tau)$ and $(\mathcal{B},\nu)$ are von Neumann algebras and $\Phi : \mathcal{A} \to \mathcal{B}$ is a normal Jordan $*$-homomorphism. If there exists a $k > 0$ such that $\|x\| \le k\|\Phi(x)\|$ for all $x \in \mathcal{A}$, then $\Phi(A)$ is WOT-closed.*

Proof. Since Φ is positive, linear and normal, it is therefore continuous with respect to the ultra-weak operator topologies on $\mathcal{A}$ and $\mathcal{B}$. Suppose $b \in \overline{\Phi(\mathcal{A})}^{WOT}$ with $\|b\| = 1$. It follows from the Kaplansky density theorem ([23, Theorem 5.3.5]) that there exists a net $\{b_\lambda\}_{\lambda\in\Lambda} \subseteq \Phi(\mathcal{A})$ such that $\|b_\lambda\| \leq 1$ for each λ and $b_\lambda \overset{WOT}{\to} b$. Let $\{a_\lambda\}_{\lambda\in\Lambda} \subseteq \mathcal{A}$ be such that $\Phi(a_\lambda) = b_\lambda$ for each λ. Then

$$\|a_\lambda\| \leq k\|\Phi(a_\lambda)\| = k\|b_\lambda\| \leq k$$

for each λ. The unit ball (and hence any multiple of the unit ball) in $\mathcal{A}$ is WOT-compact and so there exists a subnet $\{a_{\lambda'}\}_{\lambda'\in\Lambda'\subseteq\Lambda} \subseteq \{a_\lambda\}_{\lambda\in\Lambda}$ such that $a_{\lambda'} \overset{WOT}{\to} a$ for some $a \in \mathcal{A}$ with $\|a\| \leq k$. Since the weak operator topology and ultra-weak operator topology coincide on norm-bounded sets, $a_{\lambda'} \overset{UWOT}{\to} a$. Using the continuity of Φ discussed earlier, we have that $\Phi(a_{\lambda'}) \overset{UWOT}{\to} \Phi(a)$. It follows that $\Phi(a_{\lambda'}) \overset{WOT}{\to} \Phi(a)$, since the ultra-weak operator topology is stronger than the weak operator topology. However, $\Phi(a_{\lambda'}) \overset{WOT}{\to} b$ and so $\Phi(a) = b$, since the weak operator topology is Hausdorff. □

We are now in a position to describe the conditions which ensure that a normal Jordan $*$-homomorphism is surjective.

Proposition 3.4.2. *Suppose $(\mathcal{A}, \tau)$ and $(\mathcal{B}, \nu)$ are semi-finite von Neumann algebras and $\Phi : \mathcal{A} \to \mathcal{B}$ is a unital normal Jordan $*$-homomorphism which is isometric on $\mathcal{A}^{sa}$. If $\Phi(p)\mathcal{B}\Phi(p) \subseteq \Phi(\mathcal{A})$ for all $p \in \mathcal{P}(\mathcal{A})^f$, then Φ is a Jordan $*$-isomorphism.*

Proof. Suppose $b \in \mathcal{B}$. Since Φ is normal and unital, we have that $\Phi(p) \uparrow_{p\in\mathcal{D}} \Phi(\mathbf{1}) = \mathbf{1}$. Therefore $\Phi(p) \overset{SOT}{\to} \mathbf{1}$, by Corollary B.1.9. By Remark B.1.4, $\Phi(p)b\Phi(p) \overset{SOT}{\to} \mathbf{1}b\mathbf{1} = b$ and therefore $\Phi(p)b\Phi(p) \overset{WOT}{\to} b$. Since $\Phi(p)b\Phi(p) \in \Phi(\mathcal{A})$ for all $p \in \mathcal{P}(\mathcal{A})^f$, it follows that

$$b \in \overline{\Phi(\mathcal{A})}^{\mathrm{WOT}}$$

We wish to apply Lemma 3.4.1 to show that $\overline{\Phi(\mathcal{A})}^{\mathrm{WOT}} = \Phi(\mathcal{A})$ and hence that $b \in \Phi(\mathcal{A})$. We therefore show that there exists a $k > 0$ such that $\|x\| \leq k\|\Phi(x)\|$ for all $x \in \mathcal{A}$. Let $x \in \mathcal{A}$. Then $x = x_1 + ix_2$ for some $x_1, x_2 \in \mathcal{A}^{sa}$. Φ is Jordan $*$-homomorphism and so $\Phi(x^*) = \Phi(x)^*$ for all $x \in \mathcal{A}$. It follows that $\Phi(x_1)$ and $\Phi(x_2)$ are the real and imaginary parts of $\Phi(x)$. Furthermore,

$$\begin{aligned}
\|x\|_{\mathcal{A}} &= \|x_1 + ix_2\|_{\mathcal{A}} \\
&\leq \|x_1\|_{\mathcal{A}} + \|x_2\|_{\mathcal{A}} \\
&= \|\Phi(x_1)\|_{\mathcal{B}} + \|\Phi(x_2)\|_{\mathcal{B}} \qquad \text{since } \Phi \text{ is isometric on } \mathcal{A}^{sa} \\
&= \|\mathrm{Re}(\Phi(x))\|_{\mathcal{B}} + \|\mathrm{Im}(\Phi(x))\|_{\mathcal{B}} \\
&\leq \|\Phi(x)\|_{\mathcal{B}} + \|\Phi(x)\|_{\mathcal{B}} \\
&= 2\|\Phi(x)\|_{\mathcal{B}}
\end{aligned}$$

This shows that Φ is injective and allows application of Lemma 3.4.1 to show that $\overline{\Phi(\mathcal{A})}^{\mathrm{WOT}} = \Phi(\mathcal{A})$ and hence that $b \in \Phi(\mathcal{A})$. □

CHAPTER 4

The inverse of a positive isometry

It will be useful to show that a positive surjective isometry is in fact an order isomorphism under appropriate conditions. We start by presenting two properties of positive maps.

PROPOSITION 4.1.1. *Suppose* $(\mathcal{A},\tau)$ *and* $(\mathcal{B},\nu)$ *are semi-finite von Neumann algebras and* $E \subseteq S(\mathcal{A},\tau)$ *and* $F \subseteq S(\mathcal{B},\nu)$ *are symmetric spaces. If* $U : E \to F$ *is an injective positive linear map, then* x *is self-adjoint whenever* $U(x)$ *is self-adjoint.*

PROOF. Suppose first that $y \in E^{sa}$. Then y can be written in the form $y = y_1 - y_2$, where $y_1, y_2 \geq 0$. It follows that $U(y) = U(y_1) - U(y_2)$ is self-adjoint, since $U(y_1), U(y_2) \geq 0$. It follows that U maps self-adjoint elements to self-adjoint elements. If $z \in E$, then a direct calculation using $z = \frac{1}{2}(z + z^*) + \frac{1}{2i}(z - z^*)$, the linearity of U and the fact that U maps self-adjoint elements to self-adjoint elements shows that

$$U(z^*) = U(z)^*. \tag{4.1.1}$$

Suppose $U(x)$ is a self-adjoint element. Then

$$\begin{aligned} U(x^*) &= U(x)^* && \text{by (4.1.1)} \\ &= U(x) && \text{since } U(x) \text{ is self-adjoint} \end{aligned}$$

This implies that $x^* = x$, since U is injective. □

PROPOSITION 4.1.2. *Suppose* $(\mathcal{A},\tau)$ *and* $(\mathcal{B},\nu)$ *are semi-finite von Neumann algebras,* $E \subseteq S(\mathcal{A},\tau)$ *and* $F \subseteq S(\mathcal{B},\nu)$ *are symmetric spaces, and* $U : E \to F$ *is a bijective linear positive map. If* U^{-1} *is positive, then*

$$U(x_\lambda) \uparrow U(x),$$

whenever $\{x_\lambda\}_{\lambda\in\Lambda} \cup \{x\} \subseteq E^{sa} \cap \mathcal{A}$ *is such that* $x_\lambda \uparrow x$.

PROOF. Suppose $\{x_\lambda\}_{\lambda\in\Lambda} \cup \{x\} \subseteq E^{sa} \cap \mathcal{A}$ is such that $x_\lambda \uparrow x$. Since U is linear, we can assume, without loss of generality, that all these elements are positive. Since U is positive, we have that $\{U(x_\lambda)\}_\lambda$ is an increasing net and

$$U(x_\lambda) \leq U(x) \leq \left\|U(x)\right\|_{\mathcal{B}} \mathbf{1} \qquad \forall \lambda,$$

where the last inequality follows by Proposition B.1.6. Therefore $U(x_\lambda) \uparrow y$ for some $y \in \mathcal{B}^+$, by Lemma B.1.8. Since $U(x_\lambda) \leq U(x)$ for all λ, we have that

$$y \leq U(x) \tag{4.1.2}$$

Since F is a symmetric space and $U(x) \in F$, this implies that $y \in F$, by Proposition B.3.1(5). It follows that $U^{-1}(y)$ is defined and

$$x_\lambda = U^{-1}(U(x_\lambda)) \leq U^{-1}(y) \qquad \forall \lambda,$$

since U^{-1} is positive. It follows that $x \leq U^{-1}(y)$, since x is the supremum of $\{x_\lambda\}_{\lambda\in\Lambda}$. Therefore

$$U(x) \leq U(U^{-1}(y)) = y \tag{4.1.3}$$

Combining (4.1.2) and (4.1.2) we obtain $y = U(x)$. □

In the context of symmetric spaces associated with trace-finite von Neumann algebras, one method (see [4]) of showing the positivity of the inverse of a positive operator is to start by proving the following result.

LEMMA 4.1.3. [4, p.528] *Suppose $(\mathcal{A}, \tau)$ is a trace-finite von Neumann algebra. If $x, y \in L_1(\tau)^+$, then*

$$(x-y) \ll (x+y).$$

We present a partial generalization of the previous result. It holds for spaces associated with more general von Neumann algebras, but yields a weaker conclusion.

LEMMA 4.1.4. *Let $(\mathcal{A}, \tau)$ be a semi-finite von Neumann algebra and suppose $E \subseteq S(\mathcal{A}, \tau)$ is a strongly symmetric space with absolutely continuous norm. If $x, y \in E^+$, then*

$$\|x-y\|_E \leq \|x+y\|_E.$$

PROOF. Let $x, y \in E^+$ and let $\mathcal{G}_f$ denote the set of all finite linear combinations of projections with finite trace. By Corollary 1.6.8(2), $\mathcal{G}_f^+$ is dense in E^+ and so there exist sequences $(x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty \subseteq \mathcal{G}_f^+$ such that $x_n \xrightarrow{E} x$ and $y_n \xrightarrow{E} y$. Let $p_n := s(x_n) \vee s(y_n)$. Since $\tau(s(x_n)), \tau(s(y_n)) < \infty$, we have that $\tau(p_n) < \infty$, by Proposition B.1.21(3). Let ϕ_n denote the isometric isomorphism from $p_n E p_n$ onto the reduced space E_{p_n}. It follows from Proposition 1.6.3(4) and the fact that $\mathcal{A}_{p_n}$ has finite trace, that $E_{p_n} \subseteq (L_1)_{p_n}$. Therefore

$$\phi_n(x_n) - \phi_n(y_n) \ll \phi_n(x_n) + \phi_n(y_n),$$

by Lemma 4.1.3. E is strongly symmetric and so E_{p_n} is also strongly symmetric, by Proposition 1.6.11(3). It follows that for any $n \in \mathbb{N}^+$,

$$\begin{aligned} \|\phi_n(x_n) - \phi_n(y_n)\|_{E_{p_n}} &\leq \|\phi_n(x_n) + \phi_n(y_n)\|_{E_{p_n}} \\ \implies \|\phi_n(x_n - y_n)\|_{E_{p_n}} &\leq \|\phi_n(x_n + y_n)\|_{E_{p_n}} \\ \implies \|x_n - y_n\|_E &\leq \|x_n + y_n\|_E \qquad \text{since } \phi_n \text{ is an isometry} \end{aligned} \tag{4.1.4}$$

Furthermore, $x_n - y_n \xrightarrow{E} x - y$ and $x_n + y_n \xrightarrow{E} x + y$ and so $\|x_n - y_n\|_E \to \|x-y\|_E$ and $\|x_n + y_n\|_E \to \|x+y\|_E$. Since (4.1.4) holds for all $n \in \mathbb{N}^+$, we therefore have that

$$\|x-y\|_E \leq \|x+y\|_E.$$

□

In the next result we will show that if the conclusion of Lemma 4.1.4 holds in the codomain of a positive isometry U, then $U(x)$ is positive if and only if x is positive. The reason we state this result separately is that we will see later (Section 8.4) that the conclusion of Lemma 4.1.4 also holds in certain spaces which do not have absolutely continuous norm.

LEMMA 4.1.5. *Suppose $(\mathcal{A}, \tau)$ and $(\mathcal{B}, \nu)$ are semi-finite von Neumann algebras; $E(\tau)$ and $F(\nu)$ are symmetric spaces, and $U : E \to F$ is a positive isometry. If F is such that $\|x-y\|_F \leq \|x+y\|_F$, whenever $x, y \in F^+$, then $z \geq 0$, whenever $z \in E$ and $U(z) \geq 0$.*

PROOF. Let $x \in E$ be such that $U(x) \geq 0$. Then x is self-adjoint by Proposition 4.1.1. Let x_+ and x_- be the positive and negative parts of x respectively. Then $x_+, x_- \in E$, by Proposition B.3.1(3). If $x_+ = 0$, then

$x \le 0$ and so $U(x) \le 0$, since U is positive. In this case $U(x) = 0$ and hence $x = 0$, since U is injective. If $x_+ \neq 0$, then let $b_1 := U(x_+)$, $b_2 := U(x_-)$ and $b := b_1 - b_2$. Since U is positive, $b_1, b_2 \ge 0$. Furthermore,

$$\begin{aligned}
\|b_1 + b_2\|_F &= \|U(x_+ + x_-)\|_F && \text{since } U \text{ is linear}\\
&= \|U(|x|)\|_F && \text{by Proposition B.2.2(2)}\\
&= \||x|\|_E && \text{since } U \text{ is an isometry}\\
&= \|x\|_E && \text{by Proposition B.3.1(1)} \qquad (4.1.5)
\end{aligned}$$

We show, using induction, that

$$\|b_1 + kb_2\|_F \le \|x\|_E \qquad (4.1.6)$$

for all $k \in \mathbb{N}^+$. It follows by 4.1.5 that 4.1.6 holds for $k = 1$. Suppose 4.1.6 holds for $k = n$, for some $n \in \mathbb{N}^+$. Note that $b_1 - b_2 = U(x) \ge 0 \ge -b_1$ and hence,

$$\begin{aligned}
-(b_1 + nb_2) &\le b_1 - b_2 - nb_2\\
&= b_1 - (n+1)b_2\\
&\le b_1 + nb_2 && \text{since } b_2 \ge 0
\end{aligned}$$

Since $b, nb_2 \in F(\nu)^+$, we have (by using the assumption on F) that

$$\|b - nb_2\|_F \le \|b + nb_2\|_F \qquad (4.1.7)$$

Since $b = b_1 - b_2$ and $b_2 \ge 0$, we have $b \le b_1$ and hence

$$0 \le b + nb_2 \le b_1 + nb_2 \qquad (4.1.8)$$

Therefore

$$\begin{aligned}
\|b_1 - (n+1)b_2\|_F &= \|b - nb_2\|_F\\
&\le \|b + nb_2\|_F && \text{by 4.1.7}\\
&\le \|b_1 + nb_2\|_F && \text{using 4.1.8 and the symmetry of } F\\
&\le \|x\|_E && \text{using the induction assumption} \qquad (4.1.9)
\end{aligned}$$

Furthermore, $(x_+)^*(n+1)x_- = (n+1)x_+x_- = 0$, by Proposition B.2.2(1) and so

$$|x_+ + (n+1)x_-| = |x_+ - (n+1)x_-| \qquad \text{by Proposition B.2.12(3)} \qquad (4.1.10)$$

It follows that

$$\begin{aligned}
\|b_1 + (n+1)b_2\|_F &= \|U(x_+ + (n+1)x_-)\|_F && \text{since } U \text{ is linear}\\
&= \|x_+ + (n+1)x_-\|_F && \text{since } U \text{ is an isometry}\\
&= \|x_+ - (n+1)x_-\|_F && \text{using (4.1.10) and Proposition B.3.1(1)}\\
&= \|U(x_+ - (n+1)x_-)\|_F && \text{since } U \text{ is an isometry}\\
&= \|b_1 - (n+1)b_2\|_F && \text{since } U \text{ is linear}\\
&\le \|x\|_E && \text{by 4.1.9}
\end{aligned}$$

Thus 4.1.6 holds for $k = n+1$ and hence for all $k \in \mathbb{N}^+$, by induction. Furthermore, for all $k \in \mathbb{N}^+$, $0 \leq kx_- \leq x_+ + kx_-$ and therefore

$$\begin{aligned} k\|x_-\|_E &\leq \|x_+ + kx_-\|_E && \text{since } E \text{ is symmetric} \\ &= \|U(x_+ + kx_-)\|_F && \text{since } U \text{ is an isometry} \\ &= \|b_1 + kb_2\|_F && \text{using the definitions of } b_1, b_2 \text{ and the linearity of } U \\ &\leq \|x\|_E && \text{by 4.1.6} \end{aligned}$$

It follows that $\|x_-\|_E = 0$ and hence $x \geq 0$. □

COROLLARY 4.1.6. *Suppose $(\mathcal{A}, \tau)$ and $(\mathcal{B}, \nu)$ are semi-finite von Neumann algebras, $E(\tau)$ is a symmetric space and $F(\nu)$ is a strongly symmetric space with absolutely continuous norm. If $U : E \to F$ is a positive surjective isometry, then U is an order-isomorphism.*

CHAPTER 5

Positive surjective isometries between symmetric spaces

In [4] it is shown that any positive surjective isometry between a symmetric space and a fully symmetric space is in fact a weighted non-commutative composition operator (see Theorem 2.2.7). This result was proven in the setting of finite von Neumann algebras. In this chapter we develop a partial generalization of this result in the context of semi-finite von Neumann algebras. In order to facilitate this extension we assume that both spaces have absolutely continuous norm and that the isometry U has the property that $s(U(p))$ has finite trace whenever p is a projection of finite trace (we will call such an isometry finiteness-preserving). More specifically, throughout this chapter we will assume that $\mathcal{A} \subseteq \mathcal{B}(H)$ and $\mathcal{B} \subseteq \mathcal{B}(K)$ are semi-finite von Neumann algebras equipped with semi-finite faithful normal traces τ and ν respectively; $E \subseteq S(\mathcal{A}, \tau)$ is a strongly symmetric space with absolutely continuous norm and $F \subseteq S(\mathcal{B}, \nu)$ is a fully symmetric space with absolutely continuous norm; and $U : E \to F$ is a positive surjective isometry such that $\nu(s(U(q))) < \infty$ whenever $q \in \mathcal{P}(\mathcal{A})^f$. A map with this final property will be called *finiteness-preserving*. We will show that there exists a positive operator a, affiliated with the center of $\mathcal{B}$, and a Jordan $*$-isomorphism Φ of $\mathcal{A}$ onto $\mathcal{B}$ such that

$$U(x) = a\Phi(x) \qquad \forall x \in \mathcal{A} \cap E.$$

The proof is lengthy and has therefore been divided into three sections and numerous lemmas. Before describing the structure of the proof we recall and introduce some notation to be used throughout this chapter. Recall that $\mathcal{P}(\mathcal{A})^f$ denotes the net of all projections in $\mathcal{A}$ with finite trace (when dealing with subscripts we will use $\mathcal{D}$ to denote $\mathcal{P}(\mathcal{A})^f$). Since E has absolutely continuous norm, $\mathcal{P}(\mathcal{A})^f$ is in fact the net of all projections in E, by Corollary 1.6.8(3). We recall the following notations and properties of reduced spaces (see Section 1.6). If $p \in \mathcal{P}(\mathcal{A})^f$, then $S(\mathcal{A}_p, \tau_p)$ denotes the reduced space of τ_p-measurable operators on $p(H)$ and is also given by the set $\{x_p : x \in S(\mathcal{A}, \tau)\}$, where $x_p := (pxp)|_{p(H)}$. Let ϕ_p denote the canonical map from $pS(\mathcal{A}, \tau)p$ onto $S(\mathcal{A}_p, \tau_p)$. Recall that ϕ_p is a $*$-isomorphism and that the restrictions of ϕ_p to $p\mathcal{A}p$ and pEp respectively are isometries onto the reduced spaces $\mathcal{A}_p = \{x_p : x \in \mathcal{A}\}$ and $E_p = \{x_p : x \in E\}$. Define $q_{(p)} = s(U(p))$ and let ψ_p denote the canonical map from $q_{(p)}S(\mathcal{B}, \nu)q_{(p)}$ onto $S(\mathcal{B}_{q_{(p)}}, \nu_{q_{(p)}})$. If there is no danger of confusion, we will denote ϕ_p by ϕ and ψ_p by ψ. We will regularly make use of the fact that if $x \in pS(\mathcal{A}, \tau)^{sa}p$ and $f \in \mathcal{B}_{bc}(\mathbb{R})$, then $f(\phi(x)) = \phi(f(x))$ (see Remark 1.6.9) and a similar relationship holds for elements in $q_{(p)}S(\mathcal{B}, \nu)^{sa}q_{(p)}$.

The strategy of the proof is as follows. In section 5.1, we will show that for each $p \in \mathcal{P}(\mathcal{A})^f$, U induces a positive isometry $\widetilde{U}_p$ from E_p into $F_{q_{(p)}}$. The finiteness-preserving assumption on the isometry U will ensure that each $\widetilde{U}_p$ acts between spaces associated with finite von Neumann algebras. It is problematic to show that $\widetilde{U}_p$ is surjective, which would have enabled access to the result by Chilin et al. ([4] - see Theorem 2.2.7) for the finite setting. Nonetheless, it is possible to describe the structure of $\widetilde{U}_p$ in terms of a Jordan $*$-isomorphism $\tilde{\Phi}_p$ from $\mathcal{A}_p$ onto $\mathcal{B}_{q_{(p)}}$ and a positive operator $\tilde{a}_{(p)}$ using a similar technique to the one employed in the proof of Theorem 2.2.7. In section 5.2, we convert each $\tilde{\Phi}_p$ to a Jordan $*$-isomorphism Φ_p from $p\mathcal{A}p$ onto $q_{(p)}\mathcal{B}q_{(p)}$. These maps can then be combined to define a map from $\mathcal{F}(\tau)$ into $\mathcal{B}$, which will be extended in section 5.3 to all of $\mathcal{A}$ and shown to be a Jordan $*$-isomorphism. In section 5.4 we prove a partial converse to the result obtained in the first three sections.

5.1. Converting to the finite setting

We will start this section by demonstrating that for any $p \in \mathcal{P}(\mathcal{A})^f$, U maps pEp into $q_{(p)}Fq_{(p)}$. This will enable us to show that U induces an isometry $\widetilde{U}_p$ from E_p into $F_{q_{(p)}}$. We will then proceed to define $\tilde{a}_{(p)}$, $\tilde{\Phi}_p$ and show that $\tilde{\Phi}_p$ is a Jordan $*$-isomorphism such that $\widetilde{U}_p(x) = \tilde{a}_{(p)}\tilde{\Phi}_p(x)$ for all $x \in \mathcal{A}_p$.

LEMMA 5.1.1. *If $p \in \mathcal{P}(\mathcal{A})^f$, then*

$$U(pEp) \subseteq q_{(p)}Fq_{(p)}.$$

PROOF. U is positive and therefore $U(p) \geq 0$. It follows that $r(U(p)) = s(U(p)^*) = s(U(p)) = q_{(p)}$. This implies that $q_{(p)}U(p)q_{(p)} = U(p)$ and hence $U(p) \in q_{(p)}Fq_{(p)}$. If $q \in \mathcal{P}(\mathcal{A})$, then

$$\begin{aligned} & 0 \leq pqp \leq p\mathbf{1}p && \text{by Proposition B.2.2(4)} \\ \implies\ & 0 \leq U(pqp) \leq U(p) && \text{since } U \text{ is positive} \\ \implies\ & U(pqp) \in q_{(p)}Fq_{(p)} && \text{by Proposition B.3.7(2)} \end{aligned} \tag{5.1.1}$$

Let $\mathcal{G}$ denote the set of all finite linear combinations of projections in $\mathcal{A}$. It follows from (5.1.1) that if $x = \sum_{i=1}^{n} \alpha_i q_i \in \mathcal{G}$, then $U(pxp) \in q_{(p)}Fq_{(p)}$, since U is linear and $q_{(p)}Fq_{(p)}$ is a subspace of F. Let

$$\mathcal{G}_p := \phi(\mathcal{G}) = \{x_p : x \in \mathcal{G}\}.$$

We have that $U(\phi^{-1}(\mathcal{G}_p)) \subseteq q_{(p)}Fq_{(p)}$. Furthermore, E is a strongly symmetric space with absolutely continuous norm and therefore E_p is a strongly symmetric space with absolutely continuous norm, by Proposition 1.6.11. It is easily checked that $\mathcal{G}_p$ is the set of all finite linear combinations of projections in E_p. It follows that $\mathcal{G}_p$ is dense in E_p, by Corollary 1.6.8(1). Let $y \in pEp$. Then $x := \phi(y) \in E_p$ and let $(x_n)_{n=1}^{\infty} \subseteq \mathcal{G}_p$ be such that $x_n \xrightarrow{E_p} x$. Then $U(\phi^{-1}(x_n)) \xrightarrow{F} U(\phi^{-1}(\phi(y))) = U(y)$, since U and ϕ^{-1} are isometries. $q_{(p)}Fq_{(p)}$ is closed in F, by Proposition B.3.9 and $U(\phi^{-1}(x_n)) \in q_{(p)}Fq_{(p)}$ for all $n \in \mathbb{N}^+$; therefore $U(y) \in q_{(p)}Fq_{(p)}$. □

We can therefore define $\widetilde{U}_p : E_p \to F_{q_{(p)}}$ by

$$\widetilde{U}_p = \psi \circ U \circ \phi^{-1}.$$

Note that $\widetilde{U}_p$ is a composition of isometries and is therefore an isometry. We also define the element

$$\tilde{a}_{(p)} := \widetilde{U}_p(\phi(p)) = \psi(U(p)) \in F_{q_{(p)}}.$$

It follows that $\tilde{a}_{(p)} \geq 0$. We have defined $q_{(p)}$ to be the support projection of $U(p)$. It follows by Proposition B.3.7(5) that $\psi(q_{(p)})$ is the support projection of $\psi(U(p)) = \tilde{a}_{(p)}$. Since $\psi(q_{(p)})$ is the identity of $\mathcal{B}_{q_{(p)}}$ and $\nu(q_{(p)}) = \nu(s(U(p))) < \infty$ (U has been assumed to be finiteness-preserving), it follows that $\mathcal{B}_{q_{(p)}}$ is trace-finite. Therefore $\tilde{a}_{(p)}$ is invertible in $S(\mathcal{B}_{q_{(p)}}, \nu_{q_{(p)}})$, by Proposition 1.3.5(3). Furthermore, $\tilde{a}_{(p)}^{-1} \geq 0$ and $(\tilde{a}_{(p)}^{-1})^{1/2} = (\tilde{a}_{(p)}^{1/2})^{-1}$, by Proposition B.2.2(6) and Proposition B.2.5, respectively. Let a_p denote $\psi^{-1}(\tilde{a}_{(p)}) = U(p)$. Note that if $q_p \neq \mathbf{1}_{\mathcal{B}}$, then a_p is not invertible in $S(\mathcal{B}, \nu)$. However, $\tilde{a}_{(p)} = \psi(a_p)$ does have an inverse in $S(\mathcal{B}_{q_{(p)}}, \nu_{q_{(p)}})$, as mentioned earlier. We will therefore use $a_{(p)}^{-1\cdot}$ to denote $\psi^{-1}(\tilde{a}_{(p)}^{-1})$ to show that it may not be a true inverse. However, ψ^{-1} is a homomorphism and so,

$$a_{(p)}a_{(p)}^{-1\cdot} = \psi^{-1}(\tilde{a}_{(p)})\psi^{-1}(\tilde{a}_{(p)}^{-1}) = \psi^{-1}(\tilde{a}_{(p)}\tilde{a}_{(p)}^{-1}) = \psi^{-1}(q_p) = q_{(p)} \tag{5.1.2}$$

Similarly $a_{(p)}^{-1\cdot}a_{(p)} = q_{(p)}$. Furthermore, $\tilde{a}_{(p)}^{-1}$, and hence $(\tilde{a}_{(p)}^{-1})^{1/2} = \tilde{a}_{(p)}^{-1/2}$, are positive. Therefore, $\psi^{-1}(\tilde{a}_{(p)}^{-1/2})$ is positive and

$$(a_{(p)}^{-1\cdot})^{1/2} = \left(\psi^{-1}(\tilde{a}_{(p)}^{-1})\right)^{1/2} = \psi^{-1}(\tilde{a}_{(p)}^{-1/2}), \tag{5.1.3}$$

where the functional calculi have been applied in $S(\mathcal{B}, \nu)$ and $S(\mathcal{B}_{q_{(p)}}, \nu_{q_{(p)}})$, respectively.

Note that since $\mathcal{A}_p$ is trace-finite, $\mathcal{A}_p \subseteq E_p$ by Proposition 1.6.3(4). It follows that $\widetilde{U}_p$ is defined on all of $\mathcal{A}_p$ and hence U is defined on all of $p\mathcal{A}p$. Consider the map $\tilde{\Phi}_p$ from $\mathcal{A}_p$ into $S(\mathcal{B}_{q_{(p)}}, \nu_{q_{(p)}})$ defined by

$$\tilde{\Phi}_p(x) = \tilde{a}_{(p)}^{-1/2} \widetilde{U}_p(x) \tilde{a}_{(p)}^{-1/2} \qquad x \in \mathcal{A}_p$$

LEMMA 5.1.2. *If $p \in \mathcal{P}(\mathcal{A})^f$, then $\tilde{\Phi}_p$ is a Jordan $*$-isomorphism of $\mathcal{A}_p$ onto $\mathcal{B}_{q_{(p)}}$.*

PROOF. To show that $\tilde{\Phi}_p$ is a Jordan $*$-isomorphism, we will show that $\tilde{\Phi}_p$ is a unital order isomorphism of $\mathcal{A}_p$ onto $\mathcal{B}_{q_{(p)}}$. The result will then follow by Proposition 1.8.9. Let $\tilde{p} := \phi(p)$ and $\tilde{q} := \psi(q_{(p)})$. Note that $\tilde{p}$ and $\tilde{q}$ are the identities of $\mathcal{A}_p$ and $\mathcal{B}_{q_{(p)}}$ respectively. We start by showing that $\tilde{\Phi}_p$ is unital.

$$\begin{aligned}
\tilde{\Phi}_p(\tilde{p}) &= \tilde{a}_{(p)}^{-1/2} \widetilde{U}_p(\phi(p)) \tilde{a}_{(p)}^{-1/2} \\
&= \tilde{a}_{(p)}^{-1/2} \psi(U(p)) \tilde{a}_{(p)}^{-1/2} \\
&= \tilde{a}_{(p)}^{-1/2} \tilde{a}_{(p)} \tilde{a}_{(p)}^{-1/2} \\
&= \tilde{q}
\end{aligned}$$

Next, we show that $\tilde{\Phi}_p$ is positive. Note that $\widetilde{U}_p$ is a composition of positive maps and hence positive. It follows that if $x \in \mathcal{A}_p^+$, then $\widetilde{U}_p(x) \geq 0$. Using Proposition B.2.2(4), this implies that

$$\tilde{\Phi}_p(x) = \tilde{a}_{(p)}^{-1/2} \widetilde{U}_p(x) \tilde{a}_{(p)}^{-1/2} \geq 0.$$

To show that $\tilde{\Phi}_p$ maps $\mathcal{A}_p$ into $\mathcal{B}_{q_{(p)}}$ we start by showing that $\tilde{\Phi}_p(y) \in \mathcal{B}_{q_{(p)}}$ if $y \in \mathcal{A}_p^+$. Note that if $y \in \mathcal{A}_p^+$, then

$$\begin{aligned}
0 \leq \quad & y && \leq \|y\|_{\mathcal{A}_p} \tilde{p} && \text{by Proposition B.1.6} \\
\implies 0 \leq \quad & \tilde{\Phi}_p(y) && \leq \|y\|_{\mathcal{A}_p} \tilde{\Phi}_p(\tilde{p}) && \text{since } \tilde{\Phi}_p \text{ is positive and linear} \\
& && = \|y\|_{\mathcal{A}_p} \tilde{q} && \text{since } \tilde{\Phi}_p \text{ is unital}
\end{aligned}$$

It follows that $\tilde{\Phi}_p(y) \in \mathcal{B}_{q_{(p)}}$, since $\|y\|_{\mathcal{A}_p} \tilde{q} \in \mathcal{B}_{q_{(p)}}$ and $\mathcal{B}_{q_{(p)}}$ is an absolutely solid subspace of $S(\mathcal{B}_{q_{(p)}}, \nu_{q_{(p)}})$. Since any element of $\mathcal{A}_p$ can be written as a linear combination of positive elements, and using the fact that $\tilde{\Phi}_p$ is linear and $\mathcal{B}_{q_{(p)}}$ is a vector space, we have that $\tilde{\Phi}_p(\mathcal{A}_p) \subseteq \mathcal{B}_{q_{(p)}}$. Next we show that $\tilde{\Phi}_p$ is surjective. Let $b \in \mathcal{B}_{q_{(p)}}^+$ and define $c = \tilde{a}_{(p)}^{1/2} b \tilde{a}_{(p)}^{1/2}$. Then

$$\begin{aligned}
0 &\leq c && \text{by Proposition B.2.2(4)} \\
&\leq \tilde{a}_{(p)}^{1/2} \|b\|_{\mathcal{B}_{q_{(p)}}} \tilde{q} \tilde{a}_{(p)}^{1/2} && \text{using } 0 \leq b \leq \|b\|_{\mathcal{B}_{q_{(p)}}} \tilde{q} \text{ and Proposition B.2.2(4)} \\
&= \|b\|_{\mathcal{B}_{q_{(p)}}} \tilde{a}_{(p)} && \text{since } \tilde{q} \text{ is the identity of } S(\mathcal{B}_{q_{(p)}}, \nu_{q_{(p)}})
\end{aligned} \tag{5.1.4}$$

Since F is fully symmetric, $F_{q_{(p)}}$ is also fully symmetric, by Proposition 1.6.11(3). This, combined with (5.1.4), implies that $c \in F_{q_{(p)}}$, since $\|b\| \tilde{a}_{(p)} = \|b\| \psi(U(p)) \in F_{q_{(p)}}$. Furthermore, ψ^{-1} is a positive linear map and so

$$0 \leq \psi^{-1}(c) \leq \|b\| \psi^{-1}(\tilde{a}_{(p)}) = \|b\| U(p)$$

By Corollary 4.1.6, U^{-1} is positive and therefore

$$0 \leq U^{-1}(\psi^{-1}(c)) \leq \|b\| p$$

It follows by Proposition B.3.6(5) that $d := U^{-1}(\psi^{-1}(c)) \in p\mathcal{A}p$. Therefore $\phi(d)$ is defined and $\phi(d) \in \mathcal{A}_p$. Furthermore,

$$\begin{aligned}
\tilde{\Phi}_p(\phi(d)) &= \tilde{a}_{(p)}^{-1/2}\widetilde{U}_p(\phi(d))\tilde{a}_{(p)}^{-1/2} \qquad \text{by definition of } \tilde{\Phi}_p \\
&= \tilde{a}_{(p)}^{-1/2}(\psi \circ U \circ \phi^{-1})(\phi(d))\tilde{a}_{(p)}^{-1/2} \qquad \text{by definition of } \widetilde{U}_p \\
&= \tilde{a}_{(p)}^{-1/2}(\psi \circ U \circ \phi^{-1})(\phi \circ U^{-1} \circ \psi^{-1})(c))\tilde{a}_{(p)}^{-1/2} \qquad \text{by definition of } d \\
&= \tilde{a}_{(p)}^{-1/2} c \tilde{a}_{(p)}^{-1/2} \\
&= b \qquad \text{since } c = \tilde{a}_{(p)}^{1/2} b \tilde{a}_{(p)}^{1/2}
\end{aligned}$$

It follows that $\tilde{\Phi}_p$ is surjective. Furthermore, it is clear from the above calculation that for $y \in \mathcal{B}_{q_{(p)}}$,

$$\tilde{\Phi}_p^{-1}(y) = (\phi \circ U^{-1} \circ \psi^{-1})(\tilde{a}_{(p)}^{1/2} y \tilde{a}_{(p)}^{1/2}) \tag{5.1.5}$$

All that remains to be shown is that $\tilde{\Phi}_p^{-1}$ is positive. If $y \in \mathcal{B}_{q_{(p)}}^+$, then $\tilde{a}_{(p)}^{1/2} y \tilde{a}_{(p)}^{1/2} \geq 0$, by Proposition B.2.2(4). Since ψ^{-1}, U^{-1} and ϕ are positive maps, it follows, using (5.1.5), that

$$\tilde{\Phi}_p^{-1}(y) = (\phi \circ U^{-1} \circ \psi^{-1})(\tilde{a}_{(p)}^{1/2} y \tilde{a}_{(p)}^{1/2}) \geq 0.$$

We have shown that $\tilde{\Phi}_p$ is a unital order isomorphism of $\mathcal{A}_p$ onto $\mathcal{B}_{q_{(p)}}$ and therefore $\tilde{\Phi}_p$ is a Jordan $*$-isomorphism, by Proposition 1.8.9. □

In order to show that $\widetilde{U}_p(x) = \tilde{a}_{(p)}\tilde{\Phi}_p(x)$ for all $x \in \mathcal{A}_p$, we will need to show that $\tilde{a}_{(p)}$ commutes with $\tilde{\Phi}_p(x)$ for every $x \in \mathcal{A}_p$. To this end we prove the following lemma.

Lemma 5.1.3. *If $p \in \mathcal{P}(\mathcal{A})^f$, then $\tilde{a}_{(p)} \in S(Z(\mathcal{B}_{q_{(p)}}), \nu_{q_{(p)}})$*

Proof. We have $\tilde{a}_{(p)} = \psi(U(p)) \in F_{q_{(p)}}$ and so $\tilde{a}_{(p)} \in S(\mathcal{B}_{q_{(p)}}, \nu_{q_{(p)}})$. Let $\{e(\lambda)\}$ be the spectral projection family of $\tilde{a}_{(p)}$ and $z(\lambda) = z(e(\lambda))$ the central support projection of $e(\lambda)$. Note that $\tilde{a}_{(p)}\eta\mathcal{B}_{q_{(p)}}$ implies that $e(\lambda) \in \mathcal{B}_{q_{(p)}}$ for all $\lambda > 0$, by Proposition B.2.1(2). We show that $z(\lambda) = e(\lambda)$ for all $\lambda > 0$. Suppose there exists a $\gamma > 0$ such that $0 \neq z(\gamma) - e(\gamma) = z(\gamma)e^{\perp}(\gamma)$, where the last equality follows using $z(\gamma)e(\gamma) = e(\gamma)$. Furthermore, $z(\gamma)$ and $e^{\perp}(\gamma)$ commute; therefore

$$\begin{aligned}
0 &< z(\gamma)e^{\perp}(\gamma) \qquad \text{by Theorem B.1.5} \\
&= z(\gamma)e^{\perp}(\gamma)z(\gamma) \qquad \text{since } z(\gamma) \text{ is a central projection} \\
&\leq z(\gamma)(z(e^{\perp}(\gamma)))z(\gamma) \qquad \text{using } e^{\perp}(\gamma) \leq z(e^{\perp}(\gamma)) \text{ and Proposition B.2.2(4)} \\
&= z(\gamma)(z(e^{\perp}(\gamma))) \qquad \text{since } z(\gamma) \text{ is a central projection}
\end{aligned} \tag{5.1.6}$$

Since $e^{\perp}(\gamma) = \sup_{\beta>\gamma} e^{\perp}(\beta)$ (see Remark 1.3.2), we have $z(e^{\perp}(\gamma)) = \sup_{\beta>\gamma} z(e^{\perp}(\beta))$, by Proposition B.1.17. Using this information and (5.1.6), there exists a $\beta > \gamma$ such that $z(e(\gamma))z(e^{\perp}(\beta) \neq 0$. By Proposition B.1.18 and using the fact that $\mathcal{B}_{q_{(p)}}$ is a von Neumann algebra, $e(\gamma)$ and $e^{\perp}(\beta)$ have non-zero subprojections m and n respectively such that $m \sim n$. By Lemma 5.1.2 $\tilde{\Phi}_p$ is a Jordan $*$-isomorphism from $\mathcal{A}_p$ onto $\mathcal{B}_{q_{(p)}}$. It follows, by Proposition 1.8.8(5), that $\tilde{\Phi}_p^{-1}$ is a Jordan $*$-isomorphism. Therefore $r := \tilde{\Phi}_p^{-1}(m)$ and $l := \tilde{\Phi}_p^{-1}(n)$ are projections and $\tilde{\Phi}_p^{-1}(m) \sim \tilde{\Phi}_p^{-1}(n)$, by Proposition 1.8.6(3). Since E and hence E_p is a symmetric space (see Proposition 1.6.11(3)), this implies that $\|r\|_{E_p} = \|l\|_{E_p}$, by Proposition B.3.1(7). Since $\widetilde{U}_p$ is an isometry, this implies that $\|\widetilde{U}_p(r)\|_{F_{q_{(p)}}} = \|\widetilde{U}_p(l)\|_{F_{q_{(p)}}}$ and therefore $\|\tilde{a}_{(p)}^{1/2}\tilde{\Phi}_p(r)\tilde{a}_{(p)}^{1/2}\|_{F_{q_{(p)}}} = \|\tilde{a}_{(p)}^{1/2}\tilde{\Phi}_p(l)\tilde{a}_{(p)}^{1/2}\|_{F_{q_{(p)}}}$, using the definition of $\tilde{\Phi}_p$. By considering the definitions of r and l, this implies that

$$\|\tilde{a}_{(p)}^{1/2} m \tilde{a}_{(p)}^{1/2}\|_{F_{q_{(p)}}} = \|\tilde{a}_{(p)}^{1/2} n \tilde{a}_{(p)}^{1/2}\|_{F_{q_{(p)}}}. \tag{5.1.7}$$

Furthermore, since $n \leq e^{\perp}(\beta)$ and $\tilde{a}_{(p)}e^{\perp}(\beta) \geq \beta e^{\perp}(\beta)$, by Theorem B.1.11, we have

$$|\tilde{a}_{(p)}^{1/2}n|^2 = n\tilde{a}_{(p)}n = n\tilde{a}_{(p)}e^{\perp}(\beta)n \geq n\beta e^{\perp}(\beta)n = \beta n. \tag{5.1.8}$$

$\mathcal{B}_{q_{(p)}}$ is a trace-finite von Neumann algebra and so by Proposition B.2.13 there exists a unitary operator $u \in \mathcal{B}_{q_{(p)}}$ such that $\tilde{a}_{(p)}^{1/2}n = u|\tilde{a}_{(p)}^{1/2}n|$. It follows that

$$\begin{aligned}
\tilde{a}_{(p)}^{1/2}n\tilde{a}_{(p)}^{1/2} &= |n\tilde{a}_{(p)}^{1/2}|^2 \\
&= u|\tilde{a}_{(p)}^{1/2}n|^2u^* \qquad \text{by Remark B.1.31(5)} \\
&\geq \beta unu^* \qquad \text{using (5.1.8) and Proposition B.2.2(4)} \\
&\geq 0 \qquad \text{by Proposition B.2.2(4)}
\end{aligned}$$

This implies that

$$\begin{aligned}
\|\tilde{a}_{(p)}^{1/2}n\tilde{a}_{(p)}^{1/2}\|_{F_{q_{(p)}}} &\geq \beta\|unu^*\|_{F_{q_{(p)}}} \qquad \text{since } F_{q_{(p)}} \text{ is symmetric} \\
&= \beta\|n\|_{F_{q_{(p)}}} \qquad \text{by Proposition B.3.1(6)}
\end{aligned} \tag{5.1.9}$$

Similarly, using the inequality

$$|\tilde{a}_{(p)}^{1/2}m|^2 = m\tilde{a}_{(p)}m = m\tilde{a}_{(p)}e(\gamma)m \leq m\gamma e(\gamma)m = \gamma m$$

we get

$$\|\tilde{a}_{(p)}^{1/2}m\tilde{a}_{(p)}^{1/2}\|_{F_{q_{(p)}}} \leq \gamma\|m\|_{F_{q_{(p)}}} \tag{5.1.10}$$

We therefore have that

$$\begin{aligned}
\beta\|n\|_{F_{q_{(p)}}} &\leq \|\tilde{a}_{(p)}^{1/2}n\tilde{a}_{(p)}^{1/2}\|_{F_{q_{(p)}}} \qquad \text{by (5.1.9)} \\
&= \|\tilde{a}_{(p)}^{1/2}m\tilde{a}_{(p)}^{1/2}\|_{F_{q_{(p)}}} \qquad \text{by (5.1.7)} \\
&\leq \gamma\|m\|_{F_{q_{(p)}}} \qquad \text{by (5.1.10)} \\
&= \gamma\|n\|_{F_{q_{(p)}}} \qquad \text{by Proposition B.3.1(7)}
\end{aligned}$$

Since $n \neq 0$, it follows that $\beta \leq \gamma$. This contradicts the choice of β and so $z(\lambda) = e(\lambda)$ for all $\lambda > 0$. We therefore have that $e(\lambda) \in Z(\mathcal{B}_{q_{(p)}})$ for all $\lambda > 0$ and hence $\tilde{a}_{(p)}\eta Z(\mathcal{B}_{q_{(p)}})$, by Proposition B.2.1(2). Since $\mathcal{B}_{q_{(p)}}$ and hence $Z(\mathcal{B}_{q_{(p)}})$ are trace-finite von Neumann algebras, $\tilde{a}_{(p)} \in S(Z(\mathcal{B}_{q_{(p)}}), \nu_{q_{(p)}})$ by Proposition 1.3.5. □

Corollary 5.1.4. *$\widetilde{U}_p(x) = \tilde{a}_{(p)}\tilde{\Phi}_p(x)$ for all $x \in \mathcal{A}_p$.*

Proof. Note first that if $b \in \mathcal{B}_{q_{(p)}}$, then $\tilde{a}_{(p)}b = b\tilde{a}_{(p)}$, by Lemma 5.1.3 and Proposition B.2.9. By Corollary B.2.8, this implies that

$$f(\tilde{a}_{(p)})b = bf(\tilde{a}_{(p)}) \qquad \forall f \in \mathcal{B}_{bc}(\sigma(\tilde{a}_{(p)})) \tag{5.1.11}$$

Let $x \in \mathcal{A}_p$. We have by definition of $\tilde{\Phi}_p$ that $\tilde{\Phi}_p(x) = \tilde{a}_{(p)}^{-1/2}\widetilde{U}_p(x)\tilde{a}_{(p)}^{-1/2}$ and therefore $\widetilde{U}_p(x) = \tilde{a}_{(p)}^{1/2}\tilde{\Phi}_p(x)\tilde{a}_{(p)}^{1/2}$. Since $\tilde{\Phi}_p(x) \in \mathcal{B}_{q_{(p)}}$ and $f(t) := t^{1/2} \in \mathcal{B}_{bc}(\sigma(\tilde{a}_{(p)}))$, it follows by (5.1.11) that

$$\widetilde{U}_p(x) = \tilde{a}_{(p)}\tilde{\Phi}_p(x).$$

□

5.2. Extension to $\mathcal{F}(\tau)$

We have shown that for each $p \in \mathcal{P}(\mathcal{A})^f$, the isometry U induces an isometry $\widetilde{U}_p$ between the reduced spaces E_p and $F_{q_{(p)}}$, which can be represented in terms of a Jordan $*$-isomorphism $\tilde{\Phi}_p$ from $\mathcal{A}_p$ onto $\mathcal{B}_{q_{(p)}}$ and a positive operator $\tilde{a}_{(p)}$. In this section we convert each $\tilde{\Phi}_p$ to a Jordan $*$-isomorphism from $p\mathcal{A}p$ onto $q_{(p)}\mathcal{B}q_{(p)}$ and each $\tilde{a}_{(p)}$ to a positive operator $a_{(p)}$ associated with $\mathcal{A}$. We will show that the Φ_p's can be used to define a map Φ_1 from $\mathcal{F}(\tau)$ into $\mathcal{B}$ and that the $a_{(p)}$'s can be used to define a positive operator a which will be used in the representation of U. Some of the properties of Φ_1 and some of the relationships between Φ_1 and a will be discussed.

We start by converting the Jordan $*$-isomorphism $\tilde{\Phi}_p$ between the reduced von Neumann algebras to a Jordan $*$-isomorphism between the corresponding subalgebras of $\mathcal{A}$ and $\mathcal{B}$. Define

$$\Phi_p(x) = (\psi^{-1} \circ \tilde{\Phi}_p \circ \phi)(x) \qquad x \in p\mathcal{A}p$$

and

$$a_{(p)} = U(p) = \psi^{-1}(\tilde{a}_{(p)})$$

Note that ϕ and ψ^{-1} are $*$-isomorphisms, and hence Jordan $*$-isomorphisms, from $p\mathcal{A}p$ onto $\mathcal{A}_p$, and $\mathcal{B}_{q_{(p)}}$ onto $q_{(p)}\mathcal{B}q_{(p)}$ respectively. Since the composition of Jordan $*$-isomorphisms yields a Jordan $*$-isomorphism, Φ_p is a Jordan $*$-isomorphism of $p\mathcal{A}p$ onto $q_{(p)}\mathcal{B}q_{(p)}$. Furthermore, if $x \in p\mathcal{A}p$, then

$$\begin{aligned}
a_{(p)}\Phi_p(x) &= \psi^{-1}(\tilde{a}_{(p)})\psi^{-1}(\tilde{\Phi}_p(\phi(x))) && \text{by definition of } \Phi_p \text{ and } a_{(p)} \\
&= \psi^{-1}(\tilde{a}_{(p)}\tilde{\Phi}_p(\phi(x))) && \text{since } \psi^{-1} \text{ is a } *\text{-isomorphism} \\
&= \psi^{-1}(\widetilde{U}_p(\phi(x))) && \text{by Corollary 5.1.4} \\
&= \psi^{-1}((\psi \circ U \circ \phi^{-1})(\phi(x))) && \text{by definition of } \widetilde{U}_p \\
&= U(x) && (5.2.1)
\end{aligned}$$

We wish to define a map Φ_1 on $\mathcal{F}(\tau)$. If $x \in \mathcal{F}(\tau)^+$, then $s(x) = r(x)$, $\tau(s(x)) < \infty$ and $x \in s(x)\mathcal{A}s(x)$. For any $p \in \mathcal{P}(\mathcal{A})^f$ with $p \geq s(x)$, $x \in p\mathcal{A}p$, by Proposition B.3.6(4) and so $\Phi_p(x)$ is defined for all $p \in \mathcal{P}(\mathcal{A})^f$ with $p \geq s(x)$. We will show that if $x \in \mathcal{F}(\tau)^+$, then $\{\Phi_p(x)\}_{p \in \mathcal{D}, p \geq s(x)}$ is a decreasing net, which is bounded below. This would imply the existence of the SOT-limit of this net and hence enable us to define $\Phi_1(x)$ as this limit. Since any $y \in \mathcal{F}(\tau)$ can be written as a finite linear combination of elements from $\mathcal{F}(\tau)^+$, we can then extend Φ_1 to $\mathcal{F}(\tau)$ in the natural way. In the next lemma we collect together some technical results that will be used to show that the aforementioned net is in fact decreasing. Recall that for any

LEMMA 5.2.1. *Suppose $p \in \mathcal{P}(\mathcal{A})^f$*

(1) *If $b \in q_{(p)}\mathcal{B}q_{(p)}$ and $f \in \mathcal{B}_{bc}(\sigma(a_{(p)}))$, then $f(a_{(p)})b = bf(a_{(p)})$*
(2) *If $x \in (p\mathcal{A}p)^+$, then $U(x)^{1/2} = a_{(p)}^{1/2}\Phi_p(x)^{1/2}$*
(3) *If $x \in (p\mathcal{A}p)^+$, then $(a_{(p)}^{-1\cdot})^{1/2}U(x)^{1/2} = U(x)^{1/2}(a_{(p)}^{-1\cdot})^{1/2}$, where $a_{(p)}^{-1\cdot} := \psi^{-1}(\tilde{a}_{(p)}^{-1})$*
(4) *If $e \in \mathcal{P}(\mathcal{A})^f$ with $e \leq p$, then $q_{(e)}a_{(p)}^{-1\cdot}q_{(e)} \leq (a_{(e)}^{-1\cdot})^{1/2}q_{(p)}(a_{(e)}^{-1\cdot})^{1/2}$*
(5) *If $x \in p\mathcal{A}^+p$, then $U(x)^{1/2}q_{(e)} = U(x)^{1/2} = q_{(e)}U(x)^{1/2}$ whenever $e \in \mathcal{P}(\mathcal{A})^f$ with $e \geq p$.*
(6) *If $x \in p\mathcal{A}^+p$, then $\Phi_p(x) = a_{(p)}^{-1\cdot}U(x)$*

PROOF. Recall that $\mathbb{B}(\mathbb{C})$ denotes the family of all Borel subsets of $\mathbb{C}$.
1) It follows from Lemma 5.1.3 that $\tilde{a}_{(p)}\eta Z(\mathcal{B}_{q_{(p)}})$ and hence $e^{\tilde{a}_{(p)}}(A) \in Z(\mathcal{B}_{q_{(p)}})$ for all $A \in \mathbb{B}(\mathbb{C})$, by Proposition

B.2.1(2). Therefore

$$\begin{aligned}
e^{\tilde{a}_{(p)}}(A)\psi(b) &= \psi(b)e^{\tilde{a}_{(p)}}(A) \quad \text{for all } A \in \mathbb{B}(\mathbb{C}), \text{ since } \psi(b) \in \mathcal{B}_{q_{(p)}} \\
\implies \tilde{a}_{(p)}\psi(b) &= \psi(b)\tilde{a}_{(p)} \quad \text{by Corollary B.2.8} \\
\implies \psi(a_{(p)}b) &= \psi(ba_{(p)}) \quad \text{since } \tilde{a}_{(p)} = \psi(a_{(p)}) \text{ and } \psi \text{ is a homomorphism} \\
\implies a_{(p)}b &= ba_{(p)} \quad \text{since } \psi \text{ is injective} \\
\implies f(a_{(p)})b &= bf(a_{(p)}) \quad \text{by Corollary B.2.8}
\end{aligned}$$

2) Note that $(a_{(p)}^{1/2}\Phi_p(x)^{1/2})^2 = a_{(p)}\Phi_p(x) = U(x)$, using Lemma 5.2.1(1) and (5.2.1). Furthermore, $\tilde{\Phi}_p$ is positive and so $\tilde{\Phi}_p(\phi(x)) \geq 0$. It follows from $x \in p\mathcal{A}p$ that $\tilde{\Phi}_p(\phi(x)) \in \mathcal{B}_{q_{(p)}}$ and therefore $\tilde{\Phi}_p(\phi(x))^{1/2} \in \mathcal{B}_{q_{(p)}}$, since $\mathcal{B}_{q_{(p)}}$ is a von Neumann algebra. It follows that $\Phi_p(x)^{1/2} = \left(\psi^{-1}(\tilde{\Phi}_p(\phi(x)))\right)^{1/2} = \psi^{-1}\left(\tilde{\Phi}_p(\phi(x))^{1/2}\right) \in q_{(p)}\mathcal{B}q_{(p)}$ and

$$\begin{aligned}
a_{(p)}^{1/2}\Phi_p(x)^{1/2} &= a_{(p)}^{1/4}\Phi_p(x)^{1/2}a_{(p)}^{1/4} \quad \text{by Lemma 5.2.1(1)} \\
&\geq a_{(p)}^{1/4}.0.a_{(p)}^{1/4} \quad \text{by Proposition B.2.2(4), since } \Phi_p(x)^{1/2} \geq 0 \\
&= 0
\end{aligned}$$

Therefore $U(x)^{1/2} = a_{(p)}^{1/2}\Phi_p(x)^{1/2}$, using the uniqueness of positive square roots.

3) Note that

$$\begin{aligned}
(a_{(p)}^{-1\cdot})^{1/2}U(x)^{1/2} &= (a_{(p)}^{-1\cdot})^{1/2}a_{(p)}^{1/2}\Phi_p(x)^{1/2} \quad \text{by Lemma 5.2.1(2)} \\
&= \psi^{-1}(\tilde{a}_{(p)}^{-1/2})\left(\psi^{-1}(\tilde{a}_{(p)})\right)^{1/2}\Phi_p(x)^{1/2} \quad \text{by (5.1.3)} \\
&= \psi^{-1}(\tilde{a}_{(p)}^{-1/2}\tilde{a}_{(p)}^{1/2})\Phi_p(x)^{1/2} \quad \text{since } \left(\psi^{-1}(\tilde{a}_{(p)})\right)^{1/2} = \psi^{-1}(\tilde{a}_{(p)}^{1/2}) \text{ and } \psi^{-1} \text{ is a homomorphism} \\
&= \psi^{-1}(\tilde{q})\Phi_p(x)^{1/2} \quad \text{since } \tilde{q} := \psi(q_{(p)}) \text{ is the identity of } \mathcal{B}_{q_{(p)}} \\
&= q_{(p)}\Phi_p(x)^{1/2} \\
&= \Phi_p(x)^{1/2} \quad \text{by Proposition B.3.7(4), since } \Phi_p(x) \in q_{(p)}\mathcal{B}q_{(p)} \implies \Phi_p(x)^{1/2} \in q_{(p)}\mathcal{B}q_{(p)}
\end{aligned}$$

One can similarly show that $U(x)^{1/2}(a_{(p)}^{-1\cdot})^{1/2} = \Phi_p(x)^{1/2}$, thus completing the proof of (3).

4) $e \leq p$ implies that $a_{(e)} = U(e) \leq U(p) = a_{(p)}$, since U is positive. By Proposition B.2.2(4), this implies that

$$\begin{aligned}
(a_{(p)}^{-1\cdot})^{1/2}a_{(e)}(a_{(p)}^{-1\cdot})^{1/2} &\leq (a_{(p)}^{-1\cdot})^{1/2}a_{(p)}(a_{(p)}^{-1\cdot})^{1/2} \\
&= \psi^{-1}(\tilde{a}_{(p)}^{-1/2})\psi^{-1}(\tilde{a}_{(p)})\psi^{-1}(\tilde{a}_{(p)}^{-1/2}) \quad \text{by (5.1.3)} \\
&= \psi^{-1}(\tilde{a}_{(p)}^{-1/2}\tilde{a}_{(p)}\tilde{a}_{(p)}^{-1/2}) \quad \text{since } \psi^{-1} \text{ is a homomorphism} \\
&= q_{(p)} \qquad (5.2.2)
\end{aligned}$$

Furthermore, using the definition of $a_{(e)}$, noting that $e \in (e\mathcal{A}e)^+ \subseteq (p\mathcal{A}p)^+$ and applying Lemma 5.2.1(3), we obtain

$$(a_{(p)}^{-1\cdot})^{1/2}a_{(e)}^{1/2} = (a_{(p)}^{-1\cdot})^{1/2}U(e)^{1/2} = U(e)^{1/2}(a_{(p)}^{-1\cdot})^{1/2} = a_{(e)}^{1/2}(a_{(p)}^{-1\cdot})^{1/2}.$$

Therefore $(a_{(p)}^{-1\cdot})^{1/2}a_{(e)}(a_{(p)}^{-1\cdot})^{1/2} = a_{(e)}^{1/2}a_{(p)}^{-1\cdot}a_{(e)}^{1/2}$. Combining this with (5.2.2), we obtain $a_{(e)}^{1/2}a_{(p)}^{-1\cdot}a_{(e)}^{1/2} \leq q_{(p)}$. By Proposition B.2.2(4), this implies that

$$\begin{aligned}
(a_{(e)}^{-1\cdot})^{1/2}a_{(e)}^{1/2}a_{(p)}^{-1\cdot}a_{(e)}^{1/2}(a_{(e)}^{-1\cdot})^{1/2} &\leq (a_{(e)}^{-1\cdot})^{1/2}q_{(p)}(a_{(e)}^{-1\cdot})^{1/2} \\
\implies \psi^{-1}(\tilde{a}_{(e)}^{-1/2})\psi^{-1}(\tilde{a}_{(e)}^{1/2})a_{(p)}^{-1\cdot}\psi^{-1}(\tilde{a}_{(e)}^{1/2})\psi^{-1}(\tilde{a}_{(e)}^{-1/2}) &\leq (a_{(e)}^{-1\cdot})^{1/2}q_{(p)}(a_{(e)}^{-1\cdot})^{1/2} \quad \text{as in (5.1.3)} \\
\implies q_{(e)}a_{(p)}^{-1\cdot}q_{(e)} &\leq (a_{(e)}^{-1\cdot})^{1/2}q_{(p)}(a_{(e)}^{-1\cdot})^{1/2} \quad \text{as in (5.1.2)}
\end{aligned}$$

5) Note first that $x \in p\mathcal{A}p$ implies that $\tilde{\Phi}_p(\phi(x)) \in \mathcal{B}_{q_{(p)}}$ and therefore $\tilde{\Phi}_p(\phi(x))^{1/2} \in \mathcal{B}_{q_{(p)}}$. $\Phi_p(x)^{1/2} = \left(\psi^{-1}(\tilde{\Phi}_p(\phi(x)))\right)^{1/2} = \psi^{-1}\left(\tilde{\Phi}_p(\phi(x))^{1/2}\right)$. It follows that $\Phi_p(x)^{1/2} \in q_{(p)}\mathcal{B}q_{(p)} \subseteq q_{(e)}\mathcal{B}q_{(e)}$, by Proposition B.3.6(4) and hence $\Phi_p(x)^{1/2}q_{(e)} = \Phi_p(x)^{1/2}$, by Proposition B.3.7(4). Furthermore,

$$\begin{aligned} U(x)^{1/2}q_{(e)} &= (a_{(p)}^{1/2}\Phi_p(x)^{1/2})q_{(e)} \qquad \text{by Lemma 5.2.1(2)} \\ &= a_{(p)}^{1/2}\Phi_p(x)^{1/2} \qquad \text{as shown above} \\ &= U(x)^{1/2} \qquad \text{by Lemma 5.2.1(2)} \end{aligned}$$

Since $a_{(p)}^{1/2}\Phi_p(x)^{1/2} = \Phi_p(x)^{1/2}a_{(p)}^{1/2}$, by Lemma 5.2.1(1), we can similarly show that $q_{(e)}U(x)^{1/2} = U(x)^{1/2}$.
6) Note that

$$\begin{aligned} \Phi_p(x) &= \psi^{-1}(\tilde{\Phi}_p(\phi(x))) \qquad \text{by definition of } \Phi_p \\ &= \psi^{-1}(\tilde{a}_{(p)}^{-1}\widetilde{U}_p(\phi(x))) \qquad \text{using Corollary 5.1.4 and the invertibility of } \tilde{a}_{(p)} \\ &= \psi^{-1}(\tilde{a}_{(p)}^{-1})\psi^{-1}(\widetilde{U}_p(\phi(x))) \qquad \text{since } \psi^{-1} \text{ is a } *\text{-isomorphism} \\ &= a_{(p)}^{-1\cdot}U(x) \end{aligned}$$

□

Lemma 5.2.2. *Suppose $e, p \in \mathcal{P}(\mathcal{A})^f$ are such that $p \geq e$. If $x \in (e\mathcal{A}e)^+$, then*

$$\Phi_p(x) \leq \Phi_e(x).$$

Proof. Since $x \in (e\mathcal{A}e)^+ \subseteq (p\mathcal{A}p)^+$, we have by Lemma 5.2.1(6) that

$$\begin{aligned} \Phi_p(x) &= a_{(p)}^{-1\cdot}U(x) \\ &= (a_{(p)}^{-1\cdot})^{1/2}(a_{(p)}^{-1\cdot})^{1/2}U(x)^{1/2}U(x)^{1/2} \\ &= U(x)^{1/2}a_{(p)}^{-1\cdot}U(x)^{1/2} \qquad \text{by Lemma 5.2.1(3)} \\ &= U(x)^{1/2}q_{(e)}a_{(p)}^{-1\cdot}q_{(e)}U(x)^{1/2} \qquad \text{by Lemma 5.2.1(5)} \\ &\leq U(x)^{1/2}(a_{(e)}^{-1\cdot})^{1/2}q_{(p)}(a_{(e)}^{-1\cdot})^{1/2}U(x)^{1/2} \qquad \text{by Lemma 5.2.1(4) and Proposition B.2.2(4)} \\ &= (a_{(e)}^{-1\cdot})^{1/2}U(x)^{1/2}q_{(p)}U(x)^{1/2}(a_{(e)}^{-1\cdot})^{1/2} \qquad \text{by Lemma 5.2.1(3)} \\ &= (a_{(e)}^{-1\cdot})^{1/2}U(x)^{1/2}U(x)^{1/2}(a_{(e)}^{-1\cdot})^{1/2} \qquad \text{by Lemma 5.2.1(5)} \\ &= a_{(e)}^{-1\cdot}U(x) \qquad \text{by Lemma 5.2.1(3)} \\ &= \Phi_e(x) \qquad \text{by Lemma 5.2.1(6)} \end{aligned}$$

□

We are now in a position to define Φ_1 on $\mathcal{F}(\tau)$. Let $x \in \mathcal{F}(\tau)^+$. Then $s(x) = r(x) \in \mathcal{P}(\mathcal{A})^f$ and $x \in s(x)\mathcal{A}s(x)$. Since $s(x)\mathcal{A}s(x) \subseteq p\mathcal{A}p$ for all $p \in \mathcal{P}(\mathcal{A})^f$ with $p \geq s(x)$, we have $x \in p\mathcal{A}p$ for all $p \in \mathcal{P}(\mathcal{A})^f$ with $p \geq s(x)$. By Lemma 5.2.2, $\{\Phi_p(x)\}_{p\in\mathcal{D}:p\geq s(x)}$ is a decreasing net of self-adjoint operators in $\mathcal{B}$ bounded below by 0 (since Φ_p is positive for every $p \in \mathcal{P}(\mathcal{A})^f$). It follows by Lemma B.1.8 that $\Phi_p(x) \overset{SOT}{\to} y$ for some self-adjoint operator $y \in \mathcal{B}$ and $y = \bigwedge_{p\in\mathcal{D}:p\geq s(x)} \Phi_p(x)$. Define

$$\Phi_1(x) = \text{SOT} \lim_{p\in\mathcal{D}:p\geq s(x)} \Phi_p(x).$$

Φ_p is linear for each $p \in \mathcal{P}(\mathcal{A})^f$ and scalar multiplication and addition are continuous with respect to the strong operator topology and so Φ_1 is linear on $\mathcal{F}(\tau)^+$. Since any $x \in \mathcal{G}$ can be written as a linear combination of positive elements from $\mathcal{F}(\tau)$, we can extend Φ_1 to all of $\mathcal{F}(\tau)$ by linearity (we will also denote this extension by

Φ_1. Note that if $x \in \mathcal{F}(\tau)$ and W and $p \in \mathcal{P}(\mathcal{A})^f$ is such that $p \geq s(x) \vee r(x)$, then $\Phi_p(pxp) = \Phi_p(x)$. It is therefore easily checked that

$$\text{SOT}\lim_{p\in\mathcal{D}} \Phi_p(pxp) = \Phi_1(x) = \text{SOT}\lim_{p\in\mathcal{D}:p\geq s(x)\vee r(x)} \Phi_p(x) \tag{5.2.3}$$

We note that if $e \in \mathcal{F}(\tau)$ is a projection, then $\Phi_1(e) = \text{SOT}\lim_{p\in\mathcal{D}:p\geq e} \Phi_p(e)$ is a projection, since $\{\Phi_p(e)\}_{p\in\mathcal{D}:p\geq e}$ is a decreasing net of projections. In the next lemma we present some of the properties of Φ_1.

LEMMA 5.2.3. *Φ_1 is a positive map and is square preserving on self-adjoint elements.*

PROOF. To show that Φ_1 is positive, suppose $x, y \in \mathcal{F}(\tau)^{sa}$ with $x \leq y$. Then

$$\begin{aligned}
x \leq y \quad &\Longrightarrow \quad pxp \leq pyp \qquad \forall p \in \mathcal{P}(\mathcal{A})^f \qquad \text{by Proposition B.2.2(4)} \\
&\Longrightarrow \quad \Phi_p(pxp) \leq \Phi_p(pyp) \qquad \forall p \in \mathcal{P}(\mathcal{A})^f \qquad \text{since } \Phi_p \text{ is positive for all } p \in \mathcal{P}(\mathcal{A})^f \\
&\Longrightarrow \quad \text{SOT}\lim_{p\in\mathcal{D}} \Phi_p(pxp) \leq \text{SOT}\lim_{p\in\mathcal{D}} \Phi_p(pyp) \qquad \text{since both of these limits exist} \\
&\Longrightarrow \quad \Phi_1(x) \leq \Phi_1(y) \qquad \text{by (5.2.3)}
\end{aligned}$$

To show that Φ_1 is square-preserving on self-adjoint elements, we will use the fact that multiplication is jointly SOT-continuous, provided the first variable is restricted to a bounded set. We therefore start by showing that if $x \in \mathcal{F}(\tau)^{sa}$, then $\{\Phi_p(x)\}_{p\in\mathcal{D}:p\geq s(x)}$ is bounded in $\mathcal{B}$. By Proposition B.1.6, $-\|x\|_{\mathcal{A}}\mathbf{1} \leq x \leq \|x\|_{\mathcal{A}}\mathbf{1}$ and therefore $-\|x\|_{\mathcal{A}} s(x) \leq x \leq \|x\|_{\mathcal{A}} s(x)$, by Proposition B.2.2(4). This implies that if $p \in \mathcal{P}(\mathcal{A})^f$ with $p \geq s(x)$, then

$$\begin{aligned}
-\|x\|_{\mathcal{A}}\Phi_p(s(x)) \quad &\leq \quad \Phi_p(x) \leq \|x\|_{\mathcal{A}}\Phi_p(s(x)) \qquad \text{since } \Phi_p \text{ is positive} \\
\Longrightarrow -\|x\|_{\mathcal{A}}\mathbf{1}_{\mathcal{B}} \quad &\leq \quad \Phi_p(x) \leq \|x\|_{\mathcal{A}}\mathbf{1}_{\mathcal{B}} \qquad \text{since } \Phi_p(s(x)) \text{ is a projection}
\end{aligned}$$

It follows that $\{\Phi_p(x)\}_{p\in\mathcal{D}:p\geq s(x)}$ is bounded in $\mathcal{B}$. Note that since $x \in p\mathcal{A}p$ for $p \geq s(x)$, $x^2 \in p\mathcal{A}p$ for $p \geq s(x)$, since each $p\mathcal{A}p$ is a subalgebra of $\mathcal{A}$. Furthermore,

$$\begin{aligned}
\Phi_1(x^2) \quad &= \quad \text{SOT}\lim_{p\in\mathcal{D}:p\geq s(x)} \Phi_p(x^2) \\
&= \quad \text{SOT}\lim_{p\in\mathcal{D}:p\geq s(x)} (\Phi_p(x))^2 \qquad \text{since each } \Phi_p \text{ is a Jordan } *\text{-isomorphism} \\
&= \quad (\text{SOT}\lim_{p\in\mathcal{D}:p\geq s(x)} \Phi_p(x))^2 \qquad \text{by Remark B.1.4, since } \{\Phi_p(x)\}_{p\in\mathcal{D}:p\geq s(x)} \text{ is bounded} \\
&= \quad (\Phi_1(x))^2
\end{aligned}$$

□

Next we define a. Let $a_{(p)} = \int_0^\infty \lambda de^{a_{(p)}}(\lambda)$ be the spectral decomposition of $a_{(p)}$. Suppose $e \in \mathcal{P}(\mathcal{A})^f$ with $e \geq p$. Then $a_{(p)} = U(p) \leq U(e) = a_{(e)}$, since U is positive. Recall that $\psi(a_{(p)}) \in F_{q_{(p)}}$ and so $\psi(a_{(p)})\eta\mathcal{B}_{q_{(p)}}$. It follows by Proposition B.2.1(2) that $e^{\psi(a_{(p)})}(A) \in \mathcal{B}_{q_{(p)}}$ for all $A \in \mathbb{B}(\mathbb{R})$. Since $e^{\psi(a_{(p)})}(A) = \psi(e^{a_{(p)}}(A))$, by Remark 1.6.9, we have that $e^{a_{(p)}}(A) \in q_{(p)}\mathcal{B}q_{(p)}$ for all $A \in \mathbb{B}(\mathbb{R})$ and therefore,

$$\begin{aligned}
e^{a_{(p)}}(A_1)f(a_{(e)}) \quad &= \quad f(a_{(e)})e^{a_{(p)}}(A_1) \qquad \forall f \in \mathcal{B}_{bc}(\sigma(U(e))), \forall A_1 \in \mathbb{B}([0,\infty)) \qquad \text{by Lemma 5.2.1(1)} \\
\Longrightarrow e^{a_{(p)}}(A_1)e^{a_{(e)}}(A_2) \quad &= \quad e^{a_{(e)}}(A_2)e^{a_{(p)}}(A_1) \qquad \forall A_1, A_2 \in \mathbb{B}([0,\infty)) \\
\Longrightarrow e^{a_{(p)}}(\lambda,\infty) \quad &\leq \quad e^{a_{(e)}}(\lambda,\infty) \qquad \forall \lambda \geq 0 \qquad \text{by Proposition B.2.10}
\end{aligned}$$

Fix $\lambda > 0$. By Proposition B.1.19, $\{e^{a_{(p)}}(\lambda,\infty)\}_{p\in\mathcal{D}}$ converges in the strong operator topology. Define

$$e^a(\lambda,\infty) := \text{SOT}\lim_{p\in\mathcal{D}} e^{a_{(p)}}(\lambda,\infty).$$

One can show that $\{e^a(\lambda,\infty)\}_{\lambda\geq 0}$ yields a resolution of the identity and can therefore be used to define a spectral measure. It follows by Remark B.2.4 that letting

$$a = \int_0^\infty \lambda de^a(\lambda)$$

yields a closed and densely defined positive operator. It will be shown later that a is in fact also affiliated with the center of $\mathcal{B}$. The next few results detail the relationship between a and Φ_1.

LEMMA 5.2.4. *If $x \in \mathcal{F}(\tau)$, then*

$$\Phi_1(x)a = a\Phi_1(x).$$

PROOF. Suppose $x \in \mathcal{F}(\tau)^+$. Then $x \in e\mathcal{A}^+e$ for some $e \in \mathcal{P}(\mathcal{A})^f$. For any $p \in \mathcal{P}(\mathcal{A})^f$ with $p \geq e$, we have that $\Phi_p(x) \in q_{(p)}\mathcal{B}q_{(p)}$ and so for every $\lambda \geq 0$,

$$\Phi_p(x)e^{a_{(p)}}(\lambda,\infty) \quad = \quad e^{a_{(p)}}(\lambda,\infty)\Phi_p(x) \qquad \text{by Lemma 5.2.1(1)} \tag{5.2.4}$$

Fix $\lambda > 0$. We want to show that $\Phi_p(x)e^{a_{(p)}}(\lambda,\infty) \overset{SOT}{\to} \Phi_1(x)e^a(\lambda,\infty)$ and $e^{a_{(p)}}(\lambda,\infty)\Phi_p(x) \overset{SOT}{\to} e^a(\lambda,\infty)\Phi_1(x)$. Note that $\Phi_p(x) \overset{SOT}{\to} \Phi_1(x)$ using (5.2.3), and $e^{a_{(p)}}(\lambda,\infty) \overset{SOT}{\to} e^a(\lambda,\infty)$ by definition of $e^a(\lambda,\infty)$. Furthermore, for any $p \in \mathcal{P}(\mathcal{A})^f$ with $p \geq e$, we have $0 \leq \Phi_p(x) \leq \|x\|_{\mathcal{A}}\mathbf{1}_{\mathcal{B}}$, by Proposition B.1.6, and $0 \leq e^{a_{(p)}}(\lambda,\infty) \leq \mathbf{1}_{\mathcal{B}}$. Therefore $\Phi_p(x)e^{a_{(p)}}(\lambda,\infty) \overset{SOT}{\to} \Phi_1(x)e^a(\lambda,\infty)$ and $e^{a_{(p)}}(\lambda,\infty)\Phi_p(x) \overset{SOT}{\to} e^a(\lambda,\infty)\Phi_1(x)$, by Remark B.1.4. Using (5.2.4), this implies that $e^a(\lambda,\infty)\Phi_1(x) = \Phi_1(x)e^a(\lambda,\infty)$. Since this holds for every $\lambda \geq 0$, it follows by Corollary B.2.8 that $a\Phi_1(x) = \Phi_1(x)a$. The result can be extended to all of $\mathcal{F}(\tau)$ using the linearity of Φ_1. $\square$

LEMMA 5.2.5. *For $\lambda > 0$ and $p \in \mathcal{P}(\mathcal{A})^f$,*

$$e^a(\lambda,\infty)\Phi_1(p) = e^{a_{(p)}}(\lambda,\infty).$$

PROOF. For $e \in \mathcal{P}(\mathcal{A})^f$ with $e \geq p$, we have that $p \in p\mathcal{A}p \subseteq e\mathcal{A}e$ and so $\Phi_e(p)$ is defined. Using the definition of $a_{(p)}$ and applying (5.2.1), we obtain

$$a_{(p)} = U(p) = a_{(e)}\Phi_e(p) \tag{5.2.5}$$

Furthermore, $\Phi_e(p)$ is a projection (since Φ_e is a Jordan $*$-isomorphism) and $a_{(e)}\Phi_e(p) = \Phi_e(p)a_{(e)}$, by Lemma 5.2.1(1). It follows by Proposition B.2.11 that

$$\begin{aligned}
\int_0^\infty \lambda d(e^{a_{(e)}}(\lambda)\Phi_e(p)) \quad &= \quad (\int_0^\infty \lambda de^{a_{(e)}}(\lambda))\Phi_e(p) \\
&= \quad a_{(e)}\Phi_e(p) \\
&= \quad a_{(p)} \qquad \text{by (5.2.5)} \\
&= \quad \int_0^\infty \lambda de^{a_{(p)}}(\lambda) \qquad (5.2.6)
\end{aligned}$$

It follows from (5.2.6) and the uniqueness of the spectral decomposition that $e^{a_{(p)}}(\lambda,\infty) = e^{a_{(e)}}(\lambda,\infty)\Phi_e(p)$ for every $\lambda \geq 0$. As in the proof of Lemma 5.2.4, we can show that $e^{a_{(e)}}(\lambda,\infty)\Phi_e(p) \overset{SOT}{\to} e^a(\lambda,\infty)\Phi_1(p)$. Therefore,

$$e^{a_{(p)}}(\lambda,\infty) = e^a(\lambda,\infty)\Phi_1(p).$$

$\square$

COROLLARY 5.2.6. *If $p \in \mathcal{P}(\mathcal{A})^f$, then*

$$a_{(p)} = a\Phi_1(p)$$

PROOF.

$$\begin{aligned} a\Phi_1(p) &= \Big(\int_0^\infty \lambda de^a(\lambda)\Big)\Phi_1(p) \\ &= \int_0^\infty \lambda d(e^a(\lambda)\Phi_1(p)) \qquad \text{by Lemma 5.2.4 and Proposition B.2.11} \\ &= \int_0^\infty \lambda de^{a_{(p)}}(\lambda) \qquad \text{by Lemma 5.2.5} \\ &= a_{(p)} \end{aligned}$$

□

We finish this section with a technical lemma that will be used in the sequel to show that Φ_1 is normal.

LEMMA 5.2.7. *If $p \in \mathcal{P}(\mathcal{A})^f$ and $x \in p\mathcal{A}p$, then*

$$\Phi_1(p)\Phi_p(x) = \Phi_1(x) = \Phi_p(x)\Phi_1(p).$$

PROOF. For $e \in \mathcal{P}(\mathcal{A})^f$ with $e \geq p$ we have,

$$\begin{aligned} a_{(e)}\Phi_e(x) &= U(x) \qquad \text{by (5.2.1), since } p\mathcal{A}p \subseteq e\mathcal{A}e \\ \Longrightarrow a_{(e)}^{-1\cdot}a_{(e)}\Phi_e(x) &= a_{(e)}^{-1\cdot}U(x) \\ \Longrightarrow q_{(e)}\Phi_e(x) &= a_{(e)}^{-1\cdot}U(x) \qquad \text{by (5.1.2)} \\ \Longrightarrow \Phi_e(x) &= a_{(e)}^{-1\cdot}U(x) \qquad \text{by Proposition B.3.7(4), since } \Phi_e(x) \in q_{(e)}\mathcal{B}q_{(e)} \\ &= a_{(e)}^{-1\cdot}a_{(p)}\Phi_p(x) \qquad \text{by (5.2.1)} \\ &= a_{(e)}^{-1\cdot}(a\Phi_1(p))\Phi_p(x) \qquad \text{by Lemma 5.2.5} \\ &= a_{(e)}^{-1\cdot}(a\Phi_1(e)\Phi_1(p))\Phi_p(x) \qquad \text{since } \Phi_1(e), \Phi_1(p) \in \mathcal{P}(\mathcal{B}) \text{ and } \Phi_1(p) \leq \Phi_1(e) \\ &= a_{(e)}^{-1\cdot}(a_{(e)}\Phi_1(p))\Phi_p(x) \qquad \text{by Lemma 5.2.5} \\ &= q_{(e)}\Phi_e(p)\Phi_p(x) \qquad \text{by (5.1.2)} \\ &= \Phi_e(p)\Phi_p(x) \qquad \text{by Proposition B.3.7(4), since } \Phi_e(x) \in q_{(e)}\mathcal{B}q_{(e)} \end{aligned} \tag{5.2.7}$$

Furthermore, $\Phi_e(x) \overset{SOT}{\to} \Phi_1(x)$ and $\Phi_e(p)\Phi_p(x) \underset{e}{\overset{SOT}{\to}} \Phi_1(p)\Phi_p(x)$. Combining this with (5.2.7), we obtain

$$\Phi_1(x) = \Phi_1(p)\Phi_p(x).$$

Noting that for $e \in \mathcal{P}(\mathcal{A})^f$ with $e \geq p$, we have

$$\begin{aligned} \Phi_e(x)a_{(e)} &= a_{(e)}\Phi_e(x) \qquad \text{by Lemma 5.2.1(1)} \\ &= U(x) \qquad \text{using (5.2.1)} \end{aligned}$$

we can show, using a similar argument to the one employed above, that

$$\Phi_1(x) = \Phi_p(x)\Phi_1(p).$$

□

5.3. Extension to $\mathcal{A}$

In this section we show that Φ_1 can be extended to a Jordan $*$-isomorphism Φ from $\mathcal{A}$ onto $\mathcal{B}$ and that $U(x) = a\Phi(x)$ for all $x \in \mathcal{A} \cap E$. We have shown that $\Phi_1 : \mathcal{F}(\tau) \to \mathcal{B}$ is linear, positive and square-preserving on self-adjoint elements (Lemma 5.2.3). To use Proposition 3.3.6 to extend Φ_1 to a normal Jordan $*$-homomorphism Φ from $\mathcal{A}$ into $\mathcal{B}$ we need to show that Φ_1 is normal and that for $p \in \mathcal{P}(\mathcal{A})^f$, $\Phi_1(p) = 0$ if and only if $p = 0$. Note that if $p = 0$, then $\Phi_1(p) = 0$, since Φ_1 is linear. If $p > 0$, then

$$\begin{aligned} 0 &\neq U(p) && \text{since } U \text{ is an isometry and hence injective} \\ &= a_{(p)} && \text{by definition of } a_{(p)} \\ &= a\Phi_1(p) && \text{by Corollary 5.2.6} \end{aligned}$$

It follows that $\Phi_1(p) \neq 0$. Next, we show that Φ_1 is normal. Since Φ_1 is linear, it suffices to show that $\Phi_1(x_\gamma) \uparrow_\gamma \Phi_1(x)$, whenever $x_\gamma \uparrow x$ in $\mathcal{F}(\tau)^+$. Suppose $x_\gamma \uparrow x$ in $\mathcal{F}(\tau)^+$. Φ_1 is positive and so $\{\Phi_1(x_\gamma)\}_\gamma$ is increasing; it therefore suffices by Corollary B.1.9 to show that $\Phi_1(x_\gamma) \overset{SOT}{\underset{\gamma}{\to}} \Phi_1(x)$. Since $x \in \mathcal{F}(\tau)^+$, we have that $x \in e\mathcal{A}^+e$ for some $e \in \mathcal{P}(\mathcal{A})^f$. Let $p \in \mathcal{P}(\mathcal{A})^f$ with $p \geq e$. By Proposition B.3.6(5), $x_\gamma \in p\mathcal{A}^+p$ for all γ, since $x \in p\mathcal{A}p$ and $0 \leq x_\gamma \leq x$ for all γ. Since Φ_p is a Jordan $*$-isomorphism and hence normal (Proposition 1.8.8(4)), we have that $\Phi_p(x_\gamma) \uparrow_\gamma \Phi_p(x)$. By Corollary B.1.9, this implies that

$$\Phi_p(x_\gamma) \overset{SOT}{\underset{\gamma}{\to}} \Phi_p(x) \tag{5.3.1}$$

Furthermore, since $p \in \mathcal{P}(\mathcal{A})^f$ with $p \geq e$, we have

$$\begin{aligned} \Phi_1(x_\gamma) &= \Phi_1(p)\Phi_p(x_\gamma) && \text{by Lemma 5.2.7} \\ &\overset{SOT}{\underset{\gamma}{\to}} \Phi_1(p)\Phi_p(x) && \text{using (5.3.1) and Remark B.1.4} \\ &= \Phi_1(x) && \text{by Lemma 5.2.7} \end{aligned}$$

It follows that Φ_1 is normal. We can therefore use Proposition 3.3.6 to extend Φ_1 uniquely to a normal Jordan $*$-homomorphism Φ from $\mathcal{A}$ into $\mathcal{B}$. Furthermore,

$$\Phi(x) = \mathrm{SOT}\lim_{p \in \mathcal{D}} \Phi_1(pxp) \qquad \forall x \in \mathcal{A}$$

and

$$\left\|\Phi(x)\right\|_{\mathcal{B}} = \left\|x\right\|_{\mathcal{A}} \qquad \forall x \in \mathcal{A}^{sa}.$$

We will show that $U(x) = a\Phi(x)$ for all $x \in \mathcal{A} \cap E$ and that Φ is in fact surjective. To do so, we will use the following result.

Lemma 5.3.1. *If $x \in \mathcal{A}$ and $f \in \mathcal{B}_{bc}(\sigma(a))$, then*

$$f(a)\Phi(x) = \Phi(x)f(a).$$

Proof. Suppose $x \in \mathcal{A}^+$. For any $p \in \mathcal{P}(\mathcal{A})^f$, $pxp \in \mathcal{F}(\tau)$ and so by Lemma 5.2.4,

$$\Phi_1(pxp)a = a\Phi_1(pxp) \tag{5.3.2}$$

Furthermore $\Phi_1(pxp) \overset{SOT}{\to} \Phi(x)$ and $0 \leq \Phi_1(pxp) \leq \left\|x\right\|_{\mathcal{A}} \mathbf{1}_{\mathcal{B}}$ for all $p \in \mathcal{P}(\mathcal{A})^f$. It follows by Remark B.1.4 that $\Phi_1(pxp)a \overset{SOT}{\to} \Phi(x)a$ and $a\Phi_1(pxp) \overset{SOT}{\to} a\Phi(x)$. Therefore, using (5.3.2), we obtain $\Phi(x)a = a\Phi(x)$ and hence, by Corollary B.2.8,

$$\Phi(x)f(a) = f(a)\Phi(x) \qquad \forall f \in \mathcal{B}_{bc}(\sigma(a)).$$

The result can be extended from $\mathcal{A}^+$ to $\mathcal{A}$ in the usual way. $\square$

PROPOSITION 5.3.2. *If $x \in \mathcal{A} \cap E$, then*

$$U(x) = a\Phi(x).$$

PROOF. Suppose $x \in \mathcal{F}(\tau)$. Then $x \in e\mathcal{A}e$ for some $e \in \mathcal{P}(\mathcal{A})^f$. For $p \in \mathcal{P}(\mathcal{A})^f$ with $p \geq e$, we have $x \in p\mathcal{A}p$ and

$$\begin{aligned}
U(x) &= a_{(p)}\Phi_p(x) && \text{by (5.2.1)} \\
&= a\Phi_1(p)\Phi_p(x) && \text{by Corollary 5.2.6} \\
&= a\Phi_1(p)\Phi_p(pxp) && \text{by Proposition B.3.7(3), since } x \in p\mathcal{A}p \\
&\overset{SOT}{\to} a\Phi(\mathbf{1})\Phi_1(x) && \text{using (5.2.3)} \\
&= a\Phi(\mathbf{1})\Phi(x) && \text{since } \Phi \text{ extends } \Phi_1 \\
&= a\Phi(x) && \text{by Proposition 1.8.2(7)} \qquad (5.3.3)
\end{aligned}$$

Next, suppose that $x \in \mathcal{A}^+ \cap E$. By Proposition B.2.3, there exists an increasing net $\{x_\lambda\}_{\lambda\in\Lambda} \subseteq \mathcal{F}(\tau)^+$ such that $x_\lambda \uparrow x$. By (5.3.3),

$$U(x_\lambda) = a\Phi(x_\lambda) \qquad \text{for all } \lambda \tag{5.3.4}$$

Furthermore, by Corollary 4.1.6, U is an order isomorphism and so, by Proposition 4.1.2,

$$U(x_\lambda) \uparrow U(x) \tag{5.3.5}$$

Furthermore, if we let $f(t) := t^{1/2}$, then $f \in \mathcal{B}_{bc}(\sigma(a))$ and

$$\begin{aligned}
a\Phi(x_\lambda) &= f(a)f(a)\Phi(x_\lambda) \\
&= f(a)\Phi(x_\lambda)f(a) && \text{by Lemma 5.3.1} \\
&\uparrow f(a)\Phi(x)f(a) && \text{using Proposition B.2.2(7) and the normality of } \Phi \\
&= f(a)f(a)\Phi(x) && \text{by Lemma 5.3.1} \\
&= a\Phi(x) && (5.3.6)
\end{aligned}$$

It follows from (5.3.4), (5.3.5) and (5.3.6) that

$$U(x) = a\Phi(x).$$

The result can be extended to arbitrary elements of $\mathcal{A} \cap E$, since any such element can be written as a linear combination of elements from $\mathcal{A}^+ \cap E$ and U and Φ are linear. $\square$

PROPOSITION 5.3.3. *Φ is a Jordan $*$-isomorphism of $\mathcal{A}$ onto $\mathcal{B}$.*

PROOF. Thus far we have shown that Φ is a normal Jordan $*$-homomorphism from $\mathcal{A}$ into $\mathcal{B}$. In order to apply Proposition 3.4.2 to show that Φ is a Jordan $*$-isomorphism, we need to show that Φ is unital and $\Phi(p)\mathcal{B}\Phi(p) \subseteq \Phi(A)$ for all $p \in \mathcal{P}(\mathcal{A})^f$. We start by showing that Φ is unital. Assume that $\mathbf{1} - \Phi(\mathbf{1}) \neq 0$. Since $(\mathcal{B}, \nu)$ is semi-finite, there exists a $q \in \mathcal{P}(\mathcal{B})$ such that $0 < q \leq \mathbf{1} - \Phi(\mathbf{1})$ and $\nu(q) < \infty$. This implies that $q \in F$ and hence there exists an $x \in E$ such that $U(x) = q$, since U is surjective. E has absolutely continuous norm and therefore $\mathcal{A} \cap E$ is dense in E, by Theorem 1.6.7. Let $(x_n)_{n=1}^\infty \subseteq \mathcal{A} \cap E$ be such that $x_n \overset{E}{\to} x$. Then

$$\begin{aligned}
a\Phi(x_n) &= U(x_n) && \text{by Proposition 5.3.2} \\
&\overset{F}{\to} U(x) && \text{since } U \text{ is an isometry and hence continuous} \\
&= q
\end{aligned}$$

However, we also have that

$$\begin{aligned} a\Phi(x_n) &= a\Phi(x_n)\Phi(\mathbf{1}) && \text{for all } n, \text{ by Proposition 1.8.2(7)} \\ &= U(x_n)\Phi(\mathbf{1}) && \text{by Proposition 5.3.2} \\ &\xrightarrow{F} U(x)\Phi(\mathbf{1}) && \text{by Proposition B.3.1(11)} \\ &= q\Phi(\mathbf{1}) \end{aligned}$$

It follows that $q = q\Phi(\mathbf{1})$. However, since $q \leq \mathbf{1} - \Phi(\mathbf{1})$, we have that $q(\mathbf{1} - \Phi(\mathbf{1})) = q$, and so

$$q = q\Phi(\mathbf{1}) = \Big(q(\mathbf{1} - \Phi(\mathbf{1}))\Big)\Phi(\mathbf{1}) = 0.$$

This is a contradiction and so Φ is unital. Next we show that if $e \in \mathcal{P}(\mathcal{A})^f$, then

$$\Phi(e)\mathcal{B}\Phi(e) \subseteq \Phi(\mathcal{A}).$$

Recall that for for $p \in \mathcal{P}(\mathcal{A})^f$, $\Phi_1(p) = \mathrm{SOT}\lim_{q\in\mathcal{D}:q\geq p} \Phi_q(p)$ and $\Phi_q(p) \leq \Phi_p(p)$ for all $q \in \mathcal{P}(\mathcal{A})^f$ with $q \geq p$ (see Lemma 5.2.2). It follows that $\Phi(p) = \Phi_1(p) \leq \Phi_p(p)$. Therefore, if $p \in \mathcal{P}(\mathcal{A})^f$ with $p \geq e$, then

$$\begin{aligned} \Phi(e) &\leq \Phi(p) && \text{since } \Phi \text{ is positive} \\ &\leq \Phi_p(p) \\ \Longrightarrow \Phi(e)\mathcal{B}\Phi(e) &\subseteq \Phi_p(p)\mathcal{B}\Phi_p(p) && \text{by Proposition B.3.6} \end{aligned} \tag{5.3.7}$$

Suppose $b \in (\Phi(e)\mathcal{B}\Phi(e))^+$ and $p \in \mathcal{P}(\mathcal{A})^f$ with $p \geq e$. Then $b \in (\Phi_p(p)\mathcal{B}\Phi_p(p))^+$, using (5.3.7), and so there exists an $x_{(p)} \in p\mathcal{A}p$ such that $\Phi_p(x_{(p)}) = b$, since $\tilde{\Phi}_p$ is a surjective map from $\mathcal{A}_p$ onto $\mathcal{B}_{q_{(p)}}$. We wish to show that $\{x_{(p)}\}_{p\in\mathcal{D}:p\geq e}$ is increasing, that $x_{(p)} \uparrow x$ for some $x \in \mathcal{A}^+$ and that $\Phi(x) = b$. We start by showing that $\{\Phi(x_{(p)})\}_{p\in\mathcal{D}:p\geq e}$ is increasing. Note that $\Phi_1(p)\Phi_p(x_{(p)}) = \Phi_p(x_{(p)})\Phi_1(p)$, by Lemma 5.2.7. Furthermore,

$$\begin{aligned} \Phi_1(x_{(p)}) &= \Phi_1((p))\Phi_p(x_{(p)}) && \text{by Lemma 5.2.7} \\ &= (\Phi_p(x_{(p)}))^{1/2}\Phi_1(p)(\Phi_p(x_{(p)}))^{1/2} && \text{by Corollary B.2.8} \\ &= b^{1/2}\Phi_1(p)b^{1/2} && \text{since } \Phi_p(x_{(p)}) = b \quad (5.3.8) \\ &\uparrow b^{1/2}\Phi(\mathbf{1})b^{1/2} && \text{by Proposition B.2.2(7), since } \Phi_1(p) \uparrow \Phi(\mathbf{1}) \\ &= b && \text{since } \Phi \text{ is unital} \quad (5.3.9) \end{aligned}$$

To show that this implies that $\{x_{(p)}\}_{p\in\mathcal{D}:p\geq e}$ is increasing we show that if $x, y \in E\cap\mathcal{A}$ are such that $\Phi(x) \leq \Phi(y)$, then $x \leq y$. Note that if $\Phi(x) \leq \Phi(y)$, then $a^{1/2}\Phi(x)a^{1/2} \leq a^{1/2}\Phi(y)a^{1/2}$, by Proposition B.2.2(4). By Lemma 5.3.1, this implies that $a\Phi(x) \leq a\Phi(y)$ and hence $U(x) \leq U(y)$, by Lemma 5.3.2. It follows by Corollary 4.1.6, that $x \leq y$. In particular, since $p\mathcal{A}p \subseteq \mathcal{F}(\tau) \subseteq E \cap \mathcal{A}$, for each $p \in \mathcal{P}(\mathcal{A})^f$ and $\{\Phi(x_{(p)})\}_{p\in\mathcal{D}:p\geq e}$ is increasing, this implies that $\{x_{(p)}\}_{p\in\mathcal{D}:p\geq e}$ is increasing. Moreover, it follows from (5.3.8) that each $\Phi_1(x_{(p)})$ is positive and hence each $x_{(p)}$ is positive. In order to show that $x_{(p)} \uparrow x$ for some $x \in \mathcal{A}^+$, we need to show that $\{x_{(p)}\}_{p\in\mathcal{D}:p\geq e}$ is bounded. Since Φ isometric on self-adjoint elements, we have that

$$\left\|x_{(p)}\right\|_{\mathcal{A}} = \left\|\Phi(x_{(p)})\right\|_{\mathcal{B}} \leq \left\|b\right\|_{\mathcal{B}},$$

using (5.3.9) and the monotonicity of the norm. Therefore $0 \leq x_{(p)} \leq \left\|x_{(p)}\right\|_{\mathcal{A}}\mathbf{1} \leq \left\|b\right\|_{\mathcal{B}}\mathbf{1}$ for all $p \in \mathcal{P}(\mathcal{A})^f$ with $p \geq e$, by Proposition B.1.6. Since $\{x_{(p)}\}_{p\in\mathcal{D}:p\geq e}$ is also increasing, this implies, by Lemma B.1.8, the existence of an $x \in \mathcal{A}^+$ such that $x_{(p)} \uparrow x$. Therefore $\Phi_1(x_{(p)}) = \Phi(x_{(p)}) \uparrow \Phi(x)$, since Φ is normal. However, $\Phi_1(x_{(p)}) \uparrow b$ by (5.3.9) and so

$$\Phi(x) = b.$$

Therefore $\Phi(e)\mathcal{B}^+\Phi(e) \subseteq \Phi(A)$ and hence $\Phi(e)\mathcal{B}\Phi(e) \subseteq \Phi(\mathcal{A})$, since any $c \in \Phi(e)\mathcal{B}\Phi(e)$ can be written as a finite linear combination of elements from $\Phi(e)\mathcal{B}^+\Phi(e)$. Φ is therefore surjective by Proposition 3.4.2.

□

Note that since Φ is surjective, $ab = ba$ for all $b \in \mathcal{B}$, by Lemma 5.3.1. In particular, $au = ua$ for every unitary $u \in Z(\mathcal{B})' = \mathcal{B}$. It follows that a is a positive operator affiliated with the center of $\mathcal{B}$. We have therefore proven the following result.

THEOREM 5.3.4. *Let $\mathcal{A} \subseteq \mathcal{B}(H)$ and $\mathcal{B} \subseteq \mathcal{B}(K)$ be semi-finite von Neumann algebras equipped with semi-finite faithful normal traces τ and ν respectively. Let $E(\tau)$ be a strongly symmetric space with absolutely continuous norm and let $F(\nu)$ be a fully symmetric space with absolutely continuous norm. If $U : E \to F$ is a positive surjective linear isometry such that $\nu(s(U(q))) < \infty$ whenever $q \in \mathcal{P}(\mathcal{A})^f$, then there exists a positive operator a, affiliated with the center of $\mathcal{B}$, and a Jordan $*$-isomorphism Φ of $\mathcal{A}$ onto $\mathcal{B}$ such that*

$$U(x) = a\Phi(x) \qquad \forall x \in \mathcal{A} \cap E(\tau)$$

We finish this section with a few remarks regarding the additional assumptions made in Theorem 5.3.4. Firstly, we note that all L_p-spaces ($1 \le p < \infty$), Orlicz spaces whose Orlicz function satisfy the Δ_2-condition globally and many Lorentz spaces (eg. $L_{p,q}$-spaces and $L_{w,1}$-spaces) have absolutely continuous norm. The applicability of Theorem 5.3.4 to these spaces therefore depends on whether isometries on these spaces satisfy the finiteness-preserving condition. It will be shown in Corollary 7.2.11 that positive surjective isometries between certain types of Lorentz spaces are finiteness-preserving and can therefore be described using Theorem 5.3.4.

5.4. Sufficient conditions

In the first three sections of this chapter we showed that the structure of a positive surjective isometry between a symmetric space and a fully symmetric space can be described in terms of a Jordan $*$-isomorphism and a positive operator. In this section we prove a partial converse to this result. More specifically, we will show that if $\Phi : \mathcal{A} \to \mathcal{B}$ is a Jordan $*$-isomorphism (between semi-finite von Neumann algebras), which, in addition, is trace-preserving, then $\Phi \restriction \mathcal{A} \cap E(\tau)$ is a positive surjective isometry from $\mathcal{A} \cap E(\tau)$ onto $\mathcal{B} \cap E(\nu)$, whenever $E(0, \infty)$ is a fully symmetric Banach function space. We need one preliminary result.

PROPOSITION 5.4.1. *Suppose $(\mathcal{A}, \tau)$, $(\mathcal{B}, \nu)$ are semi-finite von Neumann algebras. If $\Phi : \mathcal{A} \to \mathcal{B}$ is a trace-preserving Jordan $*$-isomorphism, then Φ maps $\mathcal{A} \cap L_1(\tau)$ onto $\mathcal{B} \cap L_1(\nu)$ and Φ^{-1} is also trace-preserving.*

PROOF. Suppose $x \in \mathcal{A} \cap L_1(\tau)^+$. Then $\Phi(x) \ge 0$ and so

$$\nu(|\Phi(x)|) = \nu(\Phi(x)) = \tau(x) = \tau(|x|) < \infty.$$

Therefore $\Phi(x) \in \mathcal{A} \cap L_1(\nu)$. Since Φ is linear and any element in $\mathcal{A} \cap L_1(\tau)$ can be written as a linear combination of elements from $x \in \mathcal{A} \cap L_1(\tau)^+$, we have $\Phi(\mathcal{A} \cap L_1(\tau)) \subseteq \mathcal{B} \cap L_1(\nu)$. It follows by Proposition 1.8.8(5) that Φ^{-1} is a Jordan $*$-isomorphism from $\mathcal{B}$ onto $\mathcal{A}$. Note that if $b \in \mathcal{B}$, then

$$\tau(\Phi^{-1}(b)) = \nu(\Phi(\Phi^{-1}(b))) = \nu(b),$$

since Φ is trace-preserving. It follows that Φ^{-1} is trace-preserving and hence that $\Phi^{-1}(\mathcal{B} \cap L_1(\nu)) \subseteq \mathcal{A} \cap L_1(\tau)$. □

THEOREM 5.4.2. *Suppose $(\mathcal{A}, \tau)$, $(\mathcal{B}, \nu)$ are semi-finite von Neumann algebras and $E(0, \infty)$ is a fully symmetric Banach function space. If $\Phi : \mathcal{A} \to \mathcal{B}$ is a trace-preserving Jordan $*$-isomorphism, then $\Phi(\mathcal{A} \cap E(\tau)) = \mathcal{B} \cap E(\nu)$ and*

$$\left\|\Phi(x)\right\|_{E(\nu)} = \left\|x\right\|_{E(\tau)} \qquad \forall x \in \mathcal{A} \cap E(\tau)$$

PROOF. We start by noting that $E(\tau)$ and $E(\nu)$ are fully symmetric spaces, by Theorem 1.6.1. It follows by Proposition 5.4.1 that Φ maps $\mathcal{A} \cap L_1(\tau)$ onto $\mathcal{B} \cap L_1(\nu)$ and by Remark 2.2.5

$$\left\|\Phi(x)\right\|_{L_1(\nu)} = \left\|x\right\|_{L_1(\tau)} \qquad \forall x \in \mathcal{A} \cap L_1(\tau).$$

Since $L_1(\tau)$ has order continuous norm, it follows from Theorem 1.6.7 that $\mathcal{A} \cap L_1(\tau)$ is dense in $L_1(\tau)$, and therefore $\Phi \restriction \mathcal{A} \cap L_1(\tau)$ has an isometric extension Φ_1 from $L_1(\tau)$ onto $L_1(\nu)$. For $y \in L_1 + L_\infty(\tau)$, define

$$\Phi_2(y) = \Phi_1(y_1) + \Phi(y_2),$$

where $y = y_1 + y_2$, $y_1 \in L_1(\tau)$ and $y_2 \in L_\infty(\tau) = \mathcal{A}$. We show that Φ_2 is well-defined. Suppose $y_1 + y_2 = y_3 + y_4$, where $y_1, y_3 \in L_1(\tau)$ and $y_2, y_4 \in \mathcal{A}$. Then

$$\begin{aligned} y_1 - y_3 &= y_4 - y_2 \in L_1 \cap L_\infty(\tau) \\ \Longrightarrow \Phi(y_1 - y_3) &= \Phi(y_4 - y_2) \\ \Longrightarrow \Phi_1(y_1) - \Phi_1(y_3) &= \Phi(y_4) - \Phi(y_2) \qquad \text{since } \Phi_1(x) = \Phi(x) \text{ for all } x \in L_1 \cap L_\infty(\tau) \\ \Longrightarrow \Phi_1(y_1) + \Phi(y_2) &= \Phi_1(y_3) + \Phi(y_4) \end{aligned}$$

and hence Φ_2 is well-defined. Furthermore, $\Phi_2 \restriction L_1(\tau) = \Phi_1$ and $\Phi_2 \restriction \mathcal{A} = \Phi$. Note that $\left\|\Phi_1(x)\right\|_{L_1(\nu)} = \left\|x\right\|_{L_1(\tau)}$ for every $x \in L_1(\tau)$ and, by Proposition 1.8.8, $\left\|\Phi(x)\right\|_{\mathcal{B}} = \left\|x\right\|_{\mathcal{A}}$ for every $x \in \mathcal{A}$. If $x \in E(\tau) \subseteq L_1 + L_\infty(\tau)$, then $\Phi_2(x) \ll x$, by Theorem B.3.5. Therefore $\mu_{\Phi_2(x)} \ll \mu_x$ and hence $\mu_{\Phi_2(x)} \in E(0,\infty)$ (i.e. $\Phi_2(x) \in E(\nu)$), using the full symmetry of $E(0,\infty)$. Furthermore,

$$\begin{aligned} \left\|\Phi_2(x)\right\|_{E(\nu)} &= \left\|\mu_{\Phi_2(x)}\right\|_{E(0,\infty)} \\ &\leq \left\|\mu_x\right\|_{E(0,\infty)} \qquad \text{since } \mu_{\Phi_2(x)} \ll \mu_x \text{ and } E(0,\infty) \text{ is fully symmetric} \\ &= \left\|x\right\|_{E(\tau)} \end{aligned} \tag{5.4.1}$$

Since Φ_2 and Φ agree on $\mathcal{A}$, we have that $\Phi(\mathcal{A} \cap E(\tau)) \subseteq \mathcal{B} \cap E(\nu)$ and

$$\left\|\Phi(x)\right\|_{E(\nu)} \leq \left\|x\right\|_{E(\tau)} \qquad \forall x \in \mathcal{A} \cap E(\tau).$$

Since Φ^{-1} is also a trace-preserving Jordan $*$-isomorphism, we similarly have that $\Phi^{-1}(\mathcal{B} \cap E(\nu)) \subseteq \mathcal{A} \cap E(\tau)$ and

$$\left\|\Phi^{-1}(y)\right\|_{E(\tau)} \leq \left\|y\right\|_{E(\nu)} \qquad \forall y \in \mathcal{B} \cap E(\nu).$$

□

REMARK 5.4.3. Theorem 5.4.2 is a generalization of [4, Lemma 2.5] to more general fully symmetric spaces associated with more general von Neumann algebras. Furthermore, every AFD factor of type II_1 or II_∞ is a semi-finite von Neumann algebra. Theorem 5.4.2 is therefore also a generalization of [42, Theorem 4.1] (see Theorem 2.2.14).

CHAPTER 6

Surjective isometries between symmetric spaces

In the previous chapter we showed that under certain conditions a positive surjective isometry can be represented as a weighted non-commutative composition operator. We will use this result to show that we can obtain a similar representation for a surjective isometry, which is not necessarily positive, if it satisfies a certain disjointness-preserving condition. More specifically, we will prove the following result.

THEOREM 6.1.1. *Suppose $(\mathcal{A} \subseteq \mathcal{B}(H_1), \tau)$ and $(\mathcal{B} \subseteq \mathcal{B}(H_2), \nu)$ are semi-finite von Neumann algebras; $E(\tau)$ is a strongly symmetric space with absolutely continuous norm and $F(\nu)$ is a fully symmetric space with absolutely continuous norm. If $V : E \to F$ is a surjective isometry such that*

- *$\nu(s(V(p)) < \infty$ whenever $p \in \mathcal{P}(\mathcal{A})^f$ and*
- *$V(p)^*V(q) = 0 = V(p)V(q)^*$ whenever $p, q \in \mathcal{P}(\mathcal{A})^f$ with $pq = 0$,*

then there exists a unitary operator v, a positive operator a affiliated with the centre of $\mathcal{B}$ and a Jordan $*$-isomorphism Φ from $\mathcal{A}$ onto $\mathcal{B}$ such that

$$V(x) = va\Phi(x) \qquad \forall x \in \mathcal{A} \cap E.$$

The proof has been divided into several lemmas and is structured as follows. A map Ψ will be defined on projections with finite trace. The disjointness-preserving property of the surjective isometry will allow us to show that Ψ is additive on orthogonal projections and hence to apply Proposition 3.2.1 to obtain a map (also denoted Ψ) from $\mathcal{F}(\tau)$ into $\mathcal{B}$. At this point it would be desirable to show that this extension Ψ is normal on $\mathcal{F}(\tau)$ in order to apply Proposition 3.3.6 to extend Ψ to all of $\mathcal{A}$. This has proven to be problematic and so we follow a different approach. We will construct a unitary operator v and show that letting $U(\cdot) := v^*V(\cdot)$ defines a positive surjective isometry U from E onto F. Application of Theorem 5.3.4 will then yield the desired positive operator and Jordan $*$-isomorphism. It is worth noting that the map Ψ defined on $\mathcal{F}(\tau)$ will play a significant role in showing that U is positive and, as will be shown at the end of this section, the Jordan $*$-isomorphism obtained using Theorem 5.3.4 will turn out to be an extension of Ψ.

For $p \in \mathcal{P}(\mathcal{A})^f$, we will write $V(p) = v_{(p)}b_{(p)}$, where $v_{(p)}$ is the partial isometry and $b_{(p)}$ is the positive operator in the polar decomposition of $V(p)$. We start by defining a map Ψ on projections with finite trace by letting

$$\Psi(p) = s(V(p)) = v_{(p)}^*v_{(p)}, \qquad p \in \mathcal{P}(\mathcal{A})^f.$$

Note that, by Remark B.1.31(3), $\Psi(p) = s(b_{(p)}) = r(b_{(p)}) = s(v_{(p)})$ and therefore

$$\Psi(p)b_{(p)} = b_{(p)} = b_{(p)}\Psi(p) \qquad \text{and} \qquad v_{(p)}\Psi(p) = v_{(p)}. \tag{6.1.1}$$

To show that Ψ is additive on orthogonal projections, suppose $p, q \in \mathcal{P}(\mathcal{A})^f$ with $pq = 0$. Since $V(p)^*V(q) = 0 = V(p)V(q)^*$, we have, by Proposition B.1.32, that $v_{(p)}^*v_{(q)} = 0 = v_{(p)}v_{(q)}^*$. Furthermore, $v_{(p)} + v_{(q)}$ is a partial isometry, $|V(p) + V(q)| = b_{(p)} + b_{(q)}$ and $V(p) + V(q) = (v_{(p)} + v_{(q)})(b_{(p)} + b_{(q)})$ is the polar decomposition of $V(p + q) = V(p) + V(q)$. Therefore

$$v_{(p)} + v_{(q)} = v_{(p+q)} \qquad \text{and} \qquad b_{(p)} + b_{(q)} = b_{(p+q)} \tag{6.1.2}$$

and so

$$\Psi(p+q) = v^*_{(p+q)}v_{(p+q)} = (v_{(p)} + v_{(q)})^*(v_{(p)} + v_{(q)}) = v^*_{(p)}v_{(p)} + v^*_{(q)}v_{(q)} = \Psi(p) + \Psi(q).$$

It follows by Proposition B.1.16(1), that $\Psi(p)\Psi(q) = 0$. Furthermore, if $0 \neq p \in \mathcal{P}(\mathcal{A})^f$, then $V(p) \neq 0$, since V is injective. It follows that $\Psi(p) = s(V(p)) \neq 0$. Furthermore, by Corollary B.3.4, V has the property that $V(x_n) \overset{\mathcal{T}_m}{\to} V(x)$ whenever $(x_n)_{n=1}^\infty \cup \{x\} \subseteq \mathcal{F}(\tau)$ is such that $x_n \overset{\mathcal{A}}{\to} x$ and $s(x_n) \leq s(x)$ for all $n \in \mathbb{N}^+$. By Proposition 3.2.1, Ψ can therefore be extended to a positive linear map (also denoted by Ψ) from $\mathcal{F}(\tau)$ into $\mathcal{B}$ such that

$$\left\|\Psi(x)\right\|_{\mathcal{B}} = \left\|x\right\|_{\mathcal{A}} \qquad \text{and} \qquad \Psi(x^2) = \Psi(x)^2$$

for all $x \in \mathcal{F}(\tau)^{sa}$. To construct the unitary operator mentioned in the introduction, we start by proving a lemma that will enable us to show that $\{v_{(p)}\}_{p\in\mathcal{D}}$ is strong operator convergent (recall that when dealing with subscripts, we will often denote $\mathcal{P}(\mathcal{A})^f$ by $\mathcal{D}$).

Lemma 6.1.2. *If $x \in \mathcal{F}(\tau)$ and $p \in \mathcal{P}(\mathcal{A})^f$ with $p \geq s(x)$, then*

$$V(x) = V(p)\Psi(x) = v_{(p)}b_{(p)}\Psi(x).$$

Proof. Since $V(p) = v_{(p)}b_{(p)}$ is the polar decomposition of $V(p)$, the second equality in the statement of the lemma follows automatically. Suppose $x = q \in \mathcal{P}(\mathcal{A})^f$ and $p \in \mathcal{P}(\mathcal{A})^f$ with $p \geq q$. Then $p - q \in \mathcal{P}(\mathcal{A})^f$ and $q(p-q) = 0$. Note that

$$\begin{aligned} b_{p-q}\Psi(q) &= (b_{p-q}\Psi(p-q))\Psi(q) \qquad \text{using (6.1.1)} \\ &= 0 \qquad \text{since } q(p-q) = 0 \text{ implies that } \Psi(q)\Psi(p-q) = 0 \end{aligned} \tag{6.1.3}$$

Similarly, we have that

$$v_{p-q}b_{(q)} = v_{p-q}\Psi(p-q)\Psi(q)b_{(q)} = 0 \tag{6.1.4}$$

Furthermore,

$$\begin{aligned} V(p)\Psi(q) &= v_{(p)}b_{(p)}\Psi(q) \\ &= v_{q+(p-q)}b_{(q+(p-q))}\Psi(q) \\ &= (v_{(q)} + v_{p-q})(b_{(q)} + b_{p-q})\Psi(q) \qquad \text{using (6.1.2)} \\ &= v_{(q)}b_{(q)}\Psi(q) \qquad \text{using (6.1.3) and (6.1.4)} \\ &= V(q) \end{aligned}$$

Let $\mathcal{G}_f$ denote the set of all finite linear combinations of mutually orthogonal projections, each with finite trace. Using the linearity of V and Ψ, we have that

$$V(x) = V(p)\Psi(x) \qquad \text{if } x \in \mathcal{G}_f \text{ and } p \in \mathcal{P}(\mathcal{A})^f \text{ with } p \geq s(x) \tag{6.1.5}$$

Suppose $x \in \mathcal{F}(\tau)^{sa}$ and $p \in \mathcal{P}(\mathcal{A})^f$ with $p \geq s(x)$. By Remark B.1.12, we can find a sequence $(x_n)_{n=1}^\infty \subseteq \mathcal{G}_f^{sa}$ such that $x_n \overset{\mathcal{A}}{\to} x$ and $s(x_n) \leq s(x) \leq p$ for all $n \in \mathbb{N}^+$. By Proposition B.3.3, this implies that $x_n \overset{E}{\to} x$. Therefore $V(x_n) \overset{F}{\to} V(x)$ and $\Psi(x_n) \overset{\mathcal{B}}{\to} \Psi(x)$, since V is an isometry and Ψ is linear, and isometric on self-adjoint elements in $\mathcal{F}(\tau)$. Furthermore, since F is a normed $\mathcal{B}$-bimodule,

$$\left\|V(p)(\Psi(x_n) - \Psi(x))\right\|_F \leq \left\|V(p)\right\|_F \left\|\Psi(x_n) - \Psi(x)\right\|_{\mathcal{B}} \to 0$$

and so $V(p)\Psi(x_n) \overset{F}{\to} V(p)\Psi(x)$. However, using (6.1.5) and the fact that V is an isometry, we have that

$$V(p)\Psi(x_n) = V(x_n) \overset{F}{\to} V(x)$$

Since limits are unique, we have that $V(x) = V(p)\Psi(x)$. Finally, since any element in $\mathcal{F}(\tau)$ can be written as a linear combination of self-adjoint elements and V and Ψ are linear, we have that

$$V(x) = V(p)\Psi(x)$$

whenever $x \in \mathcal{F}(\tau)$ and $p \in \mathcal{P}(\mathcal{A})^f$ with $p \geq s(x)$. $\square$

LEMMA 6.1.3. *$\{v_{(p)}\}_{p\in\mathcal{D}}$ converges in the strong operator topology to a unitary operator $v \in \mathcal{B}$ and*

$$v\Psi(p) = v_{(p)} \qquad \forall p \in \mathcal{P}(\mathcal{A})^f.$$

PROOF. We start by noting that if $p, q \in \mathcal{P}(\mathcal{A})^f$ are such that $0 < q \leq p$, then

$$v_{(q)} = v_{(p)}\Psi(q) \tag{6.1.6}$$

To show this, note that if $p = q$, then (6.1.6) holds using (6.1.1). If $p > q$, then $0 \neq p - q \in \mathcal{P}(\mathcal{A})^f$ and $q(p-q) = 0$. Therefore, $v_{(q)} + v_{(p-q)} = v_{(p)}$, using (6.1.2). It follows that

$$\begin{aligned} v_{(p)}\Psi(q) &= (v_{(q)} + v_{(p-q)})\Psi(q) \\ &= v_{(q)} + v_{(p-q)}\Psi(p-q)\Psi(q) \qquad \text{using (6.1.1)} \\ &= v_{(q)} \qquad \text{since } (p-q)q = 0 \text{ implies that } \Psi(p-q)\Psi(q) = 0 \end{aligned}$$

Next, we note that if $p, q \in \mathcal{P}(\mathcal{A})^f$ with $q \leq p$, then $\Psi(q) \leq \Psi(p)$, since Ψ is positive. It follows that $\{\Psi(p)\}_{p\in\mathcal{D}}$ is an increasing net of projections and therefore converges in the strong operator topology to a projection $y \in \mathcal{P}(\mathcal{B})$. Let $\eta \in H_2$ and suppose $\epsilon > 0$. Since $\Psi(p) \overset{SOT}{\to} y$ as $p \uparrow_{p\in\mathcal{D}} \mathbf{1}$, there exists a $p_\epsilon \in \mathcal{P}(\mathcal{A})^f$ such that $p, q \in \mathcal{P}(\mathcal{A})^f$ with $p, q \geq p_\epsilon$ implies that

$$\big\|(\Psi(p) - \Psi(q))\eta\big\| < \epsilon \tag{6.1.7}$$

Let $p, q \in \mathcal{P}(\mathcal{A})^f$ be such that $p, q \geq p_\epsilon$. $\mathcal{P}(\mathcal{A})^f$ is a directed set and so there exists an $r \in \mathcal{P}(\mathcal{A})^f$ with $r \geq p, q$. Then

$$\begin{aligned} \big\|(v_{(p)} - v_{(q)})\eta\big\| &= \big\|v_{(r)}(\Psi(p) - \Psi(q))\eta\big\| \qquad \text{using (6.1.6)} \\ &\leq \big\|v_{(r)}\big\|_{\mathcal{B}}\big\|(\Psi(p) - \Psi(q))\eta\big\| \\ &\leq \big\|(\Psi(p) - \Psi(q))\eta\big\| \qquad \text{since } v_{(r)} \text{ is a partial isometry and hence } \big\|v_{(r)}\big\|_{\mathcal{B}} \leq 1 \\ &< \epsilon \qquad \text{using (6.1.7)} \end{aligned}$$

Therefore $\{v_{(p)}(\eta)\}_{p\in\mathcal{D}}$ is Cauchy in H_2. Since this holds for every $\eta \in H_2$, we have that $\{v_{(p)}\}_{p\in\mathcal{D}}$ is SOT-Cauchy. Furthermore, $\{v_{(p)}\}_{p\in\mathcal{D}}$ is contained in the unit ball of $\mathcal{B}(H_2)$ and so $v_{(p)} \overset{SOT}{\to} v$ for some $v \in \mathcal{B}(H_2)$, since norm-closed balls in $\mathcal{B}(H_2)$ are SOT-complete by Proposition B.1.3. $\mathcal{B}$ is SOT-closed and so $v \in \mathcal{B}$. Furthermore, for any $q \in \mathcal{P}(\mathcal{A})^f$ with $q \geq p$, we have

$$\begin{aligned} v_{(p)} &= v_{(q)}\Psi(p) \qquad \text{by (6.1.6)} \\ &\overset{SOT}{\to} v\Psi(p) \qquad \text{as } q \uparrow_{q\in\mathcal{D}} \mathbf{1}, \text{ by Proposition B.1.1} \\ \implies v_{(p)} &= v\Psi(p) \end{aligned}$$

We show that v is a partial isometry and $s(v) = y$. Note that

$$\begin{aligned} v_{(p)} \overset{SOT}{\to} v &\implies v_{(p)} \overset{WOT}{\to} v \qquad \text{since the WOT is coarser than the SOT} \\ &\implies v_{(p)}^* \overset{WOT}{\to} v^* \qquad \text{by Proposition B.1.2} \\ &\implies v_{(p)}^* v_{(p)} \overset{WOT}{\to} v^* v \qquad \text{by Remark B.1.4} \end{aligned} \tag{6.1.8}$$

Furthermore, $v_{(p)}^* v_{(p)} = |v_{(p)}| = \Psi(p) \overset{SOT}{\to} y$ and so $v_{(p)}^* v_{(p)} \overset{WOT}{\to} y$, since the weak operator topology is coarser than the strong operator topology. Using (6.1.8) and the uniqueness of weak operator topology limits, this implies that $y = v^* v$. It follows by Proposition B.1.28 that v is a partial isometry and $s(v) = y$. We show that $y = \mathbf{1}$ and hence that v is unitary. Suppose $x \in \mathcal{F}(\tau)$. For $p \in \mathcal{P}(\mathcal{A})^f$ with $p \geq s(x) \vee r(x)$ we have that $px = x = xp$ and hence, by Proposition 1.8.3(7),

$$\Psi(x) = \Psi(xp) = \Psi(x)\Psi(p) \tag{6.1.9}$$

Furthermore,

$$\begin{aligned} \Psi(x)y &= \Psi(x)\mathrm{SOT}\lim_{p\in\mathcal{D}}\Psi(p) \\ &= \mathrm{SOT}\lim_{p\in\mathcal{D}:p\geq s(x)\vee r(x)}[\Psi(x)\Psi(p)] \qquad \text{by Proposition B.1.1} \\ &= \Psi(x) \qquad \text{using (6.1.9)} \end{aligned} \tag{6.1.10}$$

It follows that if $p \geq s(x) \vee r(x)$, then

$$\begin{aligned} V(x) &= b_{(p)} v_{(p)} \Psi(x) \qquad \text{by Lemma 6.1.2} \\ &= b_{(p)} v_{(p)} \Psi(x) y \qquad \text{using (6.1.10)} \\ &= V(x)y \qquad \text{by Lemma 6.1.2} \end{aligned} \tag{6.1.11}$$

Assume that $\mathbf{1} - y \neq 0$. Since, $(\mathcal{B}, \nu)$ is semi-finite, there exists a $q \in \mathcal{P}(\mathcal{B})$ such that $0 < q \leq \mathbf{1} - y$ and $\nu(q) < \infty$. This implies, using Proposition 1.6.3(2) that $q \in F$ and hence there exists an $x \in E$ such that $V(x) = q$, since V is surjective. E has absolutely continuous norm and therefore $\mathcal{F}(\tau)$ is dense in E (see Theorem 1.6.7). Let $(x_n)_{n=1}^\infty \subseteq \mathcal{F}(\tau)$ be such that $x_n \overset{E}{\to} x$. Then $V(x_n) \overset{F}{\to} V(x) = q$. However, using (6.1.11) we have that $V(x_n) = V(x_n)y \overset{F}{\to} V(x)y = qy$. It follows that $q = qy = 0$, since $q \leq \mathbf{1} - y$. This is a contradiction and so $y = \mathbf{1}$. □

Lemma 6.1.4. *The map $U : E \to F$ defined by $U(x) = v^* V(x)$ is a positive surjective isometry.*

Proof. Since v, and hence v^*, is a unitary operator, it is easily checked that U is a surjective isometry. We show that U is positive. To do so, we start by showing that if $x \in \mathcal{F}(\tau)$ and $p \in \mathcal{P}(\mathcal{A})^f$ with $p \geq s(x)$, then

$$b_{(p)}\Psi(x) = \Psi(x) b_{(p)} \tag{6.1.12}$$

Suppose $x = q$ and $p \in \mathcal{P}(\mathcal{A})^f$ with $p \geq q$. Then

$$\begin{aligned} b_{(p)}\Psi(q) &= b_{(q+(p-q))}\Psi(q) \\ &= (b_{(q)} + b_{p-q})\Psi(q) \qquad \text{using (6.1.2)} \\ &= b_{(q)}\Psi(q) \qquad \text{since } b_{p-q}\Psi(q) = b_{p-q}\Psi(p-q)\Psi(q) = 0 \\ &= \Psi(q) b_{(q)} \qquad \text{using (6.1.1)} \\ &= \Psi(q)(b_{(q)} + b_{p-q}) \qquad \text{since } \Psi(q) b_{p-q} = \Psi(q)\Psi(p-q) b_{p-q} = 0 \\ &= \Psi(q) b_{(p)} \qquad \text{using (6.1.2)} \end{aligned}$$

Using the linearity of Ψ, we have that

$$b_{(p)}\Psi(x) = \Psi(x) b_{(p)} \tag{6.1.13}$$

whenever $x \in \mathcal{G}_f$ and $p \in \mathcal{P}(\mathcal{A})^f$ with $p \geq s(x)$. Suppose $x \in \mathcal{F}(\tau)^{sa}$ and $p \in \mathcal{P}(\mathcal{A})^f$ with $p \geq s(x)$. By Remark B.1.12, there exists a sequence $(x_n)_{n=1}^\infty \subseteq \mathcal{G}_f^{sa}$ such that $x_n \overset{\mathcal{A}}{\to} x$ and $s(x_n) \leq s(x) \leq p$ for all $n \in \mathbb{N}^+$.

Since Ψ is isometric on self-adjoint elements, we have that $\Psi(x_n) \xrightarrow{\mathcal{B}} \Psi(x)$. Note that $V(p) = v_{(p)}b_{(p)}$ and so $b_{(p)} = v_{(p)}^* V(p)$. It follows that $b_{(p)} \in F$, since $V(p) \in F$, $v_{(p)}^* \in \mathcal{B}$ and F is a bimodule. Therefore

$$\left\|b_{(p)}(\Psi(x_n) - \Psi(x))\right\|_F \leq \left\|b_{(p)}\right\|_F \left\|\Psi(x_n) - \Psi(x)\right\|_{\mathcal{B}} \to 0$$

and so $b_{(p)}\Psi(x_n) \xrightarrow{F} b_{(p)}\Psi(x)$. We can similarly show that $\Psi(x_n)b_{(p)} \xrightarrow{F} \Psi(x)b_{(p)}$, but $\Psi(x_n)b_{(p)} = b_{(p)}\Psi(x_n)$ for all $n \in \mathbb{N}^+$, using (6.1.13), and so $b_{(p)}\Psi(x) = \Psi(x)b_{(p)}$. Since Ψ is linear, we can extend this result to all of $\mathcal{F}(\tau)$ in the usual way. We are now in a position to show that U is positive. Suppose $x \in \mathcal{F}(\tau)^+$ and let $p = s(x)$. Then $p \in \mathcal{P}(\mathcal{A})^f$ and

$$\begin{aligned} V(x) &= v_{(p)}b_{(p)}\Psi(x) && \text{by Lemma 6.1.2} \\ &= v\Psi(p)b_{(p)}\Psi(x) && \text{by Lemma 6.1.3} \\ &= vb_{(p)}\Psi(x) && \text{using (6.1.1)} \end{aligned}$$

It follows that

$$\begin{aligned} v^*V(x) &= b_{(p)}\Psi(x) \\ &= b_{(p)}^{1/2}\Psi(x)b_{(p)}^{1/2} && \text{using (6.1.12) and Corollary B.2.8 (with } f(t) := t^{1/2}) \\ &\geq 0 && \text{using Proposition B.2.2(4) and the fact that } \Psi \text{ is positive} \end{aligned} \tag{6.1.14}$$

Suppose $x \in E^+$. Since E has absolutely continuous norm, there exists, by Corollary 1.6.8(2), a sequence $(x_n)_{n=1}^\infty \subseteq \mathcal{F}(\tau)^+$ such that $x_n \xrightarrow{E} x$. U is an isometry and so $v^*V(x_n) = U(x_n) \xrightarrow{F} U(x)$. By (6.1.14), $v^*V(x_n) \geq 0$ for all $n \in \mathbb{N}^+$ and therefore $U(x) \geq 0$, since F^+ is closed by Proposition B.3.1(10). □

We have proven the existence of a unitary operator v such that the map $U := v^*V$ is a positive surjective isometry from E onto F. In order to apply Theorem 5.3.4 to describe the structure of U we need to show that $\nu(s(U(p))) < \infty$ whenever $p \in \mathcal{P}(\mathcal{A})^f$. To this end, suppose that $p \in \mathcal{P}(\mathcal{A})^f$. Then

$$\begin{aligned} v^*V(p) &= v^*v_{(p)}b_{(p)} \\ &= v^*v\Psi(p)b_{(p)} && \text{by Lemma 6.1.3} \\ &= b_{(p)} && \text{using (6.1.1) and the fact that } v^*v = \mathbf{1} \end{aligned} \tag{6.1.15}$$

It follows from the above, Remark B.1.31 and the finiteness-preserving assumption on V that

$$\nu(s(U(p))) = \nu(s(b_{(p)})) = \nu(s(V(p))) < \infty.$$

By Theorem 5.3.4, there exists a positive operator a affiliated with the centre of $\mathcal{B}$ and a Jordan $*$-isomorphism Φ from $\mathcal{A}$ onto $\mathcal{B}$ such that

$$U(x) = a\Phi(x) \qquad \forall x \in \mathcal{A} \cap E.$$

It follows that

$$V(x) = va\Phi(x) \qquad \forall x \in \mathcal{A} \cap E.$$

This concludes the proof of Theorem 6.1.1.

We finish this chapter by considering the relationship between $\Phi : \mathcal{A} \to \mathcal{B}$ and the map $\Psi : \mathcal{F}(\tau) \to \mathcal{B}$ (as defined earlier), and the relationship between a and the positive operator $b_{(p)}$ appearing the polar decomposition of $V(p)$. We will show that Φ is the unique normal extension of Ψ and that $b_{(p)} = a_{(p)}$ for every $p \in \mathcal{P}(\mathcal{A})^f$, where the $a_{(p)}$'s are the positive operators used to construct a as in Theorem 5.3.4. Recall from the proof of Theorem 5.3.4 that $a_{(p)} = U(p) = \int_0^\infty \lambda de_{(p)}(\lambda)$, $e(\lambda) := \mathrm{SOT}\lim_{p \in \mathcal{D}} e_{(p)}(\lambda)$ and $a = \int_0^\infty \lambda de(\lambda)$. However,

$b_{(p)} = v^*V(p) = U(p)$ using (6.1.15) and so $b_{(p)} = a_{(p)}$ for every $p \in \mathcal{P}(\mathcal{A})^f$. To demonstrate the relationship between Φ and Ψ, recall that

$$\Phi_1(x) = \text{SOT} \lim_{q \in \mathcal{D}: q \geq s(x) \vee r(x)} \Phi_q(x) \qquad x \in \mathcal{F}(\tau) \qquad \text{and} \tag{6.1.16}$$

$$\Phi_p(p) = s(U(p)) \tag{6.1.17}$$

We start by showing that if $p \in \mathcal{P}(\mathcal{A})^f$, then $\Phi(p) = \Psi(p)$. Note that for $q \in \mathcal{P}(\mathcal{A})^f$ with $q \geq p$, we have, by Lemma 5.2.2, that $\Phi_q(p) \leq \Phi_p(p) = s(U(p))$. Using (6.1.16), it follows that

$$\Phi_1(p) \leq s(U(p)) \tag{6.1.18}$$

Furthermore, using Corollary 5.2.6 and the fact that $\Phi_1(p)$ is a projection, we have that

$$a_{(p)}\Phi_1(p) = (a\Phi_1(p))\Phi_1(p)) = a\Phi_1(p) = a_{(p)}.$$

It follows that $s(U(p)) = s(a_{(p)}) \leq \Phi_1(p)$. Combining this with (6.1.18), we obtain

$$\begin{aligned} \Phi_1(p) &= s(U(p)) \\ &= s(V(p)) \qquad \text{using (6.1.15)} \\ &= \Psi(p) \qquad \text{by definition of } \Psi(p) \end{aligned}$$

Φ extends Φ_1 and so $\Phi(p) = \Psi(p)$. Since Φ and Ψ are linear, they agree on $\mathcal{G}_f$. Furthermore, $\mathcal{G}_f^{sa}$ is dense in $\mathcal{F}(\tau)^{sa}$ and so Φ and Ψ agree on $\mathcal{F}(\tau)^{sa}$, since both these maps are isometric on $\mathcal{F}(\tau)^{sa}$. Using the linearity of Φ and Ψ, we have that

$$\Phi(x) = \Psi(x) \qquad \forall x \in \mathcal{F}(\tau).$$

Suppose $\tilde{\Phi}$ is another normal extension of Ψ and let $x \in \mathcal{A}^+$. By Proposition B.2.3, there exists an increasing net $\{x_\lambda\}_{\lambda \in \Lambda} \subseteq \mathcal{F}(\tau)^+$ such that $x_\lambda \uparrow x$. Therefore

$$\Phi(x_\lambda) \uparrow \Phi(x) \qquad \text{and} \qquad \tilde{\Phi}(x_\lambda) \uparrow \tilde{\Phi}(x)$$

using the normality of Φ and $\tilde{\Phi}$. However, $\Phi(x_\lambda) = \tilde{\Phi}(x_\lambda)$ for each λ, since both these maps extend Ψ. Therefore $\Phi(x) = \tilde{\Phi}(x)$. A similar argument can be used to show that $\tilde{\Phi}$ is linear, since it agrees with a linear map on $\mathcal{F}(\tau)$. Since any element in $\mathcal{A}$ can be written as a linear combination of elements from $\mathcal{A}^+$, the linearity of Φ and $\tilde{\Phi}$ can then be used to show that $\Phi = \tilde{\Phi}$.

Remark 6.1.5. A reasonable concern regarding Theorem 6.1.1 is that we have placed what may be restrictive conditions on the surjective isometry in order to describe its structure. In the next chapter we show that any surjective isometry between Lorentz spaces (of a particular type) satisfies these conditions and in the process we develop a semi-finite generalization of Theorem 2.2.10.

CHAPTER 7

Surjective isometries between Lorentz spaces

Throughout this chapter $w : (0,\infty) \to (0,\infty)$ will be used to denote a weight function, i.e. a decreasing function satisfying $\lim_{t\to 0} w(t) = \infty$, $\lim_{t\to\infty} w(t) = 0$, $\int_0^1 w(t)dt = 1$ and $\int_0^\infty w(t)dt = \infty$. Recall that if $(\mathcal{A},\tau)$ is a semi-finite von Neumann algebra, then the space $L_{w,1}(\tau)$ is defined as the set of all τ-measurable operators x affiliated with $\mathcal{A}$ such that $\int_0^\infty \mu_x(t)w(t)dt < \infty$. Recall further that if we let $\psi(t) := \int_0^t w(s)ds$, then $L_{w,1}(\tau) = \Lambda_\psi(\tau)$, with equality of norms. Surjective isometries on these types of Lorentz spaces have been studied in the commutative setting ([2]) and in the non-commutative setting for such spaces associated with finite von Neumann algebras ([4]). Our aim in this chapter is to characterize surjective isometries on Lorentz spaces associated with semi-finite von Neumann algebras. This chapter will also illustrate how extreme point methods can be used in the characterization of surjective isometries. The basic idea is that the characterization of the extreme points of the unit ball of a Lorentz space can be used to show that surjective isometries map projections onto scalar multiples of partial isometries. In the finite setting, this can be used to convert the surjective isometry to a positive surjective isometry and hence to apply Theorem 2.2.7. In attempting to modify this approach to the semi-finite setting, we observe that this mapping of projections to scalar multiples of partial isometries enables us to show that surjective isometries of Lorentz spaces are in fact disjointness-preserving and therefore allow application of Theorem 6.1.1. We start by describing the structure of extreme points of the unit balls of Lorentz spaces.

7.1. Extreme points of Lorentz spaces

If E is a normed space, then we will denote

$$B_E := \{x \in E : \|x\| \leq 1\} \qquad \text{and} \qquad S_E := \{x \in E : \|x\| = 1\}.$$

Before characterizing the extreme points in the Lorentz spaces under consideration, we present a result (semi-finite generalization of ([4, Lemma 2.3]) regarding extreme points in more general spaces.

Lemma 7.1.1. *Suppose $E(\tau)$ is a symmetric space and $x \in B_{E(\tau)}$ is such that $\tau(r(x)) < \infty$ (or equivalently $\tau(s(x)) < \infty$). Then x is an extreme point of $B_{E(\tau)}$ if and only if $|x|$ is one.*

Proof. Suppose $x \in B_{E(\tau)}$ is such that $\tau(r(x)) < \infty$. By Proposition B.2.13, there exists a unitary operator $u \in \mathcal{A}$ such that $x = u|x|$. Suppose $x \in \operatorname{extr}(B_{E(\tau)})$ and $|x| = \frac{1}{2}(y+z)$, with $y, z \in B_{E(\tau)}$. Then $x = u|x| = \frac{1}{2}(uy + uz)$ and $uy, uz \in B_{E(\tau)}$, since u is unitary. x is an extreme point and so $x = uy = uz$. Therefore $|x| = u^*x = u^*uy = u^*uz$, i.e. $|x| = y = z$ and hence $|x|$ is an extreme point. The converse is proved similarly. $\square$

In order to describe the extreme points of these Lorentz spaces, we present the following result which enables us to use characterizations of extreme points in the commutative setting to obtain analogous results in the non-commutative setting.

THEOREM 7.1.2. [5][7] *Suppose* $(\mathcal{A},\tau)$ *is a semi-finite von Neumann algebra and* $E(0,\infty)$ *is a symmetric space. An operator* $x \in S_{E(\tau)}$ *is an extreme point of* $B_{E(\tau)}$ *if and only if* μ_x *is an extreme point of* $B_{E(0,\infty)}$ *and one of the following conditions holds:*

(1) $\mu_x(\infty) = 0$
(2) $n(x)\mathcal{A}n(x^*) = 0,\ |x| \geq \mu_x(\infty)s(x)$,

where $\mu_x(\infty)$ *denotes* $\lim\limits_{t\to\infty}\mu_x(t)$.

The following is a non-commutative analogue of [2, Proposition 2.2] (see Proposition 2.1.5) and a semi-finite extension of [4, Theorem 4.1] (see Theorem 2.2.9). Recall that $\mathcal{V}(\mathcal{A})^f$ denotes the set of partial isometries in $\mathcal{A}$ with finite-trace support projections.

PROPOSITION 7.1.3. *Suppose* $(\mathcal{A},\tau)$ *is a semi-finite von Neumann algebra,* w *is a strictly decreasing weight function and* $E = L_{w,1}(\tau)$*.* x *is an extreme point of* B_E *if and only if* $x = \frac{v}{\psi(\tau(|v|))}$ *for some* $v \in \mathcal{V}(\mathcal{A})^f$*.*

PROOF. Suppose $x = \frac{1}{\psi(\tau(|v|))}v$ for some partial isometry $v \in \mathcal{A}$ with $\tau(|v|) < \infty$. Then $\mu_x = \frac{1}{\psi(\tau(|v|))}\chi_{[0,\tau(|v|))}$, by Proposition 1.4.5. It follows that μ_x is an extreme point of $B_{L_{w,1}(0,\infty)}$, by Proposition 2.1.5. Furthermore, $\mu_x(\infty) = 0$ and so x is an extreme point of $B_{L_{w,1}(\tau)}$, by Theorem 7.1.2. Conversely, suppose x is an extreme point of $B_{L_{w,1}(\tau)}$. By Theorem 7.1.2, μ_x is an extreme point of $B_{L_{w,1}(0,\infty)}$ and so $\mu_x = \frac{1}{\psi(m(A))}\chi_A$ for some $A \subset [0,\infty)$ with $0 < m(A) < \infty$, by Proposition 2.1.5. Let $t = m(A)$. Since μ_x is decreasing, we have that $\mu_x = \frac{1}{\psi(t)}\chi_{[0,t)}$ and hence $|x| = \frac{1}{\psi(t)}p$ for some $p \in \mathcal{P}(\mathcal{A})$ with $\tau(p) = t$, by Proposition 1.4.5. By Proposition B.2.13 there exists a unitary operator u such that $x = u|x|$. Since $up \in \mathcal{V}(\mathcal{A})$ and $|up|^2 = (up)^*(up) = p = p^2$, we have that $|up| = p$ and therefore $\tau(p) = \tau(|up|)$. It follows that $x = \frac{v}{\psi(\tau(|v|))}$ for some $v \in \mathcal{V}(\mathcal{A})$ with $\tau(|v|) < \infty$. $\square$

REMARK 7.1.4. When a first attempt was made at characterizing the extreme points of Lorentz spaces associated with semi-finite von Neumann algebras, the author was unaware of Theorem 7.1.2. The technique initially employed was a combination of deriving non-commutative analogues of results contained in [2] and extending results in [4] to Lorentz spaces associated with semi-finite von-Neumann algebras. This process is more direct and yielded a result, of independent interest, relating the additivity of the norm to the additivity of the generalized singular value in the context of non-commutative Lorentz spaces associated with semi-finite von Neumann algebras. This technique has been included in Appendix A.

The following semi-finite extension of [4, Lemma 2.3] follows immediately from Proposition 7.1.3 and Lemma 7.1.1.

COROLLARY 7.1.5. x *is an extreme point of the unit ball of* $L_{w,1}(\tau)$ *if and only if* $|x|$ *is one.*

We finish this section with a technical lemma (extension of [4, Lemma 2.6] to the semi-finite setting) that will be used later in the process of showing that a particular operator is a projection.

LEMMA 7.1.6. *Suppose* $f \in L_{w,1}(0,\infty)$ *and* $0 < s < \infty$*. If* $f \ll \chi_{(0,s]}$ *and* $f^* \neq \chi_{(0,s]}$*, then*

$$\|f\|_{L_{w,1}} < \|\chi_{(0,s]}\|_{L_{w,1}}.$$

PROOF. We start by showing that $f^*(t) \leq 1$ for all $t \in (0,s]$. Note that if $0 < t_0 \leq s$, then

$$\begin{aligned}\int_0^{t_0} f^*(t)dt &\leq \int_0^{t_0} (\chi_{(0,s]})^*(t)dt \qquad \text{since } f \ll \chi_{(0,s]} \\ &= t_0 \qquad \text{since } (\chi_{(0,s]})^* = \chi_{(0,s]}\end{aligned} \tag{7.1.1}$$

Assume that $f^*(t_1) > 1$, for some $0 < t_1 \leq s$. Then $f^*(t_1) = 1 + \epsilon$ for some $\epsilon > 0$ and $f^*(t) \geq 1 + \epsilon$ for all $t \in [0, t_1]$, since f^* is decreasing. Therefore

$$\begin{aligned} \int_0^{t_1} f^*(t)dt &\geq t_1(1+\epsilon) \\ &> t_1 \end{aligned}$$

This contradicts (7.1.1) and so $f^*(t) \leq 1$ for all $t \in (0, s]$. Next, we show that $f^*(s) < 1$. Let $\alpha := f^*(s)$. Assume $\alpha = 1$. Since f^* is continuous from the right (see Proposition 1.4.2), there exists a $t_0 > s$ such that $f^*(t_0) > 0$. Therefore,

$$\begin{aligned} \int_0^{t_0} f^*(t)dt &= \int_0^s f^*(t)dt + \int_s^{t_0} f^*(t)dt \\ &\geq s.f^*(s) + f^*(t_0).(t_0 - s) \qquad \text{since } f^* \text{ is decreasing} \\ &> s \qquad \text{since } f^*(s) = 1 \text{ and } f^*(t_0) > 0 \\ &= \int_0^{t_0} (\chi_{(0,s]})^*(t)dt \end{aligned}$$

This contradicts the fact that $f \ll \chi_{(0,s]}$ and hence $\alpha < 1$. If $\alpha = 0$, then $f^* = 0$ and thus

$$\|f\|_{L_{w,1}} = 0 < \|\chi_{(0,s]}\|_{L_{w,1}}.$$

If $\alpha > 0$, then let $r := \frac{1}{\alpha}\int_s^\infty f^*(t)dt + s$. Note that for $t_0 > s$

$$\begin{aligned} \int_s^{t_0} f^*(t)dt &\leq \int_0^{t_0} f^*(t)dt \qquad \text{since } f^* \geq 0 \\ &\leq \int_0^{t_0} (\chi_{(0,s]})^*(t)dt \qquad \text{since } f \ll \chi_{(0,s]} \\ &= s \end{aligned}$$

Since this holds for all $t_0 > s$, we have that $\int_s^\infty f^*(t)dt \leq s < \infty$. Put

$$f_1(t) := \begin{cases} f^*(t) & \text{if } t \in (0, s] \\ \alpha & \text{if } t \in (s, r] \\ 0 & \text{if } t \in (r, \infty) \end{cases}$$

Note that f_1 is positive and decreasing. Therefore $f_1^* = f_1$ almost everywhere. We show that $f \ll f_1$. If $0 < t_0 \leq s$, then

$$\int_0^{t_0} f_1(t)dt = \int_0^{t_0} f^*(t)dt \tag{7.1.2}$$

If $s < t_0 \leq r$, then

$$\begin{aligned} \int_0^{t_0} f_1(t)dt &= \int_0^s f^*(t)dt + (t_0 - s)\alpha \\ &\geq \int_0^s f^*(t)dt + \int_s^{t_0} f^*(t)dt \qquad \text{since } \alpha = f^*(s) \geq f^*(t) \text{ for all } t \geq s \\ &= \int_0^{t_0} f^*(t)dt \qquad (7.1.3) \end{aligned}$$

If $t_0 > r$, then

$$\begin{aligned}
\int_0^{t_0} f_1(t)dt &= \int_0^{r} f_1(t)dt \qquad \text{since } f_1(t)=0 \text{ for all } t\geq t_0 \\
&= \int_0^{s} f^*(t)dt + (r-s)\alpha \\
&= \int_0^{s} f^*(t)dt + (\frac{1}{\alpha}\int_s^{\infty} f^*(t)dt + s - s)\alpha \\
&= \int_0^{\infty} f^*(t)dt \\
&\geq \int_0^{t_0} f^*(t)dt \qquad (7.1.4)
\end{aligned}$$

Using (7.1.2), (7.1.3) and (7.1.4) and the fact that $f_1^*(t) = f_1(t)$ a.e., we obtain

$$\int_0^{t_0} f^*(t)dt \leq \int_0^{t_0} f_1^*(t)dt \qquad \forall t_0 > 0$$

and hence $f \ll f_1$.

Next, we show that $f_1 \ll \chi_{(0,s]}$. If $0 < t_0 \leq s$, then

$$\int_0^{t_0} f_1^*(t)dt = \int_0^{t_0} f^*(t)dt \leq \int_0^{t_0} \chi_{(0,s]}^*(t)dt.$$

If $s < t_0 \leq r$, then

$$\begin{aligned}
\int_0^{t_0} f_1^*(t)dt &= \int_0^{s} f^*(t)dt + (t_0 - s)\alpha \\
&= \int_0^{s} f^*(t)dt + (t_0 - s)(\frac{1}{r-s})\int_s^{\infty} f^*(t)dt \qquad \text{using the definition of } r \\
&\leq \int_0^{\infty} f^*(t)dt \qquad \text{since } t_0 - s \leq r - s \\
&\leq \int_0^{\infty} \chi_{(0,s]}^*(t)dt \qquad \text{since } \int_0^{w} f^*(t)dt \leq \int_0^{w} \chi_{(0,s]}^*(t)dt \text{ for every } w > 0 \\
&= \int_0^{t_0} \chi_{(0,s]}(t)dt \qquad \text{since } t_0 > s \qquad (7.1.5)
\end{aligned}$$

If $t_0 > r$, then

$$\begin{aligned}
\int_0^{t_0} f_1^*(t)dt &= \int_0^{r} f_1^*(t)dt \qquad \text{since } f_1^* = f_1 \text{ almost everywhere and } f_1(t) = 0 \text{ for } t \geq r \\
&\leq \int_0^{r} \chi_{(0,s]}^*(t)dt \qquad \text{using (7.1.5)} \\
&= \int_0^{t_0} \chi_{(0,s]}^*(t)dt
\end{aligned}$$

It follows that $f_1 \ll \chi_{(0,s]}$. Since $f^* \neq \chi_{(0,s]}$ and $f \ll \chi_{(0,s]}$, it follows that there is an $0 < s_0 < s$ such that $f_1(t) = f^*(t) < 1$ for all $t \in [s_0, s]$. It is therefore possible to construct a decreasing function f_2 on $(0,s]$ such that $f_1(t) \leq f_2(t) \leq 1$ for $t \in (0,s]$, but which has the property that $\int_0^s f_2(t)dt > \int_0^s f_1(t)dt$. Choose $0 < \beta < \alpha$ such that letting $f_2(t) = \beta$ for all $t \in (s,r]$ yields $\int_0^r f_2(t)dt = \int_0^r f_1(t)dt$. Let $f_2(t) = 0$ for all $t > r$. It follows from the way we have constructed f_2 that f_2 is decreasing and hence $f_2^*(t) = f_2(t)$ a.e. Furthermore, $\int_0^\infty f_2(t)dt = \int_0^\infty f_1(t)dt$. We show that $f_1 \ll f_2$. If $0 < t_0 \leq s$, then

$$\int_0^{t_0} f_1(t)dt \leq \int_0^{t_0} f_2(t)dt \qquad \text{since } 0 \leq f_1(t) \leq f_2(t) \text{ on } (0,s] \qquad (7.1.6)$$

If $t_0 > r$, then

$$
\begin{aligned}
\int_0^{t_0} f_1(t)dt &= \int_0^r f_1(t)dt \qquad \text{since } f_1(t) = 0 \text{ for } t \geq t_0 > r \\
&= \int_0^r f_2(t)dt \\
&= \int_0^{t_0} f_2(t)dt \qquad \text{since } f_2(t) = 0 \text{ for } t \geq t_0 > r
\end{aligned} \tag{7.1.7}
$$

If $s < t_0 \leq r$, then we note first that

$$
\begin{aligned}
\int_0^s f_2(t)dt &= \int_0^\infty f_2(t)dt - \int_s^\infty f_2(t)dt \\
&= \int_0^\infty f_1(t)dt - (r-s)\beta
\end{aligned}
$$

and

$$
\int_0^s f_1(t)dt = \int_0^\infty f_1(t)dt - (r-s)\alpha
$$

Therefore

$$
\begin{aligned}
\int_0^{t_0} f_2(t)dt - \int_0^{t_0} f_1(t)dt &= \int_0^s f_2(t)dt - \int_0^s f_1(t)dt + \int_s^{t_0} f_2(t)dt - \int_s^{t_0} f_1(t)dt \\
&= (\alpha-\beta)(r-s) + (\beta-\alpha)(t_0-s) \\
&= (\alpha-\beta)(r-t_0) \geq 0
\end{aligned} \tag{7.1.8}
$$

Since $f_1^* = f_1$ and $f_2 = f_2^*$, it follows from (7.1.6), (7.1.7) and (7.1.8) that $f_1 \ll f_2$. Put $\epsilon = \min\{\frac{\alpha-\beta}{2}, \frac{\beta}{2}\}$, $\theta = \frac{1}{2}(s+r)$ and define $f_3 = f_2 + \epsilon\chi_{(s,\theta]} - \epsilon\chi_{(\theta,r]}$. Since f_2 is decreasing, f_3 is decreasing on $[0,s]$. Let $t_1 \in (s,\theta]$, $t_2 \in (\theta, r]$ and $t_3 \in (r,\infty)$. Then $f_3(t_3) = 0 < \frac{\beta}{2} \leq \beta - \epsilon = f_3(t_2)$. Furthermore, $f_3(t_2) < f_3(t_1) \leq \beta + \frac{\alpha-\beta}{2} < \alpha = f_1(s) \leq f_2(s) = f_3(s)$. It follows that f_3 is positive and decreasing, and hence $f_3^* = f_3$ a.e. As before, it can be shown that $f_3 \ll \chi_{(0,s]}$. Furthermore,

$$
\begin{aligned}
\|f_3\|_{L_{w,1}} &= \int_0^\infty f_2 d\psi(t) + \epsilon\int_s^\theta d\psi(t) - \epsilon\int_\theta^r d\psi(t) \\
&= \|f_2\|_{L_{w,1}} + \epsilon(\psi(\tfrac{1}{2}(s+r)) - \psi(s)) - \epsilon(\psi(r) - \psi(\tfrac{1}{2}(s+r))) \qquad \text{since } \theta = \tfrac{1}{2}(s+r) \\
&> \|f_2\|_{L_{w,1}} + \epsilon(\tfrac{1}{2}\psi(s) + \tfrac{1}{2}\psi(r) - \psi(s)) - \epsilon(\psi(r) - \tfrac{1}{2}\psi(s) - \tfrac{1}{2}\psi(r)) \qquad \text{since } \psi \text{ is strictly concave} \\
&= \|f_2\|_{L_{w,1}}
\end{aligned}
$$

Therefore, using the fact that $L_{w,1}(0,\infty)$ is fully symmetric and $f \ll f_1 \ll f_2 \ll f_3 \ll \chi_{(0,s]}$, we obtain

$$
\|f\|_{L_{w,1}} \leq \|f_2\|_{L_{w,1}} < \|f_3\|_{L_{w,1}} \leq \|\chi_{(0,s]}\|_{L_{w,1}}
$$

□

COROLLARY 7.1.7. *Suppose $(\mathcal{A},\tau)$ is a semi-finite von Neumann algebra and $x, y \in L_{w,1}(\tau)$. If $\mu_y = \chi_{(0,s]}$ $(0 < s < \infty)$, $x \ll y$ and $\mu_x \neq \mu_y$, then*

$$
\|x\|_{L_{w,1}(\tau)} < \|y\|_{L_{w,1}(\tau)}.
$$

PROOF. Since $\mu_x \in L_{w,1}(0,\infty)$ and $\mu_x^* = \mu_x$, we have by Lemma 7.1.6 that

$$
\|x\|_{L_{w,1}(\tau)} = \|\mu_x\|_{L_{w,1}(0,\infty)} < \|\chi_{(0,s]}\|_{L_{w,1}(0,\infty)} = \|y\|_{L_{w,1}(\tau)}.
$$

□

7.2. Characterization of surjective isometries between Lorentz spaces

Throughout this section we will assume that $(\mathcal{A}, \tau)$ and $(\mathcal{B}, \nu)$ are semi-finite von Neumann algebras and that w is a strictly decreasing weight function. The two benefits of working with a strictly decreasing weight function are that one has access to the characterization of the extreme points (Proposition 7.1.3) and that the Lorentz spaces under consideration have strictly monotone norm (see [4]). It will be shown that a linear map $U : L_{w,1}(\tau) \to L_{w,1}(\nu)$ is a surjective isometry if and only if there exists a Jordan $*$-isomorphism $\Phi : \mathcal{A} \to \mathcal{B}$, a unitary $u \in \mathcal{B}$ and a $\delta > 0$ such that

$$U(x) = \frac{1}{\delta} u\Phi(x) \qquad \forall x \in \mathcal{A} \cap L_{w,1}(\tau) \tag{7.2.1}$$

and

$$\psi(\nu(\Phi(p))) = \delta\psi(\tau(p)) \qquad \forall p \in \mathcal{P}(\mathcal{A})^f.$$

Suppose $U : L_{w,1}(\tau) \to L_{w,1}(\nu)$ is a surjective isometry. Recall that in the finite setting (see Theorem 2.2.10), the proof that U has the desired representation involves using the characterization of the extreme points of the unit ball of a Lorentz space to show that

$$U(\mathbf{1}) = \frac{1}{\psi(\nu(|a|))} a$$

for some partial isometry $a \in \mathcal{B}$. In the finite setting one can then find a unitary operator $u \in \mathcal{B}$ such that $a = u|a|$. The most substantial part of the proof is then showing that the surjective isometry $T(x) := u^* U(x)$ is positive and that $|a| = \mathbf{1}$. Theorem 2.2.7 can then be used to obtain the desired representation. In the semi-finite setting, we will show that the techniques used to show that T is positive and $|a| = \mathbf{1}$ can be adapted to show that U is disjointness-preserving. Furthermore, the characterization of the extreme points of the unit ball of a Lorentz space will be used to show that U is finiteness-preserving. This will enable us to use Theorem 6.1.1, whose interpretation in the context of Lorentz spaces will yield the representation given in (7.2.1).

Let $\psi(t) := \int_0^t w(s)ds$ for $t > 0$. Note that if $0 \neq p \in \mathcal{P}(\mathcal{A})^f$, then $\frac{p}{\psi(\tau(p))}$ is an extreme point of the unit ball of $L_{w,1}(\tau)$, by Proposition 7.1.3. U is a surjective isometry and so the image of this element is also an extreme point. It follows by Proposition 7.1.3 that $U(p)$ can be written in the form $\alpha_p v_{(p)}$, where $v_{(p)} \in \mathcal{V}(\mathcal{B})^f$ and $\alpha_p = \frac{\psi(\tau(p))}{\psi(\nu(|v_{(p)}|))}$. By Proposition B.2.13, there exists a unitary operator $u_{(p)} \in \mathcal{B}$ such that $v_{(p)} = u_{(p)}|v_{(p)}|$. Define $T_p : L_{w,1}(\tau) \to L_{w,1}(\nu)$ by

$$T_p(x) := u_{(p)}^* U(x) \qquad x \in L_{w,1}(\tau).$$

Note that if $y \in L_{w,1}(\nu)$, then $u_{(p)}y \in L_{w,1}(\nu)$. Since U is surjective, there exists an $x \in L_{w,1}(\tau)$ such that $U(x) = u_{(p)}y$. It follows that $T_p(x) = u_{(p)}^* U(x) = u_{(p)}^* u_{(p)} y = y$ and hence T_p is surjective. Furthermore, if $x \in L_{w,1}(\tau)$, then using Proposition B.3.1(6) and the fact that U is an isometry, we have that

$$\|T_p(x)\|_{L_{w,1}} = \|u_{(p)}^* U(x)\|_{L_{w,1}} = \|U(x)\|_{L_{w,1}} = \|x\|_{L_{w,1}}$$

It follows that T_p is an isometry. If $0 \neq q \in \mathcal{P}(\mathcal{A})^f$ with $q \leq p$, then $T_p(q)$ will be scalar multiple of a partial isometry, since T_p is a surjective isometry and hence preserves extreme points. If we can show, in addition, that $T_p(q) \geq 0$, then this will mean that $T_p(q)$ is a scalar multiple of a projection, by Remark B.1.29. Application of Proposition B.1.24 to the equation

$$T_{p+q}(p) + T_{p+q}(q) = T_{p+q}(p+q) \qquad 0 \neq p, q \in \mathcal{P}(\mathcal{A})^f,\ pq = 0$$

will then enable us to show that U is disjointness-preserving. We will therefore show that

$$T_p(pL_{w,1}(\tau)p) \subseteq |v_{(p)}| L_{w,1}(\nu) |v_{(p)}|$$

and

$$T_p(x) \geq 0 \qquad \forall x \in pL_{w,1}(\tau)^+ p.$$

To accomplish this we start by showing that if $0 \neq q \in \mathcal{P}(\mathcal{A})^f$ with $q \leq p$, then $|T_p(q)| \leq |v_{(p)}|$ and hence that $T_p(q) \in |v_{(p)}| L_{w,1}(\nu) |v_{(p)}|$, by Proposition B.3.7(2). Since T_p is a surjective isometry, we have that $T_p(q) = \beta v$ for some $v \in \mathcal{V}(\mathcal{B})^f$, where $\beta = \frac{\psi(\tau(q))}{\psi(\nu(|v|))}$. Let $y = T_p(p+q)$. Then

$$y = u_{(p)}^* U(p) + T_p(q) = \alpha_p |v_{(p)}| + \beta v$$

It follows that $s(y) \leq s(|v_{(p)}|) \vee s(v) = |v_{(p)}| \vee |v|$. Furthermore, $v_{(p)}, v \in \mathcal{V}(\mathcal{B})^f$ and so $\nu(|v_{(p)}|), \nu(|v|) < \infty$. It follows by Proposition B.1.21(3) that $|v_{(p)}| \vee |v|$ and hence $s(y)$ has finite trace. By Proposition B.2.13 this implies that there exists a unitary operator $w \in \mathcal{B}$ such that $wy = |y|$. Define $W : L_{w,1}(\tau) \to L_{w,1}(\nu)$ by

$$W(x) = wT_p(x) \qquad x \in L_{w,1}(\tau).$$

It is easily checked that W is a surjective isometry. Furthermore,

$$|y| = wy = wT_p(p+q) = W(p+q) = \alpha_p w|v_{(p)}| + \beta wv \tag{7.2.2}$$

Let $m := s(w|v_{(p)}|) \vee s(wv) \vee r(w|v_{(p)}|) \vee r(wv)$. It is easily checked that $w|v_{(p)}|$ and wv are partial isometries and that $s(w|v_{(p)}|) = |v_{(p)}|$, $s(wv) = s(v)$, $r(w|v_{(p)}|) = w|v_{(p)}|w^*$ and $r(wv) = wr(v)w^*$ (see Proposition B.1.28). It follows that $\nu(s(w|v_{(p)}|)), \nu(s(wv)) < \infty$ and hence $\nu(r(w|v_{(p)}|)), \nu(r(wv)) < \infty$, by Proposition B.1.21(1). Therefore $\nu(m) < \infty$, by Proposition B.1.21(3). This implies that the identity of the reduced von Neumann algebra $\mathcal{B}_m$ has finite trace. Let φ denote the canonical mapping from $mS(\mathcal{B}, \nu)m$ onto $S(\mathcal{B}_m, \nu_m)$ and recall that the restriction of φ to $m\mathcal{B}m$ is a $*$-isometric isomorphism from $m\mathcal{B}m$ onto $\mathcal{B}_m$. Hence, if $F = L_{w,1}(\nu)$, then $\|\varphi(x)\|_{F_{q_{(p)}}} = \|x\|_F$ for all $x \in q_{(p)} F q_{(p)}$. Note that $|y|, w|v_{(p)}|, wv \in m\mathcal{B}m$ and so $\varphi(|y|), \varphi(w|v_{(p)}|), \varphi(wv) \in \mathcal{B}_m$. Let $\mathcal{M}$ be an abelian von Neumann subalgebra of $\mathcal{B}_m$ containing the spectral family of $\varphi(|y|) = |\varphi(y)|$. By Theorem B.4.1 there exists a conditional expectation E from $L_1(\mathcal{B}_m)$ onto $L_1(\mathcal{M})$. Note that $E(\mathcal{B}_m) = \mathcal{M}$, $\|E(x)\|_\infty \leq \|x\|_\infty$ for all $x \in \mathcal{B}_m$ and $\|E(z)\|_{L^1(\mathcal{M})} \leq \|z\|_{L^1(\mathcal{B}_m)}$ for all $z \in L^1(\mathcal{B}_m)$, by Theorem B.4.1. Note that $L_1(\mathcal{B}_,) + \mathcal{B}_m = L_1(\mathcal{B}_m)$ and $L_1(\mathcal{M}) + \mathcal{M} = L_1(\mathcal{M})$, by the finiteness of the restricted trace and so

$$E(x) \quad \ll \quad x \qquad \forall x \in L^1(\mathcal{B}_m), \tag{7.2.3}$$

by Theorem B.3.5. Therefore,

$$\|E(x)\|_{L_{w,1}(\mathcal{M})} \quad \leq \quad \|x\|_{L_{w,1}(\mathcal{B}_m)} \qquad \forall x \in L_{w,1}(\mathcal{B}_m) \tag{7.2.4}$$

since $L_{w,1}(0,\infty)$ is a fully symmetric space. We prove a sequence of lemmas that will enable us to show that $|\frac{1}{\beta} T_p(q)| = |v| \leq |v_{(p)}|$ and hence that $T_p(q) \in |v_{(p)}| L_{w,1}(\nu) |v_{(p)}|$.

LEMMA 7.2.1. *$E(\varphi(w|v_{(p)}|))$ and $E(\varphi(wv))$ are projections.*

PROOF. We start by showing that $E(\varphi(w|v_{(p)}|))$ and $E(\varphi(wv))$ are self-adjoint. Note that

$$\begin{aligned}
\varphi(|y|) &= E(\varphi(|y|)) && \text{by Theorem B.4.1(3), since } \varphi(|y|) \in \mathcal{M} \subseteq L^1(\mathcal{M}) \\
&= E(\varphi(\alpha_p w|v_{(p)}| + \beta wv)) && \text{by (7.2.2)} \\
&= \alpha_p E(\varphi(w|v_{(p)}|)) + \beta E(\varphi(wv)) && \text{since } E \text{ and } \varphi \text{ are linear} \qquad (7.2.5)
\end{aligned}$$

It is easily checked, using the fact that E preserves adjoints and $\alpha_p, \beta \in \mathbb{R}$, that

$$\operatorname{Re}(\alpha_p E(\varphi(w|v_{(p)}|)) + \beta E(\varphi(wv))) = \alpha_p \operatorname{Re}(E(\varphi(w|v_{(p)}|))) + \beta \operatorname{Re}(E(\varphi(wv)))$$

and so (7.2.5) implies that

$$\varphi(|y|) \quad = \quad \alpha_p \operatorname{Re} E(\varphi(w|v_{(p)}|)) + \beta \operatorname{Re} E(\varphi(wv)) \tag{7.2.6}$$

Furthermore, $\mathcal{M}$ is abelian and so $S(\mathcal{M},\nu_m)$ is also abelian. Therefore $E(\varphi(w|v_{(p)}|))$ is normal, since

$$E(\varphi(w|v_{(p)}|)), E(\varphi(w|v_{(p)}|))^* \in L^1(\mathcal{M}) \subseteq S(\mathcal{M},\nu_m).$$

By Proposition B.2.12(4) this implies that

$$|\operatorname{Re} E(\varphi(w|v_{(p)}|))| \quad \leq \quad |E(\varphi(w|v_{(p)}|))| \tag{7.2.7}$$

One can similarly show that

$$|\operatorname{Re} E(\varphi(wv))| \quad \leq \quad |E(\varphi(wv))| \tag{7.2.8}$$

Note the following:

$$\begin{aligned}
\alpha_p\big\|E(\varphi(w|v_{(p)}|))\big\|_{L_{w,1}(\mathcal{M})} + \beta\big\|E(\varphi(wv))\big\|_{L_{w,1}(\mathcal{M})} &\leq \alpha_p\big\|\varphi(w|v_{(p)}|)\big\|_{L_{w,1}(\mathcal{B}_m)} + \beta\big\|\varphi(wv)\big\|_{L_{w,1}(\mathcal{B}_m)} && \text{by (7.2.4)}\\
&= \big\|\alpha_p w|v_{(p)}|\big\|_{L_{w,1}(\mathcal{B})} + \big\|\beta wv\big\|_{L_{w,1}(\mathcal{B})} && \text{since } \varphi \text{ is an isometry}\\
&= \big\|W(p)\big\|_{L_{w,1}(\mathcal{B})} + \big\|W(q)\big\|_{L_{w,1}(\mathcal{B})}\\
&= \big\|p\big\|_{L_{w,1}(\mathcal{A})} + \big\|q\big\|_{L_{w,1}(\mathcal{A})} && \text{since } W \text{ is an isometry}\\
&= \int_0^\infty \mu_p(t)w(t)dt + \int_0^\infty \mu_q(t)w(t)dt\\
&= \int_0^\infty \mu_{p+q}(t)w(t)dt && \text{since } q \leq p \implies \mu_{p+q} = \mu_p + \mu_q,\\
& \quad \text{by Example 1.4.1(3)}\\
&= \big\|p+q\big\|_{L_{w,1}(\mathcal{A})}\\
&= \big\|T_p(p+q)\big\|_{L_{w,1}(\mathcal{B})} && \text{since } T_p \text{ is an isometry}\\
&= \big\|y\big\|_{L_{w,1}(\mathcal{B})} && \text{by definition of } y
\end{aligned} \tag{7.2.9}$$

Assume that the inequality in (7.2.7) is strict. Then

$$\big\|\operatorname{Re} E(\varphi(w|v_{(p)}|))\big\|_{L_{w,1}(\mathcal{M})} < \big\|E(\varphi(w|v_{(p)}|))\big\|_{L_{w,1}(\mathcal{M})}, \tag{7.2.10}$$

since $L_{w,1}(\mathcal{M})$ has strictly monotone norm. Furthermore,

$$\begin{aligned}
\big\|y\big\|_{L_{w,1}(\mathcal{B})} &= \big\||y|\big\|_{L_{w,1}(\mathcal{B})}\\
&= \big\|\varphi(|y|)\big\|_{L_{w,1}(\mathcal{B}_m)} && \text{since } \varphi \text{ is an isometry}\\
&= \big\|\varphi(|y|)\big\|_{L_{w,1}(\mathcal{M})} && \text{since } |y| \in \mathcal{M} \subseteq \mathcal{B}_m\\
&= \big\|\alpha_p \operatorname{Re} E(\varphi(w|v_{(p)}|)) + \beta \operatorname{Re} E(\varphi(wv))\big\|_{L_{w,1}(\mathcal{M})} && \text{using (7.2.6)}\\
&\leq \alpha_p\big\|\operatorname{Re} E(\varphi(w|v_{(p)}|))\big\|_{L_{w,1}(\mathcal{M})} + \beta\big\|\operatorname{Re} E(\varphi(wv))\big\|_{L_{w,1}(\mathcal{M})}\\
&< \alpha_p\big\|E(\varphi(w|v_{(p)}|))\big\|_{L_{w,1}(\mathcal{M})} + \beta\big\|\operatorname{Re} E(\varphi(wv))\big\|_{L_{w,1}(\mathcal{M})} && \text{using (7.2.10)}\\
&\leq \alpha_p\big\|E(\varphi(w|v_{(p)}|))\big\|_{L_{w,1}(\mathcal{M})} + \beta\big\|E(\varphi(wv))\big\|_{L_{w,1}(\mathcal{M})} && \text{using (7.2.8)}\\
& \quad \text{and the fact that } L_{w,1}(\mathcal{M}) \text{ is fully symmetric}\\
&\leq \big\|y\big\|_{L_{w,1}(\mathcal{B})} && \text{using (7.2.9)}
\end{aligned}$$

This contradiction shows that

$$|\operatorname{Re} E(\varphi(w|v_{(p)}|))| \quad = \quad |E(\varphi(w|v_{(p)}|))| \tag{7.2.11}$$

As mentioned before, $E(\varphi(w|v_{(p)}|))$ is normal and so $E(\varphi(w|v_{(p)}|))$ is self-adjoint, by Proposition B.2.12(5). We can similarly show that $E(\varphi(wv))$ is self-adjoint.

Next, we show that $E(\varphi(w|v_{(p)}|))$ and $E(\varphi(wv))$ are positive. Note that $E(\varphi(w|v_{(p)}|)) = E(\varphi(w|v_{(p)}|))^+ - E(\varphi(w|v_{(p)}|))^-$, where $E(\varphi(w|v_{(p)}|))^+, E(\varphi(w|v_{(p)}|))^- \geq 0$. $E(\varphi(wv))$ has a similar decomposition. Assume that $E(\varphi(w|v_{(p)}|))^- \neq 0$ or $E(\varphi(wv))^- \neq 0$. Then

$$\begin{aligned} \alpha_p E(\varphi(w|v_{(p)}|))^+ + \beta E(\varphi(wv))^+ &> \alpha_p E(\varphi(w|v_{(p)}|)) + \beta E(\varphi(wv)) \\ &= \varphi(|y|) \qquad \text{by (7.2.5)} \end{aligned} \tag{7.2.12}$$

Therefore,

$$\begin{aligned} \|y\|_{L_{w,1}(\mathcal{B})} &= \|\varphi(|y|)\|_{L_{w,1}(\mathcal{M})} \qquad \text{as before} \\ &< \left\|\alpha_p E(\varphi(w|v_{(p)}|))^+ + \beta E(\varphi(wv))^+\right\|_{L_{w,1}(\mathcal{M})} \qquad \text{using (7.2.12) and the strict monotonicity} \\ &\quad \text{of } \|\cdot\|_{L_{w,1}(\mathcal{M})} \\ &\leq \left\|\alpha_p E(\varphi(w|v_{(p)}|))^+\right\|_{L_{w,1}(\mathcal{M})} + \left\|\beta E(\varphi(wv))^+\right\|_{L_{w,1}(\mathcal{M})} \\ &\leq \left\|\alpha_p E(\varphi(w|v_{(p)}|))\right\|_{L_{w,1}(\mathcal{M})} + \left\|\beta E(\varphi(wv))\right\|_{L_{w,1}(\mathcal{M})} \qquad \text{using Proposition B.2.2(3)} \\ &\quad \text{and the monotonicity of the norm} \\ &\leq \|y\|_{L_{w,1}(\mathcal{B})} \qquad \text{using (7.2.9)} \end{aligned}$$

This contradiction shows that $E(\varphi(w|v_{(p)}|))^- = 0$ and $E(\varphi(wv))^- = 0$. Therefore, $E(\varphi(w|v_{(p)}|))$ and $E(\varphi(wv))$ are positive.

Finally, to show that $E(\varphi(w|v_{(p)}|))$ and $E(\varphi(wv))$ are projections, we will show that each of these elements has spectrum contained in $\{0,1\}$. This will be achieved by calculating the singular value functions of $E(\varphi(w|v_{(p)}|))$ and $E(\varphi(wv))$ and using the relationship between the range of a singular value function of an element and the spectrum of that element. Recall that $w|v_{(p)}|$ and wv are partial isometries. Since φ is a $*$-isomorphism and hence a Jordan $*$-homomorphism, $\varphi(w|v_{(p)}|)$ and $\varphi(wv)$ are also partial isometries, by Proposition 1.8.6(2). It follows, by Proposition 1.4.5, that the generalized singular value functions of $\varphi(w|v_{(p)}|)$ and $\varphi(wv)$ are characteristic functions. Note that by (7.2.3), $E(\varphi(w|v_{(p)}|) \ll \varphi(w|v_{(p)}|)$. Furthermore, by Proposition 1.4.5, $\mu_{\varphi(w|v_{(p)}|)} = \chi_{[0,s)}$, where $s = \nu_{\mathcal{B}_m}(\varphi(|w|v_{(p)}||)$. Assume $\mu_{E(\varphi(w|v_{(p)}|)} \neq \mu_{\varphi(w|v_{(p)}|)} = \chi_{[0,s)}$. Then,

$$\begin{aligned} \|y\|_{L_{w,1}(\mathcal{B})} &\leq \alpha_p \left\|E(\varphi(w|v_{(p)}|))\right\|_{L_{w,1}(\mathcal{M})} + \beta \left\|E(\varphi(wv))\right\|_{L_{w,1}(\mathcal{M})} \qquad \text{as before} \\ &< \alpha_p \left\|\varphi(w|v_{(p)}|)\right\|_{L_{w,1}(\mathcal{B}_m)} + \beta \left\|E(\varphi(wv))\right\|_{L_{w,1}(\mathcal{M})} \qquad \text{by Corollary 7.1.7} \\ &\leq \alpha_p \left\|\varphi(w|v_{(p)}|)\right\|_{L_{w,1}(\mathcal{B}_m)} + \beta \left\|\varphi(wv)\right\|_{L_{w,1}(\mathcal{B}_m)} \qquad \text{using (7.2.4)} \\ &= \|y\|_{L_{w,1}(\mathcal{B})} \qquad \text{using (7.2.9)} \end{aligned}$$

This contradiction shows that $\mu_{E(\varphi(w|v_{(p)}|)} = \mu_{\varphi(w|v_{(p)}|)}$. We can similarly show that $\mu_{E(\varphi(wv)} = \mu_{\varphi(wv)}$ and hence these are all characteristic functions. Note that, by Proposition 1.3.5(1), $S_c(\mathcal{B}_m, \nu_m) = S(\mathcal{B}_m, \nu_m)$, since $\mathcal{B}_m$ is trace-finite. It follows that $E(\varphi(wv)) \in S_c(\mathcal{B}_m, \tau)$ and hence $\sigma(E(\varphi(wv))) = \{0,1\}$, by Proposition 1.4.2(7) (since we also have that $E(\varphi(wv)) \geq 0$). It follows, by Theorem B.1.10 that $E(\varphi(wv)) \in \mathcal{M}$ is a projection. We can similarly show that $E(\varphi(w|v_{(p)}|))$ is a projection. $\square$

Lemma 7.2.2. *$E(\varphi(w|v_{(p)}|) = \varphi(|v_{(p)}|)$ and $E(\varphi(wv)) = \varphi(|v|)$.*

PROOF. We start by showing that $E(\varphi(wv)) = E(\varphi(|v|))$ and hence that $E(\varphi(|v|))$ is a projection, by Lemma 7.2.1. Note that

$$\begin{aligned}\|E(\varphi(wv))\|_{L_{w,1}(\mathcal{M})} &\leq \|\varphi(wv)\|_{L_{w,1}(\mathcal{B}_m)} && \text{by (7.2.4)}\\ &= \|wv\|_{L_{w,1}(\mathcal{B})} && \text{since } \varphi \text{ is an isometry}\\ &= \|v\|_{L_{w,1}(\mathcal{B})} && \text{by Proposition B.3.1(6)}\end{aligned} \tag{7.2.13}$$

Similarly,

$$\|E(\varphi(w|v_{(p)}|))\|_{L_{w,1}(\mathcal{M})} \leq \||v_{(p)}|\|_{L_{w,1}(\mathcal{B})} \tag{7.2.14}$$

Assume we have strict inequality in (7.2.13). Then

$$\begin{aligned}\|y\|_{L_{w,1}(\mathcal{B})} &\leq \alpha_p\|E(\varphi(w|v_{(p)}|))\|_{L_{w,1}(\mathcal{M})} + \beta\|E(\varphi(wv))\|_{L_{w,1}(\mathcal{M})} && \text{as before}\\ &< \alpha_p\||v_{(p)}|\|_{L_{w,1}(\mathcal{B})} + \beta\|v\|_{L_{w,1}(\mathcal{B})} && \text{using (7.2.14) and our assumption}\\ &= \alpha_p\|w|v_{(p)}|\|_{L_{w,1}(\mathcal{B})} + \beta\|wv\|_{L_{w,1}(\mathcal{B})} && \text{by Proposition B.3.1(6)}\\ &= \|y\|_{L_{w,1}(\mathcal{B})} && \text{as in (7.2.9)}\end{aligned}$$

This contradiction shows that

$$\|E(\varphi(wv))\|_{L_{w,1}(\mathcal{M})} = \|v\|_{L_{w,1}(\mathcal{B})} \tag{7.2.15}$$

We can similarly show that

$$\|E(\varphi(w|v_{(p)}|))\|_{L_{w,1}(\mathcal{M})} = \||v_{(p)}|\|_{L_{w,1}(\mathcal{B})} \tag{7.2.16}$$

Furthermore,

$$\begin{aligned}0 &\leq E(\varphi(wv))\\ &= E(\varphi(wv))^*E(\varphi(wv)) && \text{since } E(\varphi(wv)) \text{ is a projection by Lemma 7.2.1}\\ &\leq E(\varphi(wv)^*\varphi(wv)) && \text{by Theorem B.4.1(5)}\\ &= E(\varphi(v^*w^*)\varphi(wv)) && \text{since } \varphi \text{ is a } *\text{-isomorphism}\\ &= E(\varphi(v^*w^*wv)) && \text{since } \varphi \text{ is multiplicative on } m\mathcal{B}m\\ &= E(\varphi(|v|)) && \text{since } w \text{ is unitary and } v \text{ is a partial isometry}\end{aligned} \tag{7.2.17}$$

Assume $E(\varphi(wv)) \neq E(\varphi(|v|))$. This implies that $0 \leq E(\varphi(wv)) < E(\varphi(|v|))$, by considering (7.2.17). In this case, we have

$$\begin{aligned}\|v\|_{L_{w,1}(\mathcal{B})} &= \|E(\varphi(wv))\|_{L_{w,1}(\mathcal{M})} && \text{by (7.2.15)}\\ &< \|E(\varphi(|v|))\|_{L_{w,1}(\mathcal{M})} && \text{using } E(\varphi(wv)) < E(\varphi(|v|)) \text{ and the strict monotonicity of the norm}\\ &\leq \|\varphi(|v|)\|_{L_{w,1}(\mathcal{B}_m)} && \text{by (7.2.4)}\\ &= \|v\|_{L_{w,1}(\mathcal{B})} && \text{since } \varphi \text{ is an isometry}\end{aligned}$$

This contradiction shows that

$$E(\varphi(wv)) = E(\varphi(|v|) \tag{7.2.18}$$

Next, we use the properties of the conditional expectation to show that $E(\varphi(|v|)) \leq \varphi(|v|)$. If we knew that $\varphi(|v|) \in \mathcal{M} \subseteq L_1(\mathcal{M})$, then it would automatically follow by Theorem B.4.1 that $E(\varphi(|v|)) = \varphi(|v|)$. Unfortunately, we only know that $\varphi(|v|) \in \mathcal{B}_m$. Note that $|v|$ is a projection in $m\mathcal{B}m$ and so $\varphi(|v|)$ is a projection in

$\mathcal{B}_m$, since φ is a $*$-isomorphism. It follows by Proposition B.2.2(4) that

$$E(\varphi(|v|))\varphi(|v|)E(\varphi(|v|)) \leq E(\varphi(|v|))\mathbf{1}_{\mathcal{B}_m}E(\varphi(|v|)) = E(\varphi(|v|)) \tag{7.2.19}$$

Furthermore,

$$\begin{aligned}
E\Big(E(\varphi(|v|))\Big(\varphi(|v|)E(\varphi(|v|))\Big)\Big) &= E(\varphi(|v|))E\Big(\varphi(|v|)E(\varphi(|v|))\Big) \quad \text{by Theorem B.4.1(6)}\\
&= E(\varphi(|v|))E(\varphi(|v|))E(\varphi(|v|))) \quad \text{by Theorem B.4.1(6)}\\
&= E(\varphi(|v|)) \quad \text{since } E(\varphi(|v|)) \text{ is a projection}\\
&= E(E(\varphi(|v|))) \quad \text{by Theorem B.4.1(3)}\\
\implies 0 &= E\Big(E(\varphi(|v|)) - E(\varphi(|v|))\varphi(|v|)E(\varphi(|v|))\Big) \quad \text{since } E \text{ is linear}
\end{aligned}$$

Using (7.2.19) and Theorem B.4.1(2), it follows that $0 = E(\varphi(|v|)) - E(\varphi(|v|))\varphi(|v|)E(\varphi(|v|))$. By Proposition B.1.22(4), this implies that

$$E(\varphi(|v|)) \quad \leq \quad \varphi(|v|) \tag{7.2.20}$$

Recall that $E(\varphi(wv)) = E(\varphi(|v|))$ (see (7.2.18)) and so, using (7.2.15), we have that

$$\|E(\varphi(|v|))\|_{L_{w,1}(\mathcal{M})} = \|E(\varphi(wv))\|_{L_{w,1}(\mathcal{M})} = \|\varphi(|v|)\|_{L_{w,1}(\mathcal{B}_m)}.$$

Since we are dealing with a space with strictly monotone norm, this implies that the inequality in (7.2.20) cannot be strict and hence

$$\varphi(|v|) = E(\varphi(|v|)).$$

The equality $E(\varphi(w|v_{(p)}|)) = \varphi(|v_{(p)}|)$ is proved similarly. □

LEMMA 7.2.3. *$wv = |v|$ and $w|v_{(p)}| = |v_{(p)}|$.*

PROOF. If we can show that wv and $w|v_{(p)}|$ are projections, then we will have

$$w|v_{(p)}| = (w|v_{(p)}|)^*(w|v_{(p)}|) = |v_{(p)}|w^*w|v_{(p)}| = |v_{(p)}|,$$

since w is unitary. We would similarly have $wv = |v|$. We start by showing that $\varphi(|v_{(p)}|) = E(\varphi(w|v_{(p)}|w^*))$. Note that $|v_{(p)}|$ is a projection and so $\varphi(|v_{(p)}|)$ is a projection. Therefore

$$\begin{aligned}
\varphi(|v_{(p)}|) &= \varphi(|v_{(p)}|)\varphi(|v_{(p)}|)^*\\
&= E(\varphi(w|v_{(p)}|))\Big(E(\varphi(w|v_{(p)}|))\Big)^* \quad \text{by Lemma 7.2.2}\\
&= \Big(E\Big(\varphi(w|v_{(p)}|)^*\Big)\Big)^* E\Big((\varphi(w|v_{(p)}|))^*\Big) \quad \text{by Theorem B.4.1(1)}\\
&= \Big(E\Big(\varphi(|v_{(p)}|w^*)\Big)\Big)^* E\Big(\varphi(|v_{(p)}|w^*)\Big) \quad \text{since } \varphi \text{ is a } *\text{-isomorphism}\\
&\leq E\Big(\Big(\varphi(|v_{(p)}|w^*)\Big)^*\Big(\varphi(|v_{(p)}|w^*)\Big)\Big) \quad \text{by Theorem B.4.1(5)}\\
&= E\Big(\varphi\Big((w|v_{(p)}|)(|v_{(p)}|w^*)\Big)\Big) \quad \text{since } \varphi \text{ is a } *\text{-isomorphism}\\
&= E(\varphi(w|v_{(p)}|w^*))
\end{aligned}$$

Assume that $\varphi(|v_{(p)}|) \neq E(\varphi(w|v_{(p)}|w^*))$. Then

$$\begin{aligned}
\big\||v_{(p)}|\big\|_{L_{w,1}(\mathcal{B})} &= \big\|\varphi(|v_{(p)}|)\big\|_{L_{w,1}(\mathcal{B}_m)} \\
&< \big\|E(\varphi(w|v_{(p)}|w^*))\big\|_{L_{w,1}(\mathcal{B}_m)} \quad \text{using our assumption and the strict monotonicity of the norm} \\
&\leq \big\|\varphi(w|v_{(p)}|w^*)\big\|_{L_{w,1}(\mathcal{B}_m)} \quad \text{using (7.2.4)} \\
&= \big\|w|v_{(p)}|w^*\big\|_{L_{w,1}(\mathcal{B})} \\
&= \big\||v_{(p)}|\big\|_{L_{w,1}(\mathcal{B})} \quad \text{by Proposition B.3.1(6)}
\end{aligned}$$

This is a contradiction and so

$$\varphi(|v_{(p)}|) \;=\; E(\varphi(w|v_{(p)}|w^*)) \tag{7.2.21}$$

In particular, $E(\varphi(w|v_{(p)}|w^*))$ is a projection. Next, we show that $w|v_{(p)}|$ is normal by showing that $\varphi(|v_{(p)}|) = E(\varphi(w|v_{(p)}|w^*) = \varphi(w|v_{(p)}|w^*)$ (and hence that $|v_{(p)}| = w|v_{(p)}|w^*$). Note that $w|v_{(p)}|w^* = (w|v_{(p)}|w^*)^* = (w|v_{(p)}|w^*)^2$, i.e., $w|v_{(p)}|w^*$ is a projection. Therefore $\varphi(w|v_{(p)}|w^*)$ is also a projection (using the multiplicativity and $*$-preserving properties of φ). Replacing $|v|$ with $w|v_{(p)}|w^*$ in the proof of (7.2.20), we obtain

$$E(\varphi(w|v_{(p)}|w^*)) \;\leq\; \varphi(w|v_{(p)}|w^*) \tag{7.2.22}$$

Furthermore, using (7.2.21) and Proposition B.3.1(6), we have that

$$\big\|E(\varphi(w|v_{(p)}|w^*))\big\|_{L_{w,1}(\mathcal{B}_m)} = \big\|\varphi(|v_{(p)}|)\big\|_{L_{w,1}(\mathcal{B}_m)} = \big\||v_{(p)}|\big\|_{L_{w,1}(\mathcal{B})} = \big\|w|v_{(p)}|w^*\big\|_{L_{w,1}} = \big\|\varphi(w|v_{(p)}|w^*)\big\|_{L_{w,1}(\mathcal{B}_m)}$$

Since $L_{w,1}(\mathcal{B}_m)$ has strictly monotone norm, this implies that the inequality in (7.2.22) cannot be strict and therefore $\varphi(w|v_{(p)}|w^*) = E(\varphi(w|v_{(p)}|w^*))$. However, $E(\varphi(w|v_{(p)}|w^*)) = \varphi(|v_{(p)}|)$, by (7.2.21) and so $w|v_{(p)}|w^* = |v_{(p)}|$, since φ is injective. It follows that

$$w|v_{(p)}| \;=\; |v_{(p)}|w \tag{7.2.23}$$

It is easily checked that (7.2.23) implies that $w|v_{(p)}|$ is normal. Next, we show that $w|v_{(p)}|$ is self-adjoint by showing that $|\mathrm{Re}\, w|v_{(p)}|| = |v_{(p)}|$. Since $w|v_{(p)}|$ is normal, we have by Proposition B.2.12(4) that

$$|\mathrm{Re}\,(w|v_{(p)}|)| \leq |w|v_{(p)}|| = (|v_{(p)}|w^*w|v_{(p)}|)^{1/2} = |v_{(p)}| \tag{7.2.24}$$

Furthermore,

$$\begin{aligned}
\varphi(|v_{(p)}|) &= E(\varphi(w|v_{(p)}|)) \quad \text{by Lemma 7.2.2} \\
&= \mathrm{Re}\, E(\varphi(w|v_{(p)}|)) \quad \text{since } E(\varphi(w|v_{(p)}|)) \text{ is self-adjoint} \\
&= E(\varphi(\mathrm{Re}\, w|v_{(p)}|)) \quad \text{since } E \text{ and } \varphi \text{ are linear}
\end{aligned}$$

Using the above and (7.2.4), we therefore have that

$$\big\|\varphi(|v_{(p)}|)\big\|_{L_{w,1}(\mathcal{B}_m)} \;=\; \big\|E(\varphi(\mathrm{Re}\, w|v_{(p)}|))\big\|_{L_{w,1}(\mathcal{B}_m)} \leq \big\|\varphi(\mathrm{Re}\, w|v_{(p)}|)\big\|_{L_{w,1}(\mathcal{B}_m)}$$

Using the fact that φ is an isometry, (7.2.24) and the monotonicity of the norm, we have that

$$\big\||v_{(p)}|\big\|_{L_{w,1}(\mathcal{B})} \leq \big\|\mathrm{Re}\, w|v_{(p)}|\big\|_{L_{w,1}(\mathcal{B})} \leq \big\||v_{(p)}|\big\|_{L_{w,1}(\mathcal{B})}$$

We therefore get

$$\big\||v_{(p)}|\big\|_{L_{w,1}(\mathcal{B})} \;=\; \big\|\mathrm{Re}\, w|v_{(p)}|\big\|_{L_{w,1}(\mathcal{B})} \tag{7.2.25}$$

Using (7.2.24), (7.2.25) and the strict monotonicity of the norm we obtain

$$|v_{(p)}| \;=\; |\mathrm{Re}\,(w|v_{(p)}|)| \tag{7.2.26}$$

However, $|v_{(p)}| = |w|v_{(p)}||$ (see (7.2.24)) and so $w|v_{(p)}|$ is self-adjoint, by Proposition B.2.12(5). Furthermore, by (7.2.2),

$$wv = \frac{1}{\beta}(|y| - \alpha_p w|v_{(p)}|)$$

and so wv is also self-adjoint. Next, we show that $w|v_{(p)}|$ and wv are positive. Assume that $(wv)^- \neq 0$. Then

$$|y| = \alpha_p w|v_{(p)}| + \beta wv < \alpha_p(w|v_{(p)}|)^+ + \beta(wv)^+$$

Using the strict monotonicity of the norm and the triangle inequality, this implies that

$$\begin{aligned}
\|y\|_{L_{w,1}(\mathcal{B})} &< \alpha_p\|(w|v_{(p)}|)^+\|_{L_{w,1}(\mathcal{B})} + \beta\|(wv)^+\|_{L_{w,1}(\mathcal{B})} \\
&\leq \alpha_p\||w|v_{(p)}||\|_{L_{w,1}(\mathcal{B})} + \beta\||wv|\|_{L_{w,1}(\mathcal{B})} \qquad \text{using Proposition B.2.2(3) and} \\
&\qquad \text{the monotonicity of the norm} \\
&= \|y\|_{L_{w,1}(\mathcal{B})} \qquad \text{using (7.2.9)}
\end{aligned}$$

This contradiction shows that $(wv)^- = 0$ and hence wv is positive. We can similarly show that $w|v_{(p)}|$ is positive. Since, in addition, wv and $w|v_{(p)}|$ are partial isometries, it follows that they are projections, by Remark B.1.29. □

LEMMA 7.2.4. $|v| \leq |v_{(p)}|$

PROOF. Recall that $0 < q \leq p$ and $T_p(q) = \beta v$. Furthermore, by Lemma 7.2.3,

$$W(q) = \beta wv = \beta|v| \qquad \text{and} \qquad W(p) = \alpha_p w|v_{(p)}| = \alpha_p|v_{(p)}| \tag{7.2.27}$$

If $q = p$, then $\beta|v| = W(q) = W(p) = \alpha_p|v_{(p)}|$ and so $|v| = |v_{(p)}|$. Suppose $q < p$. Then $p - q$ is a non-zero projection and $\tau(p - q) \leq \tau(p) < \infty$. W is a surjective isometry and so $W(p - q) = \gamma b$ for some $b \in \mathcal{V}(\mathcal{B})^f$, where $\gamma = \frac{\psi(\tau(p-q))}{\psi(\nu(|b|))}$. Put $s = |v| \vee |b| - |v_{(p)}|(|v| \vee |b|)$. If we can show that s is a projection and $s = 0$, then we would have

$$|v| \vee |b| = |v_{(p)}|(|v| \vee |b|) \implies |v| \vee |b| \leq |v_{(p)}| \implies |v| \leq |v_{(p)}|$$

Assume that $s \neq 0$ and let $\epsilon = \frac{\beta}{\alpha_p}$ and $\delta = \frac{\gamma}{\alpha_p}$. We start by showing that $\epsilon = \delta$. To do this we will show that $|v|s = s = |b|s$. Note that, using (7.2.27), we have

$$b = \frac{1}{\gamma}W(p - q) = \frac{\alpha_p}{\gamma}|v_{(p)}| - \frac{\beta}{\gamma}|v| \tag{7.2.28}$$

It follows that b is self-adjoint. Furthermore, since $\mathcal{M}$ is abelian, $S(\mathcal{M}, \nu_m)$ is abelian. Since $E(\varphi(wv)), E(\varphi(w|v_{(p)}|)) \in L^1(\mathcal{M}) \subseteq S(\mathcal{M}, \nu_m)$, it follows that these commute and hence, by Lemma 7.2.2,

$$\varphi(|v_{(p)}|)\varphi(|v|) = E(\varphi(w|v_{(p)}|))E(\varphi(wv)) = E(\varphi(wv))E(\varphi(w|v_{(p)}|)) = \varphi(|v|)\varphi(|v_{(p)}|).$$

φ is multiplicative and so $|v_{(p)}||v| = |v||v_{(p)}|$. Using (7.2.28), this implies that $b|v| = |v|b$ and hence $|b||v| = |v||b|$, by Corollary B.2.8. We can similarly show that $|v_{(p)}|$ and $|b|$ commute and therefore s is a projection. It is then easily checked that s commutes with $|v|$ and with $|b|$, since $|v|$, $|v_{(p)}|$ and $|b|$ all commute. Furthermore,

$$\begin{aligned}
(7.2.29) \qquad && |v_{(p)}| &= \epsilon|v| + \delta b \qquad \text{using (7.2.28)} \\
&\implies & |v_{(p)}|(|v| \vee |b| - |v_{(p)}|(|v| \vee |b|)) &= (\epsilon|v| + \delta b)s \\
&& \implies 0 &= \epsilon|v|s + \delta bs \qquad \text{since } |v_{(p)}|^2 = |v_{(p)}| \\
&& \implies \epsilon^2 s|v|^2 s &= \delta^2 s|b|^2 s \\
(7.2.30) \qquad && \implies \epsilon|v|s &= \delta|b|s \qquad \text{since } |v|s = s|v|,\ |b|s = s|b| \text{ and hence these are projections} \\
&& \implies |v|s &= |b|s \qquad \text{since } |v|s, |b|s \in \mathcal{P}(\mathcal{B}) \\
&\implies & |v|(|v| \vee |b| - |v_{(p)}|(|v| \vee |b|)) &= |b|(|v| \vee |b| - |v_{(p)}|(|v| \vee |b|)) \qquad \text{recalling the definition of } s \\
(7.2.31) \qquad && \implies |v| - |v_{(p)}||v| &= |b| - |v_{(p)}||b| \qquad \text{since all the projections in the previous line commute}
\end{aligned}$$

Note also that

$$\begin{aligned}
s &= |v| + |b| - |v||b| - |v_{(p)}|(|v| + |b| - |v||b|) \\
&\qquad \text{by Proposition B.1.16(4), since } |v||b| = |b||v| \\
&= (|v| - |v_{(p)}||v|) + (|b| - |v_{(p)}||b|) - (|v| - |v_{(p)}||v|)(|b| - |v_{(p)}||b|) \\
&\qquad \text{since } |v_{(p)}||v||b| = |v||v_{(p)}||b| = |v_{(p)}||v||v_{(p)}||b| \\
&= (|v| - |v_{(p)}||v|) \vee (|b| - |v_{(p)}||b|) \\
&\qquad \text{by Proposition B.1.16(4), since } (|v| - |v_{(p)}||v|) \text{ and } (|b| - |v_{(p)}||b|) \text{ commute} \\
(7.2.32) \qquad &= (|v| - |v_{(p)}||v|) = (|b| - |v_{(p)}||b|) \qquad \text{using (7.2.31)} \\
\implies |v|s &= |v|(|v| - |v_{(p)}||v|) \\
&= (|v| - |v_{(p)}||v|) \qquad \text{since } |v||v_{(p)}| = |v_{(p)}||v| \text{ and } |v| \in \mathcal{P}(\mathcal{B}) \\
&= s
\end{aligned}$$

We can similarly show that $|b|s = s$. We therefore have that $|v|s = s = |b|s$ and hence that $\epsilon s = \delta s$, using (7.2.30). Since s is a projection and we have assumed that $s \neq 0$, this implies that $\epsilon = \delta$ and therefore $\beta = \gamma$, i.e.,

$$\frac{\psi(\tau(q))}{\psi(\nu(|v|))} = \frac{\psi(\tau(p-q))}{\psi(\nu(|b|))} \tag{7.2.33}$$

We show that if $\epsilon = 1$ (or equivalently $\delta = 1$), then $s = 0$. If $\epsilon = 1$, then $\frac{\beta}{\alpha_p} = \epsilon = 1 = \delta = \frac{\gamma}{\alpha_p}$ and so $\beta = \alpha_p = \gamma$. It will be useful to calculate the norm of $|v| - b$. To do so we first calculate the singular value function of $|v| - b$. Note that $|v_{(p)}| = |v| + b$, using (7.2.29) and $\epsilon = 1 = \delta$. Therefore

$$|v| - b = |v_{(p)}| - 2b = |v_{(p)}| - 2(|v_{(p)}| - |v|) = 2|v| - |v_{(p)}| \tag{7.2.34}$$

Furthermore,

$$\begin{aligned}
\Big(2(|v| - |v_{(p)}||v|) + |v_{(p)}|\Big)^2 &= 4|v| - 4|v||v_{(p)}| + |v_{(p)}| \qquad \text{since } |v_{(p)}||v| = |v||v_{(p)}| \\
&= (2|v| - |v_{(p)}|)^2 \\
&= ||v| - b|^2 \qquad \text{using (7.2.34)}
\end{aligned}$$

Furthermore, it is easily checked that $2(|v| - |v_{(p)}||v|) + |v_{(p)}| \geq 0$. Positive square roots are unique and therefore

$$||v| - b| = 2(|v| - |v_{(p)}||v|) + |v_{(p)}| \tag{7.2.35}$$

Note that $(|v| - |v_{(p)}||v|)$ and $|v_{(p)}|$ are projections. Furthermore $|v_{(p)}|(|v| - |v_{(p)}||v|) = 0$ and $|v_{(p)}| + (|v| - |v_{(p)}||v|) = |v_{(p)}| \vee |v|$, by Proposition B.1.16(4). Therefore, by Example 1.4.1(2),

$$\mu_{2(|v|-|v_{(p)}||v|)+|v_{(p)}|} \quad = \quad 2\chi_{[0,\nu(|v|-|v_{(p)}||v|))} + \chi_{[\nu(|v|-|v_{(p)}||v|),\nu(|v|\vee|v_{(p)}|))}. \tag{7.2.36}$$

Recall that we have assumed that $s \neq 0$. Since s is a projection, ν is faithful and ψ is strictly increasing this implies that $\psi(\nu(s)) > 0$. The following calculation will show that if $\epsilon = 1 = \delta$, then we obtain a contradiction and so $s = 0$ in this case.

$$\begin{aligned}
\psi(\tau(p)) &= \|p\|_{L_{w,1}(\mathcal{A})} \\
&= \big\||q-(p-q)|\big\|_{L_{w,1}(\mathcal{A})} \qquad \text{since } q(p-q)=0 \implies |q-(p-q)| = |q+(p-q)| = p, \\
&\quad\ \text{by Proposition B.2.12(3)} \\
&= \|W(q) - W(p-q)\|_{L_{w,1}(\mathcal{B})} \qquad \text{since } W \text{ is a linear isometry} \\
&= \|\beta|v| - \gamma b\|_{L_{w,1}(\mathcal{B})} \qquad \text{using (7.2.27)} \\
&= \alpha_p\big\||v| - b\big\|_{L_{w,1}(\mathcal{B})} \qquad \text{since } \beta = \alpha_p = \gamma \\
&= \alpha_p\Big(\int_0^\infty \mu_{2(|v|-|v_{(p)}||v|)+|v_{(p)}|}(t)d\psi(t)\Big) \qquad \text{using (7.2.35)} \\
&= \alpha_p\Big(\int_0^\infty 2\chi_{[0,\nu(|v|-|v_{(p)}||v|))}(t)d\psi(t) + \int_0^\infty \chi_{(\nu(|v|-|v_{(p)}||v|),\nu(|v|\vee|v_{(p)}|))}(t)d\psi(t)\Big) \qquad \text{using (7.2.36)} \\
&= \left(\frac{\psi(\tau(p))}{\psi(\nu(|v_{(p)}|))}\right)\Big(\psi(\nu(|v| - |v_{(p)}||v|)) + \psi(\nu(|v| \vee |v_{(p)}|))\Big) \\
&= \psi(\tau(p))\left(\frac{\psi(\nu(s))}{\psi(\nu(|v_{(p)}|))} + \frac{\psi(\nu(|v| \vee |v_{(p)}|))}{\psi(\nu(|v_{(p)}|))}\right) \qquad \text{since } |v| - |v_{(p)}||v| = s \text{ (see (7.2.32))} \\
&\geq \psi(\tau(p))\left(\frac{\psi(\nu(s))}{\psi(\nu(|v_{(p)}|))} + 1\right) \qquad \text{since } |v| \vee |v_{(p)}| \geq |v_{(p)}| \implies \nu(|v| \vee |v_{(p)}|) \geq \nu(|v_{(p)}|) \\
&\quad\ \text{and } \psi \text{ is strictly increasing} \\
&> \psi(\tau(p)) \qquad \text{since } \psi(\nu(s)) \neq 0
\end{aligned}$$

This contradiction shows that if $\epsilon = 1 = \delta$, then $s = 0$.

We consider various possibilities. Suppose $\tau(q) < \frac{1}{2}\tau(p)$. Note that $\tau(p) = \tau(q) + \tau(p-q)$, and so $\tau(p-q) > \tau(q)$ in this case. Therefore $\psi(\tau(p-q)) > \psi(\tau(q))$, since ψ is strictly increasing. Using (7.2.33), this implies that $\nu(|v|) < \nu(|b|)$. A simple proof by contradiction, using the previous inequality, shows that $|b| - |b||v| \neq 0$. Note that $|b| - |b||v| \leq |b| = s(b)$, since $|v|$ and $|b|$ commute. Therefore

$$\begin{aligned}
0 &\neq \delta b(|b| - |b||v|) \qquad \text{by Proposition B.2.12(2)} \\
&= (|v_{(p)}| - \epsilon|v|)(|b| - |b||v|) \qquad \text{using (7.2.29)} \\
&= |v_{(p)}|(|b| - |b||v|) \qquad \text{since } |v||b||v| = |v|^2|b| = |v||b|
\end{aligned} \tag{7.2.37}$$

It follows that there exits an $\eta \in H_2$ such that $|v_{(p)}|(|b| - |b||v|)(\eta) \neq 0$. Let $\eta_1 = (|b| - |b||v|)(\eta)$. Note that since b is self-adjoint, $|b| = s(b) = r(b)$. It follows using (7.2.37) that $|b||v_{(p)}|(\eta_1) = |b|\delta b(\eta_1) = \delta b(\eta_1)$. Furthermore,

$$|b||v_{(p)}|(\eta_1) = |v_{(p)}||b|(\eta_1) = |v_{(p)}||b|(|b| - |b||v|)(\eta) = |v_{(p)}|(|b| - |b||v|)(\eta) \neq 0$$

and so $b(\eta_1) \neq 0$.

$$\begin{aligned}
\big\||b||v_{(p)}|(\eta_1)\big\| &= \big\||b||v_{(p)}||b||v_{(p)}|(\eta_1))\big\| && \text{since } |v_{(p)}| \text{ and } |b| \text{ are commuting projections} \\
&= \big\||b||v_{(p)}|(\delta b(\eta_1))\big\| \\
&= |\delta|\big\|b(|b||v_{(p)}|(\eta_1))\big\| && \text{since } b,\ |b| \text{ and } |v_{(p)}| \text{ all commute} \\
&= |\delta|\big\||b||v_{(p)}|(\eta_1)\big\| && \text{since } b \text{ is a partial isometry and } |b||v_{(p)}|(\eta_1) \in (\mathrm{Ker}(b))^{\perp}
\end{aligned}$$

Since $|b||v_{(p)}|(\eta_1) \neq 0$, this implies that $|\delta| = 1$ and hence $\delta = 1$, since $\delta > 0$. This implies that $s = 0$.

We can similarly show that if $\tau(q) > \frac{1}{2}\tau(p)$, then $\nu(|v|) > \nu(|b|)$ and $|v| - |v||b| \neq 0$. Therefore

$$\begin{aligned}
0 &\neq \epsilon(|v| - |v||b|) \\
&= \epsilon(|v| - |v||b|) + \delta b|v| - \delta b|b||v| && \text{since } |b| = s(b) \\
&= \epsilon|v|(|v| - |v||b|) + \delta b(|v| - |v||b|) && \text{since } |v||b| = |b||v| \\
&= (\epsilon|v| + \delta b)(|v| - |v||b|) \\
&= |v_{(p)}|(|v| - |v||b|) && \text{using (7.2.29)}
\end{aligned}$$

Since $|v_{(p)}|(|v| - |v||b|)$ and $(|v| - |v||b|)$ are non-zero projections, it follows by Proposition B.1.22(5) that $\epsilon = 1$ and hence $s = 0$.

Next, suppose that $\tau(q) = \frac{1}{2}\tau(p)$. This implies that $\tau(p - q) = \frac{1}{2}\tau(p) = \tau(q)$. Therefore, using (7.2.33) and the injectivity of ψ,

$$\nu(|v|) = \nu(|b|) \tag{7.2.38}$$

We consider the cases $|v| \neq |b|$ and $|v| = |b|$ separately. Suppose $|v| \neq |b|$. If we assume that $|b||v| = |b|$, then in this case $|b| < |v|$ and so $\nu(|b|) < \nu(|v|)$. This contradicts (7.2.38) and so $|b| - |b||v| \neq 0$. As in the $\tau(q) < \frac{1}{2}\tau(p)$ case, the former would also show that $s = 0$. Suppose $|v| = |b|$. Let $0 \leq \rho < 1$ and put $x_\rho = q + i\rho(p - q)$. We will demonstrate that calculating the norms of x_ρ and $W(x_\rho)$ for different values of ρ will lead to a contradiction and hence show that $s = 0$, also in the case $|v| = |b|$. We start by calculating the singular value functions of x_ρ and $W(x_\rho)$. Note that, since $q(p - q) = 0$, we have, by Proposition B.1.23 that $|x_\rho| = q + \rho(p - q)$ and hence, by Proposition 1.4.2(2) and Example 1.4.1(2),

$$\mu_{x_\rho} = \mu_{|x_\rho|} = \chi_{[0,\tau(p)/2)} + \rho\chi_{[\tau(p)/2,\tau(p))}$$

Therefore,

$$\begin{aligned}
\|x_\rho\|_{L_{w,1}(\mathcal{A})} &= \int_0^\infty \Big(\chi_{[0,\tau(p)/2)} + \rho\chi_{[\tau(p)/2,\tau(p))}\Big)(t)d\psi(t) \\
&= \psi(\tau(p)/2) + \rho\Big(\psi(\tau(p)) - \psi(\tau(p)/2)\Big)
\end{aligned} \tag{7.2.39}$$

In order to calculate the singular value function of $W(x_\rho)$, it will be useful to show that $b = 2|v_{(p)}| - |v|$ under the present assumptions (i.e. $\tau(q) = \frac{1}{2}\tau(p)$ and $|v| = |b|$). Using (7.2.29) and $|v| = |b| = s(b)$, we obtain

$$|v_{(p)}||v| = (\delta b + \epsilon|v|)|v| = \delta b|v| + \epsilon|v||v| = \delta b + \epsilon|v| = |v_{(p)}|$$

This implies that $|v_{(p)}| \leq |v|$. Furthermore, since $\tau(q) = \tau(q - p)$ and $|v| = |b|$, we have

$$\beta = \frac{\psi(\tau(q))}{\psi(\nu(|v|))} = \frac{\psi(\tau(p - q))}{\psi(\nu(|b|))} = \gamma \tag{7.2.40}$$

Using (7.2.32) and $|v_{(p)}| \leq |v|$, we obtain $0 \neq s = |v| - |v_{(p)}||v| = |v| - |v_{(p)}|$. We have shown that $\epsilon = \delta$ and so

$$\epsilon b \quad = \quad |v_{(p)}| - \epsilon|v| \qquad \text{using (7.2.29)} \tag{7.2.41}$$

It follows that $\epsilon b|v_{(p)}| = |v_{(p)}|^2 - \epsilon|v||v_{(p)}|$ and, since $|v_{(p)}| \leq |v|$, this implies that

$$\epsilon b|v_{(p)}| \quad = \quad (1-\epsilon)|v_{(p)}| \tag{7.2.42}$$

Note that $|v_{(p)}| \leq |v| = |b| = s(b)$ and so for any $0 \neq \eta \in |v_{(p)}|(H)$, we have $\|b(\eta)\| = \|\eta\|$, since b is a partial isometry and $\eta \in s(b)$. Therefore, using (7.2.42), we obtain

$$\epsilon\|\eta\| = \|\epsilon b(\eta)\| = \|(1-\epsilon)\eta\| = (1-\epsilon)\|\eta\|$$

It follows that $\epsilon = 1 - \epsilon$, i.e., $\epsilon = \frac{1}{2}$. Using (7.2.41), this implies that

$$b = 2|v_{(p)}| - |v| \tag{7.2.43}$$

This expression for b will enable us to calculate the singular value function of $W(x_\rho)$.

$$\begin{aligned}
W(x_\rho) &= W(q) + i\rho W(p-q) \\
&= \beta|v| + i\rho\gamma b \qquad \text{see (7.2.27) and the discussion thereafter} \\
&= \beta(|v| + i\rho b) \qquad \text{using (7.2.40)} \\
&= \beta\Big(|v| + i\rho(2|v_{(p)}| - |v|)\Big) \qquad \text{using (7.2.43)} \\
&= \beta(|v| - |v_{(p)}| - i\rho|v| + i\rho|v_{(p)}| + |v_{(p)}| + i\rho|v_{(p)}|) \\
&= \beta(1-i\rho)(|v| - |v_{(p)}|) + \beta(1+i\rho)|v_{(p)}|
\end{aligned}$$

Since $|v_{(p)}| \leq |v|$, we have that $|v| - |v_{(p)}|$ and $|v_{(p)}|$ are orthogonal projections. By Proposition B.1.23, this implies that

$$|W(x_\rho)| = \beta|1-i\rho|(|v| - |v_{(p)}|) + \beta|1+i\rho||v_{(p)}| = \beta(1+\rho^2)^{1/2}|v|$$

Using Example 1.4.1 and the fact that $\mu_{W(x_\rho)} = \mu_{|W(x_\rho)|}$, we obtain $\mu_{W(x_\rho)} = \beta(1+\rho^2)^{1/2}\chi_{[0,\nu(|v|))}$. We therefore have that

$$\begin{aligned}
\|W(x_\rho)\|_{L_{w,1}(\mathcal{B})} &= \int_0^\infty (\beta(1+\rho^2)^{1/2}\chi_{[0,\nu(|v|))})(t)d\psi(t) \\
&= \frac{\psi(\tau(q))}{\psi(\nu(|v|))}(1+\rho^2)^{1/2}\psi(\nu(|v|)) \\
&= (1+\rho^2)^{1/2}\psi(\tau(p)/2) \qquad \text{since } \tau(q) = \tau(p)/2
\end{aligned}$$

Using (7.2.39) and the fact that $\|W(x_\rho)\|_{L_{w,1}(\mathcal{B})} = \|x_\rho\|_{L_{w,1}(\mathcal{A})}$, this implies that

$$(1+\rho^2)^{1/2}\psi(\tau(p)/2) = \psi(\tau(p)/2) + \rho\Big(\psi(\tau(p)) - \psi(\tau(p)/2)\Big) \tag{7.2.44}$$

This holds for all $0 \leq \rho < 1$. In particular, for $\rho = 0.1$ and $\rho = 0.2$, we obtain

$$(1.01)^{1/2}\psi(\tau(p)/2) \quad = \quad \psi(\tau(p)/2) + 0.1\Big(\psi(\tau(p)) - \psi(\tau(p)/2)\Big) \qquad \text{and} \tag{7.2.45}$$

$$(1.04)^{1/2}\psi(\tau(p)/2) \quad = \quad \psi(\tau(p)/2) + 0.2\Big(\psi(\tau(p)) - \psi(\tau(p)/2)\Big) \tag{7.2.46}$$

Taking $2 \times$ (7.2.45) $-$ (7.2.46), we obtain $\psi(\tau(p)/2) = \Big(2(1.01)^{1/2} - (1.04)^{1/2}\Big)\psi(\tau(p)/2)$ and hence $\tau(p)/2 = 0$, using the injectivity of ψ. Since τ is faithful, this implies that $p = 0$, which is a contradiction and hence $s = 0$.

We have shown that $s = 0$ in all possible cases and hence $|v| \leq |v_{(p)}|$. ☐

LEMMA 7.2.5. $T_p(pL_{w,1}(\tau)p) \subseteq |v_{(p)}|L_{w,1}(\nu)|v_{(p)}|$.

PROOF. The implication of Lemma 7.2.4 is that $|v_{(p)}||v||v_{(p)}| = |v|$ and hence $|v| \in |v_{(p)}|L_{w,1}(\nu)|v_{(p)}|$, by Proposition B.3.7(3). Therefore $v \in |v_{(p)}|L_{w,1}(\nu)|v_{(p)}|$, by Proposition B.3.7(1), and so $T_p(q) = \beta v \in |v_{(p)}|L_{w,1}(\nu)|v_{(p)}|$. Let $\mathcal{D}_{(p)} := \{e \in \mathcal{P}(\mathcal{A})^f : e \leq p\}$ and $\mathcal{G}_p^f := \mathrm{span}\,(\mathcal{D}_{(p)})$. Since $0 < q \leq p$ was arbitrary, T_p is linear and $|v_{(p)}|L_{w,1}(\nu)|v_{(p)}|$ is a subspace of $L_{w,1}(\nu)$, we have that $T_p(\mathcal{G}_p^f) \subseteq |v_{(p)}|L_{w,1}(\nu)|v_{(p)}|$. T_p is continuous and, by Proposition B.3.9, $|v_{(p)}|L_{w,1}(\nu)|v_{(p)}|$ is closed; therefore, $T_p(\overline{\mathcal{G}_p^f}) \subseteq |v_{(p)}|L_{w,1}(\nu)|v_{(p)}|$. This completes the proof of the lemma, since $\overline{\mathcal{G}_p^f} = pL_{w,1}(\tau)p$, by Proposition B.3.8. $\square$

LEMMA 7.2.6. *If $x \in pL_{w,1}(\tau)^+p$, then*

$$T_p(x) \geq 0.$$

PROOF. Recall that $T_p(q) = \beta v$. Since $v \in |v_{(p)}|L_{w,1}(\nu)|v_{(p)}|$, this implies by Proposition B.3.7(4) that $\beta v = \beta|v_{(p)}|v$. Application of Lemma 7.2.3 and (7.2.23) yields

$$T_p(q) = \beta|v_{(p)}|v = \beta w|v_{(p)}|v = \beta|v_{(p)}|wv = \beta|v_{(p)}||v|.$$

Since $|v| \leq |v_{(p)}|$, we therefore have that $|v_{(p)}||v| = |v|$ and hence $T_p(q) = \beta|v| \geq 0$. Since $0 < q \leq p$ was arbitrary, it is easily checked that $T_p(\mathcal{G}_p^+) \subseteq L_{w,1}(\nu)^+$. Furthermore, $\overline{\mathcal{G}_p^+} = pL_{w,1}(\tau)^+p$ (see Proposition B.3.8), T_p is continuous and $L_{w,1}(\nu)^+$ is closed (see Proposition B.3.1(10)). It follows that

$$T_p(pL_{w,1}(\tau)^+p) = T_p(\overline{\mathcal{G}_p^+}) \subseteq L_{w,1}(\nu)^+.$$

$\square$

The major part of the proof up to this point has been a semi-finite adaptation of the techniques employed in the proof of [4, Theorem 5.1] (see Theorem 2.2.10). The following lemma shows that this groundwork in fact enables us to show that any surjective isometry between Lorentz spaces (of this type) is disjointness-preserving.

LEMMA 7.2.7. *If $0 \neq r, s \in \mathcal{P}(\mathcal{A})^f$ are such that $rs = 0$, then $v_{(r)}v_{(s)}^* = 0 = v_{(r)}^*v_{(s)}$, where $v_{(r)}, v_{(s)} \in \mathcal{V}(\mathcal{B})^f$ are such that $U(r) = \alpha_r v_{(r)}$ and $U(s) = \alpha_s v_{(s)}$. Furthermore, $\alpha_r = \alpha_s = \alpha_{r+s}$, $v_{(r)} + v_{(s)} = v_{(r+s)}$ and $|v_{(r)}| + |v_{(s)}| = |v_{(r+s)}|$, where $U(r+s) = \alpha_{r+s}v_{(r+s)}$.*

PROOF. By Proposition B.1.21(3), $r+s = r \vee s \in \mathcal{P}(\mathcal{A})^f$ and so there exits a partial isometry $v_{(r+s)} \in \mathcal{V}(\mathcal{B})^f$ such that $U(r+s) = \alpha_{r+s}v_{(r+s)}$. By Proposition B.2.13, there exists a unitary operator $u_{(r+s)} \in \mathcal{B}$ such that $v_{(r+s)} = u_{(r+s)}|v_{(r+s)}|$. Let u denote $u_{(r+s)}$ and note that

$$T_{r+s}(r) = u^*U(r) = \alpha_r u^* v_{(r)}.$$

Furthermore,

$$(u^*v_{(r)})^*(u^*v_{(r)}) = (v_{(r)}^*u)(u^*v_{(p)}) = v_{(r)}^*v_{(r)} = |v_{(r)}|.$$

It follows by Proposition B.1.28 that $\frac{1}{\alpha_r}T_{r+s}(r)$ is a partial isometry. Furthermore, $\frac{1}{\alpha_r}T_{r+s}(r) \geq 0$, by Lemma 7.2.6 (using $p = r+s$). It follows, by Remark B.1.29, that $\frac{1}{\alpha_r}T_{r+s}(r)$ is a projection. We can similarly show that $\frac{1}{\alpha_s}T_{r+s}(s)$ and $\frac{1}{\alpha_{r+s}}T_{r+s}(r+s)$ are also projections. Therefore, $u^*v_{(r)}$, $u^*v_{(s)}$ and $u^*v_{(r+s)}$ are all projections. Furthermore,

$$\alpha_r u^*v_{(r)} + \alpha_s u^*v_{(s)} = T_{r+s}(r) + T_{r+s}(s) = T_{r+s}(r+s) = \alpha_{r+s}u^*v_{(r+s)}$$

By Proposition B.1.24 this implies that

$$u^*v_{(r)} = u^*v_{(s)} = u^*v_{(r+s)} \qquad \text{and} \qquad \alpha_r + \alpha_s = \alpha_{r+s}$$

or

$$(u^*v_{(r)})(u^*v_{(s)}) = 0, \qquad u^*v_{(r)} + u^*v_{(s)} = u^*v_{(r+s)} \qquad \text{and} \qquad \alpha_r = \alpha_s = \alpha_{r+s} \tag{7.2.47}$$

If we assume the former scenario, then

$$T_{r+s}\left(\frac{r}{\alpha_r}\right) = u^*v_{(r)} = u^*v_{(s)} = T_{r+s}\left(\frac{s}{\alpha_s}\right)$$

Since T_{r+s} is an isometry and hence injective, it follows that $\frac{r}{\alpha_r} = \frac{s}{\alpha_s}$ and hence $r = s$. This is a contradiction, since $rs = 0$ and $r, s \neq 0$. Therefore (7.2.47) holds. Note that $v^*_{(s)}u = (u^*v_{(s)})^* = u^*v_{(s)}$, since $u^*v_{(s)}$ is a projection. Similarly, $v^*_{(r)}u = u^*v_{(r)}$. Therefore $(u^*v_{(r)})(v^*_{(s)}u) = (u^*v_{(r)})(u^*v_{(s)}) = 0$, using (7.2.47). It follows that

$$v_{(r)}v^*_{(s)} = u.0.u^* = 0$$

Furthermore, $(u^*v_{(r)})(u^*v_{(s)}) = 0$ implies that

$$v^*_{(r)}v_{(s)} = (v^*_{(r)}u)(u^*v_{(s)}) = 0,$$

since $uu^* = \mathbf{1}$ and $v^*_{(r)}u = u^*v_{(r)}$. Note also that $u(u^*v_{(r)} + u^*v_{(s)}) = u(u^*v_{(r+s)})$, using (7.2.47). Since u is unitary, it follows that

$$v_{(r)} + v_{(s)} = v_{(r+s)}.$$

Furthermore, this implies that

$$\begin{aligned} |v_{(r+s)}| = v^*_{(r+s)}v_{(r+s)} &= (v_{(r)} + v_{(s)})^*(v_{(r)} + v_{(s)}) \\ &= v^*_{(r)}v_{(r)} + v^*_{(s)}v_{(s)} \qquad \text{since } v^*_{(r)}v_{(s)} = 0 = v^*_{(s)}v_{(r)} \\ &= |v_{(r)}| + |v_{(s)}| \end{aligned}$$

□

COROLLARY 7.2.8. *Suppose $(\mathcal{A}, \tau)$ and $(\mathcal{B}, \nu)$ are semi-finite von Neumann algebras and w is a strictly decreasing weight function. If $U : L_{w,1}(\tau) \to L_{w,1}(\nu)$ is a surjective isometry, then*

$$U(p)^*U(q) = 0 = U(p)U(q)^*$$

whenever $p, q \in \mathcal{P}(\mathcal{A})^f$ are such that $pq = 0$.

PROOF. Recall that $U(p) = \alpha_p v_{(p)}$ and $U(q) = \alpha_q v_{(q)}$. Using Lemma 7.2.7, we have that

$$U(p)^*U(q) = \alpha_p v^*_{(p)} \alpha_q v_{(q)} = 0$$

We can similarly show that $U(p)U(q)^* = 0$. □

Before proving the main result of this chapter, we show that if $0 \neq p, q \in \mathcal{P}(\mathcal{A})^f$ are arbitrary projections, then $\alpha_p = \alpha_q$. If $p = q$, then clearly $\alpha_p = \alpha_q$. If $q < p$, then $0 \neq p - q$ and $q(p-q) = 0$. It follows by Lemma 7.2.7 that $\alpha_q = \alpha_{p-q} = \alpha_{q+(p-q)} = \alpha_p$. If $p \nleq q$ and $q \nleq p$, then let $m = p \vee q$. It follows that $p < m$, $q < m$ and $m \in \mathcal{P}(\mathcal{A})^f$. By what has been shown already this implies that $\alpha_p = \alpha_m = \alpha_q$. There therefore exists an $\alpha \in \mathbb{R}^+$ such that $U(p) = \alpha v_{(p)}$ holds for any $0 \neq p \in \mathcal{P}(\mathcal{A})^f$.

THEOREM 7.2.9. *Suppose $(\mathcal{A}, \tau)$ and $(\mathcal{B}, \nu)$ are semi-finite von Neumann algebras and w is a strictly decreasing weight function. A linear map $U : L_{w,1}(\tau) \to L_{w,1}(\nu)$ is a surjective isometry if and only if there exists a Jordan $*$-isomorphism $\Phi : \mathcal{A} \to \mathcal{B}$, a unitary $u \in \mathcal{B}$ and a $\delta > 0$ such that*

$$U(x) = \frac{1}{\delta}u\Phi(x) \qquad \forall x \in \mathcal{A} \cap L_{w,1}(\tau)$$

and

$$\psi(\nu(\Phi(p))) = \delta\psi(\tau(p)) \qquad \forall p \in \mathcal{P}(\mathcal{A})^f.$$

PROOF. We have seen that if $p \in \mathcal{P}(\mathcal{A})^f$, then $U(p) = \alpha v_{(p)}$ for some $v_{(p)} \in \mathcal{V}(\mathcal{A})_f$. It follows that

$$\nu(s(U(p)) = \nu(|v_{(p)}|) < \infty$$

whenever $p \in \mathcal{P}(\mathcal{A})^f$. If $p, q \in \mathcal{P}(\mathcal{A})^f$ with $p, q = 0$, then

$$U(p)^* U(q) = 0 = U(p) U(q)^*,$$

by Corollary 7.2.8. It follows by Theorem 6.1.1, that there exist a Jordan $*$-isomorphism Φ, a unitary operator $u \in \mathcal{B}$ and a positive operator b affiliated with the centre of $\mathcal{B}$ such that

$$U(x) = ub\Phi(x) \qquad \forall x \in \mathcal{A} \cap L_{w,1}(\tau).$$

Furthermore, $\Phi(p) = s(U(p))$ for every $p \in \mathcal{P}(\mathcal{A})^f$. Recall that in the proof of Theorem 6.1.1, the positive operator b was obtained using the spectral decompositions of the positive operators $b_{(p)}$ ($p \in \mathcal{P}(\mathcal{A})^f$), where

$$U(p) = w_{(p)} b_{(p)}$$

is the polar decomposition of $U(p)$. In our present setting we have that

$$b_{(p)} = |U(p)| = |\alpha v_{(p)}| = \alpha |v_{(p)}| = \alpha s(U(p)) = \alpha \Phi(p),$$

where we have used the fact that $\alpha_p = \alpha$ for all $p \in \mathcal{P}(\mathcal{A})^f$. This implies that if $b_{(p)} = \int_0^\infty \lambda de_{(p)}(\lambda)$ is the spectral decomposition of $b_{(p)}$, then

$$e_{(p)}(\lambda, \infty) = \begin{cases} \Phi(p) & \text{if } 0 \leq \lambda < \alpha \\ 0 & \text{if } \lambda \geq \alpha \end{cases}$$

Furthermore, Φ is a Jordan $*$-isomorphism and hence normal and unital, by Proposition 1.8.8. It follows that $\mathrm{SOT}\lim_{p \in \mathcal{D}} \Phi(p) = \Phi(\mathbf{1}) = \mathbf{1}$ and so

$$e(\lambda, \infty) := \mathrm{SOT}\lim_{p \in \mathcal{D}} e_{(p)}(\lambda, \infty) = \begin{cases} \mathbf{1} & \text{if } 0 \leq \lambda < \alpha \\ 0 & \text{if } \lambda \geq \alpha \end{cases}$$

It follows that $b = \int_0^\infty \lambda de(\lambda) = \alpha \mathbf{1}$ and hence

$$U(x) = \alpha u \Phi(x) \qquad \forall x \in \mathcal{A} \cap E.$$

Furthermore, for any $p \in \mathcal{P}(\mathcal{A})^f$, we have

$$\psi(\tau(p)) = \|p\|_{L_{w,1}(\mathcal{A})} = \|U(p)\|_{L_{w,1}(\mathcal{B})} = \|\alpha u \Phi(p)\|_{L_{w,1}(\mathcal{B})} = \alpha \|\Phi(p)\|_{L_{w,1}(\mathcal{B})} = \alpha \psi(\nu(\Phi(p)))$$

Letting $\delta = \frac{1}{\alpha}$ completes the proof of the necessary condition.

To prove the sufficiency part of Theorem 7.2.9 suppose that $\Phi : \mathcal{A} \to \mathcal{B}$ is a Jordan $*$-isomorphism with the property that

$$\psi(\nu(\Phi(p))) = \delta \psi(\tau(p)) \qquad \forall p \in \mathcal{P}(\mathcal{A})^f, \tag{7.2.48}$$

for some $\delta > 0$ and suppose $u \in \mathcal{B}$ is a unitary operator. Let

$$T_0(x) := \frac{1}{\delta} u \Phi(x) \qquad \forall x \in \mathcal{A} \cap L_{w,1}(\tau).$$

We show that T_0 can be uniquely extended to a surjective isometry T from $L_{w,1}(\tau)$ onto $L_{w,1}(\nu)$. We start by showing that T_0 is an isometry with respect to the Lorentz norms. Let $\mathcal{G}_f$ denote the set of all finite linear combinations of mutually orthogonal projections in $\mathcal{P}(\mathcal{A})^f$. Suppose $x = \sum_{i=1}^n \alpha_i p_i \in \mathcal{G}_f$, with $|\alpha_1| > |\alpha_2| >$

... $> |\alpha_n|$ and $p_i \in \mathcal{P}(\mathcal{A})^f$ for every i, with $p_i p_j = 0$ if $i \neq j$. Then $|x| = \sum_{i=1}^{n} |\alpha_i| p_i$, by Proposition B.1.23. Furthermore, since $p_i p_j = 0$ if $i \neq j$, we have that

$$x^* x = (\sum_{i=1}^{n} \bar{\alpha}_i p_i)(\sum_{i=1}^{n} \alpha_i p_i) = \sum_{i=1}^{n} |\alpha_i|^2 p_i = (\sum_{i=1}^{n} \alpha_i p_i)(\sum_{i=1}^{n} \bar{\alpha}_i p_i) = x x^*.$$

$|x|$ can also be written in the form $|x| = \sum_{j=1}^{n} \beta_j q_j$, where $q_1 \leq q_2 \leq \ldots \leq q_n$ and the q_i's are projections. By Proposition 1.4.2(2) and Example 1.4.1,

$$\mu_x = \mu_{|x|} = \sum_{j=1}^{n} \beta_j \mu_{q_j} \tag{7.2.49}$$

x is normal and so, by Proposition 1.8.6(1), $|\Phi(x)| = \Phi(|x|) = \sum_{j=1}^{n} \beta_j \Phi(q_j)$. Since Φ is a Jordan $*$-isomorphism, Φ is positive and so $\Phi(q_1) \leq \Phi(q_2) \leq \ldots \leq \Phi(q_n)$. Furthermore, by Proposition 1.8.2(5), these are all projections and so by Example 1.4.1,

$$\mu_{\Phi(x)} = \mu_{|\Phi(x)|} = \sum_{j=1}^{n} \beta_j \chi_{[0, \nu(\Phi(q_j)))} \tag{7.2.50}$$

The following calculation shows that T_0 isometric on $\mathcal{G}_f$.

$$\begin{aligned}
\|T_0(x)\|_{L_{w,1}(\mathcal{B})} &= \|\frac{1}{\delta} u \Phi(x)\|_{L_{w,1}(\mathcal{B})} \\
&= \frac{1}{\delta} \|\Phi(x)\|_{L_{w,1}(\mathcal{B})} \qquad \text{since } \frac{1}{\delta} > 0 \text{ and } u \text{ is unitary (see Proposition B.3.1(6))} \\
&= \frac{1}{\delta} \int_0^{\infty} \mu_{\Phi(x)}(t) w(t) dt \\
&= \frac{1}{\delta} \sum_{j=1}^{n} \beta_j \psi(\nu(\Phi(q_j))) \qquad \text{using (7.2.50)} \\
&= \frac{1}{\delta} \sum_{j=1}^{n} \beta_j \delta \psi(\tau(q_j)) \qquad \text{using (7.2.48)} \\
&= \int_0^{\infty} \mu_x(t) w(t) dt \qquad \text{using (7.2.49)} \\
&= \|x\|_{L_{w,1}(\mathcal{A})}
\end{aligned}$$

$\mathcal{G}_f$ is dense in $L_{w,1}(\tau)$, by Corollary 1.6.8(1), and so T_0 has a unique isometric extension T to all of $L_{w,1}(\tau)$. We show that T is surjective. To do so, we will start by showing that $\Phi(\mathcal{P}(\mathcal{A})^f) = \mathcal{P}(\mathcal{B})^f$. If $p \in \mathcal{P}(\mathcal{A})^f$, then $\Phi(p) \in \mathcal{P}(\mathcal{B})$, by Proposition 1.8.2(5). Furthermore, using (7.2.48), we have

$$\psi(\nu(\Phi(p))) = \delta \psi(\tau(p)) < \infty,$$

since $\tau(p) < \infty$. It follows that $\Phi(p) \in \mathcal{P}(\mathcal{B})^f$ and $\Phi(\mathcal{P}(\mathcal{A})^f) \subseteq \mathcal{P}(\mathcal{B})^f$. Suppose $q \in \mathcal{P}(\mathcal{B})^f$. Φ^{-1} is a Jordan $*$-isomorphism, by Proposition 1.8.8(5), and therefore $p = \Phi^{-1}(q)$ is a projection, by Proposition 1.8.2(5). There exists $\{p_\lambda\}_{\lambda \in \Lambda} \subseteq \mathcal{P}(\mathcal{A})^f$ such that $p_\lambda \uparrow p$ and therefore

$$\begin{aligned}
\Phi(p_\lambda) &\uparrow \Phi(p) \qquad \text{since } \Phi \text{ is normal by Proposition 1.8.8(4)} \\
\Longrightarrow \nu(\Phi(p_\lambda)) &\uparrow \nu(\Phi(p)) \qquad \text{since } \nu \text{ is normal} \\
\Longrightarrow \psi(\nu(\Phi(p_\lambda))) &\uparrow \psi(\nu(\Phi(p))) \qquad \text{since } \psi \text{ is increasing and continuous}
\end{aligned}$$

We can similarly show that $\delta\psi(\tau(p_\lambda)) \uparrow \delta\psi(\tau(p))$, but $\psi(\nu(\Phi(p_\lambda))) = \delta\psi(\tau(p_\lambda))$ for all λ and therefore

$$\delta\psi(\tau(p)) = \psi(\nu(\Phi(p))) = \psi(\nu(q)) < \infty$$

It follows that $p \in \mathcal{P}(\mathcal{A})^f$ and therefore $\mathcal{P}(\mathcal{B})^f \subseteq \Phi(\mathcal{P}(\mathcal{A})^f)$. It follows that $\mathcal{P}(\mathcal{B})^f = \Phi(\mathcal{P}(\mathcal{A})^f)$.

Let $\mathcal{K}_f$ denote the set of all finite linear combinations of projections in $\mathcal{P}(\mathcal{B})^f$. Since $\Phi(\mathcal{P}(\mathcal{A})^f) = \mathcal{P}(\mathcal{B})^f$, it follows that $\Phi(\mathcal{G}_f) = \mathcal{K}_f$. Suppose $y \in L_{w,1}(\nu)$. Then $\delta u^* y \in L_{w,1}(\nu)$. $\mathcal{K}_f$ is dense in $L_{w,1}(\nu)$, so there exists a sequence $(y_n)_{n=1}^{\infty} \subseteq \mathcal{K}_f$ such that $y_n \overset{L_{w,1}(\nu)}{\to} \delta u^* y$. $\Phi(\mathcal{G}_f) = \mathcal{K}_f$ and so there exits a sequence $(x_n)_{n=1}^{\infty} \subseteq \mathcal{G}_f$ such that $\Phi(x_n) = y_n$ for every $n \in \mathbb{N}^+$. T_0 is isometric on $\mathcal{G}_f$ and so for every $n, m \in \mathbb{N}^+$,

$$\begin{aligned}
\|x_n - x_m\|_{L_{w,1}(\mathcal{A})} &= \|T_0(x_n - x_m)\|_{L_{w,1}(\mathcal{B})} \\
&= \|\frac{1}{\delta} u\Phi(x_n - x_m)\|_{L_{w,1}(\mathcal{B})} \\
&= \frac{1}{\delta}\|\Phi(x_n) - \Phi(x_m)\|_{L_{w,1}(\mathcal{B})} \qquad \text{since } \frac{1}{\delta} > 0 \text{ and } u \text{ is unitary (see Proposition B.3.1(6))} \\
&= \frac{1}{\delta}\|y_n - y_m\|_{L_{w,1}(\mathcal{B})}
\end{aligned}$$

Since $(y_n)_{n=1}^{\infty}$ is convergent and hence Cauchy, it follows that $(x_n)_{n=1}^{\infty}$ is Cauchy in $L_{w,1}(\tau)$. Therefore $x_n \overset{L_{w,1}(\tau)}{\to} x$, for some $x \in L_{w,1}(\tau)$. Since T is continuous, this implies that $T(x_n) \overset{L_{w,1}(\nu)}{\to} T(x)$. Furthermore, $\Phi(x_n) = y_n \overset{L_{w,1}(\nu)}{\to} \delta u^* y$ and so

$$T(x_n) = \frac{1}{\delta} u\Phi(x_n) = \frac{1}{\delta} u y_n \overset{L_{w,1}(\nu)}{\to} \frac{1}{\delta} u(\delta u^* y) = y.$$

It follows that $T(x) = y$. □

Remark 7.2.10. Suppose $(\mathcal{A}, \tau)$ and $(\mathcal{B}, \nu)$ are finite von Neumann algebras with $\tau(\mathbf{1}) = 1 = \nu(\mathbf{1})$ and suppose $w : [0,1] \to [0,\infty)$ is a strictly decreasing weight function with $\psi(1) = \int_0^1 w(t)dt = 1$. If $\Phi : \mathcal{A} \to \mathcal{B}$ is a Jordan $*$-isomorphism, then the condition

$$\psi(\nu(\Phi(p))) = \delta\psi(\tau(p)) \qquad \forall p \in \mathcal{P}(\mathcal{A})^f = \mathcal{P}(\mathcal{A}) \tag{7.2.51}$$

for some $\delta > 0$, is equivalent to Φ being trace-preserving. It is clear that if Φ is trace-preserving, then (7.2.51) holds for $\delta = 1$. Suppose (7.2.51) holds for some $\delta > 0$. In particular,

$$\begin{aligned}
\psi(\nu(\Phi(\mathbf{1}))) &= \delta\psi(\tau(\mathbf{1})) \\
\implies \delta &= \frac{\psi(\nu(\mathbf{1}))}{\psi(\tau(\mathbf{1}))} \qquad \text{since } \Phi(\mathbf{1}) = \mathbf{1} \text{ by Proposition 1.8.8(3)} \\
&= 1 \qquad \text{since } \tau(\mathbf{1}) = 1 = \nu(\mathbf{1}) \text{ and } \psi(1) = 1
\end{aligned}$$

Using (7.2.51), this implies that $\psi(\nu(\Phi(p))) = \psi(\tau(p))$ for all $p \in \mathcal{P}(\mathcal{A})$. Since ψ is strictly increasing and hence injective, it follows that $\nu(\Phi(p)) = \tau(p)$ for all $p \in \mathcal{P}(\mathcal{A})$. Since ν, Φ and τ are linear, it follows that $\nu(\Phi(x)) = \tau(x)$, whenever $x \in \mathcal{G} = \mathcal{G}_f$. Suppose $x \in \mathcal{A}^+$. By Remark B.1.12, there exists a sequence $(x_n)_{n=1}^{\infty} \subseteq \mathcal{G}^+$ such that $x_n \overset{\mathcal{A}}{\to} x$. Φ is an isometry, by Proposition 1.8.8(2), and so $\Phi(x_n) \overset{\mathcal{B}}{\to} \Phi(x)$. Furthermore, $\mathcal{A}$ is trace-finite and hence $\mathcal{A}$ is continuously embedded into $L_1(\tau)$. Similarly, $\mathcal{B}$ is continuously embedded into $L_1(\nu)$. It follows that $x_n \overset{L_1(\tau)}{\to} x$, $\Phi(x_n) \overset{L_1(\nu)}{\to} \Phi(x)$ and hence, since these are all positive elements,

$$\tau(x_n) = \|x_n\|_{L_1(\tau)} \to \|x\|_{L_1(\tau)} = \tau(x) \qquad \text{and} \qquad \nu(\Phi(x_n)) = \|\Phi(x_n)\|_{L_1(\nu)} \to \|\Phi(x)\|_{L_1(\nu)} = \nu(\Phi(x)).$$

However, $\nu(\Phi(x_n)) = \tau(x_n)$ for each $n \in \mathbb{N}^+$ and so

$$\nu(\Phi(x)) = \tau(x).$$

Since any element in $\mathcal{A}$ can be written as a linear combination of positive elements and Φ, ν and τ are linear, we have that Φ is trace-preserving.

We finish this chapter with a corollary to Theorem 5.3.4, which shows that if we are dealing with Lorentz spaces associated with different weight functions, then we can still describe the structure of a surjective isometry between these spaces, provided we know, in addition, that the isometry is positive.

Corollary 7.2.11. *Let $\mathcal{A} \subseteq \mathcal{B}(H)$ and $\mathcal{B} \subseteq \mathcal{B}(K)$ be semi-finite von Neumann algebras equipped with semi-finite faithful normal traces τ and ν respectively. Let w_1 and w_2 be strictly decreasing weights on $[0, \infty)$. If $U : L_{w_1,1}(\tau) \to L_{w_2,1}(\nu)$ is a positive surjective linear isometry, then there exists a positive operator $a \in S(B, \nu)$ and a Jordan $*$-isomorphism Φ of $\mathcal{A}$ onto $\mathcal{B}$ such that*

$$U(x) = a\Phi(x) \qquad \forall x \in \mathcal{A} \cap L_{w_1,1}(\tau)$$

Proof. Note that $E = L_{w_1,1}(\tau)$ and $F = L_{w_2,1}(\nu)$ are fully symmetric spaces with absolutely continuous norm, by Proposition 1.6.6. Furthermore, if $p \in \mathcal{P}(\mathcal{A})^f$, then $\frac{1}{\psi_1(\tau(p))}p$ is an extreme point of B_E, by Proposition 7.1.3. U is an isometry and so $U\left(\frac{1}{\psi_1(\tau(p))}p\right)$ is an extreme point of B_F. It follows by Proposition 7.1.3 that $U\left(\frac{1}{\psi_1(\tau(p))}p\right) = \frac{1}{\psi_2(\nu(|v|)}v$ for some partial isometry $v \in \mathcal{B}$ with $\nu(|v|) < \infty$. Therefore $s(U(p)) = |v|$ and hence $\nu(s(U(p))) < \infty$. The result therefore follows by Theorem 5.3.4. $\square$

CHAPTER 8

Positive surjective isometries on Orlicz spaces

In Chapter 5 we considered positive surjective isometries between a symmetric space and a fully symmetric space. We assumed that these spaces have absolutely continuous norm, which in the context of Orlicz spaces implies a restriction to Orlicz functions which satisfy the Δ_2-condition. In this chapter we consider positive surjective modular isometries on certain Orlicz spaces whose Orlicz functions do not satisfy the Δ_2-condition. Suppose $(\mathcal{A}, \tau)$ and $(\mathcal{B}, \nu)$ are semi-finite von Neumann algebras and ϕ is an Orlicz function. For $x \in S(\mathcal{A}, \tau)$, we let $I_\phi(x) := I_\phi(\mu_x)$. A map U from $L_\phi(\tau)$ into $L_\phi(\nu)$ is called a modular isometry if

$$I_\phi(U(x)) = \int_0^\infty \phi(\mu_{U(x)}(t))dt = \int_0^\infty \phi(\mu_x(t))dt = I_\phi(x) \qquad \forall x \in L_\phi(\tau).$$

Any modular isometry is an isometry (since it is easily checked that $\|x\|_\phi = \inf\{\lambda > 0 : I_\phi(x/\lambda) \leq 1\}$ for every $x \in L_\phi(\tau)$) and if U is a modular isometry from $L_\phi(\tau)$ onto $L_\phi(\nu)$, then U^{-1} is a modular isometry, since if $y \in L_\phi(\nu)$ and $x = U^{-1}(y)$, then

$$I_\phi(U^{-1}(y)) = I_\phi(x) = I_\phi(U(x)) = I_\phi(y).$$

In the commutative setting it is, in fact, also true that any surjective isometry of a complex Orlicz space is also a modular isometry (see [28, Theorem 4]). The first scenario we will consider is an Orlicz function ϕ, which has the property that $a_\phi := \sup\{t \geq 0 : \phi(t) = 0\} > 0$ and $b_\phi := \sup\{t \geq 0 : \phi(t) < \infty\} = \infty$. The second scenario will arise in the case where $\tau(\mathbf{1}) < \infty$ and the Orlicz function satisfies $b_\phi < \infty$. In both these cases $\mathcal{A} \subseteq L_\phi(\tau)$ and we will show in both these cases that if U is a positive surjective modular isometry from $L_\phi(\tau)$ onto $L_\phi(\nu)$, then the restriction of U to $\mathcal{A}$ is a Jordan $*$-isomorphism from $\mathcal{A}$ onto $\mathcal{B}$. For the final scenario we use the fact that there is an Orlicz function ϕ such that $L_\phi(\tau) = L_1 \cap L_\infty(\tau)$ with equality of norms. Then $L_\phi(\tau) \subseteq \mathcal{A}$, and we will show that if U is a positive surjective isometry from $L_1 \cap L_\infty(\tau)$ onto $L_1 \cap L_\infty(\nu)$, then U is uniquely extensible to a Jordan $*$-isomorphism from $\mathcal{A}$ onto $\mathcal{B}$. In the latter two scenarios, characterizations of extreme points will play a fundamental role. Throughout this chapter we will call a function discontinuous if it has at least one point of discontinuity.

8.1. Extreme points of Orlicz spaces

Before describing the extreme points of several Orlicz spaces, we substantiate some of the claims made in the introduction and provide a useful corollary to these results.

PROPOSITION 8.1.1. *Suppose $(\mathcal{A}, \tau)$ is a semi-finite von Neumann algebra with $\tau(\mathbf{1}) = \infty$ and ϕ is a discontinuous Orlicz function with $0 < a_\phi < \infty$. The following hold:*

(1) $\mathcal{A} \subseteq L_\phi(\tau)$
(2) *If $p \in \mathcal{P}(\mathcal{A})$ with $\tau(p) = \infty$, then* $\|p\|_{L_\phi(\tau)} = \frac{1}{a_\phi}$
(3) *If $(x_n)_{n=1}^\infty \cup \{x\} \subseteq \mathcal{A}$ is such that $x_n \xrightarrow{\mathcal{A}} x$, then $x_n \xrightarrow{L_\phi(\tau)} x$*
(4) *$L_\phi(\tau)$ does not have absolutely continuous norm.*

PROOF. 1): We start by noting that $\phi(a_\phi) = 0$, since ϕ is continuous from the left, and therefore

$$I_\phi(a_\phi \mu_{\mathbf{1}}) = \int_0^\infty \phi(a_0 \chi_{(0,\infty)}(t))dt = 0.$$

It follows that $\mathbf{1} \in L_\phi(\tau)$. Suppose $x \in \mathcal{A}^+$. Then $0 \le x \le \|x\|_{\mathcal{A}} \mathbf{1}$, by Proposition B.1.6 and hence $x \in L_\phi(\tau)$, by Proposition B.3.1(5) (since $\|x\|_{\mathcal{A}} \mathbf{1} \in L_\phi(\tau)$). Since any element in $\mathcal{A}$ can be written as a linear combination of positive elements and $L_\phi(\tau)$ is a linear space, we have that $\mathcal{A} \subseteq L_\phi(\tau)$.
2): We note that for $\lambda > 0$, $\mu_{p/\lambda} = \frac{1}{\lambda}\chi_{(0,\tau(p))}$, and so

$$I_\phi(p/\lambda) = \int_0^\infty \phi(\mu_{p/\lambda}(t))dt = \int_0^\infty \phi(1/\lambda)dt = \begin{cases} 0 & if\ 1/\lambda \le a_\phi \\ \infty & if\ 1/\lambda > a_\phi \end{cases}$$

It follows that $\|p\|_{L_\phi(\tau)} = \frac{1}{a_\phi}$.
3): Suppose $(x_n)_{n=1}^\infty \cup \{x\} \subseteq \mathcal{A}$ is such that $x_n \overset{\mathcal{A}}{\to} x$. Then, since $\mathbf{1} \in L_\phi(\tau)$, we can use the fact that $L_\phi(\tau)$ is a normed $\mathcal{A}$-bimodule to obtain

$$\|x_n - x\|_{L_\phi(\tau)} = \|(x_n - x)\mathbf{1}\|_{L_\phi(\tau)} \le \|x_n - x\|_{\mathcal{A}} \|\mathbf{1}\|_{L_\phi(\tau)} \to 0.$$

4): Since ϕ is increasing and continuous from the left $\phi(t) = 0$ for all $t \in [0, a_\phi]$. Since $a_\phi > 0$ this implies that ϕ is not invertible and therefore ϕ does not satisfy the Δ_2-condition globally, by Proposition 1.5.2. $L_\phi(\tau)$ therefore does not have absolutely continuous norm. $\square$

COROLLARY 8.1.2. *Suppose $(\mathcal{A}, \tau)$ and $(\mathcal{B}, \nu)$ are semi-finite von Neumann algebras with $\tau(\mathbf{1}) = \infty = \nu(\mathbf{1})$ and ϕ is a discontinuous Orlicz function with $0 < a_\phi < \infty$. If a linear map $T : L_\phi(\tau) \to L_\phi(\nu)$ is continuous with respect to the Orlicz space norms and is such that $T(\mathcal{A}) \subseteq \mathcal{B}$, then T is continuous with respect to the norms on $\mathcal{A}$ and $\mathcal{B}$.*

PROOF. We note that $\mathcal{A} \subseteq L_\phi(\tau)$, by Proposition 8.1.1 and so T is defined on all of $\mathcal{A}$. Suppose $(x_n)_{n=1}^\infty \cup \{x\} \subseteq \mathcal{A}$ is such that $x_n \overset{\mathcal{A}}{\to} x$ and $T(x_n) \overset{\mathcal{B}}{\to} y$ for some $y \in \mathcal{B}$. It follows from Proposition 8.1.1 that $x_n \overset{L_\phi(\tau)}{\to} x$ and $T(x_n) \overset{L_\phi(\nu)}{\to} y$. T is continuous with respect to the Orlicz space norms and so $T(x_n) \overset{L_\phi(\nu)}{\to} T(x)$. It follows that $T(x) = y$ and hence T is continuous with respect to the norms on $\mathcal{A}$ and $\mathcal{B}$, by the Closed Graph Theorem. $\square$

The extreme points of the unit balls of the types of Orlicz spaces we are considering have been characterized in the commutative setting. We wish to provide non-commutative analogues of these results. For ease of reference we include the following result which characterizes the extreme points of unit balls of commutative Orlicz spaces.

THEOREM 8.1.3. [18, p.506] *Suppose ϕ is an Orlicz function, (Ω, Σ, μ) is a measure space and*

$$Ext(\phi) := \mathbb{R}^+ \backslash \{u \in \mathbb{R}^+ : \exists w, v \in \mathbb{R}^+ : v \ne w,\ u = (v + w)/2,\ \phi((v + w)/2) = (\phi(v) + \phi(w))/2\}.$$

- *Assume ϕ is continuous and (Ω, Σ, μ) does not contain atoms of infinite measure. Then f is an extreme point of $B_{L_\phi(\mu)}$ if and only if $I_\phi(f) = 1$ and either*
  (1) *$|f(t)| \in Ext(\phi)$ for μ-a.e. $t \in \Omega$ or*
  (2) *there exists an atom A such that $|f(t)| \in Ext(\phi)$ for μ-a.e. $t \in \Omega \backslash A$ and $\phi(|f\chi_A|) \ne 0$.*
- *If ϕ is discontinuous and $I_\phi(f) = 1$, then f is an extreme point of $B_{L_\phi(\mu)}$ if and only if either* (1) *or* (2) *above holds.*
- *If ϕ is discontinuous and $I_\phi(f) < 1$, then f is an extreme point of $B_{L_\phi(\mu)}$ if and only if $|f(t)| = b_\phi$ for μ-a.e. $t \in \Omega$.*

It is easily checked that $L_1 \cap L_\infty(\mu) = L_\phi(\mu)$ with equality of norms, where ϕ is the Orlicz function

$$\phi(t) = \begin{cases} t & if\ 0 \le t \le 1 \\ \infty & if\ t > 1, \end{cases}$$

and (Ω, Σ, μ) is any measure space. One can therefore use the second condition in Theorem 8.1.3 to characterize the extreme points of the unit ball of $L_1 \cap L_\infty(\mu)$.

Corollary 8.1.4. [18, p.509] *Suppose (Ω, Σ, μ) is a non-atomic measure space such that $\mu(\Omega) = \infty$ and let $X = L_1 \cap L_\infty(\mu)$. f is an extreme point of B_X if and only if $|f| = \chi_A$ for some $A \in \Sigma$ with $\mu(A) = 1$.*

We present the following non-commutative analogue of the previous result.

Corollary 8.1.5. *Suppose $(\mathcal{A}, \tau)$ is a semi-finite von Neumann algebra and $E(0,\infty) = L_1 \cap L_\infty(0,\infty)$. x is an extreme point of $B_{E(\tau)}$ if and only if $x = v$ for some partial isometry $v \in \mathcal{A}$ with $\tau(|v|) = 1$.*

Proof. Suppose x is an extreme point of $B_{E(\tau)}$. By Theorem 7.1.2, μ_x is an extreme point of $B_{E(0,\infty)}$ and hence, by Corollary 8.1.4, $\mu_x = |\mu_x| = \chi_A$ for some $A \subseteq (0,\infty)$ with $\mu(A) = 1$. Since μ_x is decreasing, $\chi_A = \chi_{(0,1)}$ a.e. It follows by Proposition 1.4.5 that $|x| = p$ for some projection $p \in \mathcal{P}(\mathcal{A})$ with $\tau(p) = 1$. By Proposition B.2.13 there exists a unitary operator $u \in \mathcal{A}$ such that $x = u|x| = up$. It is easily checked that up is a partial isometry (with initial projection p) and hence $\tau(|up|) = \tau(p) = 1$. Conversely, suppose $x = v$ for some partial isometry $v \in \mathcal{A}$ with $\tau(|v|) = 1$. By Proposition 1.4.5 $\mu_x = \chi_{(0,1)}$ and therefore μ_x is an extreme point of $B_{E(0,\infty)}$, by Corollary 8.1.4. By Theorem 7.1.2, x is an extreme point of $B_{E(\tau)}$, since $\mu_x(\infty) = 0$. $\square$

We are also interested in considering the third possibility in Theorem 8.1.3 in the non-commutative setting.

Corollary 8.1.6. *Suppose $(\mathcal{A}, \tau)$ is a trace-finite von Neumann algebra and ϕ is a discontinuous Orlicz function with $b_\phi < \infty$. If $I_\phi(x) < 1$, then x is an extreme point of $B_{L_\phi(\tau)}$ if and only if $x = b_\phi u$ for some unitary operator $u \in \mathcal{A}$.*

Proof. Suppose $I_\phi(\mu_x) < 1$ and $x = b_\phi u$ for some unitary operator $u \in \mathcal{A}$. By Proposition 1.4.5, $\mu_x = b_\phi \chi_{[0,\tau(\mathbf{1}))}$, since $|u| = \mathbf{1}$. It follows by Theorem 8.1.3 that μ_x is an extreme point of $B_{L_\phi(0,\tau(\mathbf{1}))}$. Furthermore, $\mu_x(\infty) = 0$ and therefore x is an extreme point of $B_{L_\phi(\tau)}$, by Theorem 7.1.2. Conversely, suppose $I_\phi(\mu_x) < 1$ and x is an extreme point of $B_{L_\phi(\tau)}$. By Theorem 7.1.2, μ_x is an extreme point of $B_{L_\phi(0,\tau(\mathbf{1}))}$ and therefore $\mu_x(t) = b_\phi$ for m-a.e. $t \in (0, \tau(\mathbf{1}))$, by Theorem 8.1.3. Since μ_x is decreasing it follows that $\mu_x = b_\phi \chi_{(0,\tau(\mathbf{1}))}$ m-a.e. and hence $|x| = b_\phi p$ for some $p \in \mathcal{P}(\mathcal{A})$ with $\tau(p) = \tau(\mathbf{1})$, by Proposition 1.4.5. Furthermore, $\tau(\mathbf{1}) < \infty$ and so it follows that $\tau(\mathbf{1} - p) = 0$. Since τ is faithful and $p \le \mathbf{1}$ we have that $p = \mathbf{1}$ and hence $\frac{x}{b_\phi}$ is unitary. $\square$

Before considering characterizations of isometries on Orlicz spaces, we motivate why we confined ourselves to the finite setting for the previous result. Suppose $I_\phi(\mu_x) < 1$ and $x = b_\phi u$ for some unitary operator $u \in \mathcal{A}$. As shown in the proof of the previous result, $\mu_x = b_\phi \chi_{[0,\tau(\mathbf{1}))}$. Note that in this case

$$1 > I_\phi(x) = \int_0^{\tau(\mathbf{1})} \phi(\mu_x(t))dt = \int_0^{\tau(\mathbf{1})} \phi(b_\phi)dt$$

and hence this scenario is only possible if $\tau(\mathbf{1}) < \infty$ or $\phi(t) = \begin{cases} 0 & if\ 0 \le t \le b_\phi \\ \infty & if\ t > b_\phi, \end{cases}$. In the latter case $L_\phi(\tau) = L_\infty(\tau)$ and the norm on $L_\phi(\tau)$ is proportional to the norm on $L_\infty(\tau)$. Isometries on such spaces are covered by Kadison's Theorem (see Theorem 2.2.1) and therefore it suffices to restrict ourselves to the finite setting in the previous result.

8.2. Positive surjective isometries on Orlicz spaces ($0 < a_\phi < \infty$, $b_\phi = \infty$)

Throughout this section we will assume that $(\mathcal{A}, \tau)$ and $(\mathcal{B}, \nu)$ are semi-finite von Neumann algebras with $\tau(\mathbf{1}) = \infty = \nu(\mathbf{1})$ and that ϕ is a discontinuous Orlicz function with $0 < a_\phi < \infty$ and $b_\phi = \infty$. Note that this implies that $L_\phi(\tau)$ and $L_\phi(\nu)$ do not have absolutely continuous norm, by Proposition 8.1.1(4). Furthermore, $\mathcal{A} \subseteq L_\phi(\tau)$ and $\mathcal{B} \subseteq L_\phi(\nu)$, by Proposition 8.1.1(1). Suppose U is a positive surjective modular isometry from $L_\phi(\tau)$ onto $L_\phi(\nu)$. We will show that the restriction of U to $\mathcal{A}$ is a Jordan $*$-isomorphism from $\mathcal{A}$ onto $\mathcal{B}$. The structure of the proof is as follows. Using the fact that U is a modular isometry, we will show that U is unital. The positivity of U will then be used to show that $U(\mathcal{A}) \subseteq \mathcal{B}$. The fact that U^{-1} also preserves the modular will play a significant role in showing that the reverse inclusion also holds and that U^{-1} restricted to $\mathcal{B}$ is positive. The idea of the proof has some similarities with the proof of Theorem 2.2.7 and represents one setting where the aforementioned theorem can be extended to the semi-finite setting without being confined to spaces with absolutely continuous norm. The details are as follows.

LEMMA 8.2.1. *U is unital and $U(\mathcal{A}) \subseteq \mathcal{B}$.*

PROOF. We start by showing that U is unital. To prevent possible confusion, we will use $\mathbf{1}_\mathcal{A}$ to denote the identity of $\mathcal{A}$ and $\mathbf{1}_\mathcal{B}$ to denote the identity of $\mathcal{B}$. For $\lambda \geq 0$, we have $\mu_{\lambda \mathbf{1}_\mathcal{A}} = \lambda \chi_{(0,\infty)}$, by Proposition 1.4.5. In particular, using $\lambda = a_\phi$, we obtain

$$I_\phi(a_\phi \mathbf{1}_\mathcal{A}) = \int_0^\infty \phi(\mu_{a_\phi \mathbf{1}_\mathcal{A}}(t))dt = \int_0^\infty \phi(a_\phi)dt = 0, \tag{8.2.1}$$

since ϕ is continuous from the left and hence $\phi(a_\phi) = 0$. Since U is a modular isometry, we have that

$$0 = I_\phi(a_\phi \mathbf{1}_\mathcal{A}) = I_\phi(U(a_\phi \mathbf{1}_\mathcal{A})) = \int_0^\infty \phi(\mu_{U(a_\phi \mathbf{1}_\mathcal{A})}(t))dt$$

and hence $\phi(\mu_{U(a_\phi \mathbf{1}_\mathcal{A})}(t)) = 0$ for m-a.e. $t \geq 0$. It follows that $a_\phi \mu_{U(\mathbf{1}_\mathcal{A})}(t) = \mu_{U(a_\phi \mathbf{1}_\mathcal{A})}(t) \leq a_\phi$ for m-a.e $t \geq 0$. Since $\mu_{U(\mathbf{1}_\mathcal{A})}$ is decreasing and continuous from the right, this implies that

$$\mu_{U(\mathbf{1}_\mathcal{A})}(t) \leq 1 \qquad \forall t \geq 0 \tag{8.2.2}$$

We will show that the reverse inequality also holds. Fix $\lambda > a_\phi$. Since ϕ is increasing it follows from (8.2.2) that

$$\phi(\mu_{U(\lambda \mathbf{1}_\mathcal{A})}(t)) = \phi(\lambda \mu_{U(\mathbf{1}_\mathcal{A})}(t)) \leq \phi(\lambda) \qquad \forall t \geq 0 \tag{8.2.3}$$

Furthermore, $\phi(\lambda) > 0$ and so

$$\infty = \tau(\mathbf{1}_\mathcal{A})\phi(\lambda) = I_\phi(\lambda \mathbf{1}_\mathcal{A}) = I_\phi(U(\lambda \mathbf{1}_\mathcal{A})) = \int_0^\infty \phi(\mu_{U(\lambda \mathbf{1}_\mathcal{A})}(t))dt \tag{8.2.4}$$

Let $A := \{t \in [0, \infty) : \mu_{U(\lambda \mathbf{1}_\mathcal{A})}(t) > a_\phi\}$ and note that

$$\begin{aligned}
\infty &= \int_0^\infty \phi(\mu_{U(\lambda \mathbf{1}_\mathcal{A})}(t))dt \qquad \text{using (8.2.4)} \\
&= \int_A \phi(\mu_{U(\lambda \mathbf{1}_\mathcal{A})}(t))dt + \int_{A^c} \phi(\mu_{U(\lambda \mathbf{1}_\mathcal{A})}(t))dt \\
&= \int_A \phi(\lambda \mu_{U(\mathbf{1}_\mathcal{A})}(t))dt \qquad \text{since } \phi(\mu_{U(\lambda \mathbf{1}_\mathcal{A})}(t)) \leq \phi(a_\phi) = 0 \text{ for all } t \in A^c \\
&\leq \int_A \phi(\lambda)dt \qquad \text{using (8.2.3)} \\
&= m(A)\phi(\lambda)
\end{aligned}$$

Furthermore, $b_\phi = \infty$ and so $\phi(\lambda) < \infty$. It follows that $m(A) = \infty$. Since $\mu_{U(\lambda \mathbf{1}_\mathcal{A})}$ is decreasing, this implies that $\mu_{U(\mathbf{1}_\mathcal{A})}(t) > a_\phi/\lambda$ for all $t \geq 0$. Since $\lambda > a_\phi$ was arbitrary, we have that $\mu_{U(\mathbf{1}_\mathcal{A})}(t) \geq 1$ for all $t \geq 0$. Combining this with (8.2.2), we have that

$$\mu_{U(\mathbf{1}_\mathcal{A})} = \chi_{[0,\infty)}$$

and hence, using the positivity of U and Proposition 1.4.5, we have that $U(\mathbf{1}_\mathcal{A}) = |U(\mathbf{1}_\mathcal{A})| = q$, for some projection $q \in \mathcal{P}(\mathcal{B})$ with $\nu(q) = \infty$. Since U^{-1} is also a modular isometry, we can similarly show that $|U^{-1}(\mathbf{1}_\mathcal{B})| = p$ for some projection $p \in \mathcal{P}(\mathcal{A})$. Note that since U is positive and injective, $U^{-1}(\mathbf{1}_\mathcal{B})$ is self-adjoint by Proposition 4.1.1 and therefore $U^{-1}(\mathbf{1}_\mathcal{B}) \leq \left\|U^{-1}(\mathbf{1}_\mathcal{B})\right\|_\mathcal{A} \mathbf{1}_\mathcal{A}$, by Proposition B.1.6. Furthermore, $\left\|U^{-1}(\mathbf{1}_\mathcal{B})\right\|_\mathcal{A} = \left\||U^{-1}(\mathbf{1}_\mathcal{B})|\right\|_\mathcal{A} = \left\|p\right\|_\mathcal{A} = 1$ and so $U^{-1}(\mathbf{1}_\mathcal{B}) \leq \mathbf{1}_\mathcal{A}$. U is positive and therefore

$$\mathbf{1}_\mathcal{B} = U(U^{-1}(\mathbf{1}_\mathcal{B})) \leq U(\mathbf{1}_\mathcal{A}) = q \leq \mathbf{1}_\mathcal{B}.$$

It follows that $U(\mathbf{1}_\mathcal{A}) = \mathbf{1}_\mathcal{B}$. We are now in a position to show that $U(\mathcal{A}) \subseteq \mathcal{B}$. Suppose $x \in \mathcal{A}^+$. Then $0 \leq x \leq \left\|x\right\|_\mathcal{A} \mathbf{1}_\mathcal{A}$, by Proposition B.1.6. Furthermore, since U is positive,

$$0 \leq U(x) \leq U(\left\|x\right\|_\mathcal{A} \mathbf{1}_\mathcal{A}) = \left\|x\right\|_\mathcal{A} U(\mathbf{1}_\mathcal{A}) = \left\|x\right\|_\mathcal{A} \mathbf{1}_\mathcal{B}.$$

Therefore $U(x) \in \mathcal{B}$, by Proposition B.3.1(5) (since $(\left\|x\right\|_\mathcal{A} \mathbf{1}_\mathcal{B} \in \mathcal{B})$. Since any element in $\mathcal{A}$ can be written as a linear combination of positive elements, U is linear and $\mathcal{B}$ is a linear space we have that $U(\mathcal{A}) \subseteq \mathcal{B}$. ☐

Lemma 8.2.2. *The restriction of U^{-1} to $\mathcal{B}$ is positive and $U^{-1}(\mathcal{B}) \subseteq \mathcal{A}$.*

Proof. In Chapter 4, techniques were developed to determine the positivity of the inverse of a positive operator. Corollary 4.1.6 is not applicable here, since $L_\phi(\nu)$ does not have absolutely continuous norm. If we could show that $\left\|x - y\right\|_{L_\phi(\nu)} \leq \left\|x + y\right\|_{L_\phi(\nu)}$ for all $x, y \in L_\phi(\nu)^+$, then the positivity of U^{-1} would follow from Lemma 4.1.5. We will follow a different approach, however. Suppose $0 \neq q \in \mathcal{P}(\mathcal{B})$. Then $\mu_{a_\phi q^\perp} = a_\phi \chi_{(0,\nu(q^\perp))}$, by Proposition 1.4.5, and so

$$I_\phi(a_\phi q^\perp) = \int_0^\infty \phi(\mu_{a_\phi q^\perp}(t))dt = \int_0^{\nu(q^\perp)} \phi(a_\phi)dt = 0.$$

Since U and hence U^{-1} is a modular isometry, we have that $0 = \int_0^\infty \phi(a_\phi \mu_x(t))dt$, where $x = U^{-1}(q^\perp)$. It follows that $a_\phi \mu_x(t) \leq a_\phi$ for all $t > 0$. Since μ_x is continuous from the right, $\mu_x(0) \leq 1$. By Proposition 1.4.2(5), $x \in \mathcal{A}$ and

$$\left\|x\right\|_\mathcal{A} = \mu_x(0) \leq 1 \tag{8.2.5}$$

Note that $x = U^{-1}(q^\perp)$ and $U^{-1}(q)$ are self-adjoint, by Proposition 4.1.1. Therefore $U^{-1}(q^\perp) \leq \left\|U^{-1}(q^\perp)\right\|_\mathcal{A} \mathbf{1} \leq \mathbf{1}$, by Proposition B.1.6 and using (8.2.5). It follows that

$$U^{-1}(q) = U^{-1}(\mathbf{1}) - U^{-1}(q^\perp) \geq U^{-1}(\mathbf{1}) - \mathbf{1} = 0, \tag{8.2.6}$$

since $U^{-1}(\mathbf{1}) = \mathbf{1}$. Let $\mathcal{G}$ denote the set of all finite linear combinations of elements from $\mathcal{P}(\mathcal{B})$. It follows from the linearity of U^{-1} and the fact that $U^{-1}(q) \geq 0$ for every $q \in \mathcal{P}(\mathcal{B})$ that $U^{-1}(y) \geq 0$ for every $y \in \mathcal{G}^+$. Suppose $y \in \mathcal{B}^+$. It follows from the Spectral Theorem that there exists $(y_n)_{n=1}^\infty \subseteq \mathcal{G}^+$ such that $y_n \xrightarrow{\mathcal{B}} y$. By Corollary 8.1.2, $U^{-1}(y_n) \xrightarrow{\mathcal{A}} U^{-1}(y)$. Since $y_n \in \mathcal{G}^+$ for all $n \in \mathbb{N}^+$, we have that $U^{-1}(y_n) \in \mathcal{A}^+$ for all $n \in \mathbb{N}^+$. Since $\mathcal{A}^+$ is closed by Proposition B.3.1(10), $U^{-1}(y) \in \mathcal{A}^+$. Similarly to the proof of the previous lemma, it follows that $U^{-1}(\mathcal{B}) \subseteq \mathcal{A}$. ☐

Theorem 8.2.3. *Suppose $(\mathcal{A}, \tau)$ and $(\mathcal{B}, \nu)$ are semi-finite von Neumann algebras with $\tau(\mathbf{1}) = \infty = \nu(\mathbf{1})$ and ϕ is a discontinuous Orlicz function with $0 < a_\phi < \infty$ and $b_\phi = \infty$. If U is a positive surjective modular isometry from $L_\phi(\tau)$ onto $L_\phi(\nu)$, then the restriction of U to $\mathcal{A}$ is a Jordan $*$-isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.*

PROOF. It follows from Lemmas 8.2.1 and 8.2.2 that $U(\mathcal{A}) = \mathcal{B}$. Let Φ denote the restriction of U to $\mathcal{A}$. It follows from Lemma 8.2.1, Lemma 8.2.2 and the positivity of U that Φ is a linear order isomorphism from $\mathcal{A}$ onto $\mathcal{B}$ such that $\Phi(\mathbf{1}) = \mathbf{1}$. Therefore, Φ is a Jordan $*$-isomorphism, by Proposition 1.8.9. $\square$

REMARK 8.2.4. It follows from the fact that the restriction of U to $\mathcal{A}$ is a Jordan $*$-isomorphism that U is disjointness-preserving and maps projections onto projections. We saw in the proof of Theorem 7.2.9 that the characterization of the extreme points of the unit balls of Lorentz spaces enabled one to show that surjective isometries on such spaces map partial isometries onto partial isometries, and this played a fundamental role in showing that these isometries are disjointness-preserving. To further illustrate some of the patterns we have seen in this thesis we demonstrate that we can use the fact that $U(\mathbf{1}) = \mathbf{1}$ more directly to show that U (as in Theorem 8.2.3) is disjointness-preserving. Suppose $p, q \in \mathcal{P}(\mathcal{A})$ with $pq = 0$. Let $x = U(p)U(q)$ and assume $x \neq 0$. Since this implies that $|x| \neq 0$, there exists $\lambda_1 > 0$ such that $e^{|x|}(\lambda_1, \infty) \neq 0$. Note that $xs(U(q)) = x$ and so $s(x) \leq s(U(q))$. By Remark B.1.31, $s(x) = s(|x|)$ and so $e^{|x|}(\lambda_1, \infty) \leq s(|x|) = s(x) \leq s(U(q)) = e^{U(q)}(0, \infty)$. Therefore

$$0 \neq e^{|x|}(\lambda_1, \infty) = e^{|x|}(\lambda_1, \infty)e^{U(q)}(0, \infty).$$

Assume that $e^{|x|}(\lambda_1, \infty)e^{U(q)}(\lambda, \infty) = 0$ for all $\lambda > 0$. Let $f_n := \chi_{(1/n,\infty)}$. Then $f_n \uparrow \chi_{(0,\infty)}$ and hence $f_n(U(q)) \uparrow \chi_{(0,\infty)}(U(q))$, by Proposition B.2.5(4). Since this implies that $e^{U(q)}(1/n, \infty) \overset{SOT}{\to} e^{U(q)}(0, \infty)$, we have that

$$0 = e^{|x|}(\lambda_1, \infty)e^{U(q)}(1/n, \infty) \overset{SOT}{\to} e^{|x|}(\lambda_1, \infty)e^{U(q)}(0, \infty).$$

This is a contradiction and so there exists $\lambda_2 > 0$ such that $e^{|x|}(\lambda_1, \infty)e^{U(q)}(\lambda_2, \infty) \neq 0$. Let r denote the projection onto $e^{|x|}(\lambda_1, \infty)(H) \cap e^{U(q)}(\lambda_2, \infty)(H)$. We can use a similar argument to the one employed above to show that there exists a $\lambda_3 > 0$ such that

$$re^{U(p)}(\lambda_3, \infty) \neq 0.$$

Let r' denote the projection onto $r(H)e^{U(p)}(\lambda_3, \infty)(H)$ and let $\epsilon := \min\{\lambda_i : i = 1, 2, 3\}$. Then

$$\epsilon r' \leq \lambda_2 e^{U(q)}(\lambda_2, \infty) \leq U(q)e^{U(q)}(\lambda_2, \infty) \leq U(q)$$

Since we also have that $\epsilon r', U(q) \in \mathcal{B}^+$, this implies by Lemma 8.2.2 that $U^{-1}(\epsilon r') \leq q$ and hence $s(U^{-1}(\epsilon r')) \leq s(q) = q$. Similarly, $s(U^{-1}(\epsilon r')) \leq p$. It follows that

$$s(U^{-1}(\epsilon r')) \leq q \wedge p = 0$$

and hence $U^{-1}(\epsilon r') = 0$. U^{-1} is injective and therefore $\epsilon r' = 0$. This contradicts the fact that $r' \neq 0$.

8.3. Positive surjective isometries on Orlicz spaces ($b_\phi < \infty$)

Next, we consider the setting where we can apply the characterization of the extreme points of the unit ball of the particular type of Orlicz space given in Corollary 8.1.6. We will therefore be restricting ourselves to trace-finite von Neumann algebras, in which case the von Neumann algebra is contained in the associated Orlicz space and the structure of isometries on the specified Orlicz spaces can be described using Theorem 2.2.7. The characterization of the extreme points given in Corollary 8.1.6 will, however, enable us to refine the structural description given of such isometries. More specifically, we will show that a positive surjective modular isometry between the specified Orlicz spaces is unital and that applying this knowledge to the conclusion of Theorem 2.2.7, will enable us to show that the restriction of the aforementioned isometry to the von Neumann algebra is a Jordan $*$-isomorphism.

THEOREM 8.3.1. *Suppose $(\mathcal{A}, \tau)$ and $(\mathcal{B}, \nu)$ are trace-finite von Neumann algebras with $\tau(\mathbf{1}) = \nu(\mathbf{1})$ and ϕ is a discontinuous Orlicz function with $\tau(\mathbf{1})\phi(b_\phi) < 1$. If U is positive surjective modular isometry from $L_\phi(\tau)$ onto $L_\phi(\nu)$, then the restriction of U to $\mathcal{A}$ is a Jordan $*$-isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.*

PROOF. By Theorem 2.2.7, there exists a Jordan $*$-isomorphism Φ, from $\mathcal{A}$ onto $\mathcal{B}$, and a positive operator a, affiliated with the center of $\mathcal{B}$, such that

$$U(x) = a\Phi(x) \qquad \forall x \in \mathcal{A}.$$

Furthermore, we note from the proof of Theorem 2.2.7 that $a = U(\mathbf{1})$. We will show that $U(\mathbf{1}) = \mathbf{1}$ and hence that $U(x) = \Phi(x)$ for all $x \in \mathcal{A}$. Note that

$$I_\phi(b_\phi \mathbf{1}) = \int_0^{\tau(\mathbf{1})} \phi(\mu_{b_\phi \mathbf{1}}(t))dt = \tau(\mathbf{1})\phi(b_\phi) < 1$$

and so $b_\phi \mathbf{1}$ is an extreme point of $B_{L_\phi(\tau)}$, by Corollary 8.1.6. U is a surjective isometry and so $U(b_\phi \mathbf{1})$ is an extreme point of $B_{L_\phi(\nu)}$. Furthermore, $I_\phi(U(b_\phi \mathbf{1})) = I_\phi(b_\phi \mathbf{1}) < 1$, since U is a modular isometry. It follows by Corollary 8.1.6 that $U(b_\phi \mathbf{1}) = b_\phi u$ for some unitary $u \in \mathcal{B}$. U is positive and therefore $u = \mathbf{1}$, by Remark B.1.29. It follows that $U(\mathbf{1}) = \mathbf{1}$. ☐

8.4. Positive surjective isometries on $L_1 \cap L_\infty(\tau)$

Kadison ([22]) and Yeadon ([49]) have characterized L_∞-isometries and L_p-isometries, respectively. Suppose $(\mathcal{A}, \tau)$ and $(\mathcal{B}, \nu)$ are semi-finite von Neumann algebras. It is interesting to consider if it is possible for a map $U : L_1 \cap L_\infty(\tau) \to L_1 \cap L_\infty(\nu)$ to be an isometry with respect to the maximum norms on these spaces without being both a L_∞-isometry and a L_1-isometry. In this section we show that this is impossible if U is a positive surjective isometry. More specifically, we will show that if U is a positive surjective isometry from $L_1 \cap L_\infty(\tau)$ onto $L_1 \cap L_\infty(\nu)$, then it has to be the restriction of a Jordan $*$-isomorphism from $\mathcal{A}$ onto $\mathcal{B}$ and hence both a L_1-isometry and a L_∞-isometry. Conversely, it will also be shown that the restriction of any Jordan $*$-isomorphism from $\mathcal{A}$ onto $\mathcal{B}$ maps $L_1 \cap L_\infty(\tau)$ onto $L_1 \cap L_\infty(\nu)$ in an isometric fashion. We note that since $L_1 \cap L_\infty(\tau)$ does not have absolutely continuous norm, we cannot apply Theorem 5.3.4 to describe the structure of positive surjective isometries on $L_1 \cap L_\infty(\tau)$. Instead, the key element in showing that such isometries are in fact restrictions of Jordan $*$-isomorphisms, will be showing that such isometries map orthogonal projections of finite trace onto orthogonal projections of finite trace. This will be accomplished using a characterization of the extreme points of the unit ball of $L_1 \cap L_\infty(\tau)$.

Let $(\mathcal{A}, \tau)$ and $(\mathcal{B}, \nu)$ denote semi-finite von Neumann algebras. Suppose Φ is a trace-preserving Jordan $*$-isomorphism Φ from $\mathcal{A}$ onto $\mathcal{B}$. Then Φ maps $L_1 \cap L_\infty(\tau)$ onto $L_1 \cap L_\infty(\nu)$, by Proposition 5.4.1. Furthermore, using Proposition 1.8.8 and Remark 2.2.5, we have

$$\left\|\Phi(x)\right\|_{\mathcal{B}} = \left\|x\right\|_{\mathcal{A}} \qquad \text{and} \qquad \left\|\Phi(x)\right\|_{L_1(\nu)} = \left\|x\right\|_{L_1(\tau)}$$

for all $x \in L_1 \cap L_\infty(\tau)$. It follows that

$$\left\|\Phi(x)\right\|_{L_1 \cap L_\infty(\nu)} = \left\|x\right\|_{L_1 \cap L_\infty(\tau)} \qquad \forall x \in L_1 \cap L_\infty(\tau).$$

Conversely, suppose $(\mathcal{A}, \tau)$ and $(\mathcal{B}, \nu)$ are non-atomic semi-finite von Neumann algebras with $\tau(\mathbf{1}) = \infty = \nu(\mathbf{1})$ and suppose $U : L_1 \cap L_\infty(\tau) \to L_1 \cap L_\infty(\nu)$ is a positive surjective isometry. To show that U is the restriction of trace-preserving Jordan $*$-isomorphism, we proceed as follows. We start by using the characterization of the extreme points of the unit balls of $L_1 \cap L_\infty(\tau)$ and $L_1 \cap L_\infty(\nu)$ to show that

- $U(p)U(q) = 0$ if $p, q \in \mathcal{P}(\mathcal{A})^f$ with $pq = 0$; and

- $U(p) \in \mathcal{P}(\mathcal{B})^f$ whenever $p \in \mathcal{P}(\mathcal{A})^f$.

This will enable us to conclude that U maps orthogonal projections onto orthogonal projections. Letting $\Phi(p) = U(p)$ for $p \in \mathcal{P}(\mathcal{A})^f$ will then yield a map which can be extended (by Proposition 3.2.1) to a positive linear map on $\mathcal{F}(\tau)$ which is square-preserving and L_∞-isometric on self-adjoint elements. To show that this map can be extended to all $\mathcal{A}$ (using the extension procedure outlined in section 3.3), we will show that Φ agrees with U on $\mathcal{F}(\tau)$ and that U is normal. The details are as follows.

LEMMA 8.4.1. *If $p \in \mathcal{P}(\mathcal{A})^f$ with $\tau(p) = 1$, then $U(p)$ is a projection, with $\nu(U(p)) = 1$.*

PROOF. By Corollary 8.1.5, p is an extreme point of the unit ball of $L_1 \cap L_\infty(\tau)$. U is a surjective isometry and so $U(p)$ is an extreme point of the unit ball of $L_1 \cap L_\infty(\nu)$. It follows by Corollary 8.1.5 that $U(p) = v_{(p)}$ for some partial isometry $v_{(p)} \in \mathcal{B}$ with $\nu(|v_{(p)}|) = 1$. Note that $U(p) \geq 0$, since $p \geq 0$ and U is positive. By Remark B.1.29, $U(p) = v_{(p)}$ is therefore a projection, and

$$\nu(U(p)) = \nu(|U(p)|) = \nu(|v_{(p)}|) = 1.$$

□

LEMMA 8.4.2. *If $p, q \in \mathcal{P}(\mathcal{A})^f$ with $pq = 0$, then*

$$U(p)U(q) = 0.$$

PROOF. Suppose $p, q \in \mathcal{P}(\mathcal{A})^f$ with $pq = 0$ and $\tau(p) = 1 = \tau(q)$. By Proposition B.2.12(3), $|p-q| = |p+q|$. Since $\big\| |y| \big\|_{L_1 \cap L_\infty(\tau)} = \|y\|_{L_1 \cap L_\infty(\tau)}$ for every $y \in L_1 \cap L_\infty(\tau)$, it follows that

$$\begin{aligned}
\|p-q\|_{L_1 \cap L_\infty(\tau)} &= \|p+q\|_{L_1 \cap L_\infty(\tau)} \\
&= \max\Big\{ \|p+q\|_{L_1(\tau)}, \|p+q\|_{\mathcal{A}} \Big\} \\
&= \|p+q\|_{L_1(\tau)} \\
&= 2
\end{aligned}$$

Furthermore, U is linear and isometric and so

$$\|U(p) - U(q)\|_{L_1 \cap L_\infty(\nu)} = \|p-q\|_{L_1 \cap L_\infty(\tau)} = 2 \tag{8.4.1}$$

By Lemma 8.4.1, $U(p)$ and $U(q)$ are projections and so $-\mathbf{1} \leq U(p) - U(q) \leq \mathbf{1}$. Therefore $\|U(p) - U(q)\|_{\mathcal{B}} \leq \|\mathbf{1}\|_{\mathcal{B}} = 1$, by Proposition B.1.7. Using (8.4.1) it follows that

$$\begin{aligned}
2 &= \max\Big\{ \|U(p) - U(q)\|_{L_1(\nu)}, \|U(p) - U(q)\|_{\mathcal{B}} \Big\} \\
&= \|U(p) - U(q)\|_{L_1(\nu)} \qquad \text{since } \|U(p) - U(q)\|_{\mathcal{B}} \leq 1 \\
&\leq \|U(p) + U(q)\|_{L_1(\nu)} \qquad \text{by Lemma 4.1.4} \\
&\leq \|U(p)\|_{L_1(\nu)} + \|U(q)\|_{L_1(\nu)} \\
&= 2 \qquad \text{since } U(p) \text{ and } U(q) \text{ are projections with } \nu(U(p)) = 1 = \nu(U(q))
\end{aligned}$$

It follows that

$$\|U(p) + U(q)\|_{L_1(\nu)} + \|U(p) - U(q)\|_{L_1(\nu)} = 2 + 2 = 2\Big(\|U(p)\|_{L_1(\nu)} + \|U(q)\|_{L_1(\nu)} \Big).$$

Application of Theorem 2.2.3 yields

$$U(p)U(q) = U(p)^* U(q) = 0$$

and so $U(p)$ and $U(q)$ are orthogonal projections. Next, suppose that $p, q \in \mathcal{P}(\mathcal{A})^f$ with $pq = 0$ and $\tau(p), \tau(q) < 1$. By Proposition B.1.27, there exist $p_1, q_1 \in \mathcal{P}(\mathcal{A})^f$ such that

- $p+p_1, q+q_1 \in \mathcal{P}(\mathcal{A})^f$
- $\tau(p+p_1) = 1 = \tau(q+q_1)$ and
- $(p+p_1)(q+q_1) = 0$.

It follows by what has been shown already that $U(p+p_1)$ and $U(q+q_1)$ are orthogonal projections. Furthermore, since U is positive we have that $0 \leq U(p) \leq U(p+p_1)$ and therefore $s(U(p)) \leq s(U(p+p_1)) = U(p+p_1)$. Similarly, $r(U(q)) \leq U(q+q_1)$ and thus

$$U(p)U(p+p_1)U(q+q_1)U(q) = U(p)U(q).$$

However, $U(p+p_1)U(q+q_1) = 0$ and so $U(p)U(q) = 0$.

Finally, for general $p, q \in \mathcal{P}(\mathcal{A})^f$ with $pq = 0$, we note that since $(\mathcal{A}, \tau)$ is non-atomic, we have by Corollary B.1.26, $(p_i)_{i=1}^k, (q_j)_{j=1}^n \subseteq \mathcal{P}(\mathcal{A})^f$ such that

- $\sum\limits_{i=1}^k p_i = p$ and $\sum\limits_{j=1}^n q_j = q$;
- $p_i p_j = 0 = q_i q_j$ if $i \neq j$; and
- $\tau(p_i), \tau(q_j) < 1$ for all i, j.

Since $p_i \leq p$ for each i and $q_j \leq q$ for each j, we have that $p_i q_j = 0$ for all i, j. Therefore $U(p_i)U(q_j) = 0$ for all i, j and so

$$U(p)U(q) = \sum_{i=1}^k U(p_i) \sum_{j=1}^n U(q_j) = \sum_{i=1}^k \sum_{j=1}^n U(p_i)U(q_j) = 0.$$

□

Lemma 8.4.3. *$U(p)$ is a projection for any $p \in \mathcal{P}(\mathcal{A})^f$.*

Proof. Suppose $p \in \mathcal{P}(\mathcal{A})^f$ with $\tau(p) < 1$. Since $(\mathcal{A}, \tau)$ is non-atomic, there exists, by Lemma B.1.25, $p_1 \leq p^\perp$ such that $\tau(p_1) = 1 - \tau(p)$. It follows that $p + p_1 \in \mathcal{P}(\mathcal{A})^f$ and $\tau(p+p_1) = 1$. Therefore, by Lemma 8.4.1, $U(p+p_1)$ is a projection q with $\nu(q) = 1$. By Lemma 8.4.2, $U(p)^*U(p_1) = 0 = U(p)U(p_1)^*$ and therefore, by Proposition B.1.32, $s(U(p))s(U(p_1)) = 0$ and

$$s(U(p)) + s(U(p_1)) = s(U(p) + U(p_1)) = s(U(p+p_1)) = U(p+p_1) = q. \tag{8.4.2}$$

This implies that $s(U(p)) \leq q$ and so for $\eta \in s(U(p))(H)$ we therefore have that

$$\eta = q\eta = (U(p) + U(p_1))\eta = U(p)\eta,$$

since $U(p_1)\eta = U(p_1)s(U(p_1))s(U(p))\eta = 0$. It follows that $U(p) = s(U(p))$. Next, we consider the case where $p \in \mathcal{P}(\mathcal{A})^f$ with $\tau(p) \geq 1$. By Corollary B.1.26, there exists $(p_i)_{i=1}^k \subseteq \mathcal{P}(\mathcal{A})^f$ such that $\sum\limits_{i=1}^k p_i = p$; $p_i p_j = 0$ if $i \neq j$; and $\tau(p_i) < 1$ for all i, j. By what has been shown already we have that $U(p_i)$ is a projection for each i and these projections are mutually orthogonal, by Lemma 8.4.2. It follows that $U(p) = \sum\limits_{i=1}^k U(p_i)$ is a sum of mutually orthogonal projections and is therefore a projection, by repeated use of Proposition B.1.16(1). □

We have that U maps projections with finite trace onto projections. We know from Theorem 1.8.4 that if a continuous map from $\mathcal{A}$ into $\mathcal{B}$ maps projections onto projections, then that map is a Jordan $*$-homomorphism. This raises the question if U can be extended to a map Ψ from $\mathcal{A}$ into $\mathcal{B}$ that is continuous with respect to the von Neumann algebra norms; and if this is the case, can $U(\mathcal{P}(\mathcal{A})^f) \subseteq \mathcal{P}(\mathcal{B})$ be used to show that $\Psi(\mathcal{P}(\mathcal{A})) \subseteq \mathcal{P}(\mathcal{B})$? We will follow a different approach. We will show that the restriction of U to $\mathcal{F}(\tau)$ satisfies the conditions of Proposition 3.3.6 and can therefore be extended to a Jordan $*$-homomorphism from $\mathcal{A}$ into $\mathcal{B}$. We will achieve this indirectly by first considering $U \restriction \mathcal{P}(\mathcal{A})^f$ and showing that this map can be extended to a map on $\mathcal{F}(\tau)$ that has the necessary properties and which can be shown to agree with U on $\mathcal{F}(\tau)$.

LEMMA 8.4.4. *U can be extended to a normal Jordan $*$-homomorphism Φ from $\mathcal{A}$ into $\mathcal{B}$ such that*

$$\left\|\Phi(x)\right\|_{\mathcal{B}} = \left\|x\right\|_{\mathcal{A}} \qquad \forall x \in \mathcal{A}^{sa}.$$

PROOF. For $p \in \mathcal{P}(\mathcal{A})^f$, let $\Phi(p) := U(p)$. By Lemma 8.4.3, Φ maps $\mathcal{P}(\mathcal{A})^f$ into $\mathcal{P}(\mathcal{B})$ and using the linearity of U (or Lemma 8.4.2), $\Phi(p+q) = \Phi(p) + \Phi(q)$ if $p, q \in \mathcal{P}(\mathcal{A})^f$ with $pq = 0$. U is injective and so $\Phi(p) \neq 0$ if $0 \neq p \in \mathcal{P}(\mathcal{A})^f$. Furthermore, by Corollary B.3.4, U has the property that $U(x_n) \overset{\mathcal{T}_m}{\to} U(x)$ whenever $(x_n)_{n=1}^\infty \cup \{x\} \subseteq \mathcal{F}(\tau)$ is such that $x_n \overset{\mathcal{A}}{\to} x$ and $s(x_n) \leq s(x)$ for all $n \in \mathbb{N}^+$. By Proposition 3.2.1, we can therefore extend Φ to a positive linear map from $\mathcal{F}(\tau)$ into $\mathcal{B}$ such that for all $x \in \mathcal{F}(\tau)^{sa}$,

$$\left\|\Phi(x)\right\|_{\mathcal{B}} = \left\|x\right\|_{\mathcal{A}} \qquad \text{and} \qquad \Phi(x^2) = \Phi(x)^2$$

We show that U and Φ agree on $\mathcal{F}(\tau)$. Using the linearity of U and Φ, we have that U and Φ agree on $\mathcal{G}_f$. Suppose $x \in \mathcal{F}(\tau)^{sa}$. By remark B.1.12, there exists $(x_n)_{n=1}^\infty \subseteq \mathcal{G}_f^{sa}$ such that $x_n \overset{\mathcal{A}}{\to} x$ and $s(x_n) \leq s(x)$ for all $n \in \mathbb{N}^+$. Φ is isometric on $\mathcal{F}(\tau)$ and so

$$\Phi(x_n) \overset{\mathcal{B}}{\to} \Phi(x)$$

We show that $U(x_n) \overset{\mathcal{B}}{\to} U(x)$ and hence that $U(x) = \Phi(x)$. Note that

$$\left\|x_n - x\right\|_{L_1(\tau)} = \left\|(x_n - x)s(x)\right\|_{L_1(\tau)} \leq \left\|x_n - x\right\|_{\mathcal{A}} \left\|s(x)\right\|_{L_1(\tau)} \to 0.$$

It follows that

$$\left\|x_n - x\right\|_{L_1 \cap L_\infty(\tau)} = \max\left\{\left\|x_n - x\right\|_{L_1(\tau)}, \left\|x_n - x\right\|_{\mathcal{A}}\right\} \to 0.$$

Therefore $U(x_n) \overset{L_1 \cap L_\infty}{\to} U(x)$, since U is an isometry. Furthermore,

$$\left\|U(x) - U(x_n)\right\|_{\mathcal{B}} \leq \left\|U(x_n) - U(x)\right\|_{L_1 \cap L_\infty(\nu)} \to 0$$

and so $U(x_n) \overset{\mathcal{B}}{\to} U(x)$. Using the linearity of U and Φ we have that U and Φ agree on $\mathcal{F}(\tau)$.

Next we show that U is normal. If $x, y \in L_1 \cap L_\infty(\nu)^+$, then $\left\|x - y\right\|_{L_1(\nu)} \leq \left\|x + y\right\|_{L_1(\nu)}$, by Lemma 4.1.4. Furthermore, $-(x+y) \leq x - y \leq x + y$ and so $\left\|x - y\right\|_{\mathcal{B}} \leq \left\|x + y\right\|_{\mathcal{B}}$, by Proposition B.1.7. It follows that $\left\|x - y\right\|_{L_1 \cap L_\infty(\nu)} \leq \left\|x + y\right\|_{L_1 \cap L_\infty(\nu)}$ and hence U^{-1} is positive, by Lemma 4.1.5. It follows by Proposition 4.1.2 that U is normal. Note that if $\{x_\lambda\}_{\lambda \in \Lambda} \subseteq \mathcal{F}(\tau)^+$ is an increasing net and $x \in \mathcal{F}(\tau)^+$ is the supremum of $\{x_\lambda\}_{\lambda \in \Lambda}$ in $\mathcal{F}(\tau)$, then it is easily checked that x is the supremum of $\{x_\lambda\}_{\lambda \in \Lambda}$ in $L_1 \cap L_\infty(\tau)$ using the fact that $\mathcal{F}(\tau)$ is an absolutely solid subspace of $\mathcal{A}$. Since U is normal, it follows that $U(x)$ is the supremum of $\{U(x_\lambda)\}_\lambda$ in $L_1 \cap L_\infty(\nu)$ and therefore also in $\mathcal{B}$. Therefore, Φ is normal. By Proposition 3.3.6, we can extend Φ to a normal Jordan $*$-homomorphism from $\mathcal{A}$ into $\mathcal{B}$ such that $\left\|\Phi(x)\right\|_{\mathcal{B}} = \left\|x\right\|_{\mathcal{A}}$ for all $x \in \mathcal{A}^{sa}$. Furthermore, since U and Φ are normal and agree on $\mathcal{F}(\tau)^+$, it follows, by considering Proposition B.2.3, that U and Φ agree on $L_1 \cap L_\infty(\tau)^+$ (and hence on all of $L_1 \cap L_\infty(\tau)$). $\square$

LEMMA 8.4.5. *Φ is trace-preserving.*

PROOF. Suppose $p \in \mathcal{P}(\mathcal{A})^f$ with $\tau(p) > 1$. Then

$$\begin{aligned} \tau(p) &= \|p\|_{L_1(\tau)} \\ &= \|p\|_{L_1 \cap L_\infty(\tau)} \qquad \text{since } \|p\|_{\mathcal{A}} = 1 < \|p\|_{L_1(\tau)} \\ &= \|U(p)\|_{L_1 \cap L_\infty(\nu)} \qquad \text{since } U \text{ is an isometry} \\ &= \|\Phi(p)\|_{L_1 \cap L_\infty(\nu)} \qquad \text{since } \Phi \text{ and } U \text{ agree on } L_1 \cap L_\infty(\tau) \\ &= \|\Phi(p)\|_{L_1(\nu)} \qquad \text{since } \|\Phi(p)\|_{\mathcal{B}} = 1 < \|\Phi(p)\|_{L_1 \cap L_\infty(\nu)} \\ &= \nu(\Phi(p)) \qquad \text{since } \Phi(p) \text{ is positive} \end{aligned}$$

Suppose $p \in \mathcal{P}(\mathcal{A})^f$ with $\tau(p) \leq 1$. $(\mathcal{A}, \tau)$ is non-atomic with $\tau(\mathbf{1}) = \infty$ and so, by Lemma B.1.25, there exists $p_1 \in \mathcal{P}(\mathcal{A})$ with $p_1 \leq p^\perp$ and $1 < \tau(p_1) < \infty$. Using the linearity of Φ, ν and τ, and what has been shown already, we have that

$$\nu(\Phi(p)) = \nu(\Phi(p + p_1 - p_1)) = \nu(\Phi(p + p_1)) - \nu(\Phi(p_1)) = \tau(p + p_1) - \tau(p_1) = \tau(p)$$

It follows that

$$\nu(\Phi(x)) = \tau(x) \qquad \forall x \in \mathcal{G}_f, \tag{8.4.3}$$

using the linearity of Φ, ν and τ. Suppose $x \in \mathcal{F}(\tau)^+$. By Remark B.1.12, there exists $(x_n)_{n=1}^\infty \subseteq \mathcal{G}_f^+$ such that $x_n \xrightarrow{\mathcal{A}} x$ and $s(x_n) \leq s(x)$ for all $n \in \mathbb{N}^+$. By Proposition B.3.3, $x_n \xrightarrow{L_1(\tau)} x$ and so

$$\tau(x_n) = \|x_n\|_{L_1(\tau)} \to \|x\|_{L_1(\tau)} = \tau(x) \tag{8.4.4}$$

Furthermore, $\Phi(x_n) \xrightarrow{\mathcal{B}} \Phi(x)$, since Φ is isometric on self-adjoint elements. Note that $0 \leq x$ and therefore $xs(x) = x = s(x)x$. It follows, using Proposition 1.8.2(4), that $\Phi(x) = \Phi(xs(x)) = \Phi(x)\Phi(s(x))$ and thus $s((\Phi(x)) \leq \Phi(s(x))$. Furthermore, $0 \leq \Phi(x_n) \leq \Phi(x)$ for all n and therefore $s(\Phi(x_n)) \leq s(\Phi(x)) \leq \Phi(s(x))$ for all n. Since $\nu(\Phi(s(x))) = \tau(s(x)) < \infty$, we have, by Proposition B.3.3, that $\Phi(x_n) \xrightarrow{L_1(\nu)} \Phi(x)$ and therefore

$$\nu(\Phi(x_n)) = \|\Phi(x_n)\|_{L_1(\nu)} \to \|\Phi(x)\|_{L_1(\nu)} = \nu(\Phi(x)) \tag{8.4.5}$$

By considering (8.4.4) and (8.4.5) and using the fact that $\nu(\Phi(x_n)) = \tau(x_n)$ for each n (see (8.4.3)), we have that $\nu(\Phi(x)) = \tau(x)$. Next, suppose that $x \in \mathcal{A}^+$. By Proposition B.2.3, there exists $\{x_\lambda\}_{\lambda \in \Lambda} \subseteq \mathcal{F}(\tau)^+$ such that $x_\lambda \uparrow x$. Using the normality of τ, Φ and ν, we have that

$$\tau(x_\lambda) \uparrow \tau(x) \qquad \text{and} \qquad \nu(\Phi(x_\lambda)) \uparrow \nu(\Phi(x))$$

However, $\nu(\Phi(x_\lambda)) = \tau(x_\lambda)$ for each λ and so $\nu(\Phi(x)) = \tau(x)$. Finally, using the linearity of Φ, ν and τ, we can extend this result to all of $\mathcal{A}$. □

LEMMA 8.4.6. *Φ is surjective.*

PROOF. We start by showing that Φ is unital. $(\mathcal{B}, \nu)$ is semi-finite and so there exists an increasing net $\{q_\lambda\}_{\lambda \in \Lambda} \subseteq \mathcal{P}(\mathcal{B})^f$ such that $q_\lambda \uparrow \mathbf{1}$. Since $\mathcal{P}(\mathcal{B})^f \subseteq L_1 \cap L_\infty(\nu)$, $U^{-1}(q_\lambda)$ is defined for each λ. Furthermore, U^{-1} is a positive surjective isometry and so $\{U^{-1}(q_\lambda)\}_{\lambda \in \Lambda}$ is an increasing net of projections, by applying Lemma 8.4.3 to U^{-1}. It follows that $p_\lambda := U^{-1}(q_\lambda) \uparrow p$ for some $p \in \mathcal{P}(\mathcal{A})$. Furthermore, Φ is normal and so $\Phi(p_\lambda) \uparrow \Phi(p)$, but

$$\Phi(p_\lambda) = U(U^{-1}(q_\lambda)) = q_\lambda \uparrow \mathbf{1}$$

and so $\Phi(p) = \mathbf{1}$. Since $p \le \mathbf{1}$, we have that $\Phi(p) \le \Phi(\mathbf{1})$. By Proposition 1.8.2(5), Φ maps projections onto projections and so $\Phi(\mathbf{1}) \le \mathbf{1} = \Phi(p)$. It follows that $\Phi(\mathbf{1}) = \Phi(p) = \mathbf{1}$. Next, suppose $p \in \mathcal{P}(\mathcal{A})^f$. Then $\Phi(p) \in \mathcal{P}(\mathcal{B})^f$, since Φ is trace-preserving, by Lemma 8.4.5. It follows that

$$\Phi(p)\mathcal{B}\Phi(p) \subseteq \mathcal{F}(\nu) \subseteq L_1 \cap L_\infty(\nu) = U(L_1 \cap L_\infty(\tau)) \subseteq \Phi(\mathcal{A}).$$

Φ is therefore surjective, by Proposition 3.4.2. □

We have therefore shown the following result.

THEOREM 8.4.7. *Suppose $(\mathcal{A}, \tau)$ and $(\mathcal{B}, \nu)$ are non-atomic semi-finite von Neumann algebras with $\tau(\mathbf{1}) = \infty = \nu(\mathbf{1})$. If $U : L_1 \cap L_\infty(\tau) \to L_1 \cap L_\infty(\nu)$ is a positive surjective isometry, then U is the restriction to $L_1 \cap L_\infty(\tau)$ of a trace-preserving Jordan $*$-isomorphism Φ from $\mathcal{A}$ onto $\mathcal{B}$. Conversely, if Φ is a trace-preserving Jordan $*$-isomorphism from $\mathcal{A}$ onto $\mathcal{B}$, then Φ is positive and maps $L_1 \cap L_\infty(\tau)$ isometrically onto $L_1 \cap L_\infty(\nu)$.*

REMARK 8.4.8. We show that the conclusion of Theorem 8.4.7 holds even if the von Neumann algebras are not necessarily non-atomic. Suppose $(\mathcal{A}, \tau)$ and $(\mathcal{B}, \nu)$ are (general) semi-finite von Neumann algebras and suppose $U : L_1 \cap L_\infty(\tau) \to L_1 \cap L_\infty(\nu)$ is a positive surjective isometry. Let π_1 and π_2 denote the trace-preserving $*$-isomorphisms from $(\mathcal{A}, \tau)$ onto $(\mathbb{C}\mathbf{1} \otimes \mathcal{A}, \tau_\otimes)$ and $(\mathcal{B}, \nu)$ onto $(\mathbb{C}\mathbf{1} \otimes \mathcal{B}, \nu_\otimes)$, respectively (see Remark 1.6.2). Recall that π_1 and π_2 extend uniquely to $*$-isomorphisms $\tilde{\pi}_1$ and $\tilde{\pi}_2$ from $S(\mathcal{A}, \tau)$ onto $S(\mathbb{C}\mathbf{1} \otimes \mathcal{A}, \tau_\otimes)$ and from $S(\mathcal{B}, \nu)$ onto $S(\mathbb{C}\mathbf{1} \otimes \mathcal{B}, \nu_\otimes)$, respectively. Since the restrictions of $\tilde{\pi}_1$ and $\tilde{\pi}_2$ to $L_1 \cap L_\infty(\tau)$ and $L_1 \cap L_\infty(\nu)$, respectively are $*$-isometric isomorphisms from $L_1 \cap L_\infty(\tau)$ onto $L_1 \cap L_\infty(\tau_\otimes)$ and $L_1 \cap L_\infty(\nu)$ onto $L_1 \cap L_\infty(\nu_\otimes)$, the map

$$\widetilde{U}(y) := \tilde{\pi}_2 \circ U \circ \tilde{\pi}_1^{-1}(y) \qquad y \in L_1 \cap L_\infty(\tau_\otimes)$$

is a positive surjective isometry from $L_1 \cap L_\infty(\tau_\otimes)$ onto $L_1 \cap L_\infty(\nu_\otimes)$. $(\mathbb{C}\mathbf{1} \otimes \mathcal{A}, \tau_\otimes)$ and $(\mathbb{C}\mathbf{1} \otimes \mathcal{B}, \nu_\otimes)$ are non-atomic semi-finite von Neumann algebras and so by Theorem 8.4.7, there exists a trace-preserving Jordan $*$-isomorphism $\tilde{\Phi}$ from $(\mathbb{C}\mathbf{1} \otimes \mathcal{A}, \tau_\otimes)$ onto $(\mathbb{C}\mathbf{1} \otimes \mathcal{B}, \nu_\otimes)$, which agrees with $\widetilde{U}$ on $L_1 \cap L_\infty(\tau_\otimes)$. Letting

$$\Phi = \pi_2^{-1} \circ \tilde{\Phi} \circ \pi_1$$

yields a trace-preserving Jordan $*$-isomorphism from $\mathcal{A}$ onto $\mathcal{B}$, which agrees with U on $L_1 \cap L_\infty(\tau)$. This follows, since π_1, π_2^{-1} are trace-preserving $*$-isomorphisms (and hence Jordan $*$-isomorphisms) and the composition of trace-preserving Jordan $*$-isomorphisms yields a trace-preserving Jordan $*$-isomorphism.

APPENDIX A

Alternative proof of the characterization of the extreme points in Lorentz spaces

In Chapter 7 we used a characterization of the extreme points of the unit balls of certain Lorentz spaces to describe the structure of isometries between such Lorentz spaces. The characterization of the extreme points in the non-commutative setting follows from the corresponding result in the commutative setting using Theorem 7.1.2. As mentioned earlier, the author was initially unaware of Theorem 7.1.2 and therefore proved the characterization of the extreme points by developing non-commutative analogues of some of the techniques employed in the proof of [2, Proposition 2.2] (see Proposition 2.1.5) and adapting aspects of the proof of [4, Theorem 4.1] (see Theorem 2.2.9) to the semi-finite setting. The end result is achieved a great deal more efficiently by the proof given for Theorem 7.1.3, but it is interesting to consider a more direct proof. The details will be provided in this chapter. An important ingredient is the relationship between additivity of the norm and additivity of the generalized singular value as given in the following non-commutative analogue of [2, Lemma 2.1].

LEMMA A.1.1. *Suppose $(\mathcal{A},\tau)$ is a semi-finite von Neumann algebra and $w:(0,\infty)\to(0,\infty)$ is a continuous strictly decreasing weight function. If $x,y\in L_{w,1}(\tau)$, then*

$$\left\|x+y\right\|_{L_{w,1}}=\left\|x\right\|_{L_{w,1}}+\left\|y\right\|_{L_{w,1}} \iff \mu_{x+y}=\mu_x+\mu_y.$$

PROOF. The structure of the proof is exactly the same as the structure of the proof of [2, Lemma 2.1], with the exception that generalized singular values replace decreasing rearrangements. Many of the details have been omitted in [2] and have therefore been included here to demonstrate that there are no hidden difficulties relating to the replacement of decreasing rearrangements with generalized singular values. The author could however only provide these details under the added assumption that the weight function w is continuous.

If $\mu_{x+y}=\mu_x+\mu_y$, then

$$\begin{aligned}
\left\|x+y\right\|_{L_{w,1}} &= \int_0^\infty \mu_{x+y}(t)w(t)dt \\
&= \int_0^\infty (\mu_x(t)+\mu_y(t))w(t)dt \\
&= \int_0^\infty \mu_x(t)w(t)dt+\int_0^\infty \mu_y(t)w(t)dt \\
&= \left\|x\right\|_{L_{w,1}}+\left\|y\right\|_{L_{w,1}}.
\end{aligned}$$

For the converse, suppose $x,y\in L_{w,1}(\tau)$ are such that $\left\|x+y\right\|_{L_{w,1}}=\left\|x\right\|_{L_{w,1}}+\left\|y\right\|_{L_{w,1}}$. Let $h(t)=\mu_x(t)+\mu_y(t)-\mu_{x+y}(t)$ and let $H(t)=\int_0^t h(s)ds$. We show that $H=0$. Note that $h=H'$ a.e., by Theorem 1.1.1 and

$$\begin{aligned}
\int_0^\infty h(s)w(s)ds &= \int_0^\infty \mu_x(s)w(s)ds+\int_0^\infty \mu_y(s)w(s)ds-\int_0^\infty \mu_{x+y}(s)w(s)ds \\
&= \left\|x\right\|_{L_{w,1}}+\left\|y\right\|_{L_{w,1}}-\left\|x+y\right\|_{L_{w,1}} \\
&= 0 \qquad \text{(A.1.1)}
\end{aligned}$$

Furthermore, $|h(s)w(s)| \leq \mu_x(s)w(s) + \mu_y(s)w(s) + \mu_{x+y}(s)w(s)$ for all $s > 0$ and so

$$\begin{aligned}\int_0^\infty |h(s)w(s)|ds &\leq \int_0^\infty \mu_x(s)w(s)ds + \int_0^\infty \mu_y(s)w(s)ds + \int_0^\infty \mu_{x+y}(s)w(s)ds \\ &= \|x\|_{L_{w,1}} + \|y\|_{L_{w,1}} + \|x+y\|_{L_{w,1}} \\ &< \infty,\end{aligned}$$

i.e., $h(\cdot)w(\cdot)$ is integrable and therefore, by Theorem 1.1.2,

$$\lim_{t\to 0}\int_0^t |h(s)w(s)|ds = 0 \tag{A.1.2}$$

Furthermore, by Theorem 1.4.3, $\mu_{x+y} \ll \mu_x + \mu_y$, i.e.

$$\int_0^t \mu_{x+y}(s)ds \leq \int_0^t \mu_x(s)ds + \int_0^t \mu_y(s)ds \qquad \forall t > 0$$

Therefore $H(t) = \int_0^t h(s)ds \geq 0$ for all $t > 0$. It follows that

$$\begin{aligned}0 &\leq \int_0^t h(s)ds[w(t)] \qquad \text{since } w \geq 0 \\ &\leq \int_0^t |h(s)|ds[w(t)] \\ &\leq \int_0^t |h(s)|w(s)ds \qquad \text{since } w \text{ is decreasing} \\ &\to 0 \qquad \text{as } t \to 0, \text{ by (A.1.2)}\end{aligned}$$

We show that

$$\int_0^t h(s)ds[w(t)] \leq \int_0^t h(s)w(s)ds \qquad \forall t > 0 \tag{A.1.3}$$

Let $\alpha > 0$. Since $\int_0^t h(s)ds[w(t)] \to 0$ and $\int_0^t h(s)w(s)ds \to 0$ as $t \to 0$, there exists a $\gamma_\alpha > 0$ such that $t \leq \gamma_\alpha$ implies that $|\int_0^t h(s)ds[w(t)] - \int_0^t h(s)w(s)ds| < \alpha$ and therefore,

$$\int_0^t h(s)ds[w(t)] - \alpha < \int_0^t h(s)w(s)ds \tag{A.1.4}$$

For $n \in \mathbb{N}^+$, let

$$g_{\alpha,n}(t) := \begin{cases}\int_0^t h(s)w(s)ds & \text{if } 0 \leq t \leq \gamma_\alpha \\ \int_0^{\gamma_\alpha} h(s)w(s)ds + \sum\limits_{k=1}^n \int_{(k-1)\delta_{t,\alpha,n}+\gamma_\alpha}^{k\delta_{t,\alpha,n}+\gamma_\alpha} h(s)ds[w(k\delta_{t,\alpha,n}+\gamma_\alpha)] & \text{if } t > \gamma_\alpha,\end{cases}$$

where $\delta_{t,\alpha,n} = \frac{t-\gamma_\alpha}{n}$. Let $t_{n,k} = k\delta_{t,\alpha,n} + \gamma_\alpha$.

We start by showing that $\int_0^t h(s)w(s)ds \geq \int_0^t h(s)ds[w(t)] - \alpha$ for all $t > 0$. To do this we will show that $g_{\alpha,n}(t) \geq \int_0^t h(s)ds[w(t)] - \alpha$ for all $t > 0$ and $g_{\alpha,n}(t) \to \int_0^t h(s)w(s)ds$ pointwise as $n \to \infty$ for all $t > 0$. Fix $n \in \mathbb{N}^+$. If $0 < t \leq \gamma_\alpha$, then

$$\begin{aligned}g_{\alpha,n}(t) &= \int_0^t h(s)w(s)ds \\ &\geq \int_0^t h(s)ds[w(t)] - \alpha \qquad \text{by (A.1.4)}\end{aligned}$$

If $t > \gamma_\alpha$, we show that $g_{\alpha,n}(t) \geq \int_0^t h(s)ds[w(t)] - \alpha$ by showing that

$$\begin{aligned}
&\int_0^{\gamma_\alpha} h(s)w(s)ds + \sum_{k=1}^{p} \int_{t_{n,k-1}}^{t_{n,k}} h(s)ds[w(t_{n,k})] \\
&\qquad \geq \int_0^{t_{n,p}} h(s)ds[w(t_{n,p})] - \alpha
\end{aligned} \tag{A.1.5}$$

holds for all $1 \leq p \leq n$. Note that (A.1.5) holds for $p = 1$, since

$$\begin{aligned}
& \int_0^{\gamma_\alpha} h(s)w(s)ds + \int_{\gamma_\alpha}^{t_{n,1}} h(s)ds[w(t_{n,1})] \\
\geq\ & \int_0^{\gamma_\alpha} h(s)ds[w(\gamma_\alpha)] - \alpha + \int_{\gamma_\alpha}^{t_{n,1}} h(s)ds[w(t_{n,1})] \qquad \text{by (A.1.4)} \\
\geq\ & \int_0^{\gamma_\alpha} h(s)ds[w(t_{n,1})] - \alpha + \int_{\gamma_\alpha}^{t_{n,1}} h(s)ds[w(t_{n,1})] \\
& \text{since } w \text{ is decreasing and } \int_0^{\gamma_\alpha} h(s)ds \geq 0 \\
=\ & \int_0^{t_{n,1}} h(s)ds[w(t_{n,1})] - \alpha
\end{aligned}$$

Suppose that (A.1.5) holds for some $p = m$, $1 \leq m \leq n-1$. We show that it must then also hold for $p = m+1$. We need to consider two possibilities. If $\int_{t_{n,m}}^{t_{n,m+1}} h(s)ds \geq 0$, then

$$\begin{aligned}
& \int_0^{\gamma_\alpha} h(s)w(s)ds + \sum_{k=1}^{m+1} \int_{t_{n,k-1}}^{t_{n,k}} h(s)ds[w(t_{n,k})] \\
=\ & \int_0^{\gamma_\alpha} h(s)w(s)ds + \sum_{k=1}^{m} \int_{t_{n,k-1}}^{t_{n,k}} h(s)ds[w(t_{n,k})] + \int_{t_{n,m}}^{t_{n,m+1}} h(s)ds[w(t_{n,m+1})] \\
\geq\ & \int_0^{t_{n,m}} h(s)ds[w(t_{n,m})] - \alpha + \int_{t_{n,m}}^{t_{n,m+1}} h(s)ds[w(t_{n,m+1})] \\
& \text{since we have assumed that (A.1.5) holds for } p = m \\
\geq\ & \int_0^{t_{n,m}} h(s)ds[w(t_{n,m+1})] - \alpha + \int_{t_{n,m}}^{t_{n,m+1}} h(s)ds[w(t_{n,m+1})] \\
& \text{since } w \text{ is decreasing and } \int_0^{t_{n,m}} h(s)ds \geq 0 \\
=\ & \int_0^{t_{n,m+1}} h(s)ds[w(t_{n,m+1})] - \alpha
\end{aligned}$$

If $\int_{t_{n,m}}^{t_{n,m+1}} h(s)ds < 0$, then

$$\int_{t_{n,m}}^{t_{n,m+1}} h(s)ds[w(t_{n,m+1})] \geq \int_{t_{n,m}}^{t_{n,m+1}} h(s)ds[w(t_{n,m})].$$

Therefore,

$$
\begin{aligned}
& \int_0^{\gamma_\alpha} h(s)w(s)ds + \sum_{k=1}^{m+1} \int_{t_{n,k-1}}^{t_{n,k}} h(s)ds[w(t_{n,k})] \\
\geq \quad & \int_0^{t_{n,m}} h(s)ds[w(t_{n,m})] - \alpha + \int_{t_{n,m}}^{t_{n,m+1}} h(s)ds[w(t_{n,m})] \\
= \quad & \int_0^{t_{n,m+1}} h(s)ds[w(t_{n,m})] - \alpha \\
\geq \quad & \int_0^{t_{n,m+1}} h(s)ds[w(t_{n,m+1})] - \alpha,
\end{aligned}
$$

since w is decreasing and $\int_0^{t_{n,m+1}} h(s)ds \geq 0$. It follows that if (A.1.5) holds for $p = m$, $1 \leq m \leq n-1$, then (A.1.5) holds for $p = m+1$, but (*A*.1.5) holds for $p = 1$ and hence for all $1 \leq p \leq n$. Applying $p = n$, we obtain $g_{\alpha,n}(t) \geq \int_0^t h(s)[w(t)] - \alpha$.

Next, we show that $g_{\alpha,n}(t) \to \int_0^t h(s)w(s)ds$. For $t \leq \gamma_\alpha$ this is trivial, since $g_{\alpha,n}(t) = \int_0^t h(s)w(s)ds$ for all such t. Fix $t > \gamma_\alpha$ and let $\epsilon > 0$ be given. Note that

$$
\begin{aligned}
\int_0^t |h(s)|ds \quad &= \quad \int_0^t |\mu_x(s) + \mu_y(s) - \mu_{x+y}(s)|ds \\
&\leq \quad \int_0^t \mu_x(s)ds + \int_0^t \mu_y(s)ds + \int_0^t \mu_{x+y}(s)ds \\
&< \quad \infty \qquad \text{by Remark B.3.2}
\end{aligned}
\tag{A.1.6}
$$

Since w is continuous on $[\gamma_\alpha, t]$ and hence uniformly continuous, there exists a $\delta > 0$ such that

$$
|r - s| < \delta \quad \implies \quad |w(r) - w(s)| < \frac{\epsilon}{\int_0^t |h(s)|ds}. \tag{A.1.7}
$$

Choose $n_\epsilon \in \mathbb{N}^+$ such that $\frac{t-\gamma_\alpha}{n_\epsilon} < \delta$. Then for $n \geq n_\epsilon$,

$$
\begin{aligned}
|\int_0^t h(s)w(s)ds - g_{\alpha,n}(t)| \quad &= \quad |\int_{\gamma_\alpha}^t h(s)w(s)ds - \sum_{k=1}^{n} \int_{t_{n,k-1}}^{t_{n,k}} h(s)ds[w(t_{n,k})]| \\
& \qquad \text{by applying the definition of } g_{\alpha,n}(t) \\
&\leq \quad \sum_{k=1}^{n} \int_{t_{n,k-1}}^{t_{n,k}} |h(s)||w(s) - w(t_{n,k})|ds \\
&< \quad \int_{\gamma_\alpha}^t |h(s))|ds \frac{\epsilon}{\int_0^t |h(s)|ds} \qquad \text{by (A.1.7)} \\
&\leq \quad \epsilon
\end{aligned}
$$

Since $\int_0^t h(s)ds[w(t)] - \alpha \leq g_{\alpha,n}(t)$ for each $n \in \mathbb{N}^+$ and $g_{\alpha,n}(t) \to \int_0^t h(s)w(s)ds$, we have that

$$
\int_0^t h(s)ds[w(t)] - \alpha \quad \leq \quad \int_0^t h(s)w(s)ds \tag{A.1.8}
$$

Since $\alpha > 0$ was arbitrary, we have that

$$
\int_0^t h(s)ds[w(t)] \quad \leq \quad \int_0^t h(s)w(s)ds \qquad \text{for all } t > 0 \tag{A.1.9}
$$

It follows that

$$\begin{aligned} 0 &\leq H(t)w(t) \qquad \text{since } w, H \geq 0 \\ &= \int_0^t h(s)ds[w(t)] \\ &\leq \int_0^t h(s)w(s)ds \qquad \text{by (A.1.9)} \end{aligned} \tag{A.1.10}$$

Since we know that $\int_0^t h(s)w(s)ds \to 0$ if $t \to \infty$ (see (A.1.1)) or $t \to 0$ (see (A.1.2)) and (A.1.10) holds for all $t > 0$, we have that

$$\lim_{t\to\infty} w(t)H(t) = 0 \qquad \text{and} \tag{A.1.11}$$

$$\lim_{t\to 0} w(t)H(t) = 0 \tag{A.1.12}$$

We proceed to show that $H = 0$. Consider the following:

$$\begin{aligned} \int_0^\infty H(t)d(-w(t)) &= -\int_0^\infty H(t)w'(t)dt \qquad \text{since } -w \text{ is strictly increasing} \\ &= [-H(t)w(t)]_0^\infty + \int_0^\infty H'(t)w(t)dt \qquad \text{using integration by parts} \\ &= -\lim_{r\to\infty} H(r)w(r) + \lim_{r\to 0} H(r)w(r) + \int_0^\infty h(t)w(t)dt \qquad \text{since } H' = h \text{ a.e.} \\ &= 0 \qquad \text{by (A.1.11), (A.1.12) and (A.1.1)} \end{aligned} \tag{A.1.13}$$

Assume that $H \neq 0$. By Proposition 1.1.4, this implies that there exists $A \subseteq [0, \infty)$ and $\epsilon > 0$ such that $m(A) > 0$ and $H(t) \geq \epsilon$ for all $t \in A$ (since we know that $H \geq 0$). Note that since w is strictly decreasing, $w' \leq 0$. It follows that if $B \subseteq [0, \infty)$, then

$$\begin{aligned} \int_B H(t)d(-w(t)) &= -\int_B H(t)w'(t)dt \\ &\geq 0 \qquad \text{since } H \geq 0,\ w' \leq 0 \end{aligned} \tag{A.1.14}$$

and so

$$\begin{aligned} \int_0^\infty H(t)d(-w(t)) &= \int_A H(t)d(-w(t)) + \int_{A^c} H(t)d(-w(t)) \\ &\geq \int_A H(t)d(-w(t)) \qquad \text{by (A.1.14)} \\ &\geq \epsilon \int_A 1d(-w(t)) \\ &> 0 \qquad \text{since } -w \text{ is strictly increasing and } m(A) > 0 \end{aligned}$$

This contradicts (A.1.13) and so $H = 0$. Therefore $h = 0$ a.e., since $h = H'$ a.e. □

We are now in a position to prove the characterization of the extreme points more directly.

THEOREM A.1.2. *Suppose $(\mathcal{A}, \tau)$ is a semi-finite von Neumann algebra, w is a continuous strictly decreasing weight function and $E = L_{w,1}(\tau)$. Then x is an extreme point of B_E if and only if $x = \frac{v}{\psi(\tau(|v|))}$ for some $v \in \mathcal{V}(\mathcal{A})^f$.*

PROOF. Suppose $x = \frac{v}{\psi(\tau(|v|))}$ for some $v \in \mathcal{V}(\mathcal{A})^f$. Then $|x| = \frac{p}{\psi(\tau(p))}$ for some $p \in \mathcal{P}(\mathcal{A})^f$. To show that $|x|$ is an extreme point we will use a non-commutative analogue of the corresponding argument in the proof of

[2, Proposition 2.2]. Suppose $|x| = \frac{1}{2}(a+b)$ for some $a, b \in B_E$. Let $y = \frac{a}{2}$ and $z = \frac{b}{2}$. Then $|x| = y+z$, $y, z \neq 0$ and $\|y\|_{L_{w,1}} + \|z\|_{L_{w,1}} = 1$, since it is easily checked that $\|a\|_{L_{w,1}} = 1 = \|b\|_{L_{w,1}}$. It follows that

$$\|y+z\|_{L_{w,1}} = \|x\|_{L_{w,1}} = 1 = \|y\|_{L_{w,1}} + \|z\|_{L_{w,1}}.$$

Therefore, $\mu_{y+z} = \mu_y + \mu_z$, by Lemma A.1.1. By Proposition 1.4.5, $\mu_x = \alpha\chi_{[0,\tau(p))}$, where $\alpha = 1/\psi(\tau(p))$. We start by showing that μ_y and μ_z are scalar multiples of $\chi_{[0,\tau(p))}$. Note that $y, z \neq 0$, $\mu_y, \mu_z \geq 0$ and μ_y, μ_z are decreasing. Therefore $\mu_y(0), \mu_z(0) > 0$. Furthermore, $\mu_x(0) = \mu_y(0) + \mu_z(0)$ and so $0 < \mu_y(0), \mu_z(0) < \alpha$. Let $\mu_y(0) = \beta$. Assume that $\mu_y(t) \neq \beta$ for some $t \in (0, \tau(p))$. Then $\mu_y(t) < \beta$, since μ_y is decreasing, and so

$$\mu_z(t) = \mu_x(t) - \mu_y(t) > \alpha - \beta = \mu_z(0)$$

This contradicts the fact that μ_z is decreasing and so $\mu_y(t) = \beta$ for all $t \in [0, \tau(p))$. Furthermore, observe that if $t \geq \tau(p)$, then $\mu_y(t) = \mu_z(t) = 0$, since $0 \leq \mu_y(t) + \mu_z(t) = \mu_x(t) = 0$. It follows that $\mu_y = \beta\chi_{[0,\tau(p))}$ and therefore $\mu_z = (\alpha-\beta)\chi_{[0,\tau(p))}$, i.e. μ_y and μ_z are scalar multiples of $\chi_{[0,\tau(p))}$. Furthermore, by Proposition 1.4.5, $|y| = \beta q$ and $|z| = (\alpha-\beta)r$ for some $q, r \in \mathcal{P}(\mathcal{A})$ with $\tau(q) = \tau(p) = \tau(r)$.

Next, we show that $q = p = r$. Let $\eta \in p(H)$ and assume that $\eta \notin q(H)$. Then

$$\begin{aligned}
\alpha\|\eta\|_H &= \||x|(\eta)\|_H && \text{since } |x| = \alpha p \\
&= \|y(\eta) + z(\eta)\|_H \\
&\leq \|y(\eta)\|_H + \|z(\eta)\|_H \\
&= \||y|(\eta)\|_H + \||z|(\eta)\|_H && \text{by Proposition B.1.22(2)} \\
&= \beta\|q(\eta)\|_H + (\alpha-\beta)\|r(\eta)\|_H \\
&\leq \beta\|q(\eta)\|_H + (\alpha-\beta)\|\eta\|_H && \text{since } r \text{ is a projection} \\
&< \beta\|\eta\|_H + (\alpha-\beta)\|\eta\|_H && \text{by Proposition B.1.22(3)} \\
&= \alpha\|\eta\|_H
\end{aligned}$$

This is a contradiction and so $\eta \in q(H)$. Since $\eta \in p(H)$ was arbitrary, this implies that $p \leq q$ and hence $\tau(q-p) = \tau(q) - \tau(p) = 0$, since $\tau(q) = \tau(p)$. Therefore, $q - p = 0$, since τ is faithful. We can similarly show that $p = r$.

Let $v = \frac{y}{\beta}$ and $w = \frac{z}{\alpha-\beta}$. Then $v^*v = \frac{y^*y}{\beta^2} = \frac{|y|^2}{\beta^2} = q = p$ and $w^*w = r = p$ and so v and w are partial isometries, by Proposition B.1.28. We show that $v = w$. For $\eta \in p(H)$,

$$\begin{aligned}
\alpha\|\eta\|_H &= \||x|(\eta)\|_H && \text{since } \eta = p(\eta) \text{ and } \alpha p = |x| \\
&= \|\beta v(\eta) + (\alpha-\beta)w(\eta)\|_H && \text{since } |x| = y + z = \beta v + (\alpha-\beta)w \\
&\leq \|\beta v(\eta)\|_H + \|(\alpha-\beta)w(\eta)\|_H \\
&\leq \beta\|\eta\|_H + (\alpha-\beta)\|\eta\|_H && \text{since } v \text{ and } w \text{ are partial isometries} \\
&= \alpha\|\eta\|_H.
\end{aligned} \tag{A.1.15}$$

Therefore $\|\beta v(\eta) + (\alpha-\beta)w(\eta)\|_H = \|\beta v(\eta)\|_H + \|(\alpha-\beta)w(\eta)\|_H$ and so

$$\beta v(\eta) = \gamma(\alpha-\beta)w(\eta), \tag{A.1.16}$$

for some $\gamma \in \mathbb{C}$, by Proposition B.1.22(1). It follows that

$$\begin{aligned}
\alpha\|\eta\|_H &= \|\beta v(\eta) + (\alpha-\beta)w(\eta)\|_H && \text{by (A.1.15)}\\
&= \left\|\Big(\gamma(\alpha-\beta) + (\alpha-\beta)\Big)w(\eta)\right\|_H && \text{by (A.1.16)}\\
&= |\gamma(\alpha-\beta) + (\alpha-\beta)|\|\eta\|_H && \text{since } \eta \in p(H) = \ker(w)^\perp
\end{aligned}$$

Therefore,

$$\alpha = |\gamma(\alpha-\beta) + (\alpha-\beta)|. \tag{A.1.17}$$

Note that v and w are partial isometries, $\eta \in p(H) = \ker(w)^\perp = \ker(v)^\perp$ and $w(\eta) = \frac{\beta}{\gamma(\alpha-\beta)}v(\eta)$; therefore, $|\frac{\beta}{\gamma(\alpha-\beta)}| = 1$, i.e. $|\gamma| = \frac{\beta}{\alpha-\beta}$. It is easily checked that $0 < \mathrm{Re}\gamma = |\gamma|$ and so $\gamma = \frac{\beta}{\alpha-\beta}$. It follows that for $\eta \in p(H)$,

$$\beta v(\eta) = \Big(\frac{\beta}{\alpha-\beta}\Big)(\alpha-\beta)w(\eta) = \beta w(\eta),$$

using (A.1.16). Since this holds for every $\eta \in p(H) = \ker(w)^\perp = \ker(v)^\perp$ and $\beta \neq 0$, we have that $v = w$.

Since $v = \frac{y}{\beta}$ and $w = \frac{z}{\alpha-\beta}$, we have $y = \frac{\beta}{\alpha-\beta}z$ and therefore

$$|x| = y + z = \frac{\alpha}{\alpha-\beta}z. \tag{A.1.18}$$

We show that $|x| = a = b$. Note that

$$\frac{1}{2} = \|a/2\|_{L_{w,1}} = \|y\|_{L_{w,1}} = \||y|\|_{L_{w,1}} = \|\beta p\|_{L_{w,1}} = \beta\psi(\tau(p)).$$

Therefore $\beta = \frac{1}{2}(\frac{1}{\psi(\tau(p))}) = \frac{\alpha}{2}$ and hence, by (A.1.18),

$$|x| = \frac{\alpha}{\alpha-\beta}z = \frac{\alpha}{\alpha-\alpha/2}\Big(\frac{b}{2}\Big) = b.$$

It follows that $|x| = a = b$ and hence $|x|$ is an extreme point. By Lemma 7.1.1, x is also an extreme point, since $s(x) = |v|$ and $\tau(|v|) < \infty$.

Conversely, suppose $x \in B_E$ is such that there is no $p \in \mathcal{P}(\mathcal{A})^f$ such that $|x| = \frac{p}{\psi(\tau(p))}$. To show that x is not an extreme point, we will adapt the corresponding part of the proof of [4, Theorem 4.1] to the semi-finite setting. Let $\{e(\lambda)\}_{\lambda\in\mathbb{R}^+}$ be the spectral projection family of $|x|$, i.e. $e(\lambda) = e^{|x|}([0,\lambda])$. We show that we can choose $\lambda > 0$ such that

$$\|2(|x|e(\lambda) + \lambda e^\perp(\lambda))\|_{L_{w,1}} = 1. \tag{A.1.19}$$

Let $g_\lambda(t) := t.\chi_{[0,\lambda]} + \lambda.\chi_{(\lambda,\infty)}$ and $f(t) := t$. Then $g_\lambda(|x|) = |x|e(\lambda) + \lambda e^\perp(\lambda)$. Note that $0 \le g_n(t) \uparrow f(t)$, for all $t \ge 0$. Therefore $g_n(|x|) \uparrow f(|x|) = |x|$, by Proposition B.2.5(4). It follows that, as $n \to \infty$,

$$\|g_n(|x|)\|_{L_{w,1}} \quad \uparrow \quad \||x|\|_{L_{w,1}} = 1, \tag{A.1.20}$$

by Remark 1.6.5. Furthermore, $g_{1/n}(t) \to 0$ for all $t \ge 0$ and so $g_{1/n}(|x|) \overset{SOT}{\to} 0$ by Proposition B.2.5(3). It follows that $g_{1/n}(|x|) \downarrow 0$ as $n \to \infty$. $L_{w,1}(\tau)$ has order continuous norm and so

$$\|g_{1/n}(|x|)\|_{L_{w,1}} \quad \downarrow \quad 0 \tag{A.1.21}$$

We show that $\rho : \lambda \mapsto \left\| g_\lambda(|x|) \right\|_{L_{w,1}}$ is continuous. Suppose $\gamma > 0$ and $(\lambda_n)_{n=1}^\infty$ is such that $\lambda_n \uparrow \gamma$. Then

$$\begin{aligned} & 0 \le g_{\lambda_n(t)} && \uparrow && g_\gamma(t) && \text{for all } t \in \sigma(|x|) \\ \implies & 0 \le g_{\lambda_n}(|x|) && \uparrow && g_\gamma(|x|) && \text{by Proposition B.2.5(4)} \\ \implies & \rho(\lambda_n) = \left\| g_{\lambda_n}(|x|) \right\|_{L_{w,1}} && \uparrow && \left\| g_\gamma(|x|) \right\|_{L_{w,1}} = \rho(\gamma) && \text{by Remark 1.6.5} \end{aligned}$$

It follows that ρ is continuous from the left. If $(\lambda_n)_{n=1}^\infty$ is such that $\lambda_n \downarrow \gamma$, then $g_{\lambda_n}(t) \to g_\gamma(t)$ for all $t \ge 0$ and so $g_{\lambda_n}(|x|) \overset{SOT}{\to} g_\gamma(|x|)$, by Proposition B.2.5(3). Since $\{g_{\lambda_n}(|x|)\}$ is also decreasing, it follows that $g_{\lambda_n}(|x|) - g_\gamma(|x|) \downarrow 0$. $L_{w,1}(\tau)$ has order continuous norm and so $\left\| g_{\lambda_n}(|x|) - g_\gamma(|x|) \right\|_{L_{w,1}} \downarrow 0$. Therefore

$$\rho(\lambda_n) = \left\| g_{\lambda_n}(|x|) \right\|_{L_{w,1}} \downarrow \left\| g_\gamma(|x|) \right\|_{L_{w,1}} = \rho(\gamma)$$

and hence ρ is continuous from the right. Using (A.1.20) we can find a $\lambda_1 > 0$ such that $\rho(\lambda_1) > \frac{1}{2}$; using (A.1.21) we can find a $\lambda_2 > 0$ such that $p(\lambda_2) < \frac{1}{2}$. Since ρ is continuous, there exists a $\lambda > 0$ such that $\left\| |x|e(\lambda) + \lambda e^\perp(\lambda) \right\|_{L_{w,1}} = \rho(\lambda) = \frac{1}{2}$. Using this λ we obtain $\left\| 2(|x|e(\lambda) + \lambda e^\perp(\lambda)) \right\| = 1$, as desired.

Fix λ as above and let $x_1 = 2(|x|e(\lambda) + \lambda e^\perp(\lambda))$ and $x_2 = 2(|x| - \frac{1}{2}x_1)$. Then $|x| = \frac{1}{2}(x_1 + x_2)$ and

$$x_2 = 2(|x| - (|x|e(\lambda) + \lambda e^\perp(\lambda))) = 2(|x|e^\perp(\lambda) - \lambda e^\perp(\lambda))$$

We show that $\left\| x_2 \right\|_{L_{w,1}} = 1$. Let $f(t) = (t - \lambda).\chi_{(\lambda,\infty)}(t)$ and $g(t) = t.\chi_{[0,\lambda]}(t) + \lambda.\chi_{(\lambda,\infty)}(t)$. Then $x_1 = 2g(|x|)$ and $x_2 = 2f(|x|)$. Furthermore, by Proposition 1.4.4, $\mu_{f(|x|)} = f(\mu_{|x|}) = (\mu_{|x|} - \lambda).\chi_{\mu_{|x|}^{-1}(\lambda,\infty)}$. Similarly, $\mu_{g(|x|)} = \mu_{|x|}.\chi_{\mu_{|x|}^{-1}[0,\lambda]} + \lambda.\chi_{\mu_{|x|}^{-1}(\lambda,\infty)}$. It follows that

$$\mu_{f(|x|)} + \mu_{g(|x|)} = f(\mu_{|x|}) + g(\mu_{|x|}) = \mu_{|x|} = \mu_{g(|x|)+f(|x|)}$$

Therefore $\left\| |x| \right\|_{L_{w,1}} = \left\| f(|x|) \right\|_{L_{w,1}} + \left\| g(|x|) \right\|_{L_{w,1}}$, by Lemma A.1.1. However, $\left\| g(|x|) \right\|_{L_{w,1}} = \frac{1}{2}\left\| x_1 \right\|_{L_{w,1}} = \frac{1}{2}$, using (A.1.19), and $\left\| x \right\|_{L_{w,1}} = 1$. It follows that $\left\| f(|x|) \right\|_{L_{w,1}} = \frac{1}{2}$ and hence $\left\| x_2 \right\|_{L_{w,1}} = 1$.

Next, we show that $x_1 \neq x_2$, using a proof by contradiction. Assume that $x_1 = x_2$. Then $|x| = x_1$, since $|x| = \frac{1}{2}(x_1 + x_2)$. It follows that

$$|x|e(\lambda) = x_1 e(\lambda) = 2(|x|e(\lambda) + \lambda e^\perp(\lambda))e(\lambda) = 2|x|e(\lambda)$$

This implies that $|x|e(\lambda) = 0$ and hence $|x| = x_1 = 2(0 + \lambda e^\perp(\lambda) = 2\lambda e^\perp(\lambda)$. Furthermore, $\left\| |x| \right\| = \left\| x_1 \right\| = 1$ and so

$$1 = \int_0^\infty \mu_{|x|}(t) d\psi(t) = \int_0^\infty 2\lambda \chi_{[0,\tau(e^\perp(\lambda)))}(t) dt = 2\lambda \psi(\tau(e^\perp(\lambda))).$$

This implies that $2\lambda = \frac{1}{\psi(\tau(e^\perp(\lambda)))}$ and therefore

$$|x| = \frac{1}{\psi(\tau(e^\perp(\lambda)))} e^\perp(\lambda).$$

This is a contradiction and so $x_1 \neq x_2$. Therefore, $|x|$ is not an extreme point.

In the finite setting, it would follow immediately from Lemma 7.1.1, that x is also not an extreme point. We have to do a little more work to show that x is not an extreme point. Let $x = v|x|$ be the polar decomposition of x. We have that $|x| = \frac{1}{2}(x_1 + x_2)$ and so $x = \frac{1}{2}(vx_1 + vx_2)$. Furthermore, v is a partial isometry and so $\left\| vx_1 \right\|_{L_{w,1}} \le \left\| x_1 \right\|_{L_{w,1}} = 1$ and similarly $\left\| vx_2 \right\|_{L_{w,1}} \le 1$. Note that $v^*v = s(x) = r(|x|)$, by Remark B.1.31 and so

$$v^*vx_1 = r(|x|)x_1 = 2r(|x|)\Big(|x|e(\lambda) + \lambda e^\perp(\lambda)\Big) = 2\Big(|x|e(\lambda) + \lambda e^\perp(\lambda)\Big) = x_1$$

We can similarly show that $v^*vx_2 = x_2$. Since we also have that $x_1 \neq x_2$, a simple proof by contradiction shows that $vx_1 \neq vx_2$. It follows that x is not an extreme point. In terms of proving the converse, we have therefore shown that if x is an extreme point, then $|x| = \frac{p}{\psi(\tau(p))}$ for some $p \in \mathcal{P}(\mathcal{A})^f$. If this is the case, then by Proposition B.2.13 there exists a unitary operator u such that $x = u|x|$. Since $up \in \mathcal{V}(\mathcal{A})$ and $\tau(p) = \tau(|up|)$, we have that $x = \frac{v}{\psi(\tau(|v|))}$ for some $v \in \mathcal{V}(\mathcal{A})$ with $\tau(|v|) < \infty$. $\square$

APPENDIX B

Supplementary results

We include here results referenced in the text. In those instances where results have been adapted from those in the literature, short proofs have been included (unless the proof is particularly straightforward).

B.1. von Neumann algebras

In the main body of the text, many standard properties of the weak- and strong-operator topologies, spectral theory, order structure of von Neumann algebras, projections and partial isometries have been used. For ease of reference those have been included here. Unless stated otherwise, we will use $\mathcal{A}$ to denote a (general) von Neumann algebra. Certain of the results to be mentioned hold more generally for C^*-algebras, but since we do not require this increased generality, we will not distinguish such results.

B.1.1. Topologies. We start by listing some of the properties of the weak and strong-operator topologies and discussing certain notational conventions employed in the text. Recall that the strong operator topology (SOT) is the locally convex Hausdorff topology on $\mathcal{B}(H)$ generated by the family of semi-norms $\{\rho_\xi : \xi \in H\}$, where

$$\rho_\xi(x) := \left\|x\xi\right\|_H, \qquad x \in \mathcal{B}(H)$$

and the weak operator topology (WOT) is the locally convex Hausdorff topology on $\mathcal{B}(H)$ generated by the family of semi-norms $\{\rho_{\xi,\eta} : \xi, \eta \in H\}$, where

$$\rho_{\xi,\eta}(x) := |\left\langle x\xi, \eta\right\rangle|, \qquad x \in \mathcal{B}(H).$$

PROPOSITION B.1.1. [23, p.115,116] *Suppose H is a Hilbert space.*

(1) *The strong operator topology on $\mathcal{B}(H)$ is a vector topology, i.e. addition and scalar multiplication are SOT-continuous*
(2) *Multiplication is separately continuous in the strong operator topology and jointly continuous, provided that the first variable is restricted to a bounded set.*
(3) *However, when H is infinite dimensional, neither of the mappings*

$$\begin{aligned}(y, x) &\mapsto yx \\ x &\mapsto x^*\end{aligned}$$

is SOT-continuous in general. The latter mapping is, however, SOT-continuous on the set of normal elements.

PROPOSITION B.1.2. [25, p.207] *The mapping of $x \mapsto x^*$ of $\mathcal{B}(H)$ into $\mathcal{B}(H)$ is weak operator continuous.*

PROPOSITION B.1.3. [23, p.116] *Closed balls of $\mathcal{B}(H)$ are complete in the strong operator topology.*

REMARK B.1.4. We make a brief comment regarding the notation to be used when dealing with joint SOT-continuity. Let $(\mathcal{B}(H))_k := \{x \in \mathcal{B}(H) : \left\|x\right\| \leq k\}$. Suppose $\{x_\alpha\}_{\alpha\in A} \subseteq (\mathcal{B}(H))_k$ is such that $x_\alpha \overset{SOT}{\to} x \in (\mathcal{B}(H))_k$ and $\{y_\beta\}_{\beta\in B} \subseteq \mathcal{B}(H)$ is such that $y_\beta \overset{SOT}{\to} y \in \mathcal{B}(H)$. Let $D := A \times B$ and define a relation $\leq$ on D

by setting $(\alpha,\beta) \leq (\alpha',\beta')$ if and only if $\alpha \leq \alpha'$ and $\beta \leq \beta'$. It is easily checked that D equipped with this relation is a directed set. Let U be a SOT-neighbourhood of xy. Since multiplication is jointly SOT-continuous, provided the first variable is restricted to a bounded set, there exists a SOT-neighbourhood $V \subseteq (\mathcal{B}(H))_k$ of x and a SOT-neighbourhood $W \subseteq \mathcal{B}(H)$ of y such that $x' \in V$ and $y' \in W$ implies that $x'y' \in U$. We have that $x_\alpha \overset{SOT}{\to} x$ and so there exists an $\alpha_0 \in A$ such that $\alpha \geq \alpha_0$ implies that $x_\alpha \in V$. Similarly, there exists a $\beta_0 \in B$ such $\beta \geq \beta_0$ implies that $y_\beta \in W$. It follows that for $(\alpha,\beta) \geq (\alpha_0,\beta_0)$, we have $(x_\alpha, y_\beta) \in V \times W$ and hence $x_\alpha y_\beta \in U$. Therefore

$$\mathrm{SOT}\lim_{(\alpha,\beta)\in D} x_\alpha y_\beta = xy.$$

We will usually omit these details and write

$$x_\alpha y_\beta \overset{SOT}{\to} xy$$

or

$$\lim x_\alpha y_\beta = xy.$$

If $\{x_\alpha\}_{\alpha\in A} \subseteq \mathcal{B}(H)$ is such that $x_\alpha \overset{SOT}{\to} x \in \mathcal{B}(H)$ and $\{y_\beta\}_{\beta\in B} \subseteq \mathcal{B}(H)$ is such that $y_\beta \overset{SOT}{\to} y \in \mathcal{B}(H)$, then we will similarly write

$$x_\alpha + y_\beta \overset{SOT}{\to} x + y$$

and omit the details regarding the indexing of this net.

B.1.2. Order structure. Next, we list some of the properties of the cone of positive operators $\mathcal{A}^+$, discuss some of the relationships between the order structure and the norm on $\mathcal{A}$, and demonstrate the relationship between the SOT-convergence of an increasing net of self-adjoint elements and the existence of a supremum of such a net.

Theorem B.1.5. [23, p.245] *$\mathcal{A}^+$ has the following properties.*

1. *$\mathcal{A}^+$ is closed (with respect to the norm topology) in $\mathcal{A}$*
2. *If $a, b \in \mathcal{A}^+$ and $ab = ba$, then $ab \in \mathcal{A}^+$*
3. *If $a \in \mathcal{A}^+$ and $-a \in \mathcal{A}^+$, then $a = 0$.*

Proposition B.1.6. [23, p.246] *If x is a self-adjoint element of $\mathcal{A}$, then*

$$\|x\|_{\mathcal{A}}\mathbf{1} \pm x \geq 0.$$

Proposition B.1.7. [23, p.250] *Suppose x and y are self-adjoint elements of $\mathcal{A}$. If $-x \leq y \leq x$, then*

$$\|y\| \leq \|x\|.$$

Lemma B.1.8. [23, p.307] *Suppose $\{x_\lambda\}_{\lambda\in\Lambda} \subseteq \mathcal{A}$ is an increasing net of self adjoint operators. If there exists a $k > 0$ such that $x_\lambda \leq k\mathbf{1}$ for all λ, then there exists an $x \in \mathcal{A}^{sa}$ such that $x_\lambda \overset{SOT}{\to} x$ and x is the least upper bound of $\{x_\lambda\}_{\lambda\in\Lambda}$.*

Corollary B.1.9. *If $\{x_\lambda\}_{\lambda\in\Lambda} \subseteq \mathcal{A}$ is an increasing net of self adjoint operators such that $x_\lambda \uparrow x$ for some $x \in \mathcal{A}^{sa}$, then $x_\lambda \overset{SOT}{\to} x$. If $\{y_\lambda\}_{\lambda\in\Lambda} \subseteq \mathcal{A}$ is an increasing net of self adjoint operators such that $y_\lambda \overset{SOT}{\to} y$ for some $y \in \mathcal{A}^{sa}$, then $y_\lambda \uparrow y$*

B.1.3. Spectral theory. We present some well-known properties of the spectrum of a normal element, present the Spectral Theorem for bounded self-adjoint operators and discuss some consequences of this result.

THEOREM B.1.10. [23, p.271] *Suppose x is a normal element of $\mathcal{A}$. Then*

(1) *x is self-adjoint if and only if $\sigma(x) \subseteq \mathbb{R}$.*
(2) *x is positive if and only if $\sigma(x) \subseteq \mathbb{R}^+$.*
(3) *x is unitary if and only if $\sigma(x) \subseteq \{t \in \mathbb{C} : |t| = 1\}$.*
(4) *x is a projection if and only if $\sigma(x) \subseteq \{0, 1\}$.*

THEOREM B.1.11 (Spectral Theorem for bounded self-adjoint operators). [23, p.310,311] *If $x \in \mathcal{A}^{sa}$, then there exists a bounded resolution of the identity $\{e(\lambda)\}_{\lambda\in\mathbb{R}}$ such that $xe(\lambda) \leq \lambda e(\lambda)$ for each λ, $\lambda e(\lambda)^\perp \leq x\lambda e(\lambda)^\perp$ for each λ and*

$$x = \int_{-\|x\|}^{\|x\|} \lambda de(\lambda)$$

in the sense of norm convergence of approximating Riemann sums; and x is the norm limit of a sequence of finite linear combinations of orthogonal projections of the form $e(\lambda') - e(\lambda)$.

REMARK B.1.12. Let $\mathcal{G}$ denote the set of all finite linear combinations of mutually orthogonal projections in $\mathcal{P}(\mathcal{A})$. If $x \in \mathcal{A}^{sa}$, then it follows from the Spectral Theorem that there exits a sequence $(x_n)_{n=1}^\infty \subseteq \mathcal{G}^{sa}$ such that $x_n \stackrel{\mathcal{A}}{\to} x$; $s(x_n) \leq s(x)$ for all $n \in \mathbb{N}^+$; and the x_n's commute with each other and with x. Note that if, in addition $\tau(r(x)) < \infty$, then $x_n \in \mathcal{G}_f^{sa}$ for every $n \in \mathbb{N}^+$, where $\mathcal{G}_f$ denotes the set of all finite linear combinations of mutually orthogonal projections in $\mathcal{P}(\mathcal{A})^f$, i.e. if $x \in \mathcal{F}(\tau)^{sa}$, then there exists a sequence $(x_n)_{n=1}^\infty \subseteq \mathcal{G}_f^{sa}$ such that $x_n \stackrel{\mathcal{A}}{\to} x$; $s(x_n) \leq s(x)$ for every $n \in \mathbb{N}^+$; and the x_n's commute with each other and with x. It also clear that in either case the sequence will consist of positive elements if x is a positive element, since in this case $\sigma(x) \subseteq [0, \infty)$.

Any linear isometry defined on a dense subspace of a normed space into a Banach space has a unique extension to a linear isometry on the whole space. Using Remark B.1.12, it is easily checked that the argument employed to prove this can be modified to obtain the following two results.

PROPOSITION B.1.13. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are von Neumann algebras. Let $\mathcal{G}$ denote the set of all finite linear combinations of mutually orthogonal projections in $\mathcal{A}$. If $T : \mathcal{G}^{sa} \to \mathcal{B}$ is isometric on $\mathcal{G}^{sa}$ and real linear on commuting elements of $\mathcal{G}^{sa}$ (i.e. $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$, whenever $\alpha, \beta \in \mathbb{R}$ and $x, y \in \mathcal{G}^{sa}$ are such that $xy = yx$), then T can be extended to $\mathcal{A}^{sa}$ in an isometric fashion and this extension is real linear on commuting elements of $\mathcal{A}^{sa}$.*

PROPOSITION B.1.14. *Suppose $(\mathcal{A}, \tau)$ is a semi-finite von Neumann algebra. Let $\mathcal{G}_f$ denote the set of all finite linear combinations of mutually orthogonal projections in $\mathcal{P}(\mathcal{A})^f$. If $T : \mathcal{G}_f^{sa} \to \mathcal{B}$ is isometric on $\mathcal{G}_f^{sa}$ and real linear on commuting elements of $\mathcal{G}_f^{sa}$, then T can be extended to $\mathcal{F}(\tau)^{sa}$ in an isometric fashion and this extension is real linear on commuting elements from $\mathcal{F}(\tau)^{sa}$.*

REMARK B.1.15. Note that if T is an isometry from $\mathcal{G}^{sa}$ (or $\mathcal{G}_f^{sa}$) into $\mathcal{B}^{sa}$ and is real linear on commuting elements, then the extension also maps into $\mathcal{B}^{sa}$, since $\mathcal{B}^{sa}$ is closed in $\mathcal{B}$.

B.1.4. Projections. Recall that projections are the self-adjoint idempotents in $\mathcal{B}(H)$. We list among others, properties of orthogonal projections, central projections and projections in non-atomic von Neumann algebras.

PROPOSITION B.1.16. [6, p.40][23, p.110,111] *Suppose $p, q \in \mathcal{P}(\mathcal{A})$. Then*

(1) *$p+q \in \mathcal{P}(\mathcal{A})$ if and only if $pq = 0$*
(2) *$pq \in \mathcal{P}(\mathcal{A})$ if and only if $pq = qp$*
(3) *$pq = p$ if and only if $p \leq q$*
(4) *$p \vee q = p + q - pq$ and $p \wedge q = pq$, if $pq = qp$*

Recall that if $x \in \mathcal{A}$, then $z(x)$ denotes the central support projection of x, i.e. $z(x) = \mathbf{1} - p$, where p is the supremum of all central projections $q \in \mathcal{A}$ such that $qx = 0$.

PROPOSITION B.1.17. [23, p.333] *If $\{p_\lambda\}_{\lambda\in\Lambda}$ is a family of projections in $\mathcal{A}$ and $p = \bigvee_\lambda p_\lambda$, then $z(p) = \bigvee_\lambda z(p_\lambda)$.*

PROPOSITION B.1.18. [24, p.403] *Two projections p and q in $\mathcal{A}$ have non-zero equivalent subprojections if and only if $z(p)z(q) \neq 0$.*

PROPOSITION B.1.19. [23, p.112] *If $\{p_\lambda\}_{\lambda\in\Lambda}$ is an increasing net of projections, then $p_\lambda \overset{SOT}{\to} \vee p_\lambda$.*

Recall that if $p, q \in \mathcal{P}(\mathcal{A})$, then $p \sim q$ if there exists a partial isometry v such that $v^*v = p$ and $vv^* = q$. Next we list some properties of this relation.

PROPOSITION B.1.20. [24, p.402, 445][15]

(1) *$\sim$ is an equivalence relation on $\mathcal{P}(\mathcal{A})$.*
(2) *If $x \in \mathcal{A}$, then $r(x) \sim s(x)$*
(3) *If $x \in \mathcal{A}$, then $r(x) \sim r(x^*)$*
(4) *If $p, q \in \mathcal{P}(\mathcal{A})$, then*
$$p \vee q - q \sim p - p \wedge q$$
and so $p \precsim q^\perp$ if $p \wedge q = 0$.
(5) *If $p \sim q$ and r is a central projection, then $rp \sim rq$.*
(6) *If p and q are finite projections and $p \sim q$, then $p^\perp \sim q^\perp$*

PROPOSITION B.1.21. [15] *Suppose $\mathcal{A}$ can be equipped with a trace τ and let $p, q \in \mathcal{P}(\mathcal{A})$.*

(1) *If $p \sim q$, then $\tau(p) = \tau(q)$ and, in particular, $\tau(s(x)) = \tau(r(x))$ for any $x \in \mathcal{A}$.*
(2) *If $p \precsim q$, then $\tau(p) \leq \tau(q)$.*
(3) *If $p, q \in \mathcal{P}(\mathcal{A})^f$, then $p \vee q \in \mathcal{P}(\mathcal{A})^f$*

PROPOSITION B.1.22. *Suppose $\mathcal{A} \subseteq \mathcal{B}(H)$ is a von Neumann algebra.*

(1) *If $\eta, \xi \in H$ are such that $\|\eta + \xi\| = \|\eta\| + \|\xi\|$, then $\xi = \alpha\eta$ for some $\alpha \in \mathbb{C}$.*
(2) *If $x \in \mathcal{B}(H)$, then $\||x|\eta\| = \|x\eta\|$ for all $\eta \in H$*
(3) *Suppose $p \in \mathcal{B}(H)$ is a projection. If $\eta \notin p(H)$, then $\|p(\eta)\| < \|\eta\|$*
(4) *If $p, q \in \mathcal{P}(\mathcal{A})$ are such that $pqp = p$, then $p \leq q$*
(5) *If $0 \neq p, q \in \mathcal{P}(\mathcal{A})$, $\alpha, \beta \neq 0$ and $\alpha p = \beta q$, then $p = q$ and $\alpha = \beta$.*

PROOF. (1): If $\|\eta + \xi\| = \|\eta\| + \|\xi\|$, then
$$\|\eta\|^2 + 2\mathrm{Re}\,\langle \eta, \xi\rangle + \|\xi\|^2 = \|\eta\|^2 + 2\|\eta\|\|\xi\| + \|\xi\|^2.$$
It follows that $0 \leq \|\eta\|\|\xi\| = \mathrm{Re}\,\langle \eta, \xi\rangle \leq |\langle \eta, \xi\rangle| \leq \|\eta\|\|\xi\|$ and hence $\mathrm{Re}\,\langle \eta, \xi\rangle = \langle \eta, \xi\rangle$, since we have a complex number whose real part is positive and equal to the modulus of the complex number. Combining this with the previous set of inequalities we have that

$$\langle \eta, \xi\rangle = \|\eta\|\|\xi\|. \tag{B.1.1}$$

We can write ξ in the form $\xi = \alpha\eta + \zeta$ for some $\alpha \in \mathbb{C}$ and $\zeta \in H$ with $\zeta \perp \eta$. It follows that

$$0 \leq \langle \eta, \xi \rangle = \langle \eta, \alpha\eta + \zeta \rangle = \langle \eta, \alpha\eta \rangle \leq \|\eta\| \|\alpha\eta\|. \tag{B.1.2}$$

Furthermore, $\|\xi\|^2 = \langle \alpha\eta + \zeta, \alpha\eta + \zeta \rangle = \|\alpha\eta\|^2 + \|\zeta\|^2$ and so

$$\Big(\|\alpha\eta\|^2 + \|\zeta\|^2\Big)\|\eta\|^2 = \|\xi\|^2\|\eta\|^2 = \langle \eta, \xi \rangle^2 \leq \|\eta\|^2\|\alpha\eta\|^2,$$

using (B.1.1) and (B.1.2). It follows that $\|\zeta\|^2 = 0$ and therefore $\xi = \alpha\eta$.
(2) - (5) are easily checked. □

Proposition B.1.23. *Suppose $(p_i)_{i=1}^n \subseteq \mathcal{P}(\mathcal{A})$ is a family of mutually orthogonal projections. If $0 < p < \infty$ and $(\alpha_i)_{i=1}^n \subseteq \mathbb{C}$, then*

$$|\sum_{i=1}^n \alpha_i p_i|^p = \sum_{i=1}^n |\alpha_i|^p p_i.$$

Proof. We start by noting that

$$\begin{aligned} |\sum_{i=1}^n \alpha_i p_i|^2 &= (\sum_{i=1}^n \alpha_i p_i)^*(\sum_{i=1}^n \alpha_i p_i) \\ &= \sum_{i=1}^n |\alpha_i|^2 p_i \qquad \text{since } p_i^* = p_i = p_i^2 \text{ for every } i \text{ and } p_i p_j = 0 \text{ if } i \neq j. \end{aligned}$$

Let $x = \sum_{i=1}^n |\alpha_i|^2 p_i$, $p_{n+1} = \mathbf{1} - \sum_{i=1}^n p_i$ and $\alpha_{n+1} = 0$. Then the spectral measure e^x of x is given by $e^x = \sum_{i=1}^{n+1} \delta_{|\alpha_i|^2} p_i$, where for any Borel set $A \subseteq [0, \infty)$, $\delta_{|\alpha_i|^2}(A) = 1$ if $|\alpha_i|^2 \in A$ and 0 otherwise. Therefore

$$|\sum_{i=1}^n \alpha_i p_i|^p = (\sum_{i=1}^n |\alpha_i|^2 p_i)^{p/2} = \int_0^\infty \lambda^{p/2} de^x = \sum_{i=1}^n |\alpha_i|^p p_i$$

□

Proposition B.1.24. *If $0 \neq p, q, r \in \mathcal{P}(\mathcal{A})$ and $\alpha, \beta, \gamma > 0$, then $\alpha p + \beta q = \gamma r$ if and only if*

$$p = q = r \qquad \textit{and} \qquad \alpha + \beta = \gamma$$

$$\textit{or}$$

$$pq = 0,\ p + q = r \qquad \textit{and} \qquad \alpha = \beta = \gamma$$

Proof. It is clear that the sufficiency part of the proposition holds. To prove the converse, we start by showing that $r = p \vee q$. Note that

$$\eta \in r(H)^\perp \iff r\eta = 0 \iff \alpha p\eta + \beta q\eta = 0 \tag{B.1.3}$$

If the above holds and we assume that $p\eta \neq 0$, then $-\beta q\eta = \alpha p\eta \neq 0$ and so

$$0 < \alpha\|p\eta\|^2 = \alpha \langle p\eta, \eta \rangle = -\beta \langle q\eta, \eta \rangle = -\beta\|q\eta\|^2 < 0.$$

This is a contradiction. It follows that $\alpha p\eta + \beta q\eta = 0$ if and only if $p\eta = 0$ and $q\eta = 0$. Therefore

$$\eta \in r(H)^\perp \iff \eta \in p(H)^\perp \cap q(H)^\perp = (p \vee q)(H)^\perp.$$

This shows that $r^\perp = (p \vee q)^\perp$ and hence $r = p \vee q$. This implies that $p, q \leq r$ and therefore $r - p$ and $r - q$ are projections onto $r(H) \cap p(H)^\perp$ and $r(H) \cap q(H)^\perp$, respectively.

Next, we show that if $p \neq r$, then $r = p + q$, $pq = 0 = qp$ and $\alpha = \beta = \gamma$. If $p \neq r$, then $r - p$ is a non-zero projection and hence there exists a $0 \neq \eta \in r(H)$ such that $p\eta = 0$ (using $p \leq r$). Therefore,

$$0 \neq \frac{\gamma}{\beta}\eta = \beta^{-1}(\gamma r\eta) = \beta^{-1}(\alpha p\eta + \beta q\eta) = q\eta.$$

Multiplying through by q, we obtain

$$\frac{\gamma}{\beta}q\eta = q^2\eta = q\eta$$

and therefore $\frac{\gamma}{\beta} = 1$, since $q\eta \neq 0$. Therefore $\beta = \gamma$. Furthermore, $r \neq q$, since if we assume $r = q$, then $\gamma r = \alpha p + \beta q = \alpha p + \gamma r$, i.e. $\alpha p = 0$, but this contradicts the fact that $\alpha \neq 0$ and $p \neq 0$. As before, we can show that this implies that $\alpha = \gamma$. We therefore obtain $\gamma r = \gamma p + \gamma q$. It follows that $p + q$ is a projection and hence $pq = 0 = qp$, by Proposition B.1.16.

If $p = r$, then $\gamma r = \alpha r + \beta q$. Furthermore,

$$\begin{aligned}(\gamma - \alpha)q &= (\gamma - \alpha)rq \qquad \text{since } q \leq r \\ &= \beta q^2 \qquad \text{since } (\gamma - \alpha)r = \beta q \\ &= \beta q\end{aligned}$$

Therefore $(\gamma - \alpha - \beta)q = 0$. Since $q \neq 0$, we obtain $\gamma = \alpha + \beta$. Furthermore,

$$q = \frac{1}{\beta}(\gamma r - \alpha p) = \frac{1}{\beta}(\gamma r - \alpha r) = \frac{\gamma - \alpha}{\beta} r = r.$$

□

We finish this subsection by considering a few properties of non-atomic von Neumann algebras.

Lemma B.1.25. [15] *Suppose $(\mathcal{A}, \tau)$ is a non-atomic semi-finite von Neumann algebra and $p, q \in \mathcal{P}(\mathcal{A})$ with $p \leq q$. If $\alpha \in \mathbb{R}$ is such that $\tau(p) \leq \alpha \leq \tau(q)$, then there exists a projection $e \in \mathcal{P}(\mathcal{A})$ such that $p \leq e \leq q$ and $\tau(e) = \alpha$.*

Corollary B.1.26. [15] *Suppose $(\mathcal{A}, \tau)$ is a non-atomic semi-finite von Neumann algebra and $0 \neq p \in \mathcal{P}(\mathcal{A})$ with $\tau(p) = \alpha$. If $n \in \mathbb{N}^+$, then there exists $\{p_i\}_{i=1}^n \subseteq \mathcal{P}(\mathcal{A})$ such that $\tau(p_i) = \frac{\alpha}{n}$ for each i, $p_i p_j = 0$ if $i \neq j$, and $\sum_{i=1}^{n} p_i = p$.*

Proposition B.1.27. *Suppose $(\mathcal{A}, \tau)$ is a non-atomic semi-finite von Neumann algebra with $\tau(\mathbf{1}) = \infty$. If $p, q \in \mathcal{P}(\mathcal{A})$ with $pq = 0$ and $\tau(p), \tau(p) < 1$, then there exists $p_1, q_1 \in \mathcal{P}(\mathcal{A})$ such that*

(1) *$p_1 p = 0$ and $q_1 q = 0$, i.e $p_1 + p, q_1 + q \in \mathcal{P}(\mathcal{A})$;*
(2) *$\tau(p + p_1) = 1 = \tau(q + q_1)$; and*
(3) *$(p + p_1)(q + q_1) = 0$*

Proof. Note that $p \leq q^\perp$ and $q \leq p^\perp$, since $pq = 0$. It follows that $p^\perp - q \in \mathcal{P}(\mathcal{A})$ and $\tau(p^\perp - q) = \infty$, since $\tau(p^\perp) = \tau(\mathbf{1}) - \tau(p) = \infty$ and $\tau(q) < 1$. (A, τ) is non-atomic and so, by Lemma B.1.25, there exists $p_1 \in \mathcal{P}(\mathcal{A})$ with $p_1 \leq p^\perp - q$ and $\tau(p_1) = 1 - \tau(p)$. Note that

$$\begin{aligned}p_1 q &= p_1(p^\perp - q)q \qquad \text{since } p_1 \leq p^\perp - q \\ &= p_1(q - q) \qquad \text{since } q \leq p^\perp \\ &= 0 \end{aligned} \tag{B.1.4}$$

Furthermore, $p_1 p = 0$, since $p_1 \leq p^\perp - q \leq p^\perp$. By Proposition B.1.16, $p_1 + p \in \mathcal{P}(\mathcal{A})$ and $\tau(p + p_1) = 1$. Using (B.1.4) we have that $(p + p_1)q = pq + p_1 q = 0$ and hence $(p_1 + p) \leq q^\perp$. In a similar way to before, we can find $q_1 \in \mathcal{P}(\mathcal{A})$ with $q_1 \leq q^\perp - (p + p_1)$ and $\tau(q_1) = 1 - \tau(q)$. Note that

$$\begin{aligned}q_1 p &= q_1(q^\perp - (p + p_1))p \\ &= q_1(p - p) \qquad \text{since } p \leq q^\perp \text{ and } p \leq (p + p_1) \\ &= 0\end{aligned}$$

and

$$\begin{aligned}(p+p_1)(q+q_1) &= p_1q_1 \qquad \text{since } pq=0=pq_1=p_1q\\ &= p_1(q^\perp-(p+p_1))q_1 \qquad \text{since } q_1\le q^\perp-(p+p_1)\\ &= (p_1-p_1)q_1 \qquad \text{since } p_1\le q^\perp \text{ and } p_1\le(p_1+p)\\ &= 0\end{aligned}$$

□

B.1.5. Partial isometries. Recall that a partial isometry is an operator $w\in\mathcal{A}$ with the property that $\|w\xi\|=\|\xi\|$ for all ξ in $(\ker(w))^\perp$, the orthogonal complement of the kernel of w. The projection onto $(\ker(w))^\perp$ is called the *initial projection* of w and the projection onto the closure of the range of w is called the *final projection* of w. We present a useful characterization of partial isometries and list some of the important properties of the polar decomposition of unbounded operators.

PROPOSITION B.1.28. [24, p.400] *The operator $v\in\mathcal{B}(H)$ is a partial isometry if and only if v^*v is a projection. In this case v^*v is the initial projection of v and vv^* is the final projection of v, i.e. $s(v)=v^*v=|v|$ and $r(v)=vv^*=|v^*|$.*

REMARK B.1.29. If $w\in\mathcal{A}$ is a partial isometry and $w\ge0$, then w is a projection, namely $|w|=s(w)$.

THEOREM B.1.30. [24, p.404][45, p.5][15] *Let x be a closed, densely defined linear transformation from one Hilbert space to another. There exists a partial isometry v with initial projection $r(|x|)$ and final projection $r(x)$ such that*

$$x=v|x|.$$

If $x=wy$, where y is a positive operator and and w is a partial isometry with initial space the closure of the range of y, then $y=(x^*x)^{1/2}$ and $w=v$.

REMARK B.1.31. [43, p.4][15] Let x be closed, densely defined operator with polar decomposition $x=v|x|$ and let $\mathcal{A}$ be a von Neumann algebra. Then

(1) $\mathcal{D}(x)=\mathcal{D}(|x|)$
(2) $k(x)=k(|x|)$
(3) $s(|x|)=r(|x|)=r(x^*)=s(x)=s(v)=r(v^*)$
(4) $x^*=|x|v^*=v^*|x^*|$
(5) $|x^*|=v|x|v^*$
(6) If $x\in\mathcal{A}$, then $v,r(x),s(x)\in\mathcal{A}$.

PROPOSITION B.1.32. *Suppose x and y are closed, densely defined operators with polar decompositions $x=v|x|$ and $y=w|y|$, respectively. If $x^*y=0=xy^*$, then*

(1) *$v^*w=0=vw^*$ and $w^*v=0=wv^*$;*
(2) *$s(x)s(y)=0=r(x)r(y)$;*
(3) *$v+w$ and $v-w$ are partial isometries;*
(4) *$|x\pm y|=|x|+|y|$;*
(5) *$x+y=(v+w)(|x|+|y|)$ is the polar decomposition of $x+y$ and $x-y=(v-w)(|x|+|y|)$ is the polar decomposition of $x-y$.*
(6) *$s(x\pm y)=v^*v+w^*w=s(x)+s(y)$*

PROOF. (1): Since $xy^* = 0$, we have that

$$s(y) = r(y^*) \leq k(x) = (s(x))^\perp.$$

Therefore $(w^*w)(v^*v) = s(y)s(x) = 0$ and hence

$$w[(w^*w)(v^*v)]v^* = w0v^* = 0$$

Since $ww^* = r(w)$ and $vv^* = s(v^*)$, this implies that $wv^* = 0$ and hence

$$vw^* = (wv^*)^* = 0^* = 0.$$

One can similarly show that $v^*w = 0$.
(2) and (3): Using (1) we have that $(v \pm w)^*(v \pm w) = v^*v + w^*w$ and $s(x)s(y) = (v^*v)(w^*w) = 0$. It follows that v^*v and w^*w are orthogonal projections and hence $v^*v + w^*w$ is a projection. Therefore, $v \pm w$ is a partial isometry, by Proposition B.1.28. One can similarly show that $r(x)r(y) = 0$.
(4): It is easily checked, using (2), that

$$(|x| + |y|)^2 = |x|^2 + |y|^2.$$

Furthermore,

$$\begin{aligned} |x+y|^2 &= |x|^2 + |y|^2 \qquad \text{since } x^*y = 0 = y^*x \\ &= (|x| + |y|)^2 \end{aligned}$$

and hence $|x+y| = |x|+|y|$, since positive square roots are unique. One can similarly show that $|x-y| = |x|+|y|$.
(5): Note that

$$\begin{aligned} (v+w)(|x|+|y|) &= x + v(w^*w|y|) + w(v^*v|x|) + y \qquad \text{since } w^*w = s(|y|) = r(|y|) \text{ and } v^*v = r(|x|) \\ &= x + y \qquad \text{using (1)} \end{aligned}$$

The result follows using the uniqueness of the polar decomposition of $x+y$ and the fact that the initial projection of $v+w$ is $r(|x|+|y|)$. One can similarly show that $x-y = (v-w)(|x|+|y|)$ is the polar decomposition of $x-y$.
(6) follows from (5), (1) and (2). ☐

B.2. Trace-measurable operators

Throughout this section we will assume that H is a Hilbert space and $(\mathcal{A}, \tau)$ is a semi-finite von Neumann algebra. We mention some properties of affiliated operators.

PROPOSITION B.2.1. [45, p.6]

(1) *If x is closed and densely defined with polar decomposition $x = v|x|$, then $x\eta\mathcal{A}$ if and only if $v \in \mathcal{A}$ and $|x|\eta\mathcal{A}$.*
(2) *If x is normal, then $x\eta\mathcal{A}$ if and only if $e^x(B) \in \mathcal{A}$ for all Borel sets $B \subseteq \mathbb{C}$*
(3) *If x is normal and $x\eta\mathcal{A}$, then $f(x)\eta\mathcal{A}$ for any Borel function $f : \mathbb{C} \to \mathbb{C}$*

Next, we present two results pertaining to the order structure of the set of trace-measurable operators.

PROPOSITION B.2.2. [15] *Suppose $x, y \in S(\mathcal{A}, \tau)^{sa}$. Then*

(1) $x_+x_- = 0$
(2) $|x| = x_+ + x_-$
(3) $0 \leq x_+, x_- \leq |x|$
(4) *$z^*xz \leq z^*yz$ for all $z \in S(\mathcal{A}, \tau)$, whenever $x \leq y$*

(5) $e^x(\lambda,\infty) \precsim e^y(\lambda,\infty)$, *whenever* $0 \le x \le y$
(6) $x^{-1} \ge 0$, *whenever* $x \ge 0$ *and* x *is invertible*
(7) $z^* x_\lambda z \uparrow z^* x z$ *for all* $z \in S(\mathcal{A},\tau)$, *whenever* $x \ge 0$ *and* $\{x_\lambda\}_{\lambda\in\Lambda}$ *is an increasing net in* $S(\mathcal{A},\tau)^{sa}$ *such that* $x_\lambda \uparrow x$

PROPOSITION B.2.3. [15] *If* $0 \le x \in S(\mathcal{A},\tau)^{sa}$, *then there exists an increasing net* $\{x_\lambda\}_{\lambda\in\Lambda} \subseteq \mathcal{F}(\tau)^+$ *such that* $x_\lambda \uparrow x$.

Next, we present several results related to the functional calculus.

REMARK B.2.4. Let (Ω,Σ) be a measurable space and let $\mathcal{M}(\Omega,\Sigma)$ denote the set of all complex-valued Σ-measurable functions. If $e : \Sigma \to \mathcal{B}(H)$ is a spectral measure, then $\int f\, de$ is a closed and densely defined operator on H, for every $f \in \mathcal{M}(\Omega,\Sigma)$ (see [6, Proposition 10.4.10]).

PROPOSITION B.2.5. [15] *Suppose* $(\mathcal{A},\tau)$ *is a semi-finite von Neumann algebra and* $x \in S(\mathcal{A},\tau)^{sa}$.
(1) *The mapping* $f \mapsto f(x)$ *is a* $*$*-homomorphism from* $\mathcal{B}_{bc}(\sigma(x))$ *into* $S(\mathcal{A},\tau)$.
(2) *If* $(\mathcal{A},\tau)$ *is trace-finite, then* $f(x) \in S(\mathcal{A},\tau)$ *for all* $f \in \mathcal{B}(\sigma(x))$
(3) *If* $f \in \mathcal{B}_b(\sigma(x))$ *and* $(f_n)_{n=1}^\infty$ *is a uniformly bounded sequence in* $\mathcal{B}_b(\sigma(x))$ *such that* $f_n(t) \to f(t)$ *as* $n \to \infty$ *for all* $t \in \sigma(x)$, *then* $f_n(x) \overset{SOT}{\to} f(x)$.
(4) *If* $x \ge 0$, *then* $f_n(x) \uparrow f(x)$, *whenever* $f, f_n \in \mathcal{B}_{bc}(\sigma(x))$ $(n = 1, 2, \ldots)$ *are positive functions such that* $0 \le f_n(t) \uparrow f(t)$ *for all* $t \in \sigma(x)$.

PROPOSITION B.2.6. [15] *Let* $x \in S(\mathcal{A},\tau)^{sa}$ *be fixed. If* $\Gamma : \mathcal{B}_{bc}(\mathbb{R}) \to S(\mathcal{A},\tau)$ *is a unital* $*$*-homomorphism satisfying:*
(1) $\Gamma(\iota) = x$, *where* $\iota(t) := t$ *for all* $t \in \mathbb{R}$;
(2) $\Gamma(f_n) \overset{\mathcal{T}_m}{\to} \Gamma(f)$ *whenever* $(f_n)_{n=1}^\infty \cup \{f\} \subseteq \mathcal{B}_{bc}(\mathbb{R})$ *is such that* $f_n \to f$ *uniformly on compact subsets of* $\mathbb{R}$;
(3) $\Gamma(f_n) \uparrow \Gamma(f)$ *whenever* $(f_n)_{n=1}^\infty \cup \{f\} \subseteq \mathcal{B}_{bc}(\mathbb{R})$ *are positive functions satisfying* $f_n(t) \uparrow f(t)$ *for all* $t \in \mathbb{R}$,

then $\Gamma(f) = f(x)$ *for all* $f \in \mathcal{B}_{bc}(\mathbb{R})$.

If x and y are closed densely defined operators, then we will write $x \subseteq y$ if $\mathcal{D}(x) \subseteq \mathcal{D}(y)$ and $x\xi = y\xi$ for all $\xi \in \mathcal{D}(x)$.

THEOREM B.2.7. [15] *Suppose* $x : \mathcal{D}(x) \to H$ *is a normal operator. For* $y \in \mathcal{B}(H)$, *the following are equivalent*
(1) $ye^x(A) = e^x(A)y$ *for all Borel sets* $A \subseteq \mathbb{C}$
(2) $yf(x) \subseteq f(x)y$ *for all* $f \in \mathcal{B}_{bc}(\sigma(x))$
(3) $yx \subseteq xy$

We note that if $x, y \in S(\mathcal{A},\tau)$ with $x \subseteq y$, then $x = y$, (see [45, Corollary 15]) and so we obtain the following Corollary.

COROLLARY B.2.8. *Suppose* $\mathcal{A} \subseteq \mathcal{B}(H)$ *is a semi-finite von Neumann algebra. Let* $x \in S(\mathcal{A},\tau)$ *be a self-adjoint operator. For* $y \in \mathcal{B}(H)$, *the following are equivalent*
(1) $ye^x(A) = e^x(A)y$ *for all Borel sets* $A \subseteq \mathbb{C}$
(2) $yf(x) = f(x)y$ *for all* $f \in \mathcal{B}_{bc}(\sigma(x))$
(3) $yx = xy$

PROPOSITION B.2.9. *If $x \in S(Z(\mathcal{A}), \tau)$, then*

$$xy = yx \qquad \forall y \in \mathcal{A}.$$

PROOF. Suppose $x \in S(Z(\mathcal{A}), \tau)$. Then $x = x_1 + ix_2$, where $x_1, x_2 \in S(Z(\mathcal{A}), \tau)^{sa}$. For $i = 1, 2$, we therefore have that $e^{x_i}(A) \in Z(\mathcal{A})$ for all Borel sets $A \subseteq \mathbb{R}$, by Proposition B.2.1(2). It follows that $e^{x_i}(A)y = ye^{x_i}(A)$ for all Borel sets $A \subseteq \mathbb{R}$, since $y \in \mathcal{A}$. Therefore $x_i y = yx_i$, by Corollary B.2.8, and so $xy = yx$. □

PROPOSITION B.2.10. *Suppose $x, y \in S(\mathcal{A}, \tau)$ are such that $0 \leq x \leq y$. If $e^x(A)e^y(B) = e^y(B)e^x(A)$ for all Borel sets $A, B \subseteq [0, \infty)$, then*

$$e^x(\lambda, \infty) \leq e^y(\lambda, \infty) \qquad \forall \lambda \geq 0.$$

PROOF. The result is holds for $x, y \in \mathcal{B}(H)$ (see [15]). Adapting the proof in [15] yields the desired result. □

PROPOSITION B.2.11. *Let x be a closed, densely defined self-adjoint operator on H and let $x = \int_{-\infty}^{\infty} \lambda de^x(\lambda)$ be its spectral decomposition. If p is a projection commuting with x, then*

$$xp = \Big(\int_{-\infty}^{\infty} \lambda de^x(\lambda)\Big)p = \int_{-\infty}^{\infty} \lambda d(e^x(\lambda)p).$$

PROOF. For a closed, densely defined self-adjoint operator y, $\{e_\lambda\}_{\lambda\in\mathbb{R}}$ is the resolution of the identity for y if and only if $ye_\lambda \leq \lambda e_\lambda$ and $\lambda(\mathbf{1} - e_\lambda) \leq y(\mathbf{1} - e_\lambda)$ for all λ (see [23, Theorem 5.2.3]). Let $\{e_\lambda\}_{\lambda\in\mathbb{R}}$ denote the resolution of the identity for x. Since p commutes with x, p commutes with e_λ for each λ, by Corollary B.2.8, and so $e'_\lambda := e_\lambda p$ is a projection for each λ. It is easily checked that $\{e'_\lambda\}_{\lambda\in\mathbb{R}}$ is a resolution of the identity. Furthermore, for any λ,

$$(xp)e'_\lambda = pxe_\lambda p \leq p\lambda e_\lambda p = \lambda e'_\lambda,$$

using $xe_\lambda \leq \lambda e_\lambda$, Proposition B.2.2(4) and the commutativity of x, p and e_λ. One can similarly show that

$$\lambda(\mathbf{1} - e'_\lambda) \leq (xp)(\mathbf{1} - e'_\lambda)$$

and therefore $\{e'_\lambda\}_{\lambda\in\mathbb{R}}$ is the resolution of the identity for xp. □

We finish this section with three general results regarding trace-measurable operators.

PROPOSITION B.2.12. *Suppose $x, y \in S(\mathcal{A}, \tau)$.*

(1) *If $xy = 0$ and $r(y) \leq s(x)$, then $y = 0$.*
(2) *If p is a projection such that $0 \neq p \leq s(x)$, then $xp \neq 0$.*
(3) *If $x^*y = 0$, then $|x + y| = |x - y|$*
(4) *If x is normal, then $|x|^2 = (Re\,x)^2 + (Im\,x)^2$ and $|Re\,x|, |Im\,x| \leq |x|$*
(5) *x is self-adjoint if $|x| = |Re\,x|$*

PROOF. (1): Assume $y \neq 0$. This implies the existence of an $\eta \in \mathcal{D}(y)$ such that $y\eta \neq 0$. Since $xy = 0$, we have that $y\eta \in \ker(x)$. Furthermore, $r(y) \leq s(x) = k(x)^\perp$ and so $y\eta \in (\ker(x))^\perp$. It follows that $y\eta \in \ker(x) \cap (\ker(x))^\perp = \{0\}$, i.e. $y\eta = 0$, which is a contradiction.

(2): Assume that $xp = 0$. Note that $s(x) - p$ is a projection, since $p \leq s(x)$. Furthermore,

$$x(s(x) - p) = xs(x) - xp = x - 0 = x.$$

This contradicts the fact that $s(x)$ is the smallest projection q such that $xq = x$ (see Proposition 1.3.1) and hence $xp \neq 0$.

(3)-(5) are easily checked. □

PROPOSITION B.2.13. *If $x \in S(\mathcal{A},\tau)$ is such that $\tau(r(x)) < \infty$ (or equivalently $\tau(s(x)) < \infty$), then there exists a unitary $u \in \mathcal{A}$ such that $x = u|x|$.*

PROOF. We start by noting that $\tau(r(x)) = \tau(s(x))$, by Proposition B.1.21, and hence $\tau(r(x)) < \infty$ is equivalent to $\tau(s(x)) < \infty$. Let $x = v|x|$ be the polar decomposition of x. Let $p = v^*v$ and $q = vv^*$. Then $p \sim q$ and hence by Proposition B.1.21, $\tau(p) = \tau(q) = \tau(r(x)) < \infty$. Since any trace-finite projection is finite, p and q are finite projections. By Proposition B.1.20(6), $p^\perp \sim q^\perp$. We can therefore find a partial isometry $w \in \mathcal{A}$ such that $w^*w = p^\perp$ and $ww^* = q^\perp$. Note that, using Proposition 1.3.1 and Remark B.1.31,

$$r(w) = q^\perp = r(x)^\perp = r(v)^\perp = k(v^*)$$

and hence $v^*w = 0$. We can similarly show that $vw^* = 0$ and hence $w^*v = 0 = wv^*$. Let $u = v + w$. Then

$$u^*u = v^*v + v^*w + w^*v + w^*w = p + 0 + 0 + p^\perp = \mathbf{1}.$$

Similarly $uu^* = \mathbf{1}$, and hence u is unitary. Furthermore,

$$\begin{aligned} w|x| &= ws(w)r(|x|)|x| && \text{by Proposition 1.3.1 (1) and (2)} \\ &= wp^\perp p|x| && \text{since } r(|x|) = r(x^*) = s(x) = p \text{ and } s(w) = p^\perp \\ &= 0 \end{aligned}$$

It follows that

$$u|x| = (v + w)|x| = v|x| + 0 = x.$$

□

B.3. Symmetric spaces

We present some supplementary results on symmetric spaces. Throughout this section we will assume that $(\mathcal{A},\tau)$ is a semi-finite von Neumann algebra. The following details some of the most significant properties of symmetric spaces.

PROPOSITION B.3.1. [9] [15] *Let $E \subseteq S(\mathcal{A},\tau)$ be a symmetric space. The following assertions hold:*

(1) *If $x \in S(\mathcal{A},\tau)$, then $x \in E$ if and only if $|x| \in E$ if and only if $x^* \in E$. In addition $\|x\|_E = \||x|\|_E = \|x^*\|_E$*
(2) *If $x \in E$, then $Re\, x, Im\, x \in E$ and $\|Re\, x\|_E, \|Im\, x\|_E \leq \|x\|_E$*
(3) *If $x \in E^{sa}$, then $x_+, x_- \in E$ and $\|x_+\|_E, \|x_-\|_E \leq \|x\|_E$.*
(4) *If $x, y \in E$ and $\mu_x = \mu_y$, then $\|x\|_E = \|y\|_E$.*
(5) *If $x \in S(\mathcal{A},\tau)$ and $y \in E$ are such that $|x| \leq |y|$, then $x \in E$ and $\|x\|_E \leq \|y\|_E$*
(6) *If $x \in E$ and $u \in \mathcal{A}$ is unitary, then $\|ux\|_E = \|x\|_E = \|xu\|_E$*
(7) *If $p \in \mathcal{P}(E)$ and $q \in \mathcal{P}(\mathcal{A})$ such that $p \sim q$, then $q \in \mathcal{P}(E)$ and $\|p\|_E = \|q\|_E$*
(8) *If $p \in \mathcal{P}(E)$ and $q \in \mathcal{P}(\mathcal{A})$ such that $q \precsim p$, then $q \in \mathcal{P}(E)$ and $\|q\|_E \leq \|p\|_E$*
(9) *If $p, q \in \mathcal{P}(E)$, then $p \vee q \in \mathcal{P}(E)$ and $\|p \vee q\|_E \leq \|p\|_E + \|q\|_E$*
(10) *E^+ is closed in E.*
(11) *If $(x_n)_{n=1}^\infty \cup \{x\} \subseteq E$ is such that $x_n \xrightarrow{E} x$, then $x_n a \xrightarrow{E} xa$ and $ax_n \xrightarrow{E} ax$ for all $a \in \mathcal{A}$.*

Next, we remark on the local integrability of singular value functions for elements in strongly symmetric spaces.

REMARK B.3.2. If E is a strongly symmetric space and $x \in E$, then

$$\int_0^t \mu_x(s)ds < \infty \qquad \forall t > 0.$$

PROOF. If $x \in E$, then $x \in (L_1 + L_\infty)(\tau)$, by Proposition 1.6.3(3) and so, by (1.5.2),

$$\int_0^1 \mu_x(s)ds = \|x\|_{L_1+L_\infty} < \infty.$$

For $0 < t \leq 1$, we have

$$\int_0^t \mu_x(s)ds \leq \int_0^1 \mu_x(s)ds < \infty,$$

since $\mu_x \geq 0$. For $t > 1$, we have

$$\int_0^t \mu_x(s)ds = \int_0^1 \mu_x(s)ds + \int_1^t \mu_x(s)ds \leq \int_0^1 \mu_x(s)ds + (t-1)\mu_x(1) < \infty,$$

since μ_x is decreasing. $\square$

Next, we consider the conditions under which convergence in the von Neumann algebra yields convergence in an associated symmetrically normed space.

PROPOSITION B.3.3. *Suppose $E \subseteq S(\mathcal{A}, \tau)$ is a symmetrically normed space. If $\{x\} \cup (x_n)_{n=1}^\infty \subseteq E \cap \mathcal{A}$ is such that $x_n \stackrel{\mathcal{A}}{\to} x$ and $s(x), s(x_n) \leq p$ for all $n \in \mathbb{N}^+$ for some $p \in \mathcal{P}(\mathcal{A})^f$ (or $r(x), r(x_n) \leq p$ for all $n \in \mathbb{N}^+$ for some $p \in \mathcal{P}(\mathcal{A})^f$), then $x_n \stackrel{E}{\to} x$.*

PROOF. Suppose $s(x), s(x_n) \leq p$ for all $n \in \mathbb{N}^+$ for some $p \in \mathcal{P}(\mathcal{A})^f$ and $x_n \stackrel{\mathcal{A}}{\to} x$. Note that $p \in E$, by Proposition 1.6.3(2). Furthermore, for $n \in \mathbb{N}^+$,

$$\begin{aligned}
\|x_n - x\|_E &= \|(x_n - x)p\|_E && \text{since } s(x), s(x_n) \leq p \\
&\leq \|x_n - x\|_{\mathcal{A}} \|p\|_E && \text{since } E \text{ is a normed } \mathcal{A}\text{-bimodule and } x_n - x \in \mathcal{A} \\
&\to 0
\end{aligned}$$

If $r(x), r(x_n) \leq p$ for all $n \in \mathbb{N}^+$ for some $p \in \mathcal{P}(\mathcal{A})^f$, we can use $x - x_n = p(x - x_n)$ and a similar method to the one used before to show that $x_n \stackrel{E}{\to} x$. $\square$

COROLLARY B.3.4. *Suppose $(\mathcal{A}, \tau)$ and $(\mathcal{B}, \nu)$ are semi-finite von Neumann algebras and $E(\tau)$ and $F(\nu)$ are symmetrically normed spaces. If $U : E \to F$ is continuous map with respect to the norms on E and F, then $U(x_n) \stackrel{\mathcal{T}_m}{\to} U(x)$, whenever $(x_n)_{n=1}^\infty \cup \{x\} \subseteq \mathcal{F}(\tau)$ is such that $x_n \stackrel{\mathcal{A}}{\to} x$ and $s(x_n) \leq s(x)$ for all $n \in \mathbb{N}^+$ (or $r(x_n) \leq r(x)$ for all $n \in \mathbb{N}^+$).*

PROOF. Suppose $(x_n)_{n=1}^\infty \cup \{x\} \subseteq \mathcal{F}(\tau)$ is such that $x_n \stackrel{\mathcal{A}}{\to} x$ and $s(x_n) \leq s(x)$ for all $n \in \mathbb{N}^+$. Note that $x \in \mathcal{F}(\tau)$ implies that $s(x) \in \mathcal{P}(\mathcal{A})^f$ and so, by Proposition B.3.3, $x_n \stackrel{E}{\to} x$. It follows that $U(x_n) \stackrel{F}{\to} U(x)$, since U is continuous. Therefore $U(x_n) \stackrel{\mathcal{T}_m}{\to} U(x)$, by Proposition 1.6.3(1). $\square$

The next result provides a useful characterization of submajorization.

THEOREM B.3.5. [11, p.59] *Suppose $(\mathcal{A}, \tau)$ and $(\mathcal{B}, \nu)$ are semi-finite von Neumann algebras. Let $\Sigma(\mathcal{A}, \mathcal{B})$ denote the set of linear operators $T : L^1 + L^\infty(\tau) \to L^1 + L^\infty(\nu)$ such that $T(\mathcal{A}) \subseteq \mathcal{B}$, $T(L^1(\mathcal{A})) \subseteq L^1(\mathcal{B})$, $\|T(x)\|_1 \leq \|x\|_1$ for all $x \in L^1(\mathcal{A})$ and $\|T(x)\|_\infty \leq \|x\|_\infty$ for all $x \in \mathcal{A}$. If $x \in L^1 + L^\infty(\tau)$ and $y \in L^1 + L^\infty(\nu)$, then $y \ll x$ if and only if there exists a $T \in \Sigma(\mathcal{A}, \mathcal{B})$ such that $y = T(x)$.*

We finish this section by presenting several supplementary results regarding reduced von Neumann algebras and associated symmetric spaces. The majority of the results presented in the first two propositions are given in [15]; the rest are easily checked.

Proposition B.3.6. *Suppose A is a von Neumann algebra and $p \in \mathcal{P}(\mathcal{A})$. Then*

(1) *the center of $\mathcal{A}_p$ is equal to $(Z(\mathcal{A}))_p$*
(2) *$\mathcal{A}_p$ is a factor if $\mathcal{A}$ is a factor.*
(3) *$(p\mathcal{A}p)^+ = p\mathcal{A}^+p$.*
(4) *If $q \in \mathcal{P}(\mathcal{M})$ and $q \geq p$, then $p\mathcal{A}p \subseteq q\mathcal{A}q$.*
(5) *If $x \in S(\mathcal{A}, \tau)^+$ and $0 \leq x \leq y$ for some $y \in p\mathcal{A}p$, then $x \in p\mathcal{A}p$.*

Proposition B.3.7. *Let $E \subseteq S(\mathcal{A}, \tau)$ be an $\mathcal{A}$-bimodule and suppose $p \in \mathcal{P}(\mathcal{A})$.*

(1) *Suppose $x \in S(\mathcal{A}, \tau)$. Then $x \in pEp$ if and only if $|x| \in pEp$.*
(2) *If $x \in E$ and $y \in pEp$ are such that $|x| \leq |y|$, then $x \in pEp$.*
(3) *If $x \in E$, then $x \in pEp$ if $x = pxp$.*
(4) *If $x \in pEp$, then $px = x = xp = pxp$.*
(5) *If $x \in pEp$, then $s(x_p) = s(x)_p$.*

The following density result follows from the density results given in Corollary 1.6.8 and the properties of the canonical map from pEp onto E_p.

Proposition B.3.8. *Suppose $E \subseteq S(\mathcal{A}, \tau)$ is a strongly symmetrically normed space and suppose $p \in \mathcal{P}(\mathcal{A})$. Let $\mathcal{D}_{(p)} := \{q \in \mathcal{P}(\mathcal{A})^f : q \leq p\}$ and $\mathcal{G}^f_{(p)} := span(\mathcal{D}_{(p)})$. If E has order continuous norm, then $\mathcal{G}^f_{(p)}$ is dense in pEp and $(\mathcal{G}^f_{(p)})^+$ is dense in $(pEp)^+ = pE^+p$.*

The final result in this section is also easily checked.

Proposition B.3.9. *Let $E \subseteq S(\mathcal{A}, \tau)$ be a normed $\mathcal{A}$-bimodule. If $p \in \mathcal{P}(\mathcal{A})$, then pEp is closed in E.*

B.4. Conditional expectations

Suppose $(\mathcal{B}, \tau)$ is a semi-finite von Neumann algebra and $\mathcal{A}$ is a von Neumann subalgebra of $\mathcal{B}$. A positive linear mapping $E : \mathcal{B} \to \mathcal{A}$ satisfying $E(\mathbf{1}) = \mathbf{1}$ and $E(xyz) = xE(y)z$ for all $x, z \in \mathcal{A}$ and $y \in \mathcal{B}$ is called a conditional expectation from $\mathcal{B}$ onto $\mathcal{A}$.

Theorem B.4.1. [46, p.177] *Suppose $\mathcal{B}$ is a von Neumann algebra acting on a Hilbert space H. If $\mathcal{A}$ is a von Neumann subalgebra of $\mathcal{B}$, then there exists a positive linear map E from $L^1(\mathcal{B}, \tau)$ onto $L^1(\mathcal{A}, \tau)$ such that for any $x, y \in L^1(\mathcal{B}, \tau)$*

(1) *$E(x^*) = E(x)^*$*
(2) *$x \geq 0$ and $E(x) = 0$ implies that $x = 0$*
(3) *$E(z) = z$ for all $z \in L^1(\mathcal{A}, \tau)$ and in particular $E(E(x)) = E(x)$;*
(4) *$E(\mathcal{B}) = \mathcal{A}$ and $\|E(z)\|_\infty \leq \|z\|_\infty$ for all $z \in \mathcal{B}$*
(5) *$E(z^*)E(z) \leq E(z^*z)$ for all $z \in \mathcal{B}$*
(6) *$E(E(w)v) = E(wE(v)) = E(w)E(v)$ for $w \in L^1(\mathcal{B}, \tau)$ and $v \in \mathcal{B}$ or $v \in L^1(\mathcal{B}, \tau)$ and $w \in \mathcal{B}$*
(7) *$\|E(x)\|_1 \leq \|x\|_1$*

REMARK B.4.2. Note that if $x, z \in \mathcal{A}$ and $y \in \mathcal{B}$, then $E(x) \in \mathcal{A} \subseteq \mathcal{B}$ and $yE(z) \in L^1(\mathcal{B}, \tau)$. Therefore

$$\begin{aligned}
E(xyz) &= E(E(x)(yE(z))) \qquad \text{since } x, z \in \mathcal{A} \subseteq L^1(\mathcal{A}, \tau) \text{ implies that } E(x) = x \text{ and } E(z) = z \\
&= E(x)E(yE(z)) \qquad \text{by Theorem B.4.1(6)} \\
&= E(x)E(y)E(z) \qquad \text{by Theorem B.4.1(6)} \\
&= xE(y)z \qquad \text{by Theorem B.4.1(3)}
\end{aligned}$$

Furthermore $E(\mathcal{B}) = \mathcal{A}$, by Theorem B.4.1(4) and $E(\mathbf{1}) = \mathbf{1}$, by Theorem B.4.1(3), since $\mathbf{1} \in L^1(\mathcal{A}, \tau)$. It follows that E is the extension of a conditional expectation from $\mathcal{B}$ onto $\mathcal{A}$. We will therefore call E a conditional expectation from $L^1(\mathcal{B}, \tau)$ onto $L^1(\mathcal{A}, \tau)$.

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Symbols