

Extrinsic Uncertainty, Ergodic Chaos and Monetary Policy in Two Intertemporal Economic Models

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Abstract

The thesis examines extrinsic uncertainty and ergodic chaos in two types of intertemporal economic models. The thesis is divided into four chapters.

In the first chapter the existence of extrinsic uncertainty also known as sunspots is analysed within the framework of a single commodity two-period pure exchange overlapping generations model. Transversality techniques are utilised to show that extrinsically uncertain equilibria are locally generic in the space of endowments. An application of the methodology of the multijet transversality theorem demonstrates that equilibria are regular for a dense set of utility functions. The analysis of this paper extends and complements existence results concerning the robustness of stationary sunspot equilibria.

In the second chapter, a multi-commodity version of the model of the first chapter is analysed. The equilibrium system is divided into the set of equations defined by (i) the stochastic budget constraints and by (ii) stochastic excess demand functions a geometric equilibrium is defined. A transversality technique shows that for almost every endowment vector the manifolds generated by (i) and (ii) do not intersect each other hence geometric stationary sunspot equilibria do not exist. This result is contrasted against the fact that regular non-stochastic monetary steady state equilibria generically exist. Furthermore, the existence of such equilibria is sufficient for the existence of an intrinsically uncertain equilibria. The results answer the question of the validity of the equilibrium concept of extrinsic uncertainty within a stationary environment.

Continuing with the same framework of the first chapter, in the third chapter a single commodity overlapping generations economy is analysed. The equilibrium properties of a perfect foresight backward looking economy are examined in which the equilibrium consumption profiles are ergodically chaotic. It is shown that if the dynamical equilibrium system is unimodal and concave then the expected value of the second period consumption is no greater than the value of second period consumption at the steady state. Conversely the expected value of the first period consumption is at least as large as the value of first period consumption at the steady state. Consequently, the expected utility of a representative agent along an ergodically chaotic equilibrium path is no greater than the value of utility at the steady state. This has a significant result in that the stabilisation of the dynamical system by means of a redistribution of resources improves society's welfare.

In chapter four a model of an infinitely lived agent in which money is an argument of the utility function and the money supply grows at a constant rate is analysed. As in chapter 3, the statistical properties of ergodically chaotic equilibria are examined in relation to the fixed point of the equilibrium equation. For a non-negligible set of parameters which determine the money growth rate the dynamical system is ergodically chaotic and the expected money demand is strictly greater than the stationary money demand. Ergodic chaos

is thus a locally structurally stable phenomenon. Consequently, the role of monetary policy has a significant bearing on the manner in which equilibrium money demand evolves on average over the long run in relation to a stable equilibrium system. Welfare can be improved if the equilibrium system is stabilised by the elimination of chaotic trajectories achieved by means of the opportune choice of a contractionary monetary policy

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To my father for his continual support and encouragement

The Existence of Stationary Sunspot Equilibrium. An Intersection Based Approach

Abstract

The genericity of equilibria is analysed in a pure exchange stationary overlapping generations model in which uncertainty is extrinsic (sunspot) and there is one consumption good in each period of time. The question is asked whether extrinsic equilibria are robust in the space of parameters of the underlying economy. The primary result is that (i) there is a subset of positive Lebesgue measure of endowments for which extrinsically uncertain equilibria exist and are regular and (ii) extrinsically uncertain equilibria exist and are regular for an open and dense set of smooth utility functions. These results are derived by means of transversality theorems in the case of (i) and the multijet transversality theorem in the case of (ii). The analysis extends and complements pre-established existence results concerning the robustness of sunspot equilibria in the space of probabilities by showing that sunspot equilibria are structurally robust in the space of endowments and utility functions.

1.1. Introduction

Uncertainty can influence the behaviour of economic agents and consequently equilibria in many ways. For instance, a trader, when deciding which trading strategy to employ, may base his trading decisions on differing signals such as historical price trends, anticipated price trends, latest governmental figures, the behaviour of his cohort or even an event such as the day of the week. The trader, factoring these events into his trading strategy, will affect the resultant equilibria by such a belief. In particular, such beliefs may be entirely unrelated to the economic fundamentals. In such a case uncertainty is termed *extrinsic* or, by a coinage of Cass and Shell [21], *sunspot*. Belief in such events is a form of rational expectations in which the price system embeds the belief in an extrinsically determined event which is entirely *uncorrelated with the fundamentals of the economy*. Whilst the reasoning which accompanies the formation of such beliefs is of interest in its own right, the question addressed here is

whether equilibria do in fact reflect these beliefs. More precisely, are sunspot equilibria structurally stable phenomena or do they exist for negligible set of parameters which may arise rarely in which case one may dismiss such equilibria as irrelevant and having no sound basis?

In this Chapter the *equilibrium set of sunspot equilibria* are therefore analysed for a pure exchange stationary overlapping generations model in which there is *one consumption good* each period and a representative agent who lives for two periods. Within this framework Azariadis and Guesnerie [5] and Spear [75] show that (see Section 1.2) for a given endowment vector there is an open set of probabilities for which stationary sunspot equilibria (SSE) exist. The existence result of Azariadis and Guesnerie [5] and Spear [75] is extended by showing that *SSE exist for a set of endowment vectors of non-negligible measure and an open and dense set of utility functions*. This means that SSE exist for non-negligible sets of parameters which are not affected by the occurrence of extrinsic uncertainty thereby demonstrating that *SSE are structurally stable*¹ phenomena. Specifically, by enlarging the domain of definition of an economy from probabilities to endowments as well as utility functions, one can discuss the notion of a large set of economies each of which is close to each other and for which stationary sunspot equilibria exist. Hence, an economy which supports a sunspot equilibrium is qualitatively no different to many nearby economies².

In order to examine the genericity of SSE, the strategy employed is to divide the primitive system of equations into two subsystems of equations defined respectively as stochastic offer curves and market clearing equations. For a given parameterization (endowment vector and utility function), the stochastic offer curves and market clearing equations each generate manifolds in the space of consumption allocations for which a point of intersection is a stationary equilibrium. In particular, equilibria are defined as regular if the manifolds intersect transversally. A regular equilibrium has the property that small changes in the parameterization of the functions which give rise to the manifolds induces small changes in the equilibrium allocations and the resultant intersection of the manifolds retains the property of transversality. For instance, for a given utility function, if an equilibrium is regular then a small movement in the endowment vector will perturb the equilibrium allocation infinitesimally and the resultant equilibrium is regular.

Having established the existence of regular SSE for a non-negligible set of endowments, this result is utilised to show that there exists an open and dense set of utility functions for which stationary sunspot equilibria exist. This is carried out by taking an endowment vector for which the aforementioned manifolds are transversal. A perturbation of the utility function does not then change the property of non-empty transversal intersection of the manifolds and regular equilibria remain in existence. This result is an application of the multijet transversality theorem.

¹ Structural stability is not to be confused with dynamical stability. Structural stability means that a perturbation in the parameter space does not change a given property of the system. Structural stability can be either a local or global argument. The former is of interest in this paper.

² A similar approach was taken in Davila *et al.*[26]. The methodology of that paper differs to that of this discussion in that the properties of the underlying certainty economy are utilised whereas the approach of this paper is the manifold construct of the set of stationary sunspot equilibria.

This paper is laid out as follows. In Section 1.2 stationary sunspot equilibria are discussed. This discussion serves as a backdrop to Chapter 2 and to some degree Chapter 3. In Section 1.3 the model is presented and discussed. The equilibrium system is divided into two sets of equations the interaction of which define the set of sunspot equilibrium. The genericity of equilibria in the space of endowments is discussed in Section 1.4 whereas in Section 1.5 the genericity of equilibria in the space of utility functions is discussed. Section 1.6 contrasts the results and certain aspects of the sunspot model presented here. The Appendix (Section 1.7) contains definitions and proofs when necessary.

1.2. 2-Cycles and Stationary Sunspot Equilibria

An overlapping generations economy starts at time $t = 0$. A consumer is indexed by the date in which he enters the economy, i.e. $t = 0, 1, 2, \dots$. An agent lives for lives for two periods; t and $t + 1$, apart from the first consumer who is born old and lives for one period. As such two agents only ever coexist; one young and one old. For the purpose of simplicity and without any loss of generality, the first consumer is overlooked. There is a single perishable good each period over which utility is defined. For each generation $t \geq 1$ the consumer has the demand vector $x_t = (x_t^t, x_t^{t+1}) \in \mathbb{R}_{++}^2$ and each consumer is characterised by a utility function and an endowment vector as follows:

- i. The utility function $u : \mathbb{R}_{++}^2 \rightarrow \mathbb{R}$ possesses standard properties; $u^{-1}(a)$ is bounded from below for all $a \in \mathbb{R}_{++}$ and the closure satisfies $cl\{x' \in \mathbb{R}_{++}^2 : u(x') \geq u(x)\} \subset \mathbb{R}_{++}^2$. u is smooth and has positive derivatives, i.e. $Du \in C^\infty(\mathbb{R}_{++}^2, \mathbb{R}_{++}^2)$ as well as being strictly concave and having negative definite Hessian. u belongs to the space of utility functions U which possess the aforementioned properties and is endowed with the compact open topology³.
- ii. The endowment vector of each agent satisfies $\omega = (\omega_1, \omega_2) \in \Omega \equiv \mathbb{R}_{++}^2$.

Both the endowment vector and utility function are time independent and hence stationary and as such are the same for all agents in the economy. At each period of time fiat money is the only asset in the economy. The consumption good can be exchanged for money thereby allowing consumption in excess of the endowment in the first or second period of life but not both. An economy is defined as a pair $(\omega, u) \in \Omega \times U$.

³ See Appendix 1.7 for a discussion of the compact open topology.

For each t the price of the commodity is $p_t \in \mathbb{R}_{++}$. Each consumer chooses a consumption profile by maximising lifetime utility subject to lifetime constraint⁴:

$$\max_{x_t^t, x_t^{t+1}} u(x_t^t, x_t^{t+1}) \text{ s.t. } p_t x_t^t + p_{t+1} x_t^{t+1} \leq p_t \omega_1 + p_{t+1} \omega_2 \quad (1.2.1)$$

The resultant demand functions of the agent born at time t for t and $t+1$ are respectively $f_t^t(p_t, p_{t+1}, \omega)$ and $f_t^{t+1}(p_t, p_{t+1}, \omega)$ where f_t^t is the demand at t of the agent born at t and f_t^{t+1} is the demand at $t+1$ of the agent born at t . An equilibrium is defined as a sequence of commodity prices $\{\dots, p_t, p_{t+1}, \dots\}$ for which the demand of the old agent (born in $t-1$) f_{t-1}^t and the demand of the young agent (born in t) f_t^t sum to the supply of the endowments:

$$f_{t-1}^t(p_{t-1}, p_t, \omega) + f_t^t(p_t, p_{t+1}, \omega) = \omega_1 + \omega_2, \quad \forall t \geq 1 \quad (1.2.2)$$

Since both u and ω are stationary, if $p_{t-1} = p_t$ then $f_{t-1}^{t-1}(\cdot) = f_t^t(\cdot)$ and $f_{t-1}^t(\cdot) = f_t^{t+1}(\cdot) \quad \forall t$. In such a case the constraint of each agent is equivalent to the market clearing equation hence $p_{t-1} = p_t$ is an equilibrium price (Tuinstra and Weddepohl [78], Proposition 1, pg. 190). Such an equilibrium is termed the steady state⁵.

Another type of equilibrium is that of a periodic equilibrium of order 2 or a 2-cycle. A 2-cycle equilibrium is an equilibrium in which there is an equilibrium price sequence $\{\dots, p_t, p_{t+1}, p_{t+2}, p_{t+3}, \dots\}$ in which prices fluctuate consecutively between p_t and p_{t+1} ($p_t = p_{t+2}$ and $p_{t+1} = p_{t+3} \quad \forall t \geq 1$) and equilibrium allocations are given by (1.2.2) which fluctuate between (f_t^t, f_t^{t+1}) , $(f_{t+1}^{t+1}, f_{t+1}^{t+2})$, $\text{mod } t=2$. Since the equilibrium repeats itself every two periods, in order to examine 2-cycles it is sufficient to consider two time periods; t and $t+1$, or equivalently $t=1$ and $t=2$. Let $p^t = p^{t+2} = p^1$, and $p^{t+1} = p^{t+3} = p^2$ for all $t \geq 1$ ⁶. A 2-cycle is then an equilibrium to the maximisation problems of agent 1 who optimises $\max_{x_2^2, x_2^1} u(x_2^2, x_2^1)$ s.t. $p_1 x_1^1 + p_1 x_1^2 \leq p_1 \omega_1 + p_2 \omega_2$ and agent 2 who optimises $\max_{x_2^2, x_2^1} u(x_2^2, x_2^1)$ s.t. $p_2 x_2^2 + p_1 x_2^1 \leq p_2 \omega_1 + p_1 \omega_2$.

The question arises as to the conditions under which 2-cycles occur. The approach taken here is diagrammatic, making use of the properties of the offer curves⁷. The offer curve of agent 1 is the loci of points described by the vector $f_1 = (f_1^1(p_1, p_2, \omega), f_1^2(p_1, p_2, \omega))$ as

⁴ Money demand has been overlooked in this economy as it plays no role in the analysis of this section.

⁵ If the OLG economy converges on this point for some t then it will remain there for all $t+1$.

⁶ It is stressed that the concern of this discussion is the *comparative statics* of the 2-cycle economy and *not* the dynamic process underlying the sequence of equilibrium prices and trades. See for instance Benhabib and Day [15] and Gale [33] for a discussion concerning the occurrence of cycles and topological chaos in the OLG model presented here.

⁷ See Balasko and Ghiglino [12] for a discussion of cycles.

$(p_1, p_2) \in \mathbb{R}_{++}^2$ vary for a given $\omega \in \mathbb{R}_{++}^2$. The offer curve of agent 2 is the loci of points described by the vector $f_2 = (f_2^2(p_2, p_1, \omega), f_2^1(p_2, p_1, \omega))$ as $(p_1, p_2) \in \mathbb{R}_{++}^2$ vary for the same $\omega \in \mathbb{R}_{++}^2$. Both offer curves pass through ω and given that $p_1 = p_2$ is a steady state equilibrium both f_1 and f_2 also pass through the equilibrium allocation $(f_1^1(p_1, p_1, \omega), f_1^2(p_1, p_1, \omega))$. In particular, f_1 and f_2 are symmetrical images of each other about the line passing through ω and $(f_1^1(p_1, p_1, \omega), f_1^2(p_1, p_1, \omega))$ ⁸. Figure 1.2.1 below illustrates this. In figure 2.1 the offer curves f_1 and f_2 both pass through the endowment vector ω and the steady state at point c and lies on the line with orthogonal equilibrium price vector (p_1, p_1) . A 2-cycle exists if the offer curves intersect at a point other than c e.g. point a . At point a the orthogonal price vector is (p_1, p_2) with $p_1 \neq p_2$. Given that f_1 and f_2 are reflections of each other about the line (ω, c) , then the existence of an equilibrium at a entails the existence of an equilibrium at point b with orthogonal price vector (p_2, p_1) ⁹.

A sufficient condition for the existence of a 2-cycle is that the offer curve of agent 1 (and by symmetry agent 2) has a gradient less than 1 at the steady state (Balasko and Ghiglino [12] pg. 570 and Grandmont [38]) for which it is sufficient that the offer curve strongly bend backwards as can be seen in Figure 1.2.1 where f_1 has a gradient less than 1 at c . Since f_1 is continuously differentiable in (p_1, p_2) it will then cut f_2 at a point to the right of c , e.g. b as prices vary by which a 2-cycle obtains. A symmetrical argument can be made for the offer curve f_2 .

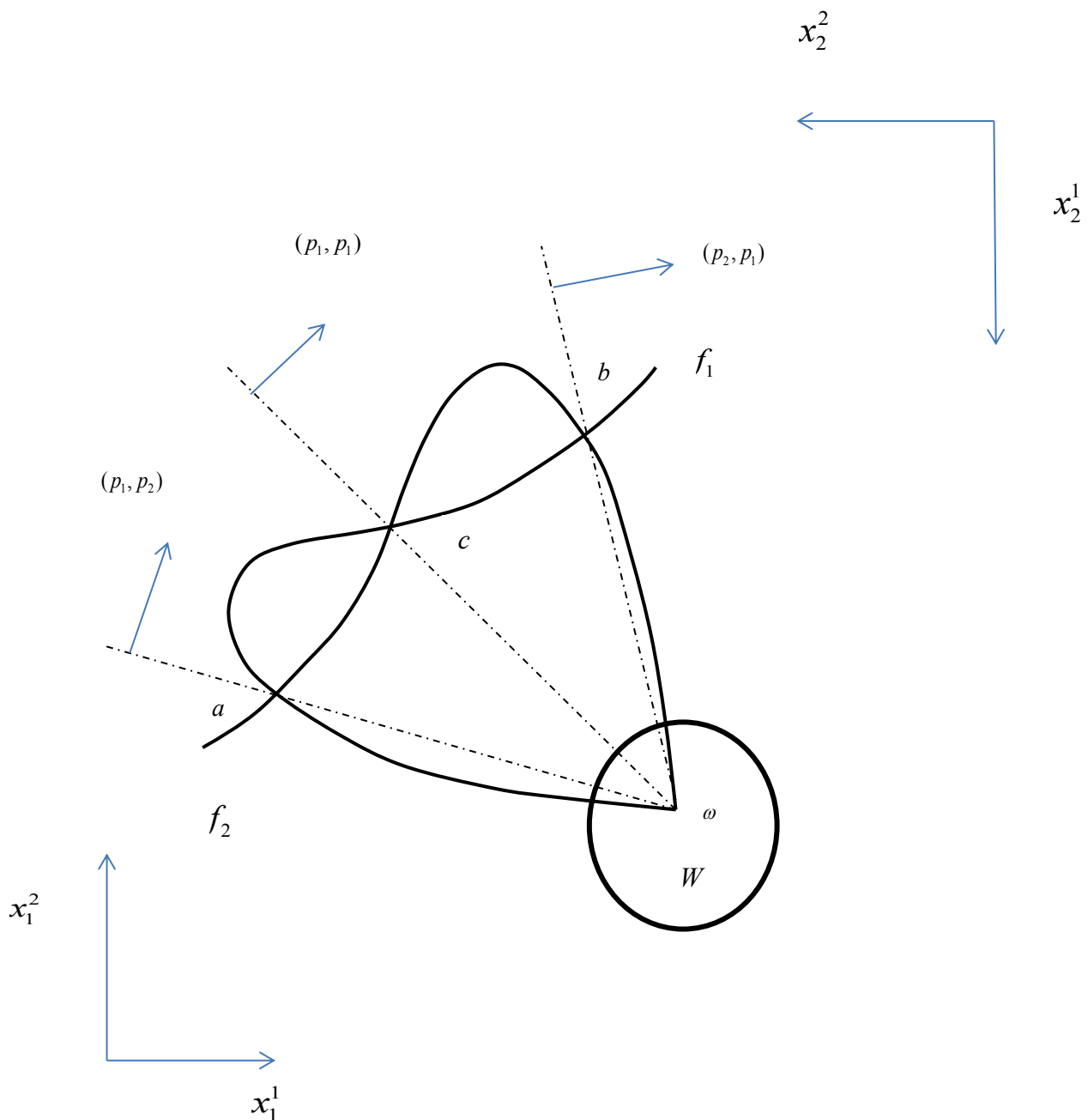
Given the construction of the offer curve that is required to generate 2-cycles, the question arises as to whether 2-cycles are robust, that is whether the offer curves intersect multiple times for a non-negligible set of endowments. Given that both f_1 and f_2 are smooth functions of not only prices but also endowments, as ω varies infinitesimally the offer curves also vary infinitesimally and thereby retain the property of multiple intersections and 2-cycles remain in existence. This point is demonstrated by Balasko and Ghiglino [12] (Proposition 1, pg. 572).

In Figure 1.2.1, for example, for all $\omega \in W$ 2-cycles are locally generic or structurally stable in a local sense; as ω is varied in W f_1 and f_2 intersect at multiple points necessarily passing through a steady state equilibrium at which point the offer curves have a gradient less than 1.

⁸ See [12] for a demonstration and discussion of this point.

⁹ A 2-cycle equilibrium price vector (p_1, p_2) implies the existence of a 2-cycle equilibrium price vector of the form (p_2, p_1) and is accompanied by an invariant equilibrium stationary steady state equilibrium price (p_1, p_1) . Thus 2-cycle equilibrium prices always appear in pairs and the total number of equilibrium prices are odd in number. This is an application of the Hopf-Poincare Theorem.

Figure 1.2.1



2-cycle equilibria are therefore locally robust in the space of endowments and can be considered a perfect foresight rational expectations equilibrium in which it is correctly anticipated that p_2 follows p_1 which follows p_2 and so on... These rational expectations equilibria are such that the intrinsic data of the economy is time invariant. The model may be extended by allowing the price formation to incorporate uncertainty. For example, price uncertainty may evolve in a manner such that p_1 follows price p_1 with probability π_{11} , p_2 follows p_1 with probability π_{12} whereas p_1 follows p_2 with probability π_{21} and p_2 follows

p_2 with probability π_{22} ¹⁰. The particular structure of any resultant equilibria is one in which the fundamentals of the economy are unaffected by the occurrence of randomness and price uncertainty is a reflection of uncertainty which is *extrinsic* to the economy. As such the randomness of the prices is due entirely to the agents' commonly held belief concerning the occurrence of the uncertainty. Azariadis [4], Azariadis and Guesnerie [5] and Cass and Shell [21] examined this type of rational expectations equilibrium termed, by a coinage of the latter two authors, *sunspot equilibria*.

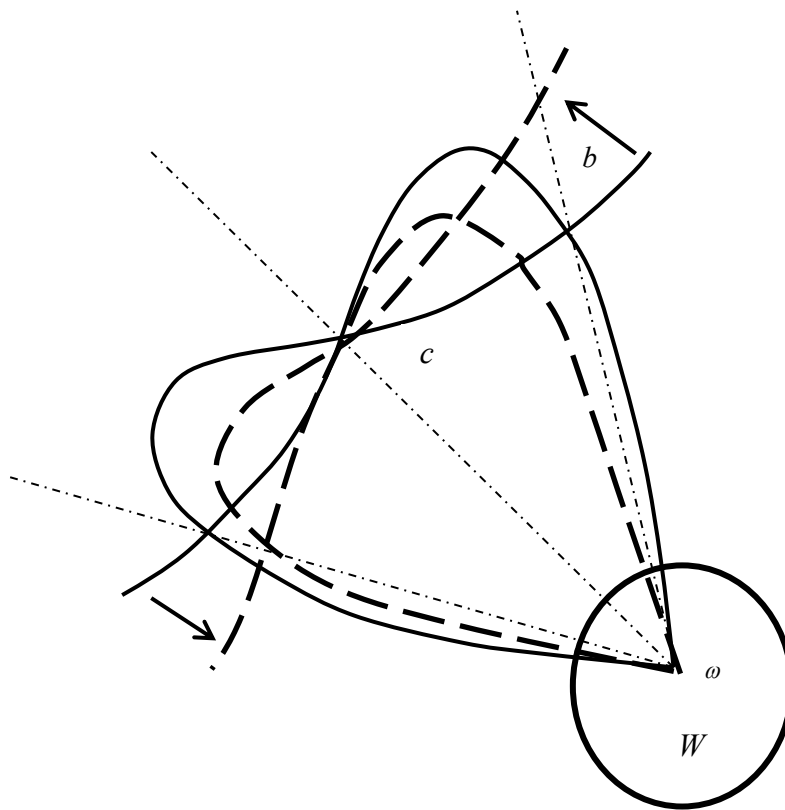
The existence of sunspot equilibria within the framework of a stationary OLG model (*stationary sunspot equilibria*) was analysed in Azariadis and Guesnerie [5] and Spear [75]. In the former it was demonstrated¹¹ that stationary sunspot equilibria exist within the vicinity of cyclical equilibria of order 2. Intuitively, SSE occur as follows. In the case of stochastic demands agent 1 has the demand vector $f_1^1(p_1, p_2, \pi_{12}, \omega)$ in the first period of life and faces contingent demands $f_2^{11}(p_1, p_2, \pi_{12}, \omega)$ and $f_2^{12}(p_1, p_2, \pi_{12}, \omega)$ in the second period of life whereas agent 2 has $f_1^2(p_1, p_2, \pi_{21}, \omega)$ in the first period of life and faces contingent demands $f_2^{21}(p_1, p_2, \pi_{21}, \omega)$ and $f_2^{22}(p_1, p_2, \pi_{21}, \omega)$ in the second period of life¹². If π_{12} and π_{21} are close to 1 then the f_2^{12} and f_2^{21} will be close to the values of the demand functions of the 2-cycle economy. Consequently, the demands derived from the sunspot model are perturbations of the 2-cycle demands or conversely 2-cycles are degenerate stochastic demands. This argument is represented diagrammatically. In Figure 1.2.2 the 2-cycle offer curves are illustrated as thick curves. The sunspot offer curves are represented as the dotted curves and intersect at multiple points hence SSE exist. Perturbing the 2-cycle offer curves does not change the existence of multiple intersection points. As such the sunspot offer curves are perturbations in the probability space of 2-cycle offer curves where the probabilities are disturbed in the vicinity of π_{12} and π_{21} close to 1. Hence the existence of 2-cycles implies the existence of stationary sunspot equilibria as the latter are derived by means of perturbations of the 2-cycle offer curves.

¹⁰ Uncertainty follows a first order Markov process.

¹¹ The relation between 2-cycles and SSE in the OLG framework was established by Azariadis and Guesnerie (Theorem 1, pg. 729 and [5]) where it was shown that in an economy that admits a periodic equilibria of order two, there is generically a neighbourhood $\nu(\pi)$ of the 2x2 Markov matrix π such that SSE exist with respect to every π in $\nu(\pi)$. Furthermore, 2-cycles were shown to be necessary and sufficient for the existence of SSE (Theorem 2 pg. 732 [12]).

¹² f_1^s is the demand in the first period of life given that the agent is born into state s and $f_2^{ss'}$ is the demand in the second period of life given that the agent is born into state s and faces state s' in the second period. See Section 3.

Figure 1.2.2



In sum, SSE are obtained by the perturbation of offer curves for which equilibria of order 2 exist. Given that 2-cycles are locally generic in the space of endowments and that 2-cycles imply that there is a neighbourhood of probabilities for which SSE exist then one would expect that there is an open set of endowments *given* such probabilities for which SSE exist. In other words, SSE are locally generic in the space of probabilities *and* endowments. The establishing of this would then augment the parameter space by extending the domain of definition of an economy from probabilities to the inclusion of endowments. By an analogous argument, one would expect that if there exists a SSE, then such an equilibrium would be robust to perturbations in the utility function which gives rise to such equilibria. By the consideration of not only the probability space but also the endowment and utility functions, the generic existence of stationary sunspot equilibria can be examined.

1.3. The Model

The model is a stochastic extension of the OLG model discussed in Section 1.2 where uncertainty is present as there exists a commonly held belief concerning the probability of the

realisation of an event s which has no bearing on the economy (u, ω) . Instead, the realisation of the event has an effect on price formation in the next period. Uncertainty is therefore *extrinsic* or *sunspot* as it does not have an impact on the fundamentals of the economy. In order to facilitate the analysis and without loss of generality it is assumed that there are two states of nature $s \in \{1, 2\}$. For each state of nature there is therefore a price p_s . Uncertainty evolves in accordance with a first order Markov process in which the transition matrix is $\pi = \{\pi_{ss'}\}_{s,s'=1,2}$ where $\pi_{ss'}$ is the probability of state s' being realised conditional upon the realisation of state s . Therefore, the price $p_{s'}$ follows the price p_s with probability $\pi_{ss'}$. Such that probabilities are non-degenerate, it is assumed that $(\pi_{12}, \pi_{21}) \in \text{int}(I^2)$ is fixed where $I = [0, 1]$; the belief in the uncertain or sunspot event occurs with non-zero probability. Hence the occurrence of $(p_1, p_2) \in \mathbb{R}_{++}^2$ takes place with non-zero probability. The triple $(u, \omega, \pi) \in U \times \Omega \times \text{int}(I^2)$ defines a *sunspot economy*. Given the stationarity of the economy as well as the Markovian structure of uncertainty, to characterise the problem it suffices to consider two agents; Agent 1 born into state 1 who faces states 1 and 2 with probabilities π_{11} and π_{12} respectively and Agent 2 born into state 2 who faces states 1 and 2 with probabilities π_{21} and π_{22} respectively. The maximisation problems are:

Problem 1.1 – Agent 1

$$\max_{x_1^1, x_2^{1j}} \pi_{11} u(x_1^1, x_2^{11}) + \pi_{12} u(x_1^1, x_2^{12})$$

$$\text{s.t. } p_1 x_1^1 + p_m x_m^1 \leq p_1 \omega_1 \text{ and}$$

$$p_j x_2^{1j} + \leq p_j \omega_2 + p_m x_m^1, \text{ for } j = 1, 2$$

Problem 1.2 – Agent 2

$$\max_{x_1^2, x_2^{2j}} \pi_{21} u(x_1^2, x_2^{21}) + \pi_{22} u(x_1^2, x_2^{22})$$

$$\text{s.t. } p_2 x_1^2 + p_m x_m^2 \leq p_2 \omega_1 \text{ and}$$

$$p_j x_2^{2j} + \leq p_j \omega_2 + p_m x_m^2, \text{ for } j = 1, 2$$

x_1^s , $s = 1, 2$ is first period consumption of an agent born into state s . $x_2^{ss'}$ ($s, s' \in \{1, 2\} \times \{1, 2\}$) is the second period consumption of an agent born into state s contingent on state s' occurring in the second period of life with probability $\pi_{ss'}$. p_m is the price of money and the demand

for money of an agent born into state s is $x_m^s = (p_s/p_m)(\omega_1 - x_1^s)$, $s=1,2$. By eliminating money demand, the constraints in Problems 1.1 and 1.2 can be written respectively as (1.3.1), (1.3.2) and (1.3.3), (1.3.4):

$$p_1 x_1^1 + p_1 x_2^{11} = p_1 \omega_1 + p_1 \omega_2 \quad (1.3.1)$$

$$p_1 x_1^1 + p_2 x_2^{12} = p_1 \omega_1 + p_2 \omega_2 \quad (1.3.2)$$

$$p_2 x_1^2 + p_1 x_2^{21} = p_2 \omega_1 + p_1 \omega_2 \quad (1.3.3)$$

$$p_2 x_1^2 + p_2 x_2^{22} = p_2 \omega_1 + p_2 \omega_2 \quad (1.3.4)$$

Equilibrium obtains when supply is equal to demand in the goods and money market across all states of nature thereby yielding four commodity market equilibrium equations (1.3.5) and two money market equilibrium equations given by (1.3.6), where $\bar{m}$ is the stationary money supply in the economy:

$$x_1^s + x_2^{1s} = \omega_1 + \omega_2 = x_1^s + x_2^{2s}, \quad s=1,2 \quad (1.3.5)$$

$$x_m^s = (p_s/p_m) \cdot (\omega_1 - x_1^s) = \bar{m}, \quad s=1,2 \quad (1.3.6)$$

Since equilibrium in the goods market implies equilibrium in the money market, the equilibrium condition (1.3.6) is redundant¹³. By (1.3.5), equilibrium in the goods market implies that $x_2^{11} = x_2^{21}$ and $x_2^{12} = x_2^{22}$. This latter condition is both sufficient and necessary for equilibrium (Spear [75] pg. 363). Taking this equilibrium condition into account and noting that the market clearing conditions (1.3.5) imply the constraints (1.3.1) and (1.3.4) and conversely for $s=s'$, then subject to the constraints (1.3.1) – (1.3.4) being satisfied, the equilibrium conditions (1.3.5) are reduced to the following two equations $x_1^1 + x_2^{21} = \omega_1 + \omega_2$ and $x_1^2 + x_2^{12} = \omega_1 + \omega_2$.

If equilibria reflect extrinsic uncertainty or sunspots then allocations will differ across states of nature; $x_1^1 \neq x_1^2$ and $x_2^{12} \neq x_2^{21}$. A stationary sunspot equilibrium is thereby characterised by the vectors $(x_1^1, x_2^{21}) \neq (x_1^1, x_2^{12}) \neq (x_1^2, x_2^{21}) \neq (x_1^2, x_2^{12})$ as well as an equilibrium price system in which $p_1 \neq p_2$. If instead $p_1 = p_2$ is an equilibrium price then (1.3.1) is equal to (1.3.2) and (1.3.3) is equal to (1.3.4), which implies that the stochastic maximisation problems 1.1 and 1.2 above collapse to certainty maximisation problems in which $x_1^1 = x_1^2$ and $x_2^{11} = x_2^{21} = x_2^{12} = x_2^{22}$ as ω is the same in both maximisation problems. Any resultant equilibrium cannot then differ across states of nature and is not a sunspot equilibrium. Such an equilibrium is termed a steady state or a certainty equilibrium. It is noted that for every

¹³ This approach is taken in Davila *et al.* [26] and Spear [75].

endowment vector there exists a steady state equilibrium where of course the equilibrium allocation depends upon the specific value of the endowment vector.

In order to examine the structure of equilibrium, the system of first order conditions and market clearing conditions are used:

$$(1 - \pi_{12})D_1u(x_1^1, x_2^{21})(x_1^1 - \omega_1) + (1 - \pi_{12})D_2u(x_1^1, x_2^{21})(x_2^{21} - \omega_2) \\ + \pi_{12}D_1u(x_1^1, x_2^{12})(x_1^1 - \omega_1) + \pi_{12}D_2u(x_1^1, x_2^{12})(x_2^{12} - \omega_2) = 0 \quad (1.3.7)$$

$$\pi_{21}D_1u(x_1^2, x_2^{21})(x_1^2 - \omega_1) + \pi_{21}D_2u(x_1^2, x_2^{21})(x_2^{21} - \omega_2) \\ + (1 - \pi_{21})D_1u(x_1^2, x_2^{12})(x_1^2 - \omega_1) + (1 - \pi_{21})D_2u(x_1^2, x_2^{12})(x_2^{12} - \omega_2) = 0 \quad (1.3.8)$$

$$x_1^1 + x_2^{21} - \omega_1 - \omega_2 = 0 \quad (1.3.9)$$

$$x_1^2 + x_2^{12} - \omega_1 - \omega_2 = 0 \quad (1.3.10)$$

A zero of the system of equations (1.3.7) – (1.3.10) is clearly a zero of the system of equations (1.3.1) – (1.3.6) subject to solving the individual utility programming problems. Hence an equilibrium of the latter is an equilibrium of the former and conversely.

Let $u \in U$ be fixed and let $X \equiv \mathbb{R}_+^4$ be the space of feasible consumption allocations, a generic element being $x = (x_1^1, x_2^{12}, x_1^2, x_2^{21})$. Since a certainty equilibrium characterised by $x_1^1 = x_1^2$ and $x_2^{21} = x_2^{12}$ cannot be sunspot equilibrium allocation, the set of feasible stationary sunspot allocations is $\bar{X} = \{x \in X : x_1^1 \neq x_1^2, x_2^{12} \neq x_2^{21}\}$. $\bar{X}$ is a submanifold of X and is locally identified with $\mathbb{R}_+^4$. It is noted that the set theoretic difference $X \setminus \bar{X}$ is the set of feasible steady state equilibria which has negligible measure in X .

Define the smooth family of maps¹⁴ $\psi_u : X \times \Omega \rightarrow \mathbb{R}^2$ by equations (1.3.7) and (1.3.8). A member of this family of maps is defined by the map $\psi_{u,\omega} : X \rightarrow \mathbb{R}^2$ parameterized by $u \in U$ and $\omega \in \Omega$. As a point of definition $\psi_{u,\omega}$ is termed the stochastic offer curve. A zero of $\psi_{u,\omega}$ is an equilibrium allocation for which the first order conditions are satisfied; if $x \in \psi_{u,\omega}^{-1}(0)$ for some $\omega \in \Omega$ then (1.3.7) and (1.3.8) are satisfied. π is held fixed in the map $\psi_{u,\omega}$ and is suppressed. Let $\phi : X \times \Omega \rightarrow \mathbb{R}^2$ be family of maps defined by (1.3.9) and (1.3.10) where again for some $\omega \in \Omega$ a member of the family of maps is the map $\phi_\omega : X \rightarrow \mathbb{R}^2$. ϕ_ω is parameterized by $\omega \in \Omega$ and is termed the market clearing equation. A zero of ϕ_ω is an

¹⁴ Smoothness follows from the assumption made on the utility function.

equilibrium in the goods market; if $x \in \phi_\omega^{-1}(0)$ for some $\omega \in \Omega$ then (1.3.9) and (1.3.10) are satisfied. For $(\omega, u, \pi) \in \Omega \times U \times \text{int}(I^2)$, a stationary sunspot equilibrium is a vector $x = (x_1^1, x_2^{12}, x_1^2, x_2^{21}) \in \bar{X}$ such that $x \in \psi_{u,\omega}^{-1}(0) \cap \phi_\omega^{-1}(0)$. An equilibrium is regular if $\psi_{u,\omega}^{-1}(0) \nparallel \phi_\omega^{-1}(0)$.

The remainder of this chapter is laid out as follows. In Section 1.4 it is shown that there is an open set W of endowments Ω of non-negligible Lebesgue measure for which regular stationary sunspot equilibria such that for every element of this set $\psi_{u,\omega}^{-1}(0)$ and $\phi_\omega^{-1}(0)$ generate smooth manifolds which intersect each other transversally, i.e. $\psi_{u,\omega}^{-1}(0) \nparallel \phi_\omega^{-1}(0)$. This implies that SSE are generically regular in W and are thereby structurally robust, standing up to small perturbations in the space of endowments. In Section 1.5 it is shown that for *a.e.* $\omega \in W$ (such that $\psi_{u,\omega}^{-1}(0) \nparallel \phi_\omega^{-1}(0)$), there is an open and dense set of U such that for all u in this set one has that $\psi_{u,\omega}^{-1}(0) \nparallel \phi_\omega^{-1}(0)$ thereby demonstrating that SSE are structurally stable in the space of utility functions. Both arguments are local in nature.

1.4. Genericity of Regular Equilibria in the Space of Endowments

It is shown that there is a non-negligible set of endowments $W \subset \Omega$ every point of which supports a regular stationary sunspot equilibria. Stated concisely, for *a.e.* $\omega \in W$:

- i. $x \in \psi_{u,\omega}^{-1}(0) \cap \phi_\omega^{-1}(0)$ and $x \in X \setminus \bar{X}$
- ii. $\psi_{u,\omega}^{-1}(0) \nparallel \phi_\omega^{-1}(0)$

i is the statement that sunspots generically exist in the set W . ii is the statement that sunspot equilibria are regular, i.e. characterised by a transversality condition. As a consequence of i and ii, SSE are structurally stable being robust to perturbations of ω . If i and ii are demonstrated then a typical stationary OLG sunspot model is “rich” enough to rebuke the claim that sunspot equilibria are not significant. In this regard and in continuation of the enquiry of Cass and Shell [21], sunspots do matter. Furthermore, by the demonstration of i and ii, the existence results of Azariadis and Guesnerie [5] and Spear [75] are augmented by extending the parameter space over which SSE are shown to exist from the probability space to the space of endowments.

In order to show the above, it is assumed that there is some endowment vector ω for which a 2-cycle exists. By the discussion of Section 2.1 there exists a stationary sunspot equilibrium. In order to show points i and ii above it is required to show that for a set of ω of

non-negligible measure, the tangent spaces of $\psi_{u,\omega}^{-1}(0)$ and of $\phi_{\omega}^{-1}(0)$ are non-tangential at every point of intersection and hence span the ambient space X thereby implying that in a vicinity of ω , every endowment vector supports a sunspot equilibrium.

It is recalled that for $x \in \psi_{u,\omega}^{-1}(0) \cap \phi_{\omega}^{-1}(0)$, if $\psi_{u,\omega}^{-1}(0) \nparallel \phi_{\omega}^{-1}(0)$ then the following condition is satisfied^{15 16}:

$$T_x \psi_{u,\omega}^{-1}(0) + T_x \phi_{\omega}^{-1}(0) = X \quad (1.4.1)$$

(1.4.1) is the condition that the tangent spaces at a point common to the two manifolds are not tangential to each other or equivalently span the ambient space and are hence *transversal*. Given that both $\psi_{u,\omega}^{-1}(0)$ and $\phi_{\omega}^{-1}(0)$ are parameterized by ω , one would expect that if $\psi_{u,\omega}^{-1}(0) \nparallel \phi_{\omega}^{-1}(0)$ then as ω is perturbed to ω' within a neighbourhood the resultant manifolds $\psi_{u,\omega'}^{-1}(0)$ and $\phi_{\omega'}^{-1}(0)$ retain the property of transversality. Figure 1.4.1 demonstrates this principle¹⁷.

In order to give rigour to the foregoing argument, the following are demonstrated.

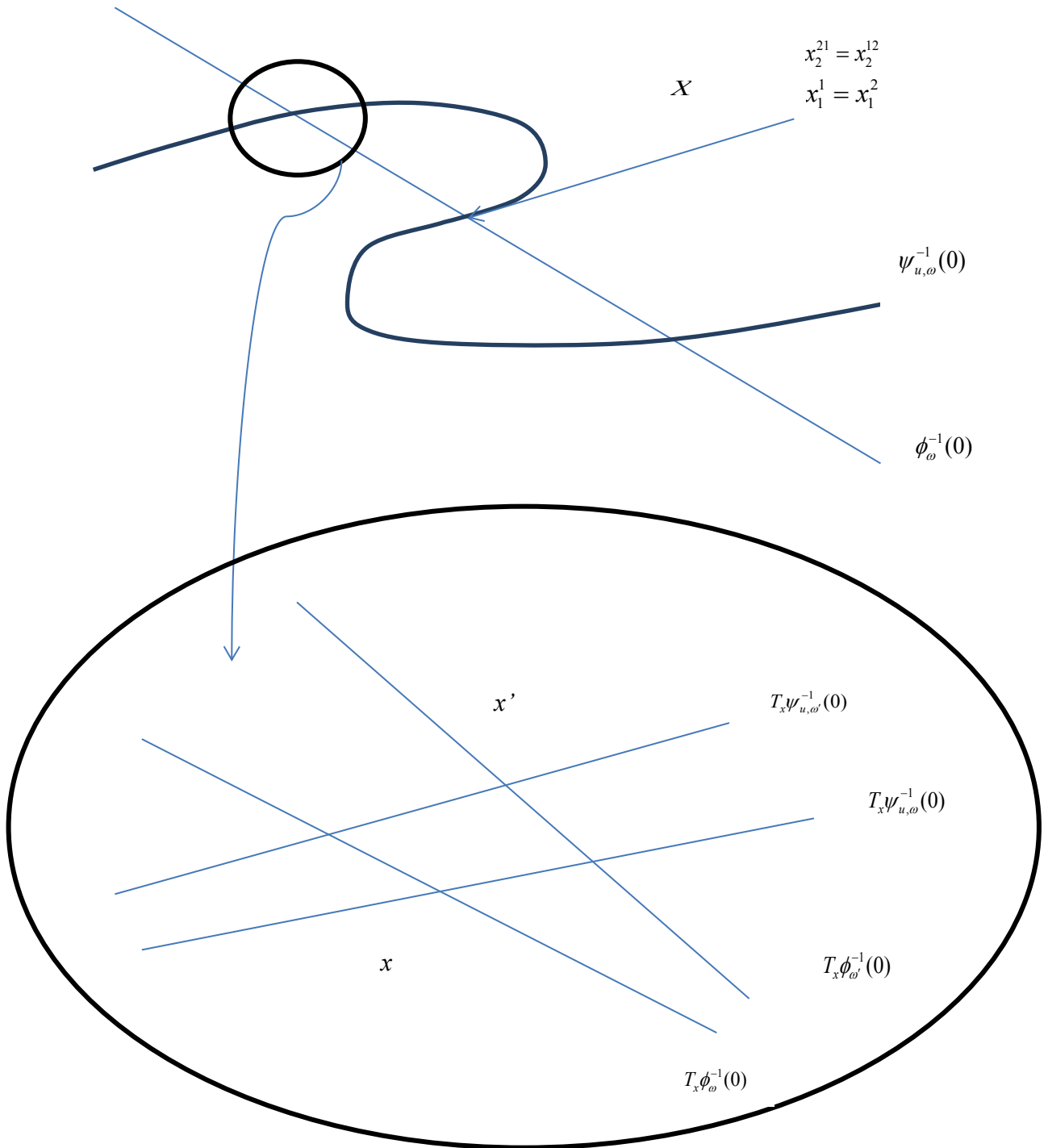
- A. $\psi_{u,\omega}^{-1}(0)$ and $\phi_{\omega}^{-1}(0)$ each generate a 2-dimensional manifold in X for a set of endowments of positive measure.
- B. $\phi_{\omega}^{-1}(0)$ is a submersion and hence transversal to any manifold in its range.
- C. $\psi_{u,\omega}^{-1}(0)$ is a submanifold in the range of $\phi_{\omega}^{-1}(0)$. Since i holds it follows that $\psi_{u,\omega}^{-1}(0) \nparallel \phi_{\omega}^{-1}(0)$ and thereby establishing the transversality condition of ii.

¹⁵ See Guillemin and Pollack [41]

¹⁶ The + in (1.4.1) denotes the span of the two subspaces $T_x \psi_{u,\omega}^{-1}(0)$ and $T_x \phi_{\omega}^{-1}(0)$. Transversality means that the tangent plane to $\psi_{u,\omega}^{-1}(0)$ and the tangent plane to $\phi_{\omega}^{-1}(0)$ at a point of intersection, is not contained in any hyperplane in the ambient space.

¹⁷ In Figure 1.4.1, for a given ω the two manifolds $\psi_{u,\omega}^{-1}(0)$ and $\phi_{\omega}^{-1}(0)$ are illustrated in the upper part as intersecting each other at multiple points from which it follows that there is some $x \in \psi_{u,\omega}^{-1}(0) \cap \phi_{\omega}^{-1}(0)$ which is a sunspot equilibrium, illustrated as being situated in the highlighted area. The point of intersection between $\psi_{u,\omega}^{-1}(0)$ and $\phi_{\omega}^{-1}(0)$ is blown in up the lower part of Figure 1.4.1. At the point of intersection of the manifolds x , the tangent spaces $T_x \psi_{u,\omega}^{-1}(0)$ and $T_x \phi_{\omega}^{-1}(0)$ span the space X in which case (1.4.1) is satisfied. Suppose that ω is perturbed to ω' . For a small enough perturbation both $\psi_{u,\omega}^{-1}(0)$ and $\phi_{\omega}^{-1}(0)$ are translated infinitesimally to $\psi_{u,\omega'}^{-1}(0)$ and $\phi_{\omega'}^{-1}(0)$ respectively and intersect at x' , a sunspot equilibrium. The resultant tangent spaces $T_{x'} \psi_{u,\omega'}^{-1}(0)$ and $T_{x'} \phi_{\omega'}^{-1}(0)$ are also perturbed infinitesimally and continue to satisfy (1.4.1). As a result, as ω varies SSE also vary in a smooth manner.

Figure 1.4.1.



A. The Manifold $\psi_{u,\omega}^{-1}(0)$

By application of the Regular Value Theorem¹⁸, it is demonstrated that $\psi_{u,\omega}^{-1}(0)$ is a two dimensional manifold in X . Consider the map $\psi_u : X \times \Omega \rightarrow \mathbb{R}^2$ for some fixed $u \in U$. The derivative matrix of ψ_u with respect to (x, ω) has the following submatrix (the matrix of derivatives of ψ_u with respect to ω):

$$\begin{bmatrix} -(1-\pi_{12})D_1u(x_1^1, x_2^{21}) - \pi_{12}D_1u(x_1^1, x_2^{12}) & -(1-\pi_{12})D_2u(x_1^1, x_2^{21}) - \pi_{12}D_2u(x_1^1, x_2^{12}) \\ -\pi_{21}D_1u(x_1^2, x_2^{21}) - (1-\pi_{12})D_1u(x_1^2, x_2^{12}) & -\pi_{21}D_2u(x_1^2, x_2^{21}) - (1-\pi_{12})D_2u(x_1^2, x_2^{12}) \end{bmatrix} \quad (1.4.2)$$

The first order conditions of the maximisation problems 1.1 and 1.2 are:

$$\begin{aligned} (1-\pi_{12})D_1u(x_1^1, x_2^{21}) + \pi_{12}D_1u(x_1^1, x_2^{12}) - \lambda_1^1 p_1 - \lambda_1^2 p_1 &= 0 \\ (1-\pi_{12})D_2u(x_1^1, x_2^{21}) - \lambda_1^1 p_1 &= 0, \quad \pi_{12}D_2u(x_1^1, x_2^{12}) - \lambda_1^2 p_2 = 0 \\ \pi_{21}D_1u(x_1^2, x_2^{21}) + (1-\pi_{12})D_1u(x_1^2, x_2^{12}) - \lambda_2^1 p_2 - \lambda_2^2 p_2 &= 0 \\ \pi_{21}D_2u(x_1^2, x_2^{21}) - \lambda_2^1 p_1 &= 0, \quad (1-\pi_{12})D_2u(x_1^2, x_2^{12}) - \lambda_2^2 p_2 = 0 \end{aligned}$$

Given the equivalence between the system of equations (1.3.7) – (1.3.10) and (1.3.1) – (1.3.6) and the equivalence between the first order conditions of the two systems, it is with no loss of generality that (1.4.2) can be written (1.4.2) in terms of the first order conditions yields the following equivalent matrix:

$$\begin{bmatrix} \lambda_1^1 p_1 + \lambda_1^2 p_1 & \lambda_1^1 p_1 + \lambda_1^2 p_2 \\ \lambda_2^1 p_2 + \lambda_2^2 p_2 & \lambda_2^1 p_1 + \lambda_2^2 p_2 \end{bmatrix} \quad (1.4.3)$$

The determinant of (1.4.3) is the polynomial

$$p_1 \lambda_1^1 \lambda_2^1 (p_1 - p_2) + p_2 \lambda_1^2 \lambda_2^2 (p_1 - p_2) + \lambda_1^2 \lambda_2^1 (p_1 p_1 - p_2 p_2) \quad (1.4.4)$$

Since $p_s > 0$, $s=1,2$ and $\lambda_s^{s'} > 0$, $(s,s') \in \{1,2\} \times \{1,2\}$ then (1.4.4) is equal to 0 if and only if $p_1 = p_2$ in which case $rank D\psi_u(x, \omega) < 2$. It follows that for all $x \in \bar{X}$, $rank D\psi_u(x, \omega) = 2$. Now let x be a steady state or certainty equilibrium allocation, i.e. $x \in X \setminus \bar{X}$. Take the derivative of ψ_u with respect to x and evaluate at $x = (x_1^1, x_2^{12}, x_1^2, x_2^{21})$ where $x_1^1 = x_1^2$ and $x_2^{21} = x_2^{12}$. Then the resultant 2×4 matrix is of the form

$$\begin{bmatrix} \Lambda_1 & \Lambda_2 & \Lambda_2 & 0 \\ 0 & \Lambda_2 & \Lambda_2 & \Lambda_1 \end{bmatrix} \quad (1.4.5)$$

¹⁸ See Appendix 1.7 for a statement of the Regular Value Theorem

where $\Lambda_i = D_i(x_1, x_2) + D_{ii}(x_1, x_2)(x_i - \omega_i) + D_{ji}(x_1, x_2)(x_j - \omega_j)$, $i \neq j$, $(i, j) \in \{1, 2\}$. There is a 2×2 submatrix which has determinant $(\Lambda_1)^2 > 0$ implying that $\text{rank} D\psi_u(x, \omega) = 2$ for $x \in X \setminus \bar{X}$. It follows that the family of maps $\psi_u : X \times \Omega \rightarrow \mathbb{R}^2$ has full rank. To show that a typical member of this family of maps has full row rank, Proposition 1.4.1 is utilised.

Proposition 1.4.1 (Mas-Colell, [56] pg. 321) Let $F : N \times B \rightarrow \mathbb{R}^m$, $N \subset \mathbb{R}^n$, $B \subset \mathbb{R}^s$ be C^r with $r > \max\{n-m, 0\}$. Suppose that 0 is a regular value of F ; that is $F(x, b) = 0$ implies that $\text{rank} DF(x, b) = m$. Then, except for a set of $b \in B$ of Lebesgue measure 0, $F_b : N \rightarrow \mathbb{R}^m$ has 0 as a regular value. ■

By application of Proposition 1.4.1, it is concluded that $\psi_{u, \omega}$ has 0 as a regular value for almost all $\omega \in \Omega$ given some $u \in U$. By repeated application of the Regular Value Theorem, for *a.e.* $\omega \in \Omega$, $\psi_{u, \omega}^{-1}(0)$ is a 2-dimensional manifold in X . Furthermore, $\psi_{u, \omega} \nrightarrow 0$ for *a.e.* $\omega \in \Omega$. Proposition 1.4.2 summarises.

Proposition 1.4.2 For *a.e.* $\omega \in \Omega$ the preimage $\psi_{u, \omega}^{-1}(0)$ forms a smooth 2-dimensional submanifold of X . ■

B. The Manifold $\phi_\omega^{-1}(0)$

It is required to show that for the family of maps $\phi : X \times \Omega \rightarrow \mathbb{R}^2$, for every $\omega \in \Omega$, a member of the family of maps $\phi_\omega^{-1}(0)$ forms a 2-dimensional submanifold of X . To show this, the derivative matrix of ϕ is¹⁹:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 1 & 0 & -1 & -1 \end{bmatrix} \quad (1.4.6)$$

It is straight forward to extract from (1.4.6) a 2×2 submatrix of rank 2 thereby showing that $\text{rank} D\phi(x, \omega) = 2$. By Proposition 1.4.1, one has that $\phi_\omega(x)$ has 0 as a regular value for almost all $\omega \in \Omega$. In fact, given that ϕ_ω does not lose rank for any $\omega \in \Omega$ as $\phi_\omega^{-1}(0)$ is a linear subspace of X it is concluded that $\phi_\omega \nrightarrow 0$ for *all* $\omega \in \Omega$. By application of the Regular Value

¹⁹ The columns pertain to the derivative of $\phi(x, \omega)$ with respect to the arguments $(x_1^1, x_2^{12}, x_1^2, x_2^{21}, \omega_1, \omega_2)$.

Theorem, for all, $\phi_\omega^{-1}(0)$ forms a 2-dimensional manifold in X . The following proposition obtains.

Proposition 1.4.3 $\phi_\omega^{-1}(0)$ forms a smooth 2-dimensional submanifold in X for all $\omega \in \Omega$.

■

C. Transversality of $\psi_{u,\omega}^{-1}(0)$ and $\phi_\omega^{-1}(0)$

Propositions 1.4.2 and 1.4.3 mean that for almost every endowment vector both $\psi_{u,\omega}^{-1}(0)$ and $\phi_\omega^{-1}(0)$ form submanifolds of X .

It is now shown that there is an open set of endowments W for which $\psi_{u,\omega}^{-1}(0)$ and $\phi_\omega^{-1}(0)$ intersect transversally by demonstrating that the tangent space to $\phi_\omega^{-1}(0)$ is surjective. This will imply that $\phi_\omega^{-1}(0)$ is a submersion and hence transversal to any manifold in its range. By showing that $\psi_{u,\omega}^{-1}(0)$ is a manifold in the range of $\phi_\omega^{-1}(0)$ and by assumption $\psi_{u,\omega}^{-1}(0) \cap \phi_\omega^{-1}(0) \neq \emptyset$ it follows that $\psi_{u,\omega}^{-1}(0) \pitchfork \phi_\omega^{-1}(0)$.

To begin, consider the map $\phi_\omega : X \rightarrow \mathbb{R}^2$. By Proposition 1.4.3 $\phi_\omega^{-1}(0)$ is a 2-dimensional manifold in X . Any point $x \in \phi_\omega^{-1}(0)$ has a tangent space $T_x \phi_\omega^{-1}(0)$ which can be characterised by a vector of the form $(a_1, 0, a_3, 0)$. Consider now the map $\zeta_\omega : \phi_\omega^{-1}(0) \times \mathbb{R}^2 \rightarrow \mathbb{R}^4$ defined by^{20 21}

$$\zeta_\omega(x, \varepsilon) = (x_1^1 + \varepsilon_1, x_2^{12}, x_1^2 + \varepsilon_2, x_2^{21}) = (x + \varepsilon)$$

where $\varepsilon = (\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2$. The vector $(x + \varepsilon)$ is a regular value of ζ if $D\zeta_\omega$ maps $T_x \phi_\omega^{-1}(0) \times \mathbb{R}^2$ onto $T_{\zeta_\omega(x, \varepsilon)} \mathbb{R}^4$, in which case the derivative map $D\zeta_\omega : T_x \phi_\omega^{-1}(0) \times \mathbb{R}^2 \rightarrow T_{\zeta_\omega(x, \varepsilon)} \mathbb{R}^4$

²⁰ This argument is similar to Nagata [69] Proposition 6.3 pg. 81.

²¹ The introduction of the map ζ may be deemed a perturbation of the projection map $proj \circ \phi_\omega^{-1}(0) : X \rightarrow X$ in that ζ translates the set of x which belong to $\phi_\omega^{-1}(0)$ by an amount ε . This is an adaptation of the argument presented in Guillemin and Pollack [41] pg. 62 which states that for $f : X \rightarrow \mathbb{R}^m$ a smooth map, if S is an open ball (or any such regular geometric object) in $\mathbb{R}^m$ then the map $F : X \times S \rightarrow \mathbb{R}^m$ defined by $F_{(x,s)} = f(x) + s$ is a translation of the ball S . F is thus a submersion of $X \times S$ and therefore transversal to any submanifold Z of $\mathbb{R}^m$. By the transversality theorem, for almost every $s \in S$ the map $f_s(x) = f(x) + s$ is transversal to Z .

is surjective²² and the map $\zeta_\omega : \phi_\omega^{-1}(0) \times \mathbb{R}^2 \rightarrow \mathbb{R}^4$ is a submersion for some $x \in \phi_{u,\omega}^{-1}(0)$. If ζ_ω is a submersion then the image of ζ_ω is transversal to any submanifold in its range.

The tangent space $T_x \phi_\omega^{-1}(0) \times \mathbb{R}^2$ yields a vector of the form $(a_1, 0, a_3, 0, a_5, a_6)$. By applying $D\zeta_\omega(x, \varepsilon)$ to the vector $(a_1, 0, a_3, 0, a_5, a_6)$ one has a vector of the form

$$(b_1, b_2, b_3, b_4) \quad (1.4.7)$$

Any vector contained in $T_{\zeta_\omega(x, \varepsilon)}(\mathbb{R}^4)$ is of the following form:

$$(\beta_1, \beta_2, \beta_3, \beta_4) \quad (1.4.8)$$

By setting $b_i = \beta_i$, $i = 1, \dots, 4$, equates (1.4.7) to (1.4.8) implying that $D\zeta_\omega$ is surjective by which $\zeta_\omega(x, \varepsilon)$ is a submersion and is transversal to any manifold in its range. Hence $\zeta_\omega(x, \varepsilon)$ is transversal to the manifold $\psi_{u,\omega'}^{-1}(0)$, i.e. $\zeta_\omega(x, \varepsilon) \nparallel \psi_{u,\omega'}^{-1}(0)$, as $\psi_{u,\omega'}^{-1}(0)$ is a submanifold of X by Proposition 1.4.2. Now, note that:

$$\begin{aligned} \zeta_\omega(\phi_\omega^{-1}(0), \varepsilon) &= \{x' \in X : x' = x + \varepsilon, x \in \phi_\omega^{-1}(0)\} \\ &= \{x' \in X : x' - \varepsilon = \omega\} \\ &= \{x' \in X : x' = \omega + \varepsilon\} \\ &= \phi_{\omega+\varepsilon}^{-1}(0) \end{aligned}$$

Therefore, $\phi_{\omega+\varepsilon}^{-1}(0)$, being equal to $\zeta_\omega(\phi_\omega^{-1}(0), \varepsilon)$, implies that $\phi_{\omega+\varepsilon}^{-1}(0)$ is a submersion and hence transversal to any submanifold in X .

It is now required to reason that given that there exists a regular stationary sunspot equilibrium for some endowment vector there exists an open neighbourhood of endowment vectors every element of which supports a regular stationary sunspot endowment.

To begin, since $\zeta_\omega : \phi_\omega^{-1}(0) \times \mathbb{R}^2 \rightarrow \mathbb{R}^4$ is transversal to $\psi_{u,\omega'}^{-1}(0)$ for all $\omega \in \Omega$ then by the transversality theorem (Guillemin and Pollack [41], pg.68) implies that for almost all ε , the image of $\zeta_\omega(\phi_\omega^{-1}(0), \varepsilon)$ being $\phi_{\omega+\varepsilon}^{-1}(0)$ is transversal to $\psi_{u,\omega'}^{-1}(0)$. Since one can write any vector in Ω as $\omega + \varepsilon$ then for all $\omega \in \Omega$ $\phi_\omega^{-1}(0)$ is transversal to $\psi_{u,\omega'}^{-1}(0)$. Furthermore, since ω' , which determines $\psi_{u,\omega'}^{-1}(0)$ is arbitrary then $\phi_\omega^{-1}(0) \nparallel \psi_{u,\omega'}^{-1}(0)$ for almost every $\omega \in \Omega$ by repeated application of the Transversality Theorem. This implies that for almost every endowment vector equilibrium is regular and for a negligible set of endowments equilibria

²² See Appendix 1.7.

are not regular. In the latter case, the equilibrium is termed *singular* as is the endowment vector.

Now, given that the steady state equilibrium exists for every $\omega \in \Omega$ then one has that for almost every $\omega \in \Omega$ $\phi_\omega^{-1}(0) \nparallel \psi_{u,\omega}^{-1}(0)$ and $\psi_{u,\omega}^{-1}(0) \cap \phi_\omega^{-1}(0) \neq \emptyset$ where the non-emptiness of the intersection follows from the assumption that there exists one regular stationary sunspot equilibrium which in turn follows from the existence of a 2-cycle. This implies that there is an open neighbourhood in the vicinity of ω such that for almost every endowment vector in this neighbourhood the manifold of the stochastic offer curve and the market clearing conditions are transverse to each other²³.

Theorem 1.4.1 For almost every ω belonging to an open of endowments W , and for some utility function $u \in U$, the two submanifolds $\psi_{u,\omega}^{-1}(0)$ and $\phi_\omega^{-1}(0)$ intersect transversally in X and such intersection yields a stationary sunspot equilibria. ■

Theorem 1.4.1 implies that for almost every ω in the set W of positive measure, $\psi_{u,\omega}^{-1}(0)$ and $\phi_\omega^{-1}(0)$ lie in general position. Since transversality is a generic property there are arbitrarily many economies within the vicinity of each other, each one of which is qualitatively no different to the other.

The topological properties of equilibrium set sheds further light on the structure of equilibria. Since the combined dimensions of the manifolds $\psi_{u,\omega}^{-1}(0)$ and $\phi_\omega^{-1}(0)$ sums to that of the dimension of the ambient space; $\dim \psi_{u,\omega}^{-1}(0) + \dim \phi_\omega^{-1}(0) = \dim X = 4$, and the manifolds have a non-empty transversal intersection for almost every endowment vector then the intersection of the two manifolds generates a manifold of dimension 0; a set of discrete points. Therefore *stationary sunspot equilibria are locally isolated*.

Furthermore, for each endowment vector $\phi_\omega^{-1}(0)$ is closed and bounded hence compact and $\psi_{u,\omega}^{-1}(0)$ is a closed set as it is the preimage of a closed set by a continuous function. The intersection of $\psi_{u,\omega}^{-1}(0)$ with $\phi_\omega^{-1}(0)$, being a 0-dimensional manifold, is consequently a closed, bounded and discrete set therefore compact and discrete. This implies that there exists an open cover of $\psi_{u,\omega}^{-1}(0) \cap \phi_\omega^{-1}(0)$ for which there is a finite sub-cover. Every

²³ The same could have been demonstrated by means of an application of the implicit function theorem. Given that almost every endowment vector generates an equilibrium which is regular, if there is some ω such that x is a sunspot equilibrium then there are open neighbourhoods Z and W in X and Ω respectively where x can be written as a function of ω and $x(\omega)$ solves the system of equations (ψ_u, ϕ) . Since the Jacobian of this system is not singular at (x, ω) then every economy in W supports a sunspot equilibrium as if a sunspot equilibrium were to cease to exist ω would have to pass through a critical set of equilibria which would necessitate that the Jacobian Matrix of the system of equations lose rank or that the equilibrium cease to be regular.

point of $\psi_{u,\omega}^{-1}(0) \cap \phi_{\omega}^{-1}(0)$ can thus be contained in an open ball each of which are disjoint to each other. By the finiteness of the open sub-cover, the set of equilibria are finite. Stationary sunspot equilibria are finite in number.

Theorem 1.4.1 as well as the properties of local finiteness and being isolated are local properties. The question is then whether these properties hold globally. It is clear that the properties of being locally finite and isolated extend beyond the case in which endowments belong to the set W . This raises the question of whether regular sunspot equilibria can exist for almost every endowment vector, i.e. whether W is equivalent to Ω ?

To see that SSE cannot exist for every $\omega \in \Omega$, let there be some $\omega \in W$ such that SSE exist. One of the set of SSE allocations must be a certainty equilibrium denoted $x^* = (x_1^*, x_2^*)$, i.e. $x_1^{1*} = x_1^{2*}$ and $x_2^{12*} = x_2^{21*}$. The certainty equilibrium is an equilibrium which is attainable by the restriction that either $(\pi_{12}, \pi_{21}) = (0, 0)$ or $(\pi_{12}, \pi_{21}) = (1, 1)$; x^* is obtainable by solving the system of equations (1.3.7) – (1.3.10) subject to this restriction. x^* is Pareto optimum as it cannot be dominated by any other vector as each agent has the same equilibrium allocation and the same utility function. Hence every certainty equilibrium is Pareto optimal.

Let P denote the set of certainty equilibria. For each equilibrium in this set the associated equilibrium price vector is a scalar of $(p_1, p_2) = (1, 1)$ (i.e. $p_1 = p_2$). Now let there be an endowment vector ω' which is equal to x^* . Suppose now that at ω' there is another equilibrium with prices $p^{**} = (p_1^{**}, p_2^{**})$ such that $p_1^{**} \neq p_2^{**}$ with an associated equilibrium allocation for agents 1 and 2 given respectively as (x_1^{1**}, x_2^{12**}) and (x_2^{1**}, x_2^{21**}) . Then in equilibrium one has that $x_1^{1**} + x_2^{21**} = \omega'_1 + \omega'_2$ and $x_1^{2**} + x_2^{12**} = \omega'_1 + \omega'_2$. But $x_1^* + x_2^* = \omega'_1 + \omega'_2$ and the vector $(x_1^* - \omega'_1, x_2^* - \omega'_2)$ is orthogonal to the vector scalar of $(p_1, p_2) = (1, 1)$ as this is a certainty equilibrium. It follows that $(x_1^{1**} - \omega'_1, x_2^{21**} - \omega'_2)$ and $(x_1^{2**} - \omega'_1, x_2^{12**} - \omega'_2)$ are also orthogonal to $(p_1, p_2) = (1, 1)$. Hence (x_1^{1**}, x_2^{12**}) and (x_2^{1**}, x_2^{21**}) are equilibrium allocations at the price vector $(p_1, p_2) = (1, 1)$ and must thereby be equal to the equilibrium vector (x_1^*, x_2^*) . It is concluded that the if $\omega \in P$ then equilibrium is unique.

Given that $\phi_{\omega}^{-1}(0) \cap \psi_{u,\omega}^{-1}(0)$ for almost every $\omega \in P$ then most economies which support a certainty equilibria are regular. Therefore, for a neighbourhood of each endowment vector in P transversality implies that there exists a non-negligible set in the neighbourhood of each endowment vector such that each endowment in that set is associated with a unique equilibrium²⁴. It is concluded that regular SSE cannot exist with full measure in Ω as there

²⁴ Balasko [10] states as for “relatively small” trade vectors, equilibrium is unique. Whilst this statement pertains to a pure exchange economy, the restriction of the probabilities to either $(\pi_{12}, \pi_{21}) = (0, 0)$ or $(\pi_{12}, \pi_{21}) = (1, 1)$ renders the sunspot economy equivalent to a pure exchange economy (see Chapter 2 of this thesis for a discussion of this point). Hence having established that equilibrium is unique for each endowment vector in the Pareto optimal set and since all such allocations are regular, then a perturbation of the endowment

exists a set of endowments with non-negligible measure such that only unique equilibria are supported.

This argument depicts the manner in which the set of equilibria is formed and in particular the manner in which the manifolds interact. Denote as B the subset of Ω for which the endowment vector supports a regular SSE allocation (where $W \subset B$) and denote as C the subset of Ω for which the endowment vector supports a unique regular equilibrium allocation (where $P \subset C$). It is noted that $B \cap C = \emptyset$. The combined measure of B and C is full in Ω as the set of singular economies is negligible. Suppose that endowments are moved in a continuous manner; $\omega(t) \in \Omega$, $t \in [0,1]$ where $\omega(0) \in B$ and $\omega(1) \in C$. By continuity of the set $\omega([0,1]) \subset \Omega$ there is some $t^* \in (0,1)$ for which there is an endowment vector $\omega(t^*)$ at which sunspot equilibria cease to exist and equilibrium is unique. For almost all t close to t^* and $t < t^*$ $\psi_{u,\omega}^{-1}(0)$ intersects $\phi_{\omega}^{-1}(0)$ transversally multiple times whereas for almost all t close to t^* and $t > t^*$ there is a unique transversal intersection of $\psi_{u,\omega}^{-1}(0)$ and $\phi_{\omega}^{-1}(0)$. Therefore, as endowments are moved in this fashion the two manifolds at first intersect multiple times and equilibrium is regular then crosses an endowment vector for which the multiplicity of equilibrium ceases is characterised by the tangential or non-transversal intersection of the manifolds. At this point of tangential intersection, a singular economy comes into existence where the set of singular economies, being not regular, has negligible measure. The existence of sunspot equilibria is related to the existence of a set of endowments for which the equilibria are singular and the equilibrium manifolds intersect non-transversally. Whilst the set of singular economies is of zero measure, the relevance of this set for the existence of sunspots goes beyond the size of the measure of that set²⁵.

The preceding provides sufficient conditions for the existence and non-existence of sunspot equilibria. If stationary sunspot equilibria do not exist then the topology of the equilibrium set being defined by the intersection of the manifolds, will be globally diffeomorphic to the set P in which case $\phi_{\omega}^{-1}(0) \cap \psi_{u,\omega}^{-1}(0)$ and $\#(\psi_{u,\omega}^{-1}(0) \cap \phi_{\omega}^{-1}(0)) = 1$ for every endowment vector.

1.5. Genericity of Equilibria in the Space of Utility Functions

In this section it is shown that there is a residual set of utility functions each member of which generates an equilibrium which is regular and sunspot. This result goes hand in hand

vector does not disrupt the property of transversal intersection of the manifolds hence no singular economy comes into existence by which it is concluded that the number of equilibria is constant over such a perturbation, i.e. one.

²⁵ The pulling of the manifolds apart in a manner such that multiple equilibria cease to exist is indicative of a catastrophe.

with Theorem 1.4.1 by which there exists a dense set of utility functions and endowments every element of which is a regular economy and the resultant equilibria are sunspot. The primary tool of this section is the multijet transversality theorem.

Let X^{26} and Y be two manifolds, possibly Euclidean spaces. Let $C^\infty(X, Y)$ be the set of smooth functions from X to Y where $C^\infty(X, Y)$ is endowed with the compact-open topology. The k -jet of $f \in C^\infty(X, Y)$ at $x \in X$ is the polynomial defined by the truncated Taylor series expansion of f of order k at x . Let $J^k(X, Y)$ be the k -jet bundle of mappings in $C^\infty(X, Y)$. This is the set of equivalence classes of maps in $C^\infty(X, Y)$ (Taylor series expansions) which are equivalent up to order k (see Golubitsky and Guillemin [10], pg. 37). The k -jet extension mapping is the map $j^k : C^\infty(X, Y) \rightarrow C^\infty(X, J^k(X, Y))$. In particular, if X and Y are Euclidean spaces then for $x \in X$ one has that $j^k f(x) = (x, f(x), Df(x), \dots, D^k f(x))$ where $D^i f(x)$ is the i^{th} order partial derivative of f at x . On the basis of the preceding, the following are defined (Golubitsky and Guillemin [36], pg. 57).

- i. $X^s = X \times X \times \dots \times X$ s times.
- ii. $X^{(s)} = \{(x_1, \dots, x_s) \in X^s : x_i \neq x_j \text{ for } 1 \leq i < j \leq s\}$.
- iii. $J^k(X, Y)^s = J^k(X, Y) \times \dots \times J^k(X, Y)$, s times.
- iv. $J_s^k(X, Y)$ is $J^k(X, Y)^s$ restricted to the base space $X^{(s)}$.
- v. $j_s^k f(x_1, \dots, x_s) = (j^k f(x_1), \dots, j^k f(x_s))$

The principle theorem that will be used is the Multijet Transversality Theorem.

Theorem 1.5.1. Multijet Transversality Theorem ([36], pg. 57) Let X and Y be smooth manifolds and V a submanifold of $J_s^k(X, Y)$. Let $T_V = \{f \in C^\infty(X, Y) : j_s^k f \not\cap V\}$. Then T_V is a residual²⁷ subset of $C^\infty(X, Y)$. ■

In that which follows, the space of smooth utility functions U is defined as the subset of the function space for which an element u is monotone increasing and concave. U is an open subset of $C^\infty(X, Y)$ and is also endowed with the compact-open topology. Consider now that an equilibrium allocation is a set $x = (x_1, x_2, x_3, x_4)$ where $x_1 = (x_1^1, x_2^{12})$, $x_2 = (x_1^1, x_2^{21})$, $x_3 = (x_1^2, x_2^{12})$ and $x_4 = (x_1^2, x_2^{21})$. Since equilibria are sunspot then $x_i \neq x_j$, $i \neq j$, $i, j \in \{1, 2, 3, 4\}$. Then, $x \in X^{(4)}$ where $X \equiv \mathbb{R}_{++}^2$ ²⁸. The k -jet bundle is $J_4^k(\mathbb{R}_{++}^2, \mathbb{R})$. Since only the first order conditions are utilised in the application of the Multijet Transversality

²⁶ With abuse of notation, in this section the space X is not the same as that of Section 1.4.

²⁷ A residual set is an open and dense set.

²⁸ Note the difference of the definition of X in this section and the previous section.

Theorem²⁹ it suffices to characterise $u \in U$ by its value and partial derivatives hence $k=1$ and the k -jet bundle is $J_4^1(\mathbb{R}_{++}^2, \mathbb{R})$. It follows that if any two manifolds intersect in $J_4^1(\mathbb{R}_{++}^2, \mathbb{R})$ then such a point of intersection x belongs to $X^{(4)}$ and is a SSE allocation; the restriction of J_4^1 to the base space $X^{(4)}$ for which $x_i \neq x_j$ naturally facilitates the analysis of SSE as such equilibria necessarily belong to $X^{(4)}$. For $u \in U \subset C^\infty(\mathbb{R}_{++}^2, \mathbb{R})$, the k -jet extension is defined as $j_4^1: u \rightarrow J_4^1(\mathbb{R}_{++}^2, \mathbb{R})$.

In order to apply the foregoing, assume that $\omega \in W$ is fixed. Define the map $\Xi: J_4^1(\mathbb{R}_{++}^2, \mathbb{R}) \rightarrow \mathbb{R}^4$ where $\Xi = (\psi_\omega, \phi_\omega)$ has the same structural form as the system of equations (1.3.7) – (1.3.10). In order to show that $\Xi \not\equiv 0$ for a non-negligible set of $u \in U$. The following lemma is utilised in this regard.

Lemma 1.5.1 Let X, Y and Z be manifolds and V be a submanifold of Z . Let $f: X \rightarrow Y$ and $h: Y \rightarrow Z$ be smooth maps. Suppose that h is a submersion. If f is transversal to $h^{-1}(V)$ then $h \circ f$ is transversal to V .

Proof See Nagata [70] pg. 123, replicated in Appendix 1.7. ■

In order to utilise Lemma 1.5.1 let $f: X \rightarrow Y$ be the map $j_4^1: u \rightarrow J_4^1(\mathbb{R}_{++}^2, \mathbb{R})$ where $j_4^1(u) \in J_4^1(\mathbb{R}_{++}^2, \mathbb{R})$ is the image of some $u \in U$. Let $h: Y \rightarrow Z$ be the map $\Xi: J_4^1(\mathbb{R}_{++}^2, \mathbb{R}) \rightarrow \mathbb{R}^4$ and $V = 0$. Since Ξ is a submersion³⁰ then $\Xi^{-1}(0)$ is a submanifold of $J_4^1(\mathbb{R}_{++}^2, \mathbb{R})$. By the multijet transversality theorem the set $T_{\Xi^{-1}(0)} = \{u \in C^\infty(\mathbb{R}_{++}^2, \mathbb{R}) : j_4^1(u) \not\in \Xi^{-1}(0)\}$ is residual in $C^\infty(\mathbb{R}_{++}^2, \mathbb{R})$. Since U is a G_δ subset³¹ of $C^\infty(\mathbb{R}_{++}^2, \mathbb{R})$ then the set $U_{\Xi^{-1}(0)} = T_{\Xi^{-1}(0)} \cap U$ is residual in U . It follows that for each $u \in U_{\Xi^{-1}(0)}$ one has that $j_4^1(u) \not\in \Xi^{-1}(0)$. By the fact that Ξ is a submersion then $\Xi \circ j_4^1(u) \not\equiv 0$. Since $\Xi = (\psi_\omega, \phi_\omega)$ then for $u \in U_{\Xi^{-1}(0)}$ and a.e. $\omega \in W$ one has that $(\psi_{u,\omega}, \phi_\omega) \not\equiv 0$. The following theorem summarises.

Theorem 1.5.2 For almost every $(u, \omega) \in U_{\Xi^{-1}(0)} \times W$ equilibrium is regular and sunspot. ■

²⁹ In this regard the assumption of concavity of the utility function does not come into play.

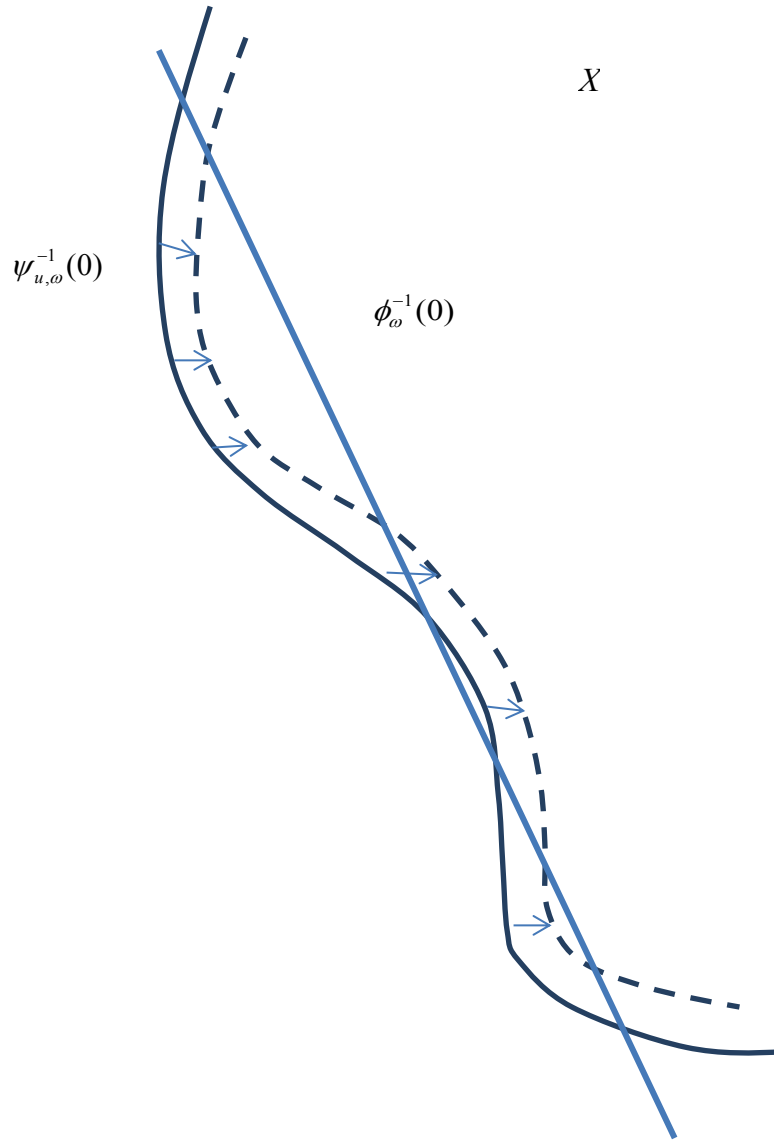
³⁰ This follows from the fact that $(\psi_u, \phi) \not\equiv 0$

³¹ Countable intersection of open sets.

The principle behind Theorem 1.5.2 is illustrated in Figure 1.5.1. The figure shows the transversal intersection of the two manifolds $\psi_{u,\omega}^{-1}(0)$ and $\phi_{\omega}^{-1}(0)$ in the ambient space X for some $(u, \omega) \in U_{\Xi^{-1}(0)} \times W$. The dotted line shows the manifold $\psi_{u',\omega}^{-1}(0)$ near $\psi_{u,\omega}^{-1}(0)$ for some $u' \in U_{\Xi^{-1}(0)}$ near u in the compact open topology. In both cases the manifolds are transversal to each other. As the utility function is perturbed infinitesimally in the set $U_{\Xi^{-1}(0)}$, stationary sunspot equilibria continue to remain in existence and are regular for almost all $\omega \in W$. It is concluded that SSE are structurally stable in not only the endowment space but also the space of utility functions.

To summarise this section, it has been established that there is a dense set of utility functions $U_{\Xi^{-1}(0)}$ for which SEE exist. The size of this set may in fact be small but does have the property of being locally generic. Combining $U_{\Xi^{-1}(0)}$ with W generates local genericity in both parameter spaces. Since $(\pi_{12}, \pi_{21}) \in \text{int}(I^2)$ has remain fixed throughout it is a reasonable assumption to make that for each $(u, \omega) \in U_{\Xi^{-1}(0)} \times W$ there is an open and dense set of (π_{12}, π_{21}) for which SSE exist and are regular. This point is discussed in Section 1.6.

Figure 1.5.1



1.6. Conclusion

By means of decomposing the primitive system of equilibrium equations into two sub-systems, stationary sunspot equilibria have been characterised as the points of intersection of two manifolds. The parameterization of each manifold by endowments and utility functions permitted the demonstration that the two manifolds lie in general position, i.e. are transversal, for almost all $(u, \omega) \in U_{\Xi^{-1}(0)} \times W$ and for some $\pi \in \text{int}(I^2)$. Regular stationary sunspot equilibria are therefore generic and are structurally stable in a local sense.

The use of topological tools as opposed to fixed point methodology is appropriate for the model considered here. In the case of employing a standard fixed point argument, one would have to transform the equilibrium system into a system of equations of the form $f : N \rightarrow N$ where N is compact and convex in order to show existence of equilibrium. Such an instrument is too blunt in order to establish equilibrium as it would not be clear whether the equilibrium in question was the steady state or sunspot. Furthermore, application of a fixed point theorem does not shed any light on the manner in which equilibria behave with respect to the changes in the parameter space.

Throughout the discussion of this Chapter it was assumed that $\pi \in \text{int}(I^2)$ was fixed. Since the two manifolds lie in general position then the tangent planes at the point x intersect transversally. The position of the manifold $\psi_{u,\omega}^{-1}(0)$, being determined in part by π for some fixed $(u, \omega) \in U_{\Xi^{-1}(0)} \times W$, can be perturbed slightly as can the tangent space by a perturbation of π within a small neighbourhood, with the property that the resultant manifold is transversal to $\phi_{\omega}^{-1}(0)$. It follows that there exists an open set of probabilities such that SSE exist and are regular.

As a criticism of this last point in the discussion of this chapter it was assumed that there exists some $\pi \in \text{int}(I^2)$ such that SSE exist. By the result of Azariadis and Guesnerie [5], such that this holds it suffices that there exists a cycle of order 2 in the certainty economy with probabilities $\pi_{12} = \pi_{21} = 1$. If there exists some endowment vector such that there exists a regular cycle of order 2 (or any higher order), then by the same argument as presented above, one can perturb the probability vector in the vicinity of $\pi_{12} = \pi_{21} = 1$ with the result that the manifolds continue to intersect transversally and SSE exist. Indeed, the existence of 2-cycles was examined in Balasko and Ghiglino [12] (Proposition 3 pg. 576), where it was demonstrated that there is a bijection between 2-cycles of the OLG model and multiple equilibria of the symmetrical two-agent two-good pure exchange economy³². It follows that for the existence of SSE it is sufficient that there exist multiple equilibria for a 2-consumer 2-good symmetrical pure exchange economy. The question then arises as to whether SSE can exist in the case in which 2-cycles or equivalently multiplicity of equilibria do not.

This question is answered in Davila [25] in which the structure of the offer curve of certainty economy is related to the existence of SSE. Crucial in the argument presented there is the fact that the offer curves bend backwards and have multiple intersection points. Consequently, SSE cannot exist if the offer curves do not intersect multiple times which in turn implies that SSE do not exist unless there is multiplicity of equilibria in symmetrical two-agent two-good pure exchange economy. One must therefore ask the question as to whether any theory which is premised on the multiplicity of equilibria can stand up to scrutiny.

³² See Mas-Colell *et al.* [57] Ch. 15 Example 15.B.2 for an example of a symmetrical pure exchange economy. See Tuinstra and Weddepohl [78] for an in-depth discussion of this type of pure exchange economy. See also Chapter 2 of this thesis.

Finally, by breaking the assumption of extrinsic uncertainty and letting endowments be state dependent (i.e. *uncertainty is intrinsic*) where the dependency is such that endowments vary within a small neighbourhood of the stationary endowment vector, transversality arguments can be employed to show that there exist a non-negligible set of stochastic endowments for which regular stochastic equilibria exist. For instance, an agent born in state s faces second period endowment vectors $(\omega_2^{s1}, \omega_2^{s2})$ with probabilities (π_{s1}, π_{s2}) where $\omega_2^{ss'}$ is within a small distance of ω_2 . Since $\psi_{u,\omega}^{-1}(0)$ and $\phi_{\omega}^{-1}(0)$ sit in general position for a non-negligible set of endowments then perturbing the endowments within this set retains the property of transversality. This argument is examined in more depth in Section 2.5.

1.7. Appendix

Lemma 1.5.1 Let X , Y and Z be manifolds and V be a submanifold of Z . Let $f: X \rightarrow Y$ and $h: Y \rightarrow Z$ be smooth maps. Suppose that h is a submersion. If f is transversal to $h^{-1}(V)$ then $h \circ f$ is transversal to V .

Proof (Nagata [70] pg. 123) $h^{-1}(V)$ is a submanifold of Y as h is transversal to V . Since f is transversal to $h^{-1}(V)$ then for any $x \in (h \circ f)^{-1}(V)$ then

$$df_x(T_x X) + T_{f(x)}(h^{-1}(V)) = T_{f(x)} Y \quad (1.7.1)$$

If $f(x)$ belongs to $h^{-1}(V)$ then

$$dh_{f(x)}(T_{f(x)} Y) + T_{h \circ f(x)} V = T_{h \circ f(x)} Z \quad (1.7.2)$$

and

$$dh_{f(x)}(T_{f(x)} h^{-1}(V)) = T_{h \circ f(x)} V \quad (1.7.3)$$

Applying $dh_{f(x)}$ to both sides of A.1 yields

$$dh_{f(x)} \circ df_x(T_x X) + dh_{f(x)}(T_{f(x)}(h^{-1}(V))) = dh_{f(x)}(T_{f(x)} Y) \quad (1.7.4)$$

By (1.7.3), (1.7.4) becomes

$$dh_{f(x)} \circ df_x(T_x X) + T_{h \circ f(x)} V = dh_{f(x)}(T_{f(x)} Y) \quad (1.7.5)$$

Adding $T_{h \circ f(x)} V$ to both sides of (1.7.2) and noting (1.7.3) then

$$dh_{f(x)} \circ df_x(T_x X) + T_{h \circ f(x)} V = T_{h \circ f(x)} Z$$

which implies that $h \circ f$ is transversal to V since $x \in (h \circ f)^{-1}(V)$ is arbitrarily chosen. ■

Manifold (Guillemin and Pollack [41] pg. 3) Let X be a subset of some Euclidean space $\mathbb{R}^n$. Then X is a k -dimensional manifold if it is locally diffeomorphic to $\mathbb{R}^k$. This means that each point x possesses a neighbourhood V in X which is diffeomorphic to an open set U of $\mathbb{R}^k$. A diffeomorphism $\rho: U \rightarrow V$ is called a parameterization of the neighbourhood V (V must be expressible as $\tilde{V} \cap X$ for some open set $\tilde{V} \subset \mathbb{R}^n$) The inverse diffeomorphism $\rho^{-1}: V \rightarrow U$ is called a coordinate system on V , where the map ρ^{-1} can be written in coordinates $\rho^{-1} = (x_1, \dots, x_k)$, the k smooth functions $x_1, \dots, x_k$ on V are called coordinate functions. The k -tuple of coordinate functions $(x_1, \dots, x_k)$ is used to identify V with U implicitly, and a point $v \in V$ is identified with its coordinates $(x_1(v), \dots, x_k(v)) \in U$.

Regular Value Theorem (Guillemin and Pollack [41] pg. 21, Hirsch [42], pg. 22, Milnor [67], pg. 11)

If $f: M \rightarrow N$ is a smooth map between manifolds of dimension $m \geq n$ and if $y \in N$ is a regular value, then the set $f^{-1}(y)$ in M is a smooth submanifold of dimension $m - n$. ■

Submersion Let $f: M \rightarrow N$. If $df_x: T_x(X) \rightarrow T_y(Y)$ is surjective, f is called a submersion at x . A map that is a submersion at every point is simply a submersion.

Compact Open Topology The r -jet space $J^r(\mathbb{R}^m, \mathbb{R}^n)$ can be identified with a Euclidean space and hence permits the Euclidean norm $d(\cdot, \cdot)$. In order to topologize $C^\infty(\mathbb{R}^m, \mathbb{R}^n)$ one must determine how close f is to g by evaluating their r -jets on some subset of $\mathbb{R}^m$. Suppose that f is an arbitrary map in $C^\infty(\mathbb{R}^m, \mathbb{R}^n)$ and δ is a continuous map from $\mathbb{R}^m$ to $\mathbb{R}_+$. Consider for a positive number k the set $B_{\delta,k}(f) = \{g \in C^\infty(\mathbb{R}^m, \mathbb{R}^n) : d(j^r f(x), j^r g(x)) < \delta(x), x \in \mathbb{R}^m, \text{ s.t. } \|x\| \leq k\}$. The family of sets $B_{\delta,k}(f)$ parameterized by δ and k form a fundamental system of neighbourhoods at f thus specifying a topology. The topology $B_{\delta,k}(f)$ as a neighbourhood basis at each f is called the compact open C^r topology for $C^\infty(\mathbb{R}^m, \mathbb{R}^n)$. Intuitively, all maps in $B_{\delta,k}(f)$ are close to f in the sense that their first r -partial derivatives are δ -close to f on some compact subset of $\mathbb{R}^m$. The compact open topology is tractable but it is weak as a topology inasmuch as it only considers the closeness of two points (maps) on a compact set. Consequently, this topology does not control the behaviour of maps near infinity.

The Interrelationship between the Non-Existence of Stationary Sunspot Equilibria and Steady State Equilibria in an Overlapping Generations Economy

Abstract

A two period stationary pure exchange overlapping generations economy is analysed. In this model there is strictly more than one commodity each period and uncertainty is extrinsic. The main result is that for almost every endowment vector there does not exist any equilibrium which reflects extrinsic or sunspot uncertainty. This is to be contrasted with the result that for almost every endowment vector there exists an equilibrium which does not incorporate extrinsic uncertainty. It is then shown that the existence of the steady state equilibrium is sufficient for the existence of stationary equilibria under intrinsic uncertainty.

2.1. Introduction

In a model of exchange, uncertainty may influence decision making in various ways. Uncertainty may appear as a determinant of the fundamentals of the economy, affecting preferences, endowments, demographic structure, and production and so on. In such a case, equilibrium prices and allocations reflect uncertainty as rational agents modify behaviour conditional upon the realisation of an uncertain event. Uncertainty is said to be intrinsic to the economic system. Uncertainty is instead extrinsic to the economic system if randomness *does not* affect any of the economic fundamentals but instead has a direct and unique bearing only on prices. That is, agents' rational decision making reflects the common belief that prices are determined by a stochastic process which does not affect the economic fundamentals. Expectations are deemed to be self-fulfilling, requiring nothing more than a common outlook on the price process. The resultant equilibrium is a rational expectations equilibrium under extrinsic uncertainty and is termed a sunspot equilibrium (Cass and Shell [21]).

Various models of extrinsic uncertainty or sunspot have been analysed, from a model in which there is restricted market participation (Balasko *et al.* [11]), to dynamic (Peck [72]) and stationary (Azariadis and Guesnerie [5], Davila *et al.* [26] and Spear [75]) overlapping generations (OLG) models. In the latter class of models in which there is a two-period lived agent, a single consumption good consumed in the second period of life produced by means the supply of labour in the first period of life, Azariadis and Guesnerie [5] show that stationary sunspot equilibria³³ (SSE) exist for a non-negligible set of probabilities³⁴ if and only if cyclical equilibrium of order 2 exist in the non-stochastic economy. As pointed out by Azariadis and Guesnerie [5] (see also Grandmont [38]), it is sufficient for the existence of cycles of order 2 that the offer curve of the representative agent bends backwards at the steady state and has a gradient less than one in absolute value. Hence the existence of 2-cycles, and by extension SSE, depends on the multiple intersection of the offer curves of the representative agents in the underlying OLG economy which is devoid of extrinsic uncertainty. In turn, the multiple intersection of the offer curves is equivalent to the existence of multiple zeroes of the excess demand functions which implies the existence of equilibrium other than the steady state. This condition is in turn sufficient for the existence of SSE; SSE are obtained by a perturbation of the excess demand function where the perturbation has the interpretation that each perturbed excess demand function pertains to a different probability vector. A standard transversality argument is sufficient to guarantee that SSE exist for a non-negligible set of probabilities (Theorem 1, [5]).

In a two period stationary pure exchange overlapping generations economy with one commodity each period and either separable or non-separable utility functions, Spear [75] shows³⁵ that for SSE to exist it suffices that the excess demand function has a positive gradient at the steady state. This condition is equivalent to the elasticity of saving of the first period consumption good evaluated at the monetary steady state being less than -1/2 when the Markovian probability structure is degenerate and equivalent to the occurrence of 2-cycles. It is thus implicit in the analysis of [75] that 2-cycle equilibria are sufficient for the existence of SSE in a pure exchange environment. Furthermore, SSE exist for a non-negligible set of probabilities³⁶.

In a similar but distinct vein, within the framework of a stationary pure exchange OLG model in which the representative agent lives for two periods, there are multiple commodities ($L > 1$) each period and in which the endowment structure is stochastic, (uncertainty is intrinsic and not extrinsic as the fundamentals of the economy depend upon the occurrence of an uncertain event), Spear [76] discusses the existence of stochastic rational expectations equilibrium. It is demonstrated that if the utility function is not time separable and there are $L > 1$ commodities each period then stationary stochastic equilibria do not generically exist³⁷ (Theorem 4.1, Appendix 1, pg. 264, [76]). The non-existence of

³³ Equilibria which are sunspot and are time invariant.

³⁴ An open neighbourhood of probabilities (Theorems 1 and 4 [5]).

³⁵ Spear [75] chronologically precedes Azariadis and Guesnerie [5]. The work of the latter is a refinement of the former and thereby similar on many accounts.

³⁶ See Chapter 1 of this thesis for a discussion of this point.

³⁷ A property is said to be generic in a space if it holds for an open and dense subset of that space.

stochastic equilibria arises as in the equilibrium system there are $4L$ market clearing equations and $2L$ prices hence more independent equations than variables. Intuitively, the solution to some subset of the system of equations, should it exist, will not generally solve the remaining system of equations, unless there is some interdependence between those equations. The precise manner in which Spear [76] demonstrates the generic non-existence of stochastic equilibrium is by means of the multijet transversality theorem³⁸ (Golubitsky and Guillemin [36] pg. 57) applied to the system of first order equations and market clearing equations thereby showing that for an open and dense set of utility functions and endowment vectors, equilibria do not exist. However, if the utility function is separable³⁹ then stochastic equilibria exist. In order to demonstrate this an argument is used to show that $2L$ of the $4L$ market clearing equations are redundant (where $L > 1$). A fixed point argument is then applied to the remaining $2L$ equations and there exists an equilibrium price vector of the form $(p_1^*, p_2^*) \in \mathbb{R}_{++}^L \times \mathbb{R}_{++}^L$. The resultant equilibrium is necessarily a stochastic rational expectations equilibrium as the endowment vector is stochastic (Spear [76] Appendix 2, pg. 268).

Motivated by the foregoing observations, the aim of this paper is to build upon the intuition of Spear [76] within an extrinsically uncertain framework. The following are demonstrated.

- A. Given both separable and non-separable utility functions, SSE do not generically exist for almost every endowment vector. This is shown by an approach reminiscent of the geometric equilibrium analysis of Balasko⁴⁰ [7], [8], [9], [10]⁴¹ in which the equilibrium system is decomposed into the set of budget constraints and equilibrium equations. The transversal intersection of the manifolds generated by the two subsystems of equations generates a regular geometric equilibrium. By means of a transversality argument it is shown that for almost every endowment vector the budget and equilibrium manifold fail to intersect. This implies that there can be no price-income pair for which the market clearing equations and budget constraints are simultaneously satisfied. Consequently, generically SSE do not exist.
- B. Whilst SSE do not exist, non-stochastic steady state equilibria do generically exist. Such equilibria exist irrespective of the separability of the utility function. This result is obtained by a topological degree argument of a pure exchange economy extended to the extrinsically uncertain economy.
- C. The existence of a stationary steady state in the extrinsically uncertain or sunspot economy is sufficient for the existence of stationary stochastic equilibrium in the intrinsically uncertain economy.

³⁸ The multijet transversality theorem is employed in Chapter 1 in order to show existence of sunspot equilibria in the case of $L=1$ commodity per period.

³⁹ i.e. utility over the two time periods has the form $u(x_1, x_2) = W(x_1) + V(x_2)$. The utility function satisfies standard assumptions.

⁴⁰ See Balasko [9] Ch. 5.

⁴¹ See also Ghiglino and Tvede [34] and Goenka and Matta [35].

This paper is laid out as follows. In Section 2.2 the stationary overlapping generations model is presented and discussed. In Section 2.3 the budget manifold and equilibrium manifold are defined and shown to be submanifolds of the price-income space for a fixed level of total resources. A transversality argument is evoked by which it is shown that the two manifolds do not intersect thereby implying that SSE do not generically exist. In Section 2.4 the certainty economy (non-stochastic counterpart to the extrinsic uncertainty economy) is defined. The set of equilibria are analysed whereby it is shown that the steady state equilibria exists for almost every endowment vector. In Section 2.5 it is shown that the existence of the steady state implies the existence of intrinsically uncertain equilibria inasmuch as the latter equilibria can be obtained by means of a perturbation of the former type of equilibrium in the space of endowments. Section 2.6 concludes.

2.2. The Stationary Sunspot Model

The model is a stationary⁴² stochastic overlapping generations model in which uncertainty enters the model by means of each agent's commonly held belief that a randomly determined signal influences prices. Such a process is termed *sunspot*⁴³. As such, randomness is *extrinsic* and does not have any bearing on the fundamentals of the system, i.e. demographic structure, preferences, endowment structure, but instead operates solely by means of a commonly held belief that prices affect consumption plans⁴⁴. This commonly held belief is expressed by means of a public forecast function $\sigma(\cdot)$ which assigns, in a one-to-one manner, a price vector p_i to a commonly observed random signal s_i ; $p_i = \sigma(s_i)$. Each signal belongs to a finite signal set. For ease of exposition and without loss of generality it is assumed that there are two signals $s \in \{1, 2\}$. In each period a representative agent is born and lives for two periods⁴⁵. At any point in time there are two agents coexisting one of whom is young the other of whom is old. Uncertainty unfolds by means of a stationary Markov process where the probability of an agent observing a sequence of events over a life-cycle is given by $\pi_{ss'}$, where $\pi_{ss'}$ is the probability of state s' being realised conditional on the realisation of signal s . Given that there are two signals then $\{s, s'\} \in \{1, 2\} \times \{1, 2\}$. The sum of all the probabilities of all the events over any given life-cycle is equal to one. It is assumed that every conditional probability is bound between (0,1) and is hence non-degenerate.

There are $L > 1$ commodities each period. An agent born in state s , has first period demand vector $x_1^s \in \mathbb{R}_{++}^L$, $x_1^s = (x_1^s, x_2^s, \dots, x_L^s) \in \mathbb{R}_{++}^L$, for $s \in \{1, 2\}$, and has second period

⁴² Time invariant.

⁴³ See *inter alia* [5] and [21]

⁴⁴ Expectations are rational.

⁴⁵ The analysis follows through if the agent lives for any finite number of periods. The assumption of a two period lived agent is made for the sake of simplicity.

contingent demand vector $x_2^{ss'} \in \mathbb{R}_{++}^L$ where $x_2^{ss'} = (x_{2,1}^{ss'}, x_{2,2}^{ss'}, \dots, x_{2,L}^{ss'}) \in \mathbb{R}_{++}^L$, for $s \in \{1, 2\}$. Over an entire life-cycle each agent has vectors of demands; $(x_1^s, x_2^{ss'}, x_2^{s's'}) \in \mathbb{R}_{++}^L \times \mathbb{R}_{++}^{2L}$. Since there are 2 diverse agent types; one born in each state, an agent is henceforth identified by the state in which he is born.

Given the definition of the price forecast function, to each event a price vector is assigned. If $\pi_{ss'}$ is the probability of the of state s' being realised conditional on the realisation of signal s then the price assigned is p_s for any $s \in \{1, 2\}$ where $p_s = (p_{1,s}, \dots, p_{L,s}) \in \mathbb{R}_{++}^L$, $s = 1, 2$. Each agent is endowed with a set of non-stochastic endowment vectors of the L commodity goods over the 2 periods of life; $\omega = (\omega_1, \omega_2) \in \Omega \equiv \mathbb{R}_{++}^{2L}$ ⁴⁶. As uncertainty is extrinsic, the utility function $u: \mathbb{R}_{++}^{2L} \rightarrow \mathbb{R}$ is state independent where $u^{-1}(a)$ is bounded from below for all $a \in \mathbb{R}_{++}$ and the closure if u is contained in $\mathbb{R}_{++}^{2L}$. u is smooth, i.e. $Du \in C^\infty(\mathbb{R}_{++}^L, \mathbb{R}_{++}^L)$, strictly concave and has negative definite Hessian, i.e. $y^T \cdot D^2u(x) \cdot y < 0$ for all $x \in \mathbb{R}_{++}^{2L}$ and $y \in \mathbb{R}^{2L}$, $y \neq 0$.⁴⁷

The maximisation problem for the agents born into states 1 and 2 are respectively:

Problem 2.1 – Agent 1

$$\begin{aligned} & \max_{x_1^1, x_2^{1j}} \pi_{11}u(x_1^1, x_2^{11}) + \pi_{12}u(x_1^1, x_2^{12}) \\ & \text{s.t. } p_1 \cdot x_1^1 + p_m x_m^1 \leq p_1 \cdot \omega_1 \text{ and} \\ & p_s \cdot x_2^{1s} \leq p_s \cdot \omega_2 + p_m x_m^1, \quad \text{for } s = 1, 2 \end{aligned} \tag{2.2.1}$$

Problem 2.2 – Agent 2

$$\begin{aligned} & \max_{x_1^2, x_2^{2j}} \pi_{21}u(x_1^2, x_2^{21}) + \pi_{22}u(x_1^2, x_2^{22}) \\ & \text{s.t. } p_2 \cdot x_1^2 + p_m x_m^2 \leq p_2 \cdot \omega_1 \text{ and} \end{aligned}$$

⁴⁶ Spear's framework [76] of intrinsic uncertainty is similar to the extrinsic uncertainty framework of this paper but differs inasmuch as the first period endowment vector is stochastic, having the form of $(\omega_1^1, \omega_1^2, \omega_2)$ where ω_1^s , is the first period endowment vector of the agent born in state $s = 1, 2$. In [76] the utility function is state independent. Therefore, if $\omega_1^1 = \omega_1^2$ then the intrinsically uncertain economy collapses to the extrinsically uncertain economy.

⁴⁷ No assumption is made concerning the separability of the utility function.

$$p_s \cdot x_2^{2s} + \leq p_s \cdot \omega_2 + p_m x_m^2, \text{ for } j = 1, 2 \quad (2.2.2)$$

where p_m is the price of money and $\pi_{s1} + \pi_{s2} = 1$, $s = 1, 2$. The demand for money of an agent born into state s is $x_m^j = (p_s / p_m) \cdot (\omega_1 - x_1^s)$, $s = 1, 2$. $p_s \in \mathbb{R}_{++}^L$, $s = 1, 2$ is the price of the commodity in state s . Denote $p = (p_1, p_2) \in S \equiv \mathbb{R}_{++}^{2L}$ as the price space. Define $I = [0, 1]$. Then $(\pi_{12}, \pi_{21}) \in \text{int}(I^2)$ defines the set of all permissible non-degenerate Markovian probabilities. As a matter of definition a pair $(\omega, \pi) \in \Omega \times \text{int}(I^2)$ is termed an extrinsically uncertain economy or a sunspot economy. Instead if $\pi_{12} \in \{0, 1\}$ or $\pi_{21} \in \{0, 1\}$ the resultant economy is termed a certainty economy.

2.3. The Budget Manifold and the Equilibrium Manifold

In this section, the budget manifold and the set of geometric equilibria are defined. These topological structures are related to the equilibrium manifold by means of the fibre bundle structure of the latter. It is shown that in the case of extrinsic uncertainty the equilibrium manifold is an empty set.

2.3.1. The Price-Income Space and the Budget Manifold

In order to facilitate the analysis, the role of money is overlooked. As such, given an interior solution, the constraints to the maximisation Problems 2.1 and 2.2 above can be written respectively as (2.3.1) – (2.3.2) and (2.3.3) – (2.3.4):

$$p_1 \cdot x_1^1 + p_1 \cdot x_2^{11} = p_1 \cdot \omega_1 + p_1 \cdot \omega_2 \quad (2.3.1)$$

$$p_1 \cdot x_1^1 + p_2 \cdot x_2^{12} = p_1 \cdot \omega_1 + p_2 \cdot \omega_2 \quad (2.3.2)$$

$$p_2 \cdot x_1^2 + p_1 \cdot x_2^{21} = p_2 \cdot \omega_1 + p_1 \cdot \omega_2 \quad (2.3.3)$$

$$p_2 \cdot x_1^2 + p_2 \cdot x_2^{22} = p_2 \cdot \omega_1 + p_2 \cdot \omega_2 \quad (2.3.4)$$

Income is stochastic and dependent upon the state into which the agent is born, the probability of the realisation of the event as well as the prevailing price and endowment vectors.

Define $p_i \cdot \omega_j = w_{ij}$, $\{i, j\} \in \{1, 2\} \times \{1, 2\}$ and define stochastic incomes as⁴⁸:

$$p_1 \cdot \omega_1 + p_1 \cdot \omega_2 = w_{11} + w_{12} \quad (2.3.5)$$

$$p_1 \cdot \omega_1 + p_2 \cdot \omega_2 = w_{11} + w_{22} \quad (2.3.6)$$

$$p_2 \cdot \omega_1 + p_1 \cdot \omega_2 = w_{21} + w_{12} \quad (2.3.7)$$

$$p_2 \cdot \omega_1 + p_2 \cdot \omega_2 = w_{21} + w_{22} \quad (2.3.8)$$

Total resources are fixed at some value $r \in \mathbb{R}_{++}^L$ where $\omega_1 + \omega_2 = r$. Given $r > 0$ the space of endowments is restricted to $\Omega(r) \equiv \{\omega \in \Omega: \omega_1 + \omega_2 = r, r > 0\}$. The total value of resources is then $(p_1, p_2) \cdot r$ (or equivalently $(p_1, p_2) \cdot (\omega_1 + \omega_2)$) for some $\omega \in \Omega(r)$. Denote $B = S \times \mathbb{R}_{++}^4$ as the *price-income* space where $(p, w) = (p_1, p_2, w_{11}, w_{12}, w_{21}, w_{22}) \in B$ is a typical element. The dimension of this space is $\dim B = 2L + 4$. For a specific endowment vector $\omega \in \Omega(r)$, the following holds:

$$\begin{aligned} (p_1, p_2) \cdot (\omega_1, \omega_2) &= p_1 \cdot (\omega_1 + \omega_2) + p_2 \cdot (\omega_1 + \omega_2) \\ &= p_1 \cdot r + p_2 \cdot r \\ &= w_{11} + w_{12} + w_{21} + w_{22} \end{aligned} \quad (2.3.9)$$

For some $\omega \in \Omega(r)$, $(p_1, p_2) \cdot (\omega_1, \omega_2) = w_{11} + w_{12} + w_{21} + w_{22}$ is a polynomial of order 1 in (p, w) . Since total resources are fixed for some $\omega \in \Omega(r)$, prices and income are restricted to the subset $B(r)$ of B such that (3.9) holds. $B(r)$ is an affine subspace or linear manifold, and hence submanifold, of B with $\dim B(r) = 2L + 3$ and is termed the *restricted price-income space*.

The *budget manifold* is the set $A(\omega)$ of price-income pairs such that (2.3.5) – (2.3.8) are satisfied. $A(\omega)$ is a linear subspace of B where $\dim A(\omega) = 2L$. Furthermore, for $(p, w) \in A(\omega)$ one has that (3.5) and (3.8) hold. By summing these two constraints one has that $p_1 \cdot \omega_1 + p_1 \cdot \omega_2 + p_2 \cdot \omega_1 + p_2 \cdot \omega_2 = w_{11} + w_{21} + w_{12} + w_{22}$. The same applies to the summation of the constraints (2.3.6) and (2.3.7). Thus if $(p, w) \in A(\omega)$ then $(p, w) \in B(r)$ so $A(\omega)$ is a $2L$ dimensional linear subspace of $B(r)$, itself a $2L + 3$ dimensional subspace of B .

⁴⁸ (2.3.5) and (2.3.6) (also (2.3.7) and (2.3.8)) are linearly independent iff $p_1 \neq p_2$. As noted subsequently, a sunspot equilibrium can exist if and only if $p_1 \neq p_2$. Conversely, if $p_1 = p_2$ then any resultant equilibrium cannot be sunspot and is characterised by linear dependence of the constraints.

2.3.2. Geometric Equilibria

The solutions to maximisation Problems 2.1 and 2.2 yield the vectors of demands $f^j : S \times \Omega \times \text{int}(I) \rightarrow \mathbb{R}_{++}^{3L}$, $j=1,2$ where $f^1 = (f_1^1, f_2^{11}, f_2^{12})$ and $f^2 = (f_1^2, f_2^{21}, f_2^{22})$ given by the following two $3L$ vectors:

$$f^1(p_1, p_2, p_1 \cdot \omega_1 + p_1 \cdot \omega_2, p_1 \cdot \omega_1 + p_2 \cdot \omega_2, \pi_{12})$$

$$f^2(p_1, p_2, p_2 \cdot \omega_1 + p_1 \cdot \omega_2, p_2 \cdot \omega_1 + p_2 \cdot \omega_2, \pi_{21})$$

where f^j , $j=1,2$ is smooth, bounded from below and satisfies desirability. Given the definition of stochastic income, for a pair $(p, \omega) \in S \times \Omega(r) \times \text{int}(I)$, each agent's vector of demand functions is the map $f^j : S \times \mathbb{R}_{++}^2 \times \text{int}(I) \rightarrow \mathbb{R}_{++}^{3L}$ defined by the vectors $f^1(p_1, p_2, w_{11} + w_{12}, w_{11} + w_{22}, \pi_{12})$ and $f^2(p_1, p_2, w_{21} + w_{12}, w_{21} + w_{22}, \pi_{21})$. Equilibrium occurs when demand is equal to supply across all goods and across all states of nature. Equilibrium is a stationary sunspot equilibrium if

$$f_1^s + f_2^{s's} = \omega_1 + \omega_2, (s, s') \in \{1, 2\} \times \{1, 2\} \quad (2.3.10)$$

If (2.3.10) holds then $(f_1^1, f_2^{11}) \neq (f_1^1, f_2^{12}) \neq (f_1^2, f_2^{21}) \neq (f_1^2, f_2^{22})$ which implies and is implied by $p_1 \neq p_2$ (i.e. prices reflect the extrinsically uncertain event as do equilibrium allocations). An equilibrium is a stationary steady state if $p_1 = p_2$. A stationary steady state is not a sunspot equilibrium and conversely., by (2.3.10) there exist $4L$ equilibrium equations. Letting $\pi \in \text{int}(I^2)$ remain fixed, dividing (2.3.10) into two subsystems each of $2L$ equations given by the smooth map $z_j : B(r) \rightarrow \mathbb{R}^{2L}$ $j=1,2$:

$$\begin{aligned} z_1(p_1, p_2, w_{11}, w_{12}, w_{21}, w_{22}, \pi) \\ = \begin{cases} f_1^1(p_1, p_2, w_{11} + w_{12}, w_{11} + w_{22}, \pi_{12}) + f_2^{11}(p_1, p_2, w_{11} + w_{12}, w_{11} + w_{22}, \pi_{12}) - r \\ f_1^2(p_1, p_2, w_{21} + w_{12}, w_{21} + w_{22}, \pi_{21}) + f_2^{22}(p_1, p_2, w_{21} + w_{12}, w_{21} + w_{22}, \pi_{21}) - r \end{cases} \end{aligned} \quad (2.3.11)$$

$$\begin{aligned} z_2(p_1, p_2, w_{11}, w_{12}, w_{21}, w_{22}, \pi) \\ = \begin{cases} f_1^1(p_1, p_2, w_{11} + w_{12}, w_{11} + w_{22}, \pi_{12}) + f_2^{21}(p_1, p_2, w_{11} + w_{12}, w_{11} + w_{22}, \pi_{21}) - r \\ f_1^2(p_1, p_2, w_{21} + w_{12}, w_{21} + w_{22}, \pi_{21}) + f_2^{12}(p_1, p_2, w_{21} + w_{12}, w_{21} + w_{22}, \pi_{12}) - r \end{cases} \end{aligned} \quad (2.3.12)$$

The set of *equilibria*, denoted $B_j(\omega)$, is the subset of $B(r)$ such that $z_j(p, w) = 0$. $B_j(\omega)$ is thus the set of $(p, w) \in B(r)$ obtained as the preimage of $z_j^{-1}(0)$ for which $\text{rank} Dz_j$ has full rank.

Since $\text{rank} Dz_j(p, w) = 2L$, $j = 1, 2$ by application of the Regular Value Theorem⁴⁹ $z_j^{-1}(0)$ or $B_j(\omega)$, $j = 1, 2$ forms a smooth submanifold of $B(r)$ of dimension 2 where the dimension of $B_j(\omega)$ is obtained as follows. For almost every $\omega \in \Omega(r)$ one has that $\text{rank} Dz_j(p, w) = 2L$. $z_j^{-1}(0)$ is a smooth manifold⁵⁰ of $B(r)$ of dimension $\dim B(r) - \dim \mathbb{R}^{2L} - 1 = 2L + 3 - 2L - 1 = 3 - 2 = 2$. Let $j = 1$ for instance. In equilibrium $f_1^1 + f_2^{11} = \omega_1 + \omega_2$ and $f_1^2 + f_2^{22} = \omega_1 + \omega_2$. Multiplying the first L equations by equilibrium price p_1 and the second L equations by the equilibrium price p_2 yields $p_1 \cdot f_1^1 + p_1 \cdot f_2^{11} = p_1 \cdot \omega_1 + p_1 \cdot \omega_2$, $p_2 \cdot f_1^2 + p_2 \cdot f_2^{22} = p_2 \cdot \omega_1 + p_2 \cdot \omega_2$. Summing the latter two expressions yields:

$$\begin{aligned} p_1 \cdot f_1^1 + p_1 \cdot f_2^{11} + p_2 \cdot f_1^2 + p_2 \cdot f_2^{22} &= p_1 \cdot \omega_1 + p_1 \cdot \omega_2 + p_2 \cdot \omega_1 + p_2 \cdot \omega_2 \\ &= w_{11} + w_{12} + w_{21} + w_{22} \end{aligned} \quad (2.3.13)$$

Thus $\dim z_j^{-1}(0) = \dim B(r) - \dim \mathbb{R}^{2L} - 1$. Therefore, if $(p, w) \in z_1^{-1}(0)$ is a price-income pair which clears the market then (3.12) holds⁵¹ and $B_1(\omega)$ is a 2-dimensional submanifold of $B(r)$ ⁵². By analogous reasoning $B_2(\omega)$ is a smooth submanifold of $B(r)$ of dimension 2.

Putting together the framework of this section and the last, the following are defined. Let $\pi \in \text{int}(I^2)$ be fixed. The *equilibrium manifold* is the set of prices and endowments for which the equilibrium equation holds and has a manifold structure.

$$E = \{(p, \omega) \in S \times \Omega : z_j(p, \omega, \pi) = 0, j = 1, 2\} \quad (2.3.14)$$

The restricted price-income space $B(r)$ is the set of prices and incomes permissible given the restriction on total resources.

⁴⁹ The Regular Value Theorem (Guillemin and Pollack [41] or Milnor [67]) states that if $f : X \rightarrow Y$ is a smooth map between smooth manifolds and $y \in Y$ is a regular value of f then the preimage $f^{-1}(y)$ is a smooth submanifold of X with $\dim f^{-1}(y) = \dim X - \dim Y$.

⁵⁰ Smoothness follows from z_j being a smooth map.

⁵¹ This is a reformulation of Walras Law.

⁵² It is noted that throughout the arguments of Sections 2.3.1 and 2.3.2 homogeneity of prices was not incorporated into the dimensions of the manifolds. If a price normalisation were to be taken then the results remain unchanged. A price normalisation is not made as to do so would break the symmetry of the stationarity of the system.

$$B(r) = \{(p, w) \in B : \omega \in \Omega(r)\} \quad (2.3.15)$$

The budget manifold $A(\omega)$ is the set of prices and incomes which satisfy the constraints given the restriction on total resources

$$A(\omega) = \{(p, w) \in B(r) : p_i \omega_1 + p_j \omega_2 = w_{i1} + w_{j2}, (i, j) \in \{1, 2\} \times \{1, 2\}, \omega \in \Omega(r)\} \quad (2.3.16)$$

The set of equilibria $B_j(\omega)$ is the set of prices and incomes for which the market clears given the constraint on resources.

$$B_j(\omega) = \{(p, w) \in B(r) : z_j(p, w, \pi) = 0\}, \quad j = 1, 2 \quad (2.3.17)$$

A *geometric equilibrium* (Balasko [7], [8], [9], [10], Ghiglino and Tvede [34] and Goenka and Matta [35]) is the subset of prices and incomes, given a constraint on resources; $(p, w) \in B(r)$ for which the *budgets are maintained and the excess demand functions are equal to zero*. A geometric equilibrium is the set of prices and incomes which belong to the budget manifold and the set of equilibria for both subsystems of equations:

$$(p, w) \in A(\omega) \cap B_1(\omega) \cap B_2(\omega) \quad (2.3.18)$$

The geometric equilibrium is dual to the Walrasian equilibrium, being nothing other than a translation of the latter into the space of economies defined over the price-income space.

A *fibre*, denoted $F(p, w)(r)$ for some $r > 0$, is the set *price-endowment* pairs such that $(p, w) \in A(\omega) \cap B_1(\omega) \cap B_2(\omega)$. Thus for a given (p, w) for some r a fibre is the set of prices and incomes which belong to the equilibrium manifold $(p, \omega) \in E$. Intuitively, for the demand functions $f(p, p \cdot \omega, \pi_{ss})$, non-linearities are involved in the dependency with respect to prices and incomes for a given probability. However, if the price vector and all stochastic incomes are kept constant then aggregate demand remains constant. As such a pair (p, ω) remains an equilibrium when ω is varied whilst prices, incomes and total resources are kept constant. *Fibres are thus linear manifolds embedded in the equilibrium manifold E*. In this sense the equilibrium manifold is the *union of the fibre bundles over all prices and incomes*⁵³. Furthermore, there is a one-to-one relationship between the equilibrium manifold and the geometric equilibria (Balasko [10], pg. 176).

⁵³ See Balasko [10], pg. 76 for a discussion of the fibre structure of the equilibrium set in a pure exchange economy.

2.3.3. Non-Existence of Geometric Equilibria

It is now shown that generically, i.e. for almost all $\omega \in \Omega(r)$ and almost all $r > 0$, stationary sunspot equilibria do not exist. This is demonstrated by showing that that geometric equilibria do not generically exist. This implies that the equilibrium manifold, being the set of all fibre bundles, is empty. The non-existence of SSE is demonstrated by means of a transversality argument.

Definition 2.3.1. (Definition 4.1 Golubitsky and Guillemin [36] pg. 50) Let X and Z be smooth manifolds and $f : X \rightarrow Z$ a smooth mapping. Let Y be a smooth submanifold of Z and x a point of X . Then f intersects Y transversally at x if either

- i. $f(x) \notin Y$, or
- ii. $f(x) \in Y$ and $T_{f(x)}W + (df)_x(T_x X) = T_{f(x)}Y$ (where $T_x X$ is the tangent space to X at x and likewise for $T_{f(x)}W$ and $T_{f(x)}Y$)

Transversality of manifolds is a generic property; almost every manifold is transversal to another manifold either by non-intersection with that manifold (Definition 2.3.1.i) or by non-empty, non-tangential intersection characterised by the tangent spaces of the manifolds at the point of intersection spanning the ambient space (Definition 2.3.2.ii). Transversality depends not only on the manner in which two manifolds interact, but also on the ambient space in which they both reside. If the two manifolds are transverse yet the dimension of the ambient space is greater than the combined dimension of the two manifolds then the manifolds can only be transversal by non-intersection⁵⁴. Formally:

⁵⁴ Two manifolds X and Y both of which belong to a space Z which may or may not be a manifold itself of some other space are said to be transversal to each other if they intersect non-tangentially. Symbolically one writes $X \pitchfork Y$. For instance, suppose that X and Y are two curves in $\mathbb{R}^2$ which cut each other. If X and Y intersect each other at a point non-tangentially then a small perturbation does not disturb the property of non-tangential intersection. In such a case, the tangent spaces of X and Y span the ambient space (Definition 2.3.1.ii). More generally, in the case in which X and Y do intersect each other tangentially then a small perturbation of either X or Y or both suffices to restore the property of transversality. Transversal intersection is a generic property (Guillemin and Pollack pp. 67 [41]). However, as pointed out in Guillemin and Pollack [41] (Golubitsky and Guillemin pg. 51 [36] and also Hirsch pg. 67 [42]) and reported as Proposition 2.3.1, the transversal intersection of X and Y depends not only on the manner in which the two surfaces or manifolds interact but also on the ambient space in which they reside. If X and Z are manifolds and $Y \subset Z$ is a submanifold with $\dim X + \dim Y < \dim Z$ (i.e. $\dim(X) < \text{codim}(Y)$). Then $X \pitchfork Y$ implies $X \cap Y = \emptyset$; if the dimensions of the manifolds X and Y is less than that of the ambient space Z then, transversality being a generic property, implies that X and Y will typically miss each other in the space Z , i.e. have empty intersection. If X and Y do intersect each other and $\dim(X) < \text{codim}(Y)$ by an infinitesimal perturbation of either X or Y the two manifolds are

Proposition 2.3.1. (Golubitsky and Guillemin [15], Proposition 4.2, pg. 51) Let X and Z be smooth manifolds and $Y \subset Z$ a submanifold. Suppose that $\dim X + \dim Y < \dim Z$ (i.e. $\dim(X) < \text{codim}(Y)$ ⁵⁵). Let $f: X \rightarrow Z$ be smooth and suppose that $f \pitchfork Y$. Then $f(X) \cap Y = \emptyset$. ■

Let f be the identity map in Proposition 2.3.1. If $X \pitchfork Y$;

$$\dim(X) < \text{codim}(Y) \Rightarrow X \cap Y = \emptyset \quad (2.3.19)$$

(2.3.19) may occur if the dimension of either X or Y is not large enough relative to the ambient space Z .

Applying the preceding proposition to $A(\omega)$ and to $B_j(\omega)$. Let $\omega \in \Omega(r)$. $(p, w) \in B(r)$ is an equilibrium if and only if $(p, w) \in A(\omega) \cap B_1(\omega) \cap B_2(\omega)$. Since $\text{codim}(A(\omega)) = \dim B(r) - \dim(A(\omega)) = 3$ and $\dim(B_j(\omega)) = 2$, $j = 1, 2$ then:

$$\dim(B_j(\omega)) < \text{codim}(A(\omega)), \quad j = 1, 2 \quad (2.3.20)$$

By Proposition 2.3.1, (2.3.20) implies that $A(\omega) \cap B_j(\omega) = \emptyset$, $j = 1, 2$ hence $A(\omega) \cap B_1(\omega) \cap B_2(\omega) = \emptyset$ and $A(\omega) \pitchfork B_j(\omega)$, $j = 1, 2$. Since $\omega \in \Omega(r)$ and $r > 0$ were arbitrarily chosen, (2.3.20) generically holds. It is concluded that for almost every endowment vector, the manifolds, $A(\omega)$ and $B_j(\omega)$ are transverse by empty intersection implying that geometric equilibria do not exist. Consequently for almost every price-endowment pair stationary sunspot equilibria do not exist. By transversality, the non-existence of SSE is robust⁵⁶.

Theorem 2.3.1 For $L > 1$ ⁵⁷ commodities each period, stationary sunspot equilibria do not generically exist in the stationary OLG model. ■

By Theorem 2.3.1, the equilibrium manifold E , being the union of fibre bundles associated with the set of geometric equilibria, is empty.

pulled off each other and are thereby disjoint. It is easy to see that X and Y will typically be disjoint to each other and by Definition 2.3.1.i transversal to each other or lie in general position

⁵⁵ $\text{codim}(Y) = \dim Z - \dim Y$

⁵⁶ If there is some $(p, w) \in A(\omega) \cap B_1(\omega) \cap B_2(\omega)$ then a slight perturbation of $\omega \in \Omega(r)$ will suffice to pull the manifolds away from each other thereby restoring the condition (2.3.20).

⁵⁷ For $L=1$ commodity, the conditions for existence are discussed in the Section 2.5.

2.4. The Interrelationship between the Steady State Equilibrium and the Non-Existence of Stationary Sunspot Equilibria

Theorem 2.3.1 states that

$$A(\omega) \cap B_1(\omega) \cap B_2(\omega) = \emptyset \text{ and } A(\omega) \cap B_j(\omega), \quad j = 1, 2 \text{ a.e. } \omega \in \Omega(r) \text{ and } r > 0 \quad (2.4.1)$$

By (2.4.1) equilibria, and in particular SSE do not exist. This result is due to the fact that there are too many equations and not enough variables as seen from the point of view of the manifold structure of the set of geometric equilibria.

This result is now contrasted with the fact that for almost every endowment vector a steady state equilibrium exists. At first sight it appears that these two statements are contradictory. This however is not the case.

A steady state or a certainty equilibrium is an equilibrium in which prices do not reflect uncertainty; i.e. there exists a pair $p^* = (p_1^*, p_2^*) \in \mathbb{R}_{++}^{2L}$ with $p_1^* = p_2^*$ such that $z_j(p, w, \pi) = 0$. The following defines a quantity consistency axiom.

Definition 2.4.1. Quantity Consistency of Stochastic Demands (Chiappori and Guesnerie [9])⁵⁸ Let f_i be agent i 's demand function. If $p_1 = p_2$ then

$$f_i(p_1, p_2, \omega, \pi_{ij}) = f_i(p_1, p_2, \omega, \pi_{ij}'), \quad \forall \pi_{ij} \in I.$$

By the quantity consistency axiom, a certainty equilibrium can be analysed by setting $p_1 = p_2$ in the maximisation problems 1 and 2 of Section 2 or equivalently by setting either $(\pi_{12}, \pi_{21}) = (0, 0)$ or $(\pi_{12}, \pi_{21}) = (1, 1)$ and solving for the equilibrium price p^* . Let the sunspot economy be degenerate by the restriction $(\pi_{12}, \pi_{21}) = (1, 1)$. The maximisation Problems 2.1 and 2.2 of Section 2.2 are:

Problem 2.1(i) – Agent 1

$$\max_{x_1^1, x_2^{12}} u(x_1^1, x_2^{12})$$

⁵⁸ By Definition 2.4.1 if $p_1 = p_2$ by inspection of the maximisation problems 2.1 and 2.2 $x_2^{s1} = x_2^{s2}$, $s = 1, 2$ and the maximisation problem reduces to a certainty maximisation.

$$\text{s.t. } p_1 \cdot x_1^1 + p_2 \cdot x_2^{12} \leq p_1 \cdot \omega_1 + p_2 \cdot \omega_2 \quad (2.4.2)$$

Problem 2.2(i) – Agent 2

$$\begin{aligned} & \max_{x_1^1, x_2^{21}} u(x_1^2, x_2^{21}) \\ & \text{s.t. } p_2 \cdot x_1^2 + p_1 \cdot x_2^{21} \leq p_2 \cdot \omega_1 + p_1 \cdot \omega_2 \end{aligned} \quad (2.4.3)$$

The resultant vector of demand functions for agents 1 and 2 are respectively $f^1 = (f_1^1, f_2^{12}) \in \mathbb{R}_{++}^{2L}$ and $f^2 = (f_1^2, f_2^{21}) \in \mathbb{R}_{++}^{2L}$. The system of excess demand functions pertaining to maximisation Problems 2.1(i) and 2.2(i) is equivalent to the system of excess demand functions (2.3.12) with the restriction that $(\pi_{12}, \pi_{21}) = (1, 1)$ and is the mapping $z_2 : S \times \mathbb{R}_{++}^{2L} \rightarrow \mathbb{R}^{2L}$ where $S \equiv \mathbb{R}_{++}^{2L}$ is the non-normalized price space⁵⁹ given by⁶⁰:

$$z_2 = \begin{pmatrix} (f_1^1(p_1, p_2, p_1 \cdot \omega_1 + p_2 \cdot \omega_2) + f_2^{21}(p_1, p_2, p_2 \cdot \omega_1 + p_1 \cdot \omega_2) - \omega_1 - \omega_2) \\ (f_1^2(p_1, p_2, p_2 \cdot \omega_1 + p_1 \cdot \omega_2) + f_2^{12}(p_1, p_2, p_1 \cdot \omega_1 + p_2 \cdot \omega_2) - \omega_1 - \omega_2) \end{pmatrix} \quad (2.4.4)$$

In [78], Tuinstra and Weddepohl show that stationary OLG economies are equivalent to pure exchange economies. This equivalence can be seen by inspection of the maximisation Problems 1(i) and 2(i) which have the form of a pure exchange model in which there are two consumers and $2L$ goods with the requirement that both consumers have the same utility functions and that the consumption and endowment vectors are permuted; agent one has the consumption and endowment vector (x_1^1, x_2^{12}) and (ω_1, ω_2) and agent two has the consumption vector (x_2^{21}, x_1^2) and (ω_2, ω_1) ⁶¹. Such an economy is termed a cyclical pure exchange economy [76]. The economy remains however a pure exchange economy to which one can apply the battery of well-established results. The following two propositions concerning pure exchange economies are stated.

⁵⁹ The normalisation of prices is permissible but simply not carried out.

⁶⁰ Given that f_2^{11} and f_2^{22} are not well-defined demand vectors as $(\pi_{11}, \pi_{22}) = (0, 0)$ then the system of excess demand functions z_1 of (2.3.11) is not well-defined.

⁶¹ By translating the OLG model into a pure exchange model, agent 1's first period consumption vector becomes the first L goods and the second period consumption vector becomes the second L goods whereas agent 2's second period consumption vector becomes the first L goods and the first period consumption vector becomes the second L goods. Thus the stationary OLG model in which there are L goods over two time periods and two agents is equivalent to a pure exchange economy in which there is one time period and $2L$ goods and two agents.

Proposition 2.4.1. (Balasko [10], Propositions 2.7.1 and 2.7.2) The set of regular economies⁶² R in Ω is open and dense with full measure. ■

Proposition 2.4.2. (Balasko [10], Corollary 4.6.4) The number of equilibria of a regular economy $\omega \in R$ is odd. ■

Under usual assumptions (strict quasi-concavity of utility functions, boundedness of demand functions and desirability), Proposition 2.4.1 states that generically (i.e. for almost every endowment vector) economies are regular⁶³. Proposition 2.4.2 states that for any regular economy the number of equilibrium price vectors are odd. Since one is an odd number then for almost every $\omega \in \Omega$ there exists at least one equilibrium price vector. By the equivalence of the degenerate stationary sunspot OLG economy to the cyclical pure exchange economy, there exists an odd number of equilibria for almost every endowment vector. This result applies equally to the case in which total resources are constrained (Balasko [10], Ch. 5). Hence for $(\pi_{12}, \pi_{21}) = (1, 1)$ regular equilibria generically exist for almost every $\omega \in \Omega(r)$ and the system (2.4.3) has at least one fixed point.

Denote as $\hat{A}(\omega)$ as the budget manifold and $\hat{B}_2(\omega)$ as the equilibrium set. The intersection of $\hat{A}(\omega)$ with $\hat{B}_2(\omega)$ ⁶⁴ is non-empty. By the fact that an equilibrium is regular if and only if $\hat{A}(\omega)$ is transversal to $\hat{B}_2(\omega)$ (Balasko [10], pg. 114) by the fact that regular equilibrium are generic in the space of endowments it is concluded:

$$\hat{A}(\omega) \cap \hat{B}_2(\omega) \neq \emptyset \text{ and } \hat{A}(\omega) \pitchfork \hat{B}_2(\omega) \text{ a.e. } \omega \in \Omega(r) \text{ and } r > 0 \quad (2.4.5)$$

In relation to (2.4.5) by the fact that equilibria are odd in number for any regular economy, $A(\omega) \cap B_2(\omega)$ generically forms a zero-dimensional manifold⁶⁵ comprised of an odd number of points.

Tuinstra and Weddepohl further show that the cyclical pure exchange economy possesses a cyclical equilibrium structure (Proposition 1, [78]); if (p_1^*, p_2^*) is an equilibrium of (2.4.3) then there exists a permuted equilibrium of the form (p_2^*, p_1^*) where if (p_1^*, p_2^*)

⁶² A regular economy is a vector $\omega \in \Omega$ for which the Jacobian of the set of excess demand functions does not lose rank. Proposition 4.1 is an application of Sard and Brown's Theorem. See Milnor [67].

⁶³ An economy which is not regular is termed singular and gives rise to a critical equilibrium. A critical equilibrium may be characterised by a continuum of price vectors. However such cases arise with measure 0. Debreu [30] was the first to demonstrate this.

⁶⁴ $\hat{B}_2(\omega)$ is the equilibrium manifold derived from the system of equations (4.3) and $\hat{A}(\omega)$ is the budget manifold derived from the constraints in Problems (2.4.1) and (2.4.2).

⁶⁵ The fact that the intersection of the two manifolds is a zero-dimensional manifold follows from the fact that generically the number of equilibria are odd hence finite in number and locally isolated.

satisfies $p_1^* \neq p_2^*$ the equilibrium is termed *cyclical*⁶⁶. It follows that every cyclical equilibrium has a permuted counterpart and cyclical equilibria, if they exist, do so in pairs. By Proposition 2.4.2, there exists an odd number of equilibria so there must exist a “symmetry breaking” cyclical equilibrium price vector which satisfies $(p_1^*, p_2^*) = (p_2^*, p_1^*)$ or $p_1^* = p_2^*$ thereby yielding the equilibrium vector (p_1^*, p_1^*) . By Propositions 2.4.1 and 2.4.2, an equilibrium of the form $p_1^* = p_2^*$ exists for almost every $\omega \in \Omega$, where the precise values of the equilibrium price vector will depend on the value of ω and are such that $A(\omega) \cap B_2(\omega)$. Therefore certainty equilibria are generic and robust in the space of endowments for the certainty or non-stochastic economy.

Corollary 2.4.1 Regular steady state equilibria exist for almost every $\omega \in \Omega$ in the certainty economy. ■

In Corollary 2.4.1 steady state equilibria are characterised by the transversal intersection of the budget manifold and the equilibrium set, i.e. (2.4.4) holds. By the quantity consistency of Definition 2.4.1, since (p_1^*, p_1^*) is an equilibrium for $(\pi_{12}, \pi_{21}) = (1, 1)$ ⁶⁷ then (p_1^*, p_1^*) is an equilibrium for all $\pi \in I^2$. It is therefore concluded that for *any* $\pi \in \text{int}(I^2)$ there exists a *steady state equilibrium* having the form (p_1^*, p_1^*) .

This however does not contradict Theorem 2.3.1, i.e. (2.4.1) and (2.4.5) are simultaneously satisfied. The reason for this is that in the examination of the certainty economy the dimensions of the budget manifold and equilibrium set are such that the sum of them is equal to the ambient price-income space in which case Proposition 2.3.1 does not apply. Furthermore, the number of unknowns and equations in the system of equations (2.4.4) is equal thereby yielding a properly defined equilibrium system. Hence $(p_1^*, p_1^*, w) \in \hat{A}(\omega) \cap \hat{B}_2(\omega)$ yet when the equilibrium (p_1^*, p_1^*, w) is considered in the extrinsically uncertain economy the dimensions of the manifolds are such that the certainty equilibrium does not exist in that system. Intuitively, the reason for this is that the equilibrium in the stochastic system will embed the hypothesis of extrinsic uncertainty and equilibrium in the certainty system will not. The two systems are not equivalent and in fact are such that the set of equilibria generate manifolds which sit in spaces of different size. Hence, the certainty equilibrium is in a sense a degenerate sunspot equilibrium but must be considered as an equilibrium of a non-stochastic system of equations. Therefore, the steady

⁶⁶ Specifically, in this context such equilibria are termed cyclical pure exchange equilibria. By the equivalence to stationary OLG models such equilibria occur if the OLG model exhibits predictable fluctuations of cycles of order 2.

⁶⁷ The same argument holds for $(\pi_{12}, \pi_{21}) = (0, 0)$ however in this case the resultant equilibrium is necessarily the steady state as $p_1 = p_2$.

state equilibrium is an equilibrium of the extrinsically uncertain economy but the certainty economy is the correct economic system to use for any comparative statics as, by the sunspot hypothesis, if $p_1 = p_2$ then uncertainty is not embedded in the price regime. The following corollary is stated and read with caution.

Corollary 2.4.2 For almost every $\omega \in \Omega$ and $\pi \in I^2$ there exists a steady state equilibrium of the form $(p_1^*, p_1^*) \in \mathbb{R}_{++}^{2L}$ in the extrinsically uncertain economy. ■

2.5. Conclusion

In this paper it has been demonstrated that for the case of multiple commodities stationary sunspot equilibria do not exist for almost every endowment vector. This result was established by showing that the manifolds defined by the budget set and equilibrium equations can intersect transversally only by non-intersection. Since transversality is a generic property, non-intersection of the manifolds occurs for almost every endowment vector for given total resources. Consequently, there cannot generically exist any price-income pair for which SSE exist. This result is to be contrasted with the generic existence of steady state equilibria in the certainty economy.

It is interesting to note that for the certainty economy obtained by $(\pi_{12}, \pi_{21}) = (1, 1)$, cyclical equilibria of the form (p_1^*, p_2^*) and (p_2^*, p_1^*) may occur. Given the generic regularity of such equilibria, if a cyclical equilibrium exists, a straight forward application of the implicit function theorem would imply that cyclical equilibria exist for a non-negligible set of endowments; there is an open neighbourhood M of ω for which cyclical equilibria exist and for each $\omega \in M$ there exists a steady state equilibrium. However, for any (π_{12}, π_{21}) in a neighbourhood N of $(1, 1)$ the only equilibria to remain in existence is the steady state. Thus an infinitesimal perturbation of the probability vector about $(\pi_{12}, \pi_{21}) = (1, 1)$ in N suffices to remove from existence the cyclical equilibria. That is, if $\pi \in N$, for all ω $A(\omega) \cap B_1(\omega) \cap B_2(\omega) = \emptyset$ and $A(\omega) \not\cap B_j(\omega)$, whereas if $(\pi_{12}, \pi_{21}) = (1, 1)$ and $\omega \in M$, $\hat{A}(\omega) \cap \hat{B}_2(\omega) \neq \emptyset$, $\hat{A}(\omega) \not\cap \hat{B}_2(\omega)$ and $\#(\hat{A}(\omega) \cap \hat{B}_2(\omega)) > 1$. Consequently, *cyclical equilibria are not robust to perturbations in the probability space* and have no bearing on the existence or non-existence of stationary sunspot equilibria in the case of multiple commodities. This is contrasted with the case in which there exists one commodity. When $L=1$, the constraints $p_s x_1^s + p_s x_2^{ss} = p_s \omega_1 + p_s \omega_2$, $s=1, 2$ are equivalent to the market clearing equations (2.3.11) thereby the equilibrium system of equations is defined by (2.3.12). It can

be shown that there is a dependence between the first and second equation of (2.3.11) so the system of equilibrium equations is reduced to the study of either $f_1^1 + f_2^{21} - r$ or $f_1^2 + f_2^{12} - r$. The equilibrium system is no longer overdetermined, being defined by a single equation which is a function of the price ratio and stochastic incomes and as pointed out in the introduction, for the existence of SSE it suffices that 2-cycles exist. In sum, whilst the flavour of the extrinsic uncertain problem of $L > 1$ and $L = 1$ commodities per period is the same, the predictions of the two models are antithetical.

As a matter of example in this regard, Burke [19] discusses the elimination of sunspot equilibria by perturbing the underlying certainty economy in which there is one commodity to a nearby disaggregated economy in which there are multiple commodities each of which are perfect substitutes. In the disaggregated economy sunspot equilibria cease to exist ([19], Theorem pg. 830)⁶⁸. Whilst the OLG model of [19] is neither a pure exchange economy nor stationary, under adaptation, the same result holds in the discussion of this paper. For instance, suppose that $L=1$ and the utility function under certainty $u: \mathbb{R}_{++}^2 \rightarrow \mathbb{R}$ is $u(x_1, x_2)$ with the constraint $p_1 x_1 + p_2 x_2 = p_1 \omega_1 + p_2 \omega_2$, $(p_1, p_2) \in \mathbb{R}_{++}^2$ and $(\omega_1, \omega_2) \in \mathbb{R}_{++}^2$. The economy defined as such is termed the *aggregated economy under certainty*. Suppose that the aggregated economy is now rendered stochastic being characterised by stochastic maximisation problems 1 and 2 and that stationary sunspot equilibria exist. Suppose further that the aggregated certainty economy is *disaggregated* into $L > 1$ commodities by treating the sole consumption good as finitely many perfect substitutes. That is, let $(x_1, x_2) \in \mathbb{R}_{++}^2$ be disaggregated into $(x_1, x_2) \in \mathbb{R}_{++}^{2L}$ where $x_1 = (x_{1,1}, \dots, x_{1,L})$ and $x_2 = (x_{2,1}, \dots, x_{2,L})$. Likewise, the endowment vector is also disaggregated; $(\omega_1, \omega_2) \in \mathbb{R}_{++}^{2L}$ with $\omega_1 = (\omega_{1,1}, \dots, \omega_{1,L})$ and $\omega_2 = (\omega_{2,1}, \dots, \omega_{2,L})$ as is the price vector; $(p_1, p_2) \in \mathbb{R}_{++}^{2L}$ where $p_1 = (p_{1,1}, \dots, p_{1,L})$ and $p_2 = (p_{2,1}, \dots, p_{2,L})$. The utility function takes on the form $u(x_{1,1} + \dots + x_{1,L}, x_{2,1} + \dots + x_{2,L})$ where the summation of the commodities reflects the assumption of perfect substitutability. The stationary stochastic OLG model for the disaggregated economy has maximisation problems of the form of 2.1 and 2.2 of Section 2.2. By Theorem 2.3.1 there does not exist a sunspot equilibrium for the disaggregated economy.

That stationary sunspots should cease to exist by disaggregation of the commodity appears a contrived result. Yet the non-existence of SSE remains an artefact of the stationarity of the economy. When viewed in this manner one must question the validity and import of stationary sunspot equilibria in the *aggregated* economy. Consequently, if one ascribes to the notion that sunspots are in fact an estimable equilibrium concept then the

⁶⁸ In [19] it is established that the disaggregated economy has no equilibrium other than a stationary perfect foresight economy and sunspots do not exist. The manner in which this is shown utilises primarily a numerical example based on the discussion of [5] in the framework of a consumption-production economy. It is noted that in [19] there is no discussion of the general non-existence of stationary sunspot equilibria as the framework there is not necessarily stationary.

stationarity aspect of overlapping generations model must be dispensed with⁶⁹ or one must remain content to examine sunspots in the one commodity case possibly treating that commodity as an aggregation of various commodities.

⁶⁹ This is the approach taken in Chiappori *et al.* [22] in which the equilibrium system is a one-step forward looking model where present each period depends upon the expectation of the price in the subsequent period. Since the price at each point in time is not dependent on the pre-existing price, then the equilibrium equations are not overdetermined and the existence of sunspot equilibrium is established.

Ergodic Chaos in the Overlapping Generations Model

Abstract

The equilibrium properties of a pure exchange overlapping generations economy are characterised in which the consumption profiles are ergodic and chaotic. Chaotic equilibria are contrasted against equilibrium consumption at the steady state. By utilising the probability distribution of ergodic consumption profiles, it is demonstrated that expected indirect utility along a typical equilibrium path is less than the indirect utility at the steady state. An argument for stabilisation is made and a stabilising rule is proposed. The stochastic properties of ergodic equilibria are then related to sunspot equilibria.

3.1. Introduction

The free functioning of the market mechanism has played an important role in the analysis of equilibrium. Proponents of the belief in such a mechanism often posit that over the long term the market is an inherently stable system exhibiting a tendency to converge to and remain at a stable or steady state equilibrium. Any movement away from the steady state equilibrium is deemed transitory. Yet in practice the economy need not be stable and fluctuations in equilibria are observed indicating a lack of convergence to a stable equilibrium. In order to explain such fluctuations stochastic or random shocks have been incorporated as explanatory variables into equilibrium models (Kydland and Prescott [51] Sargent and Wallace [74])⁷⁰ thereby allowing the model to explain movements in economic data. Dynamic stability subject to random shocks is however but one manner in which to explain equilibrium dynamics. Another competing theory (Grandmont [38]) is that the market is a dynamically unstable system even in the absence of exogenous shocks. In such a theory the equilibrium system is deterministic and the long-run stable equilibrium is not converged upon given that

⁷⁰ These are not overlapping generations models as examined in this chapter.

the market mechanism does not correct any perturbation from a steady state equilibrium. Hence fluctuations and cycles are present and will persist.

Within the framework of an overlapping generations model (OLG) the occurrence of cyclical equilibria have been analysed by various authors. Gale [33] studied the evolution of the equilibrium consumption paths utilising the OLG framework. In [33] the equilibrium dynamics are divided into two cases. The first is the Classical case in which the young exhibit impatience and borrow in the first period of life in order to consume in excess of wealth in the second period of life. The second is the Samuelson case in which the young delay consumption saving in the first period of life to consume in excess of wealth when old⁷¹. The principle findings of Gale [33] are that in the Classical economy, if (1) the initial value of an agent's consumption is greater than that of the endowment then the equilibrium sequence will move away from the initial endowment vector converges towards the steady state equilibrium or executes some form of *limit cycle* about it ([33] pg. 16). (2) If the initial first period consumption is less than the endowment then the equilibrium sequence will move to a point at which equilibrium consumption is no longer feasible; some form of economic breakdown occurs. Instead, if the economy is Samuelson then for any value of initial consumption, above or below the initial endowment, the equilibrium path will move away from the steady state and converge towards the autarky equilibrium⁷². Gale's findings indicate that on the one hand in the Classical economy the outcome depends on the starting values with different results obtaining given this choice. On the other hand the no-trade or autarky outcome of the Samuelson economy, is, as stated by Gale, difficult to explain and that while the '*Samuelson case is logically consistent, the case relevant to the real world [...] is the Classical one*' [33].

Whilst Gale's analysis was concerned with the properties of the two differing types of economies, it was noted that equilibrium paths need not exhibit convergence to a steady state. In the case of the Classical economy equilibria may not reach the steady state equilibrium and may cycle about it (Example 3, [33] pg. 27). This example is referred to as an '*... amusing example of a "business cycle" which has nothing to do with expectations ...*'. This example serves in some measure as a forerunner to the explanation of business cycles as arising from endogenous deterministic cycles. This cyclical behaviour was not however expounded upon in [33]. Nor was it indicated that the apparent failing of the Samuelson case to explain equilibrium paths could be overcome if the equilibrium sequence were to move backwards through time.

Utilising the theory of topological chaos in discrete one-dimensional dynamical systems (see Appendix for a discussion of topological chaos), Benhabib and Day [8] focus on the possibility of oscillatory equilibrium sequences in the Classical OLG economy⁷³ as

⁷¹ In the Classical case first (second) period endowment is less (greater) than the first (second) period consumption at the steady state equilibrium. In the Samuelson case first (second) period endowment is greater (less) than the first (second) period consumption at the steady state equilibrium

⁷² An equilibrium is autarkic if consumption is equal to initial endowments or wealth. In such an equilibrium no trade is effectuated.

⁷³ As in [33], Benhabib and Day [15], pg. 42 point out that the Samuelson model leads to the equilibrium sequence converging to the autarky point or breaking down.

examined by Gale [33]. Given the assumption of additive separability and stationarity of the utility functions and endowments⁷⁴, by means of transforming the equilibrium equation into a dynamical system of the form $f : X \rightarrow X$, where f is continuous and unimodal (possessing a single critical point) and X is a compact interval, it is shown that the equilibrium trajectories have the capacity to generate periodic equilibria as well as topologically chaotic sequences. These phenomena occur if the utility of the first period consumption is sufficiently concave with respect to the utility of the second period consumption. In such a case the equilibrium map f , derived from the representative agent's offer curve, is sufficiently non-linear such that equilibria may exhibit a cycle of order 3 and, by Li and Yorke's Theorem (see [39]), topological chaos coexists with cycles of every period. Benhabib and Day [15] thereby substantiate Gale's [33] example of a real business cycle providing justification for the hypothesis that equilibrium fluctuations are the result of inherently unstable equilibrium systems. Given that chaos is characterized by sensitive dependence to initial values, two equilibrium sequences will typically exhibit differing trajectories, yet irrespective of this, it is demonstrated that any erratic or chaotic trajectories are Pareto efficient ([15] Theorem 3, pg. 51).

Whilst Benhabib and Day [15] examine erratic fluctuations in a Classical economy, Grandmont [38] examines the source of erratic equilibrium cycles in a two-period OLG Samuelson economy with production⁷⁵. In order to bypass the difficulty mentioned in both [15] and [33] that the equilibrium dynamics of the Samuelson economy either converge to the autarky equilibrium or are unsustainable over the long run, Grandmont examines the backwards perfect foresight equilibrium dynamics⁷⁶. The advantage of the Samuelson economy is that money has a positive value in equilibrium. If instead, the economy were Classical then the equilibrium sequence would move forward in time but not admit money with a positive value. In fact money would have a negative value or, as proffered by Benhabib and Day, the equilibrium sequence is supported by the 'social contrivance of credit' ([15] pg. 49). Moreover, the steady state in the Classical economy is sub-optimal with respect to autarky [33] which is not the case in the Samuelson economy. Grandmont's analysis, whilst having the interpretational difficulty of going backwards in time given the forward looking optimization problem of a representative agent, does therefore have the advantage that it justifies the use of money in an intertemporal setting in which equilibrium may exhibit fluctuations or erratic behaviour. Grandmont studies not only the existence of cyclical and chaotic equilibrium sequences but the reasons for which they persist and the bifurcation route to chaos. It is demonstrated that within a stationary environment⁷⁷, fluctuating competitive monetary equilibria persist and are not uncommon. Such that cycles occur, it is sufficient that at some relative price ratio the intertemporal wealth effect strongly outweighs the substitution effect. This occurs if the degree of concavity of the utility function, as measured by the Arrow-Pratt measure of risk aversion, is sufficiently large for the second period consumption

⁷⁴ Stationarity means time invariance, i.e. does not change over time.

⁷⁵ The model with production is equivalent to the two-period pure exchange model of [15] and [33]. See [38] pg. 1000.

⁷⁶ Grandmont [38] notes that the dynamical system may be locally inverted and equilibria move forwards in time.

⁷⁷ Stationarity of utility function and upper bounds of labour supply.

with respect to the first period consumption⁷⁸. This condition is related to the offer curve of the representative agent being such that it bends backwards and has a gradient less than one in absolute value at the monetary steady state; for some price vector the intertemporal income effect strongly dominates the substitution effect. Furthermore, the higher the degree to which the wealth effect dominates the substitution effect, the greater the degree to which the offer curve bends backwards which in turn implies the more ‘non-linear’ is the equilibrium dynamical system. Therefore, not only may cycles exist but a cycle of period three and hence topological chaos may exist where the genesis of deterministic cycles is the risk aversion of the representative agent. Grandmont ([38] pg. 998) concludes that under *laissez-faire*, many long run equilibria may coexist and policies may be designed to force the economy to settle at the steady state. In light of these findings, *ad hoc* explanatory assumptions concerning the prevalence of cycles due to exogenous shocks need to be re-examined which *a fortiori* provide a strong rational for countercyclical deterministic policies to stabilise deterministic business cycles.

Whilst the backward bending offer curve in the Samuelson economy may generate an equilibrium dynamical system which has the capacity to generate cycles or chaos within the OLG model of Gale [15], this structure was shown by Azariadis and Guesnerie [5] to provide the conditions for the existence of *stationary sunspot equilibria* (SSE). By Theorem 1 it is sufficient for the existence of SSE in the OLG model if there is a cycle of order two. As a corollary of this theorem, if the equilibrium dynamics are topologically chaotic then there exists an equilibrium of which is not a period of a power of 2 and by Sarkovskii’s ordering theorem cycles of all periods exist and thereby so do SSE.

One would expect however, that the relationship between the existence of chaos and SSE goes deeper in that an unpredictable and apparently random looking equilibrium process would have an inherent stochastic process which would serve as a the probability distribution over which sunspot equilibria occur. This line of inquiry may unfortunately be quite meaningless as the set of points over which the topologically chaotic dynamics persist may in fact have a negligible Lebesgue measure (Boldrin and Woodford [16], Day and Pianigiani [28] pg., 50 and Grandmont [39] pp. 62. See Appendix 3.8 for a discussion of this point). This means that the chaotic trajectories on X will be unobservable. The importance and relevance of topological chaos is then questionable. To this end, a different definition of chaos is motivated; that of ergodic chaos. Intuitively, a dynamical system is ergodic and chaotic if the trajectories of almost all initial conditions visit infinitely often sets of non-negligible measure in X . In such a case, the set of all trajectories approximate an ergodic and absolutely continuous distribution which is invariant under f . A typical trajectory of f therefore approximates a limiting probability distribution and this distribution does not vary depending on the initial value of f ⁷⁹. Hence, rather than concern oneself with an individual trajectory (a meaningless observation in a chaotic system) one can utilise the invariant

⁷⁸ This is similar to the analysis of Benhabib and Day [15] but differs inasmuch as in Grandmont [38] the non-linearity of the equilibrium map is due to the relative degree of concavity of the *second* period utility function with respect to the *first* period whereas in [15] the non-linearity of the equilibrium map is due to the relative degree of concavity of the *first* period utility function with respect to the *second* period utility.

⁷⁹ This need not be the case if f is topologically chaotic.

probability distribution in order to make probabilistic statements concerning the typical evolution of the state variable. Within the context of an OLG model, statistical regularities of empirically observable complex dynamics can be employed to make qualitative statements concerning the evolution of equilibria. In this regard Benhabib and Day [15] note that the use of ergodic theory may indeed be fruitful in that the equilibrium consumption path may exhibit ergodic chaos in which case a limiting long-run average of net trades and real balances could be characterised even though these quantities fluctuate from period to period. This question was not pursued further in [8].

Within the same OLG production model of [38], Araujo and Maldonado [3] utilise the distribution derived the equilibrium dynamical system in order to construct stationary sunspot equilibria by means of utilising the endogenously arising probability distribution derived from the ergodically chaotic equilibrium dynamics. These authors show that not only do such equilibria exist but they can be obtained by means of learning a rule. The economy therefore moves from a deterministic chaotic equilibrium sequence which exhibits ergodic chaos to a truly stochastic equilibrium sequence.

The discussion of this chapter fills a gap in the literature by examining the implications of ergodic chaos in the Samuelson pure exchange OLG model. The focus is on the relationship between the properties of ergodically chaotic equilibrium consumption sequences and the steady state equilibrium. To give substance to this statement let $f : X \rightarrow X$ be dynamical system derived from the excess demand function which governs the backwards looking equilibrium dynamics. If f is an ergodic process by the Birkhoff Mean Ergodic Theorem (see Appendix 3.8) the time average $\langle g(x) \rangle$ of the trajectory is equal to the space average for some integrable function g :

$$\langle g(x) \rangle = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} g(f^n(x)) \rightarrow \int_X g(x) \phi(x) dx = E[g(x)]$$

Suppose that $g \equiv x$;

$$\langle x \rangle = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} f^n(x) \rightarrow \int_X x \phi(x) dx = E[x]$$

where for the purposes herein $\langle x \rangle$ is the time average of equilibrium consumption which is equal to the expected value $E[x]$ by the Birkhoff Mean Ergodic Theorem. The question is posed: how do ergodically chaotic equilibrium consumption profiles behave with respect to consumption at the monetary steady state $\bar{x}$? More precisely, let x_1 and x_2 be the equilibrium consumption in respectively the first and second period of a representative agent's life cycle. It is shown that if the dynamical system is concave over its domain then the time average obeys $\langle x_2 \rangle \leq \bar{x}_2$ for second period consumption and for first period consumption $\langle x_1 \rangle \geq \bar{x}_1$ where $\bar{x}_i$, $i=1,2$ is the steady state equilibrium. Welfare analysis elucidates the long-run equilibrium properties of equilibrium consumption by demonstrating that along an

ergodically chaotic equilibrium path the expected indirect utility of the representative agent is less than the indirect utility at the steady state; $\langle u(x_1, x_2) \rangle \leq u(\bar{x}_1, \bar{x}_2)$ ⁸⁰. As a policy implication of this result, if there were an external agent charged with maximising the welfare across all agents over time, or maximising societal welfare, the stabilisation of the system generates a level of indirect utility at the steady state the value of which is at least as large as the expected utility of a representative agent when the system is ergodically chaotic. By this measure, there is a strong argument for stabilising intervention which would force the economy to move to the steady state. The question arises as to what form of policy can stabilise the equilibrium system and eliminate chaos (and by extension cyclical equilibria) with the least disruption. It is shown that for fixed total resources a redistributive policy in which endowments are transferred from the young agent to the old agent suffices to stabilise the dynamical system such that the asymptotic outcome is the steady state and for which indirect utility at the steady state remains invariant.

The chapter is laid out as follows. In Section 3.2 the overlapping generations model is presented. In Section 3.3 the long run properties of ergodically chaotic OLG equilibria are discussed. The primary assumption of this section is that the dynamical system is strictly concave which in turn derives from the strict concavity of the representative agent's offer curve. Welfare properties of this system are analysed. In Section 3.4 it is shown that there is a redistributive policy by means of which the dynamical system is stabilised and the long-run equilibrium yields a higher level of utility than the ergodically chaotic equilibrium sequence. In Section 3.5 the link between ergodic chaos and sunspot equilibria are discussed. Section 3.6 provides an example for which the dynamical equilibrium system is both strictly concave and generates ergodic chaos. Section 3.7 concludes and Section 3.8 contains the Appendix.

3.2. The Overlapping Generations Model

This section is divided into four parts. The first part describes the overlapping generations model. In the second and third parts the equilibrium condition and equilibrium equation which details the manner in which the dynamical equilibrium equation is formulated. In the fourth part the structure of the dynamical system and the conditions under which the map forms a unimodal map of the interval is discussed.

⁸⁰ The methodology employed in this paper is not delimited to the OLG model. Indeed, the OLG model is but one of a class of one-dimensional discrete time models which has the capacity to generate ergodic chaos. Chapter 4 of this thesis examines the equilibrium properties of an infinitely lived agent using similar techniques to those of this paper.

3.2.1. The Model

The model is the pure exchange overlapping generations equilibrium model developed in Samuelson [73], and extended in Gale [33] and Cass *et al.* [20]. There is a constant population comprised of identical agents in which one agent is born every period and lives for two periods. There are therefore only ever two agents alive at any point in time; a young and an old agent. Time is indexed $\{\dots, -1, 0, 1, 2, \dots\}$ extending indefinitely into the past and into the future. There is a single consumption good each period x_t^τ where t corresponds to the period in which the agent is born and $\tau = t, t+1$ corresponds to the period of the life-cycle to which the consumption good belongs. An agent born at time t has consumption profile $(x_t^t, x_t^{t+1}) \in \mathbb{R}_{++}^2$. Intertemporal preferences of the representative agent are stationary and are defined by an additively separable utility function of the form:

$$u(x_t^t, x_t^{t+1}) = u_1(x_t^t) + u_2(x_t^{t+1}) \quad (3.2.1)$$

where $u_i(\cdot)$, $i=1,2$ is continuous, increasing in its argument and concave and at least one $u_i(\cdot)$, $i=1,2$ is strictly concave. The closure of $u^{-1}(a)$, $a > 0$ is contained within the positive orthant of $\mathbb{R}^2$. Each agent receives the same, stationary endowment of the consumption good; $(\omega_1, \omega_2) \in \mathbb{R}_{++}^2$. The price of the consumption good at time t of the agent born at time t is p_t . The expected price of the second period consumption good of the agent born in time t is p_{t+1}^e , where e denotes the expectation. The formation of price expectations is not discussed here⁸¹. It is instead assumed that each agent possesses perfect foresight⁸²; $p_{t+1}^e = p_{t+1}$. There is a fixed nominal stock of money m which is held by the young agent and exchanged with the old for the consumption good. The price of money is normalised to one; $p_m = 1$. The intertemporal budget constraints of a representative agent are;

$$p_t(x_t^t - \omega_1) + m = 0 \quad (3.2.2)$$

$$p_{t+1}(x_t^{t+1} - \omega_2) = m \quad (3.2.3)$$

Each agent maximises (3.2.1) subject to (3.2.2) and (3.2.3). The excess demands of the consumption good at t and $t+1$ of the agent born at t is defined respectively as $z_t^t = (x_t^t - \omega_1)$ and $z_t^{t+1} = (x_t^{t+1} - \omega_2)$. The demand for money is readily obtained from (3.2.2) and (3.2.3);

$$m_t = -p_t z_t^t = p_{t+1} z_t^{t+1} \quad (3.2.4)$$

Money demand can be eliminated from (3.2.2) and (3.2.3) to give;

⁸¹ See Grandmont [38] for a discussion of price formation under periodic equilibria. It is noted that in [38] there is no discussion of the formation of price expectations under chaotic sequences.

⁸² This assumption is made in Grandmont [38],

$$p_t z_t^t + p_{t+1} z_t^{t+1} = 0 \quad (3.2.5)$$

Since the economy is Samuelson then the representative agent when young saves in order to consume in excess of endowments in the second period of life. Thus $-\omega_1 < z_t^t < 0$ and $z_t^{t+1} > 0$ by which it is verified that money demand is positive. Such that the economy is well-defined it is assumed that $z_t^{t+1} < \omega_1$. Henceforth, the role of money is overlooked⁸³, being implicitly defined by (3.2.5). As such, the maximisation problem of a representative agent⁸⁴ is defined by the maximisation of (3.2.1) subject to (3.2.5). The first order conditions to this problem are:

$$\frac{u'_1(x_t^t)}{u'_2(x_t^{t+1})} = \frac{p_t}{p_{t+1}} \quad (3.2.6)$$

$$p_t(x_t^t - \omega_1) + p_{t+1}(x_t^{t+1} - \omega_2) = 0 \quad (3.2.7)$$

Rearranging (3.2.6) and (3.2.7) yields;

$$\frac{u'_1(z_t^t + \omega_1)}{u'_2(z_t^{t+1} + \omega_2)} = \frac{-z_t^{t+1}}{z_t^t} \quad (3.2.8)$$

(3.2.8) defines the loci of points (z_t^t, z_t^{t+1}) which satisfy the first order conditions and is thereby the offer curve of the representative agent. It is noted that the right hand side of (3.2.8) is always positive as z_t^t and z_t^{t+1} must always possess different signs by Walras law⁸⁵.

3.2.2. Equilibrium

Equilibrium obtains when demand and supply at time t of the coexisting young and old of generations t and $t-1$ respectively sum to zero in each period:

$$z_t^t + z_{t-1}^t = 0 \quad (3.2.9)$$

Equilibrium is characterised by a sequence of consumption profiles supported by a sequence of prices⁸⁶; $\{z_t^t, z_t^{t+1}, \frac{p_t}{p_{t+1}}\}_{t=-\infty}^{\infty}$ such that (3.2.8) and (3.2.9) are satisfied at each t ⁸⁷. There are

⁸³ This approach is employed in Benhabib and Day [15] and Grandmont [38]. The equilibrium dynamics of the commodity define the equilibrium dynamics of the money market. It suffices thus to consider the equilibrium in the former.

⁸⁴ Under the assumption of perfect foresight.

⁸⁵ A savings in youth must be paid back by a dissaving when old and vice versa. That is, Walras law must hold.

⁸⁶ Homogeneity of degree zero in prices of the demand functions allows the expression of demands in the manner suggested.

two *steady state equilibria* to (3.2.9). The first steady state is *autarky* which obtains when demand is equal to endowments⁸⁸ $z_t^t = z_{t-1}^t = 0$ ($x_t^t = \omega_1$ and $x_t^{t+1} = \omega_2$). The second steady state is the *monetary steady state*⁸⁹ denoted $(\bar{x}_1, \bar{x}_2)$ (or $(\bar{z}_1, \bar{z}_2)$ in the case of excess demand) which obtains when $z_t^t = -z_{t-1}^t < 0$. If the equilibrium is stationary and monetary as well as Samuelson then $\bar{x}_1 < \omega_1$ and $\bar{x}_2 > \omega_2$. Since $-\omega_1 < z_t^t < 0$ and $z_t^{t+1} > 0$, by a continuity argument the offer curve, written as $u_1'(z_t^t + \omega_1)z_t^t + u_2'(z_t^{t+1} + \omega_2)z_t^{t+1} = 0$, cuts the 45° line emanating from ω . At this point $(\bar{z}_1, \bar{z}_2)$ satisfies both (3.2.8) and (3.2.9) in which case $u_1'(\bar{z}_1 + \omega_1)\bar{z}_1 - u_2'(\bar{z}_2 + \omega_2)\bar{z}_1 = 0$ or $u_1'(\bar{x}_1) = u_2'(\bar{x}_2)$. It follows that the monetary steady state equilibrium exists and at this point marginal utilities of first and second period consumption are equal and $p_t = p_{t+1}$. At the autarky equilibrium, geometrical considerations show that $u_1'(\omega_1)/u_2'(\omega_2) < 1$. Furthermore, $z_t^{t+1}/z_t^t \geq u_1'(\omega_1)/u_2'(\omega_2)$ as if the converse holds then the equilibrium is necessarily autarkic⁹⁰.

3.2.3. The Equilibrium Dynamical System

The *equilibrium dynamical system* is obtained by writing (2.8) as $z_t^t u_1'(z_t^t) = -z_t^{t+1} u_2'(z_t^{t+1})$ and defining $U_1(z_t^t) \equiv z_t^t u_1'(z_t^t)$ by which $z_t^t = U_1^{-1}[-z_t^{t+1} u_2'(z_t^{t+1})]$. In equilibrium $z_{t-1}^t = -z_t^t$ hence

$$z_{t-1}^t = f(z_t^{t+1}) = -U_1^{-1}[-z_t^{t+1} u_2'(z_t^{t+1})] \quad (3.2.10)$$

(3.2.10) is the first order difference equation which generates the *backwards looking perfect foresight equilibrium dynamics* in the *second period good*; $\{z_{t-1}^t\}_{t=0}^{\infty}$. The equilibrium sequence in the second period consumption good necessarily defines an equilibrium sequence in the first period good by (2.9) and equilibrium prices are recoverable from (3.2.5) or (3.2.6). Such that (3.2.9) is well-defined it is required that U_1 is invertible which requires that $z_t^t u_1'(z_t^t)$ is monotone. No imposition is made on the invertibility of $z_t^{t+1} u_2'(z_t^{t+1})$. If $z_t^{t+1} u_2'(z_t^{t+1})$ is invertible then the backwards looking perfect foresight dynamics in the second period consumption good can be inverted to define a *forward looking* equilibrium equation in the *second period consumption good*, i.e. $z_t^{t+1} = f^{-1}(z_{t-1}^t)$ ⁹¹. If $z_t^{t+1} u_2'(z_t^{t+1})$ is not invertible then the dynamics of (3.2.10) are defined only looking backwards in which case the forwards

⁸⁷ Thereby also defining the sequence $\{p_t\}_{t=0}^{\infty}$.

⁸⁸ The autarky equilibrium is inefficient. The demand for money is zero in the autarky steady state.

⁸⁹ The demand for money is positive at the monetary steady state.

⁹⁰ It is shown below that autarky is a repelling fixed point of the equilibrium dynamical system so trajectories move away from this equilibrium implying that it is not obtained.

⁹¹ As pointed out by Gale [33] and subsequently by Benhabib and Day [15] the forward looking equilibrium sequence will converge to autarky.

looking map f^{-1} is a multivalued correspondence ascribing multiple values at time $t+1$ to second period consumption given a value at t . By the one-to-one relationship between the offer curve (3.2.8) and the dynamical system (3.2.10), the invertibility or lack of invertibility of f depends respectively on whether or not the offer curve of the representative agent is monotone or bends backwards at some value of consumption, that is whether $z_t^{t+1}u'_2(z_t^{t+1})$ is strictly monotone or not.

Since $0 \leq z_t^{t+1} \leq \omega_1$ for all t then $f: Z \rightarrow Z$ where $Z = [0, \omega_1]$ is an interval map. The two fixed points of f are $f(0) = 0$ in the case of autarky and $f(\bar{z}_2) = \bar{z}_2$ in the case of the monetary steady state.

In order to evaluate the stability of the dynamical system, f is dynamically stable at a fixed point if $|f'(z_2)| < 1$ and dynamically unstable at the fixed point if $|f'(z_2)| > 1$. Taking the derivative of (2.10):

$$f'(\bar{z}_2) = \frac{-(u'_2(\bar{z}_2 + \omega_2) + \bar{z}_2 u''_2(\bar{z}_2 + \omega_2))}{-U'_1[-\bar{z}_2 u'_2(\bar{z}_2 + \omega_2)]}$$

Since $U_1[-z_2 u'_2(z_2 + \omega_2)] = z_1 u'_1(z_1 + \omega_1)$ then $U'_1[-z_2 u'_2(z_2 + \omega_2)] = u'_1(z_1 + \omega_1) + z_1 u''_1(z_1 + \omega_1)$ which evaluated at the steady state $\bar{z}_1 = -\bar{z}_2$ yields

$$f'(\bar{z}_2) = \frac{u'_2(\bar{z}_2 + \omega_2) + \bar{z}_2 u''_2(\bar{z}_2 + \omega_2)}{u'_1(\bar{z}_1 + \omega_1) + \bar{z}_1 u''_1(\bar{z}_1 + \omega_1)} \quad (3.2.11)$$

At the autarky equilibrium (3.2.11) is $f'(0) = u'_2(\omega_2)/u'_1(\omega_2)$ which by $u'_1(\omega_1)/u'_2(\omega_2) < 1$ implies that $f'(0) > 1$. Thus whilst $z_2 = 0$ is a fixed point it is unstable and any trajectory within the vicinity of this fixed point will be repelled.

The monetary steady state is unstable if $f'(\bar{z}_2) < -1$, which by (3.2.11) implies that⁹²

$$u'_2(\bar{z}_2 + \omega_2) + \bar{z}_2 u''_2(\bar{z}_2 + \omega_2) < -(u'_1(\bar{z}_1 + \omega_1) + \bar{z}_1 u''_1(\bar{z}_1 + \omega_1)) \quad (3.2.12)$$

The Arrow-Pratt measure of risk aversion is defined as $R_i(x) = -xu''_i(x)/u'_i(x)$. This is a natural measure of the concavity of a real-valued function. Condition (3.2.12) can be redefined in terms of the Arrow-Pratt measure as⁹³:

⁹² To see this consider that $z_1 = -z_2$ and $z_2 = -U_1^{-1}[-z_2 u'_2(z_2 + \omega_2)]$ where the time subscripts have been dropped. Then $U_1[-f(z_2)] = -z_2 u'_2(z_2 + \omega_2)$ and taking the derivative yields $U'_1[-f(z_2)](-f'(z_2)) = -(u'_2(z_2 + \omega_2) + z_2 u''_2(z_2 + \omega_2))$. But $z_1 = -z_2 = -f(z_2)$ and $U'_1(z_1) = u'_1(z_1 + \omega_1) + z_1 u''_1(z_1 + \omega_1)$ by which $(u'_1(z_1 + \omega_1) + z_1 u''_1(z_1 + \omega_1))(-f'(z_2)) = -(u'_2(z_2 + \omega_2) + z_2 u''_2(z_2 + \omega_2))$ and (2.10) follows. It is noted that (3.2.13) is equivalent to expression (4.3) or (4.10) in Grandmont [38] pg. 1016 and 1019 respectively, the difference being that in [38] the excess demand functions are utilised whereas here the primitive system is used to define the dynamical system.

$$R_2(\bar{z}_2 + \omega_2) > 2 \frac{(\bar{z}_2 + \omega_2)}{\bar{z}_2} + R_1(\bar{z}_1 + \omega_1) \frac{(\bar{z}_2 + \omega_2)}{(\bar{z}_1 + \omega_1)} \quad (3.2.13)$$

where $R_i(\bar{z}_i + \omega_i) = \frac{-(\bar{z}_i + \omega_i)u_i''(\bar{z}_i + \omega_i)}{u_i'(\bar{z}_i + \omega_i)} > 0$, $i = 1, 2$. The inequality of (3.2.13) and dynamical instability obtain if the utility function of the agent when old is sufficiently concave with respect to the utility function when young. Instead in the inequality were reversed in (3.2.13) then $|f'(\bar{z}_2)| < 1$ and the backwards perfect foresight dynamics would converge to the monetary steady state.

3.2.4. The Structure of the Equilibrium Dynamical System

Assuming that (3.2.12) (or (3.2.13)) holds, by the fact that there is a one-to-one relationship between f and the offer curve of a representative agent, it follows that $f'(\bar{z}_2) < -1$ if and only if the offer curve bends backwards and has a gradient less than 1 in absolute value at the monetary steady state. By the same token, given that the offer curve bends backwards due to the high degree of risk aversion of the representative agent when old with respect to the degree of risk aversion when young, the equilibrium dynamical system f possesses a critical point z_2^* if and only if $f'(\bar{z}_2) < 0$ for which $f'(z_2^*) = 0$. Inspection of (3.2.11) and (3.2.12) shows that f possesses a critical point if either $u_2'(z_2 + \omega_2) + z_2 u_2''(z_2 + \omega_2) = 0$ or $u_1'(z_1 + \omega_1) + z_1 u_1''(z_1 + \omega_1) \rightarrow +\infty$ where $z_1^* = -z_2^*$. In order to rule out the latter case since $-\omega_1 < z_1 < 0$ let $z_1 \rightarrow 0$. The denominator is $u_1'(\omega_1) < +\infty$ as $u_1'(\omega_1)/u_2'(\omega_2) < 1 < +\infty$. Hence $z_1 = 0$ cannot be a critical point implying that, given that equilibrium holds, $z_2 = 0$ cannot be a critical point. Similarly, if $z_1 \rightarrow -\omega_1$ then the denominator is $u_1'(0) \rightarrow +\infty$. But then $x_1 = 0$ in which case the second period consumption is not well-defined. $z_1 = -\omega_1$ cannot yield a value of z_2 which is a critical point of f . Consider instead the numerator of (3.2.11). Then z_2^* solves $u_2'(z_2^* + \omega_2) + z_2^* u_2''(z_2^* + \omega_2) = 0$ or equivalently $z_2^* = -u_2'(z_2^* + \omega_2)/u_2''(z_2^* + \omega_2)$. z_2^* yields the unique critical point by monotonicity of u_2 and its derivatives. Similarly, monotonicity of u_2 implies that f is monotone of either side of z_2^* . Furthermore, by the fact that (i) $f'(0) > 1$ and

⁹³ (3.2.12) is derived by noting first that at $\bar{z}_1 = -\bar{z}_2$ then $u_1'(\bar{z}_1 + \omega_1) = u_2'(\bar{z}_2 + \omega_2)$. By (3.2.11) $-u_2'(\bar{z}_2 + \omega_2) - \bar{z}_2 u_2''(\bar{z}_2 + \omega_2) > u_1'(\bar{z}_2 + \omega_2) + \bar{z}_1 u_1''(\bar{z}_1 + \omega_1)$ from which $- \bar{z}_2 u_2''(\bar{z}_2 + \omega_2) > 2u_2'(\bar{z}_2 + \omega_2) + \bar{z}_1 u_1''(\bar{z}_1 + \omega_1) u_2'(\bar{z}_2 + \omega_2) / u_1'(\bar{z}_2 + \omega_2)$ hence $-u_2''(\bar{z}_2 + \omega_2) / u_2'(\bar{z}_2 + \omega_2) > 2/\bar{z}_2 + \bar{z}_1 u_1''(\bar{z}_1 + \omega_1) / u_1'(\bar{z}_1 + \omega_1) \bar{z}_2$. By $\bar{z}_1 = -\bar{z}_2$ then $-u_2''(\bar{z}_2 + \omega_2) / u_2'(\bar{z}_2 + \omega_2) > 2/\bar{z}_2 - u_1''(\bar{z}_1 + \omega_1) / u_1'(\bar{z}_1 + \omega_1)$ and finally $-(\bar{z}_2 + \omega_2) u_2''(\bar{z}_2 + \omega_2) / u_2'(\bar{z}_2 + \omega_2) > 2(\bar{z}_2 + \omega_2) / \bar{z}_2 - ((\bar{z}_2 + \omega_2) / (\bar{z}_1 + \omega_1)) (u_1''(\bar{z}_1 + \omega_1) / (\bar{z}_2 + \omega_2) u_1'(\bar{z}_1 + \omega_1))$ which is equal to (3.2.12).

(ii) $f'(\bar{z}_2) < -1$ then $0 < z_2^* < \bar{z}_2$. The map *has the unimodal form* as seen in Figure 2.1, i.e. f is monotone on either side of z_2^* . This implies that (iii) $0 \leq z_2 \leq \bar{z}_2 \Leftrightarrow f(z_2) \geq z_2$ and (iv) $\omega_1 \geq z_2 \geq \bar{z}_2 \Leftrightarrow f(z_2) \leq z_2$.

The *trajectory of the critical point* defines the set of values of z_t^{t+1} which are feasible, i.e. the set of trajectories which f generates. Consider that f is an interval map; $f: Z \rightarrow Z$, $Z = [0, \omega_1]$. If ω_1 is chosen in an apposite manner then by property (ii) of the last paragraph the image of the critical point is at least as large as the fixed point; $\omega_1 \geq f(z_2^*) > \bar{z}_2$. Since, $f(z_2^*) \in [0, \omega_1]$ then $f^2(z_2^*) \in [0, \omega_1]$. If the measure of concavity of u_2 relative to u_1 is large then the greater the gradient at $\bar{z}_2$. In such a case one has that $f^2(z_2^*) < z_2^*$. Let $Z = [z', z'']$ where $z' = f^2(z_2^*)$ and $z'' = f(z_2^*)$ then $f: Z \rightarrow Z$ is an interval map and any point starting in Z remains in Z under action of f . Given that $f'(\bar{z}_2) < -1$ then the fixed point is hyperbolic repelling and is thus never reached. It follows that *bounded equilibrium fluctuations* are observed⁹⁴.

The following summarise the properties of f .

Property 3.2.1 Let $f: Z \rightarrow Z$ be the continuous map of the dynamical system defined by (3.2.10), where $Z = [z', z'']$, $z' < z''$ is compact. f satisfies the following properties:

- i. f is unimodal; there is one and only one critical point $z_2^* \in (z', z'')$ such that $f'(z_2^*) = 0$ with $f'(z_2) > 0 \Leftrightarrow z_2 < z_2^*$ and $f'(z_2) < 0 \Leftrightarrow z_2 > z_2^*$. Furthermore, $f''(z_2^*) < 0$ and $z' < f(z_2^*) < z''$.
- ii. $f(z_2^*) = z''$ and $f^2(z_2^*) = f(z'') = z'$. Define $Z = [z', z''] = [f^2(z_2^*), f(z_2^*)]$. Then f is an interval map which maps back to itself; $f: Z \rightarrow Z = f(Z)$, and $f(Z) = Z$. This implies that there is a fixed point $\bar{z}_2 \in (z', z'')$ such that $f(\bar{z}_2) = \bar{z}_2$. Furthermore $|f'(\bar{z}_2)| > 1$.
- iii. f has no fixed points in the interval (z', z_2^*) . If there is a fixed point other than $\bar{z}_2$ in Z then the fixed point is equal to z' . If $f(z') = z'$ then $z' = 0$ and $f'(z') > 1$.
- iv. f has the property $0 \leq z_2 \leq \bar{z}_2 \Leftrightarrow f(z_2) \geq z_2$ and $z_2 \geq \bar{z}_2 \Leftrightarrow f(z_2) \leq z_2$.

It is noted that if $z_t^{t+1} \in [0, \omega_1] \setminus Z$ then $\lim_{t \rightarrow \infty} f(z_t^{t+1}) \in Z$ so it is with no loss of generality that the *behaviour of f is restricted to Z* . Property 3.2.1 defines a *single peaked map f that is monotone over the sub-intervals $[z', z_2^*)$ and $(z_2^*, z'']$* . The compact set Z is termed the

⁹⁴ The condition $f^2(z_2^*) < z_2^*$ follows from (3.2.13). See Figure 3.2.1.

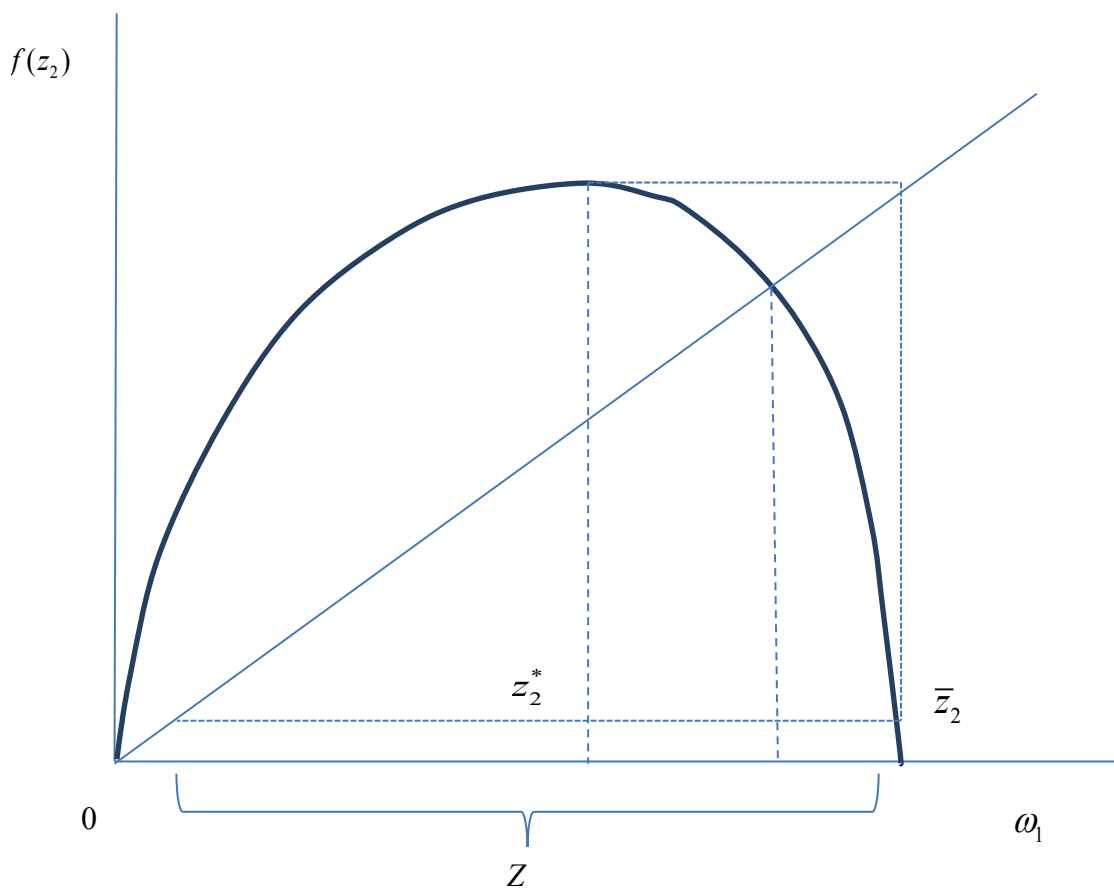
trapping set. Since $f^{-1}(Z) \supset Z$ any point in the domain is mapped onto by a point in the codomain; if $z_2 \in f^{-1}(Z)$ then $f^n(z_2) \in Z$ for all n .

Having established Property 3.2.1, it is *assumed that f is concave*. The role of this assumption comes into play in the next section where results are derived concerning the value of the fixed point with respect to the expected value of the equilibrium.

Assumption 3.2.1 $f : Z \rightarrow Z$ is concave over Z .

Having presented and discussed the model and various attendant properties, the dynamical system is analysed in the next section.

Figure 3.2.1



3.3. Ergodic Chaos and Welfare Evaluation

Let $f : Z \rightarrow Z$, be the unimodal interval map which satisfies Property 3.2.1 and Assumption 3.2.1. Suppose that f is ergodically chaotic and that the measure μ is absolutely continuous invariant measure with respect to f . Let the probability distribution be denoted φ . By the Birkhoff Theorem the time average of the typical trajectory is equal to the expected value of the distribution:

$$\langle z_2 \rangle = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} f^n(z_2) = \int_Z z_2 \varphi(z_2) dz_2 = E[z_2] \quad (3.3.1)$$

where z_2 is the second period equilibrium consumption derived from the dynamical system $f : Z \rightarrow Z$. The question arises as to the behaviour of the time average of a typical trajectory or equivalently the expected value of the probability distribution. The following theorem shows that the expected value of the second period consumption is less than or equal to the steady state value of the second period consumption in the backwards perfect foresight dynamics.

Theorem 3.3.1⁹⁵ Suppose that Property 3.2.1 and Assumption 3.2.1 hold and that $f : Z \rightarrow Z$ is ergodically chaotic. The time average of the second period excess demand in the perfect foresight backwards equilibrium dynamics is no greater than the value of excess demand at the monetary steady state, i.e. $\langle z_2 \rangle \leq \bar{z}_2$.

Proof Noting first that (3.3.1) holds, let $\varphi(z_2)$ be the unique and absolutely continuous and invariant density generated by $f : Z \rightarrow Z$ for the backwards dynamics

$z_{t-1} = f(z_t)$. Since $\int_{z'}^{z''} \varphi(z_2) dz_2 = 1$ and f is concave over Z , Jensen's inequality⁹⁶ implies

$$f \left(\int_{z_2'}^{z_2''} f(z_2) \varphi(z_2) dz_2 \right) \geq \int_{z_2'}^{z_2''} f(f(z_2)) \varphi(z_2) dz \quad (3.3.2)$$

⁹⁵ Theorem 3.3.1 follows from Theorem 1 of Huang [43] pg. 61. In [43], Huang analyses a tatonnement process in which the dynamical system is convex. Hence, in the proof of [43] the inequality is in Jensen's inequality is reversed whereby the result is that the time average is at least as large as the value of the fixed point.

⁹⁶ Jensen's inequality states that for a concave function g ; $E[g(x)] \leq g(E[x])$.

Ergodicity implies that $E[f(z_2)] = E[z_2]$ which in turn implies that the right hand side of

$$(3.3.2) \text{ is } \int_{z'_2}^{z''_2} f(f(z_2))\varphi(z_2)dz = \int_{z'_2}^{z''_2} f(z_2)\varphi(z_2)dz = \int_{z'_2}^{z''_2} z_2\varphi(z)dz = \langle z_2 \rangle. \quad (3.3.2) \text{ is then equivalent}$$

to $f\left(\int_{z'_2}^{z''_2} z_2\varphi(z_2)dz\right) \geq \int_{z'_2}^{z''_2} z_2\varphi(z_2)dz$ which in turn is equivalent to

$$f(\langle z_2 \rangle) \geq \langle z_2 \rangle \quad (3.3.3)$$

From Property 3.2.1.iv, if $f(z_2) \geq z_2$ then $z_2 \leq \bar{z}_2$. Since $\langle z_2 \rangle$ is a unique real number this property implies

$$f(\langle z_2 \rangle) \geq \langle z_2 \rangle \Leftrightarrow \langle z_2 \rangle \leq \bar{z}_2 \quad (3.3.4)$$

This ends the proof^{97 98}. ■

Since f is ergodic then z_2 can be treated as a random variable. Given that $z_2 = x_2 - \omega_2$ then x_2 is also treated as a random variable which has exactly the same probability as z_2 ; $\varphi(z_2) = \varphi(x_2)$. By Theorem 3.3.1, it follows that⁹⁹ $\langle z_2 \rangle \leq \bar{z}_2 \Leftrightarrow \langle x_2 \rangle \leq \bar{x}_2$. Furthermore, in equilibrium $x_1 + x_2 = \omega_1 + \omega_2$. The probability of observing x_2 uniquely determines the probability of observing x_1 and the probability distribution of x_1 is that of x_2 which follows from the fact that the equilibrium dynamics are determine solely in terms of x_2 .

In order to obtain the expected equilibrium value of the first period consumption take the expectations of the equilibrium equation $E[x_1 + x_2] = E[\omega_1 + \omega_2] = \omega_1 + \omega_2$. By linearity of the expectations operator $E[x_1] + E[x_2] = \omega_1 + \omega_2$. At the steady state $\bar{x}_1 + \bar{x}_2 = \omega_1 + \omega_2$ hence $E[x_1] + E[x_2] = \bar{x}_1 + \bar{x}_2$. By Theorem 3.3.1 $E[x_2] \leq \bar{x}_2$ so it follows that $E[x_1] \geq \bar{x}_1$. Therefore on average, a representative agent will consume more than the steady state allocation when young and less than the steady state allocation when old. Given that the economy is

⁹⁷ Property 3.2.1 and Assumption 3.2.1 are not necessary for Theorem 3.3.1 to hold. The same result may obtain for instance if f were to not be concave over some sub-interval of its trapping set. Hence, the assumptions made in Theorem 3.3.1 are sufficient but not necessary.

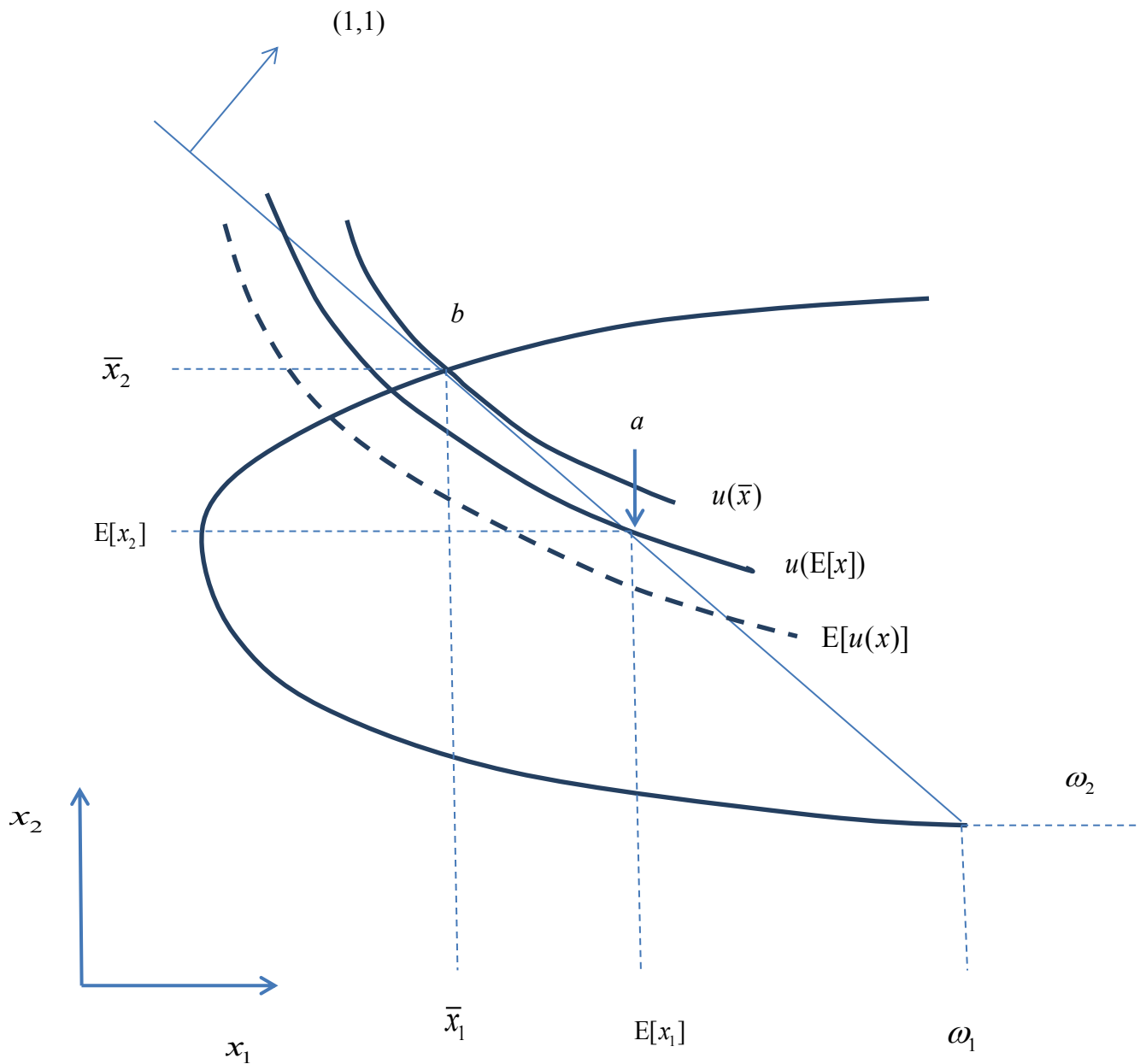
⁹⁸ If f is strictly concave then $\langle z_2 \rangle < \bar{z}_2$. This will in general be the case. The inequality is kept weak for the sake of generality.

⁹⁹ $\int_{z'_2}^{z''_2} z_2\varphi(z_2)dz \leq \bar{z}_2 \Leftrightarrow \int_{x'_2}^{x''_2} (x_2 - \omega_2)\varphi(x_2)dx \leq \bar{x}_2 - \omega_2 \Leftrightarrow \int_{x'_2}^{x''_2} x_2\varphi(x_2)dx - \omega_2 \int_{x'_2}^{x''_2} \varphi(x_2)dx \leq \bar{x}_2 - \omega_2$
 $\Leftrightarrow \int_{x'_2}^{x''_2} x_2\varphi(x_2)dx - \omega_2 \leq \bar{x}_2 - \omega_2 \Leftrightarrow \langle x_2 \rangle \leq \bar{x}_2$.

Samuelson the representative agent aims to save when young and consume in excess of the endowment when old. Treating the steady state as a benchmark, under chaos, on average the equilibrium conditions dictate that the agent does not save enough in the first period of life and consumes too little in the second period of life. This argument of course relies on the steady state having some intrinsically desirable property with respect to the chaotic equilibria. For instance, by $E[x_1] \geq \bar{x}_1$ and $E[x_2] \leq \bar{x}_2$, given that u_i , $i=1,2$ is increasing then $u_1(E[x_1]) \geq u_1(\bar{x}_1)$ and $u_2(E[x_2]) \leq u_2(\bar{x}_2)$. In order to determine the lifetime utility of expected equilibrium consumption profiles, a simple diagrammatic argument suffices. By the fact that $E[x_1] + E[x_2] = \omega_1 + \omega_2$ and $E[x_1] \geq \bar{x}_1$ as well as $E[x_2] \leq \bar{x}_2$, then the coordinate $(E[x_1], E[x_2])$ is at a point such as a in Figure 3.1. The indifference curve passing through this curve lies below the indifference curve passing through the steady state value $(\bar{x}_1, \bar{x}_2)$ at a point such as b . This implies that $u_1(E[x_1]) + u_2(E[x_2]) \leq u_1(\bar{x}_1) + u_2(\bar{x}_2)$. Since u_i , $i=1,2$ are concave functions, an application of Jensen's inequality implies that $E(u_i(x_i)) \leq u_i(E(x_i))$, $i=1,2$. It follows that $E(u_1(x_1)) + E(u_2(x_2)) \leq u_1(E[x_1]) + u_2(E[x_2]) \leq u_1(\bar{x}_1) + u_2(\bar{x}_2)$. This is consistent with the manner in which risk averse agents behave. The following corollary obtains.

Corollary 3.1 Under Property 3.2.1 and Assumption 3.2.1, if $f: Z \rightarrow Z$ is ergodically chaotic, indirect utility at the monetary steady state is at least as large as the indirect utility of the expected consumption vector which is in turn at least as large as expected value of the indirect utility; $E[u(x)] \leq u(E[x]) \leq u(\bar{x})$. ■

Figure 3.3.1



Corollary 3.3.1 states that the expected utility of each agent is no greater than the utility at the monetary steady state. It would appear that in the long run the typical consumer derives a level of expected utility which is no greater than the level of utility at steady state thereby suggesting that the steady state has an efficiency property which is absent in the chaotic system. Caution must be exercised with this conclusion as each agent is born into a deterministic system and lives for two periods only hence no agent would consider expected utility when determining whether or not steady state utility is higher¹⁰⁰.

¹⁰⁰ For instance, Boldrin and Woodford [16] pg. 214, point out that within the framework of the OLG model presented here, the duration of cycles must be at least as long as the lifecycle of the representative agent. To this end, cyclical or erratic consumption profiles may be deemed of little import over any particular agent's lifecycle

Instead, to interpret Corollary 3.3.1 suppose that there exists a social planner who is tasked with optimising the well-being of the infinite sequence of agents or society as a whole. Suppose further that each agent is identified with his consumption profile. The infinite sequence of agents can then be identified with a continuous mass defined over Z weighted by the probability of occurrence^{101 102}, i.e. the sequence of agents are identified with the probability distribution $\int_{z'_2}^{z''_2} \phi(z_2) dz_2$ ¹⁰³. For example, $pr(z_t \leq \bar{z}_2)$ would be the probability that an agent has consumption no greater than the steady state value and thus characterises an agent type. (3.3.8) is written as $\int_Z u(z)\phi(z)dz \leq u(\bar{z})$ where $A_1 = \int_Z u(z)\phi(z)dz$ and $A_2 = \int_Z u(\bar{z})\phi(z)dz = u(\bar{z})\int_Z \phi(z)dz$. Corollary 3.1 states that $\Delta A = A_1 - A_2 \geq 0$; there is a welfare improvement for society as a whole if the dynamical system were to be stabilised. It follows that if society is viewed as an infinite sequence of agents who form a mass then a social planner can on average improve societal well-being by stabilising the dynamical system and forcing convergence to the fixed point, provided of course stabilisation does not alter the value of the fixed point. This outcome is however not obtainable under *laissez-faire* and necessitates intervention.

This argument does not however consider the effect of stabilisation on agent types. Suppose that at time t an agent enters the system and faces the equilibrium consumption profile $(\bar{x}_t^t, \bar{x}_t^{t+1})$ ^{104 105}. The level of utility at this equilibrium consumption profile will be

as an agent does not live long enough to observe the full force of these chaotic or erratic consumption profiles. It could be argued that if an agent were to live for more than two periods then the erratic consumption profile may not be observed. Aiyagari [2] provides some evidence in this direction. Yet despite this argument cyclical or erratic movements do have a bearing on an agent's consumption profile as well as the sequence of equilibrium consumption profiles and consequently society's consumption patterns.

¹⁰¹ That agents form a mass is due to the probability measure being absolutely continuous. See Appendix 3.8

¹⁰² In this instance a mass of agents is defined as a non-negligible subset of the trapping set Z over which the probability distribution is defined, i.e. a type of agent is associated with the probability of observing that agent type not by the point in time in which that agent will be observed. Given that an ergodic and chaotic sequence has the property of sensitive dependence to initial conditions, it is not possible to predict *when* a given type of agent will be observed but only the probability of that agent being observed. If one were to assume that the weight given to an agent is modified by means of a discount factor, i.e. agents born closer to the present are accorded more weight, then the afore-established results would not apply

¹⁰³ Or equivalently $\int_{z'_1}^{z''_1} \phi(z_1) dz_1$

¹⁰⁴ $(\bar{x}_t^t, \bar{x}_t^{t+1})$ denotes the equilibrium consumption profile of the agent born at time t whereas $(\bar{x}_1, \bar{x}_2)$ denotes the equilibrium consumption profile at the monetary steady state.

¹⁰⁵ Recall that the equilibrium dynamics are backwards looking whilst each agent optimises looking forwards. As such, it is assumed that the backwards looking solution is also a forwards looking solution. Therefore, the deterministic equilibrium consumption profile $(\bar{x}_t^t, \bar{x}_t^{t+1})$ derived from the backwards looking dynamics are also treated as a forward looking equilibrium consumption profiles. Furthermore, $(\langle x_1 \rangle, \langle x_2 \rangle)$, being derived from

either $u(\bar{x}_t', \bar{x}_t^{t+1}) \leq u(\bar{x}_1, \bar{x}_2)$ or $u(\bar{x}_t', \bar{x}_t^{t+1}) \geq u(\bar{x}_1, \bar{x}_2)$ where in the former the agent is born into the downstate and in the latter the agent is born into the upstate¹⁰⁶. An agent born into downstate prefers the steady state whereas an agent born into the upstate prefers chaos over the steady state. Since agents exist with positive probability in both the down- and upstate¹⁰⁷ then there can be no consensus that stabilisation is preferable to chaos as those agents born into the upstate face terms of trade favourable to the terms of trade at the steady state and hence prefer chaos over the steady state. Instead, agents born into the downstate face terms of trade less favourable than the terms of trade at the steady state than those born into the upstate and hence prefer the steady state over chaos. The mass of downstate agents is $\int_{z'}^{\bar{z}} \varphi(z) dz$ and the mass of upstate agents is $\int_{\bar{z}}^{z''} \varphi(z) dz$. Let:

$$B_1 = \int_{z'}^{\bar{z}} u(z) \varphi(z) dz, \quad B_2 = \int_{z'}^{\bar{z}} u(\bar{z}) \varphi(z) dz = u(\bar{z}) \int_{z'}^{\bar{z}} \varphi(z) dz$$

$$C_1 = \int_{\bar{z}}^{z''} u(z) \varphi(z) dz, \quad C_2 = \int_{\bar{z}}^{z''} u(\bar{z}) \varphi(z) dz = u(\bar{z}) \int_{\bar{z}}^{z''} \varphi(z) dz$$

B_1 (C_1) is the utility weighted by the mass of the downstate (upstate) agents and B_2 (C_2) is the utility at the steady state weighted by the mass of downstate (upstate) agents. Given that each downstate agent prefers stability to chaos then the mass of downstate agents prefer stability to chaos; $B_1 \leq B_2$. Similarly, given that each upstate agent prefers chaos to stability then so too do the mass of upstate agents; $C_1 \geq C_2$ ¹⁰⁸. Then stabilisation will decrease the utility of the upstate agents and increase the utility of the downstate agents. The social planner will then effectuate a stabilising regime if the total change in welfare is positive; $\Delta B + \Delta C = (B_1 - B_2) + (C_1 - C_2) > 0$ where $\Delta B > 0$ and $\Delta C < 0$; the welfare gain of stabilisation is greater than the welfare loss.

This criterion for stabilisation does not however consider the mass of agents which belong to either state. It is conceivable that stabilising the dynamical system improves welfare even if the mass of downstate agents is small in comparison to the mass of upstate agents. Stabilisation may mean that there is a loss of utility for the many in favour of a gain in

the backwards looking equilibrium dynamics, is treated as the expected consumption profile of the forwards looking solution.

¹⁰⁶ The inequalities are strict as if the inequalities were not strict then $(\bar{x}_t', \bar{x}_t^{t+1}) = (\bar{x}_1, \bar{x}_2)$ for some t and hence for all t contradicting the assumption that f is ergodically chaotic

¹⁰⁷ Since f is ergodic f visits sets of non-negligible measure in Z infinitely often. Hence agents are born into both the down- and up-state.

¹⁰⁸ To see that this argument holds let x be a downstate agent. Then since utility is increasing, monotone and concave one has that $u(x) \leq u(\bar{z})$ for any $x \in [z', \bar{z}]$. Now weighting the utility of all agents does not change this inequality hence $\int_{z'}^{\bar{z}} u(z) \varphi(z) dz \leq \int_{z'}^{\bar{z}} u(\bar{z}) \varphi(z) dz$ and $B_1 \leq B_2$. A similar argument applies for the upstate agents.

utility for the few, yet the extent to which the few benefit outweighs the loss by the many. In sum, the social planner would therefore have to consider not only whether stability will improve welfare but also the relative weight of the group of agents who either gain or lose under stability.

3.4. Redistributive Policy and Stabilisation

In the previous section it was established that the steady state yields a level of utility at least as large as the expected utility. The question arises as to the manner in which the *dynamical system* can be *stabilised* to force convergence to the steady state without the level of *indirect utility at the steady state* being altered. For the endowment vector $\omega = (\omega_1, \omega_2) \in \mathbb{R}_+^2$ a policy is a redistribution of endowments from the young to the old which keeps total resources constant; let $\omega_1 + \omega_2 = r$ for some $r > 0$ and let $\lambda \in [0, 1]$ such that $\omega_\lambda = ((1 - \lambda)\omega_1, \omega_2 + \lambda\omega_1)$ is a policy. Suppose that $(\bar{x}_1, \bar{x}_2)$ is the steady state associated with the initial endowment (ω_1, ω_2) . Then for all $(\omega_1, \omega_2) \in \mathbb{R}_+^2$ which satisfy $\omega_1 + \omega_2 = r$ the fixed point $(\bar{x}_1, \bar{x}_2)$ remains invariant, i.e. does not change for any value of endowments for which total resources remain fixed¹⁰⁹. This means that as endowments are redistributed along the line emanating from (ω_1, ω_2) passing through $(\bar{x}_1, \bar{x}_2)$, this latter vector does not change.

In order to obtain bounds on the set of feasible redistributive policies, it is first noted that the economy is Samuelson so the young consume less than the endowment in order to consumer more in old age; or $0 < x_1 < \omega_1$ and $\omega_2 < x_2 < \omega_1 + \omega_2$ ¹¹⁰. Hence any policy must satisfy $0 < \bar{x}_1 < (1 - \lambda)\omega_1$ and $\omega_2 + \lambda\omega_1 < \bar{x}_2 < \omega_1 + \omega_2$. Define $\hat{\lambda} = (r - \bar{x}_2)/\omega_1$. Then any $\lambda \in [0, \hat{\lambda})$ supports a monetary steady state which is Samuelson.

¹⁰⁹ To see this let $f(p_t, p_{t+1}, p_t\omega_1 + p_{t+1}\omega_2) = (f^1(p_t, p_{t+1}, p_t\omega_1 + p_{t+1}\omega_2), f^2(p_t, p_{t+1}, p_t\omega_1 + p_{t+1}\omega_2))$ be the demand vector obtained from the maximisation problem of an agent at time t . The loci of points of this vector for $(p_t, p_{t+1}) \in \mathbb{R}_{++}^2$ for given $(\omega_1, \omega_2) \in \mathbb{R}_+^2$ define the offer curve of the representative agent born an time t . Total resources are fixed at $r > 0$. At the steady state $\bar{x} = (\bar{x}_1, \bar{x}_2)$ the equilibrium price is $p_t = p_{t+1}$. Without any loss of generality let there be a price normalisation $(p_t, p_{t+1}) = (p_1, 1)$. Then at the steady state $p_t = p_{t+1} = 1$. The income of the representative agent is then simply $\omega_1 + \omega_2 = r$ in which case the vector of demands is $\bar{x} = (\bar{x}_1, \bar{x}_2) = f(1, 1, r) = (f^1(1, 1, r), f^2(1, 1, r))$ which is the monetary steady equilibrium consumption. It follows that for any $(\omega_1, \omega_2) \in \mathbb{R}_+^2$ which satisfy $\omega_1 + \omega_2 = r$ the vector $(\bar{x}_1, \bar{x}_2)$ is invariant. See Balasko and Ghiglino [12].

¹¹⁰ Or equivalently $-\omega_1 < z_1 < 0$ and $0 < z_2 < \omega_1$.

Such that $f: Z \rightarrow Z$ exhibits ergodic chaos then it is necessary that the fixed point of the monetary steady state be repelling; $f'(\bar{z}_2) < -1$. This condition is in turn given as the inequality in (2.12) or equivalently the inequality in (3.2.13). Let $(\omega_1, \omega_2) \in \mathbb{R}_+^2$ be such that (2.12) holds. Then the dynamical system is unstable, possibly ergodically chaotic. Let $\lambda \in (0, \hat{\lambda})$. By the redistributive policy each agents has endowment vector $\omega_\lambda = ((1-\lambda)\omega_1, \omega_2 + \lambda\omega_1)$. (2.12) is then written as:

$$u'_2(\bar{x}_2) + (\bar{x}_2 - (\omega_2 + \lambda\omega_1))u''_2(\bar{x}_2) < -(u'_1(\bar{x}_1) + (\bar{x}_1 - (1-\lambda)\omega_1))u''_1(\bar{x}_1)) \quad (3.4.1)$$

As $\lambda \rightarrow \hat{\lambda}$ then $(1-\lambda)\omega_1 = \bar{x}_1$ and $(\omega_2 + \lambda\omega_1) = \bar{x}_2$ in which case the gradient of the dynamical system at the fixed point is $f'(\bar{z}_2) = u'_2(\bar{x}_2)/u'_1(\bar{x}_1) = 1$ implying that there has been a reversal of the inequality of (3.4.1). By a continuity argument, for any λ in a small neighbourhood of $\hat{\lambda}$ such that the fixed point is Samuelson, the dynamical system satisfies $f'(\bar{z}_2) > 0$ and the steady state is a hyperbolic attracting fixed point. By a further continuity argument, the set of policies for which the equilibrium system is stabilised is established by letting $\bar{\lambda}$ be such that $f'(\bar{z}_2) = -1$ or (4.1) holds with equality. It is straight forward to show:

$$\bar{\lambda} = \frac{(\bar{x}_2 - \omega_2)}{\omega_1} - \frac{2u'_2(\bar{x}_2)}{\omega_1(u''_2(\bar{x}_2) - u''_1(\bar{x}_1))} \quad (3.4.2)$$

For all $\lambda \in (\bar{\lambda}, \hat{\lambda})$ the equilibrium equation obeys $1 > f'(\bar{z}_2) > -1$ and the *backward foresight equilibrium dynamics are dynamically stable and converge to the steady state*. It is concluded that for fixed total resources, there is a non-negligible set of second period endowments for which the dynamical system is stabilised where the stabilisation occurs due to the gradient of f falling at the fixed point as endowments are redistributed intra- or intergenerationally¹¹¹. This is illustrated in Figure 3.4.1.

In order to explore the implications of a stabilisation policy let $\omega_{\bar{\lambda}}$ be the endowment vector for which (3.4.2) holds and let $[\bar{x}, \omega_{\bar{\lambda}}]$ be the interval on the line emanating from ω and passing through $\bar{x}$ (see Figure 3.4.1) where ω is such that $f: Z \rightarrow Z$ is ergodically chaotic. For every $\hat{\omega} \in (\bar{x}, \omega_{\bar{\lambda}}) = (\omega_{\hat{\lambda}}, \omega_{\bar{\lambda}})$ one has that:

$$E[u_\omega(x_1, x_2)] \leq u_{\hat{\omega}}(\bar{x}_1, \bar{x}_2) \quad (3.4.3)$$

$$= u_\omega(\bar{x}_1, \bar{x}_2) \quad (3.4.4)$$

¹¹¹ A redistributive policy may take the form of a taxation of the young and a subsidy of the old. Consider the maximisation problem; $\max_{x_t^i, x_{t+1}^i \in \mathbb{R}_+^2} u_1(x_t^i) + u_2(x_{t+1}^i)$ subject to $p_t x_t^i + p_{t+1} x_{t+1}^i = p_t \omega_1 (1 - \tau_1) + p_{t+1} \omega_2 (1 - \tau_2)$. If $\tau_1 > 0$ and $\tau_1 = -\tau_2$ the taxation policy can be sufficiently aggressive such that the equilibrium system is stabilised.

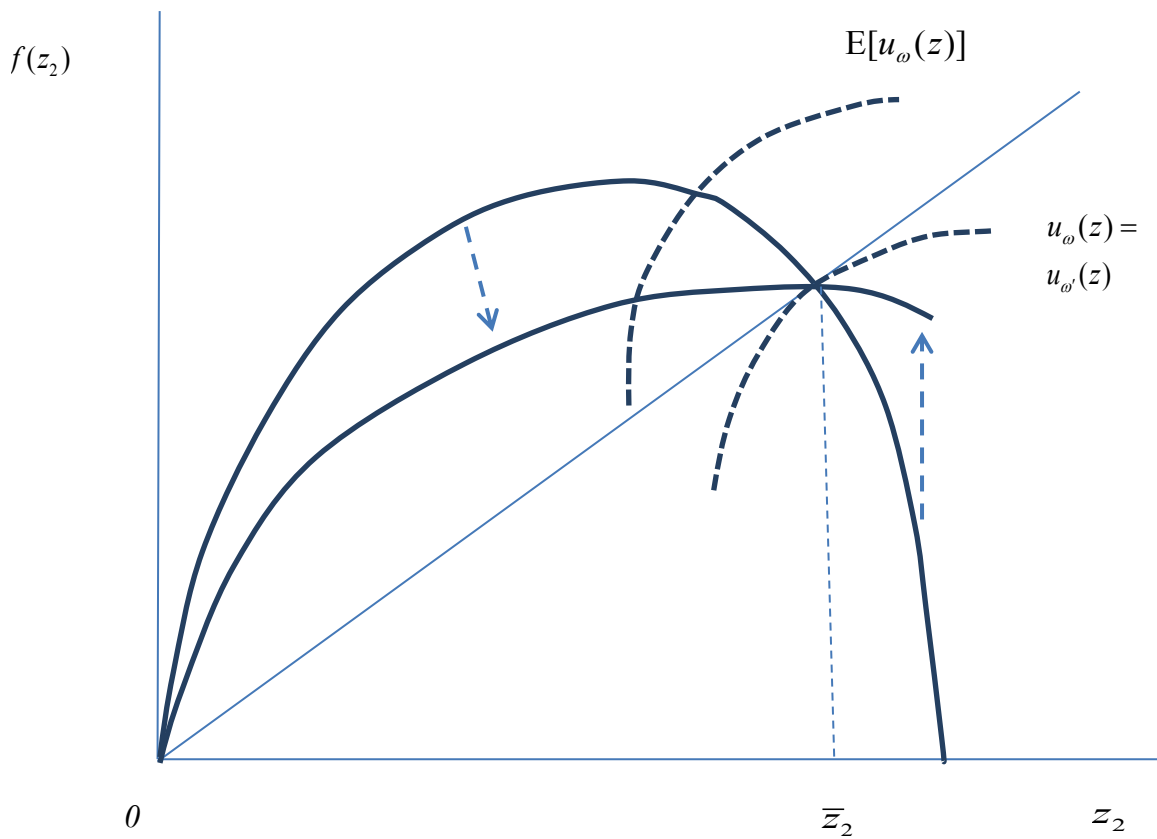
For a policy which redistributes endowments to a point in the interval $(\bar{x}, \omega_{\bar{\lambda}})$, (3.4.3) and (3.4.4) imply that the level of utility at the monetary steady state of the stabilised system is equal to the level of utility at the steady state of the original system which, by Corollary 3.3.1, is at least as large as the level of expected utility of the chaotic system. Hence, in stabilising the equilibrium system the social planner can choose any $\lambda \in (\bar{\lambda}, \hat{\lambda})$ such that $\hat{\omega} \in (\bar{x}, \omega_{\bar{\lambda}}) = (\omega_{\hat{\lambda}}, \omega_{\bar{\lambda}})$ with the result that the dynamical system is stabilised and the level of utility at the steady state remains the same¹¹².

Theorem 3.4.1 For every endowment vector ω for which f is ergodically chaotic there exists a continuum of endowment vectors $(\omega_{\hat{\lambda}}, \omega_{\bar{\lambda}})$ for each one f is dynamically stable and for each one the value of indirect utility at the corresponding monetary steady state is at least as large as the level of expected utility of the ergodically chaotic dynamical system. ■

As a matter of extending the principle of Theorem 3.4.1, it can be verified that a redistributive policy will stabilise the dynamical system irrespective of the type of equilibrium dynamics present. For instance, the same approach will stabilise either a topological chaos or periodic equilibria. As pointed out in the introduction, if the dynamical system exhibits periodicity of any order then it exhibits periodicity of order 2 which is both sufficient and necessary for the existence of *stationary sunspot equilibria*. It follows that as *endowments are redistributed to be closer to the steady state equilibrium* then the dynamical system is stabilised and the stationary sunspot equilibria cease to exist. Under the assumptions made it is concluded that if *equilibrium trades* are small and in the *vicinity of the steady state* then stationary sunspot equilibria do not exist.

¹¹² Different stabilising endowment vectors may of course be associated with different dynamical system which take differing times to reach the steady state. These itinerants hold little weight given that the expected value is summed over infinitely many observations.

Figure 3.4.1



3.5. Sunspot Equilibria and Chaos

In [5] Azariaridis and Guesnerie discuss the equilibrium properties of stationary sunspot equilibrium in an overlapping generations model. In the model of [5] utility is maximised over a Markovian probability distribution which is formulated by the agent's Bayesian beliefs in extrinsic events. As such, in the model of [5], the belief system which determines the stochastic structure of equilibrium is not endogenously determined. Instead, by treating the dynamical system as if it were a stochastic process, agent's expectations may be formed by the observation of historic economic data. The manner in which an agent incorporates these endogenously determined beliefs concerning the behaviour of the state variable gives rise to what Araujo and Maldonado [3] term a *chaotic sunspot*; extrinsic uncertainty defined on the equilibrium set which is ergodic and chaotic and hence has the property of a random variable with probability measure. Uncertainty is extrinsic as the fundamentals of the dynamical system are stationary. This means that the equilibrium path is not influenced by the endowments or preferences which in turn implies that when the set equilibrium is treated as a

stochastic process, the underlying fundamentals of the economy do not have any influence on the structure of the probability space.

Let $(Z, \mathcal{F}, \mu)$ be the probability space of the dynamical system $f: Z \rightarrow Z$ where $\mathcal{F}$ is the sigma-algebra defined over the sample space Z and μ is the probability measure. $f(Z)$ can be treated as a random variable. Since f is invariant with respect to μ then $f(Z)$ also has the distribution μ . Let $U_2: Z \rightarrow \mathbb{R}$ be a random variable defined on $(Z, \mathcal{F}, \mu)$ which is $\mathcal{F}$ -measurable defined as $U_2(z_2) = u'_2(z_2 + \omega_2)z_2$. Since $z_2 \in f(Z) = [z'_2, z''_2]$ then the set of events observed can be restricted to $[U_2(z'_2), U_2(z''_2)]$.

In order to define the equilibrium condition, suppose that at time t the market achieves an equilibrium. The young agent at time t formulates expectations as to the future expected value of consumption given the present period equilibrium consumption. In other words, the agent maximises utility subject to the constraint given that equilibrium obtains in the first period of life. Expected utility is maximised subject to the budget constraint;

$$\max E[u_1(\omega_1 - p_{t+1}/p_{t+1}(x_t^{t+1} - \omega_1)) + u_2(x_t^{t+1})] \quad (3.5.1)$$

where the expected value is taken with respect to the measure μ . Taking the first order condition of (3.5.1) and rearranging yields:

$$E[u'_1(x'_t)(x'_t - \omega_1) + u'_2(x_t^{t+1})(x_t^{t+1} - \omega_2)] \quad (3.5.2)$$

Substituting the equilibrium equation $x'_t + x_{t-1}^t = \omega_1 + \omega_2$ into (3.5.2):

$$E[-u'_1(\omega_1 - z_{t-1}^t)z_{t-1}^t + u'_2(z_t^{t+1} + \omega_2)z_t^{t+1}] \quad (3.5.3)$$

Since z_{t-1}^t is a historic equilibrium and the measure μ pertains to z_t^{t+1} . A sunspot equilibrium is a zero of

$$-u'_1(\omega_1 + \omega_2 - x_{t-1}^t)(x_{t-1}^t - \omega_2) + E_\mu[u'_2(x_t^{t+1})(x_t^{t+1} - \omega_2)] \quad (3.5.4)$$

Intuitively (3.5.4) states that at time t the equilibrium value x_{t-1}^t of the agent born in time $t-1$ was observed which determined the first period equilibrium value x_t^t of the agent born at time t . Given the probability measure derived from the data of the economy the agent at time t forms expectations concerning the future value of consumption. The resultant equilibrium is a sunspot as the formation of expectations neither bears upon nor is affected by the fundamentals of the economy, e.g. preferences or endowments.

The question of the existence of equilibrium to (3.5.4) is not taken up. However, suppose that $f: Z \rightarrow Z$ is not chaotic but is instead ergodic and periodic of order k . Periodicity of equilibria implies that the support of μ is concentrated on a negligible set and the probability of any observation is $1/k$ and z_t follows z_{t+1} . By an argument analogous to

Azariadis and Guesnerie [5], one can find a stationary k -sunspot in the vicinity of the measure μ .

3.6. A Numerical Example

A numerical example based on a quadratic utility function is examined in this section. In order to discuss the conditions under which ergodic chaos is present and hence permit a discussion of the results of the previous sections, a criterion is provided. The Schwarzian derivative of the map $f : Z \rightarrow Z$ is defined as $Sf(z)$ with $f \in C^3$ where

$$Sf(z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2} \left[\frac{f''(z)}{f'(z)} \right]^2, \text{ s.t. } f'(z) \neq 0. \quad (3.6.1)$$

The Schwarzian derivative is negative¹¹³ if $Sf(z) < 0$ for all $z \in Z$, $z \neq z^*$. The following theorem provides a sufficient condition for f to have an absolutely continuous and invariant probability measure.

Theorem 3.6.1. (Grandmont [39] Theorem D.1.8 pg. 63) Let $f : Z \rightarrow Z$ satisfy the following conditions:

- i. $f \in C^3$ and there exists $z^* \in (z', z'') = (f^2(z^*), f(z^*))$ such that $f'(z^*) = 0$ and $f''(z^*) < 0$, $f'(z) > 0 \Leftrightarrow z < z^*$, $f'(z) < 0 \Leftrightarrow z > z^*$.
- ii. $f(z) \geq z \Leftrightarrow z \leq \bar{z}$ and $f(z) \leq z \Leftrightarrow z \geq \bar{z}$. Furthermore, $Sf(z) < 0$ for all $z \in Z$, $z \neq z^*$.
- iii. There exists $n \geq 2$ such that $y = f^n(z^*)$ is an unstable fixed point of f , i.e. satisfies $f(y) = y$ and $|f'(y)| > 1$.

Then f exhibits ergodic chaos. ■

Property 3.2.1 is congruent with conditions i and ii of Theorem 3.6.1. If it can be further demonstrated that f has a negative Schwartzian derivative and an unstable fixed point then f is ergodically chaotic. Therefore, in the sequel, the demonstration of ergodic chaos will in part

¹¹³ It can be checked that if $Sf < 0$ then $|f'|^{\frac{1}{2}}$ is convex over X (Day and Pianigiani [28] pg. 50). However concavity of f is neither sufficient nor necessary to guarantee that $Sf < 0$.

depend on showing that the dynamical system satisfies the above stated conditions of Theorem 3.6.1.

The specific utility function that is used is the quadratic quasi-linear utility function:

$$u(x_t^t, x_t^{t+1}) = u_1(x_t^t) + u_2(x_t^{t+1}) = x_t^t + ax_t^{t+1} - b(1/2)(x_t^{t+1})^2 \quad (3.6.1)$$

By the considerations of Section 3.2, the equilibrium dynamical system is

$$x_{t-1}^t - \omega_2 = a(1 - (b/a)x_t^{t+1})(x_t^{t+1} - \omega_2) \quad (3.6.2)$$

In order to collapse (3.6.7) into a manageable form, let $\omega_1 > 1$, $\omega_2 = 0$ and $b = a$:

$$z_{t-1} = f_b(z_t) = bz_t(1 - z_t) \quad (3.6.3)$$

where $z_t = z_t^{t+1}$. (3.6.3) is logistic map parameterized by b (see [27] and [68] for example). It can be checked that for $f_b : [0,1] \rightarrow [0,1]$ is an interval map for $b \in [0,4]$. At $z_t = 0$ the autarky equilibrium obtains. The monetary steady state is $\bar{z} = 1 - b^{-1}$ and the critical point is $z^* = 1/2$ for all $b \in [0,4]$ ¹¹⁴. It is noted that at $z_t = 1$ satiation obtains. The gradient at the monetary steady state is $f_b'(\bar{z}) = 2 - b$. The dynamical system is unstable if $f_b'(\bar{z}) < -1$ which is the case if $b > 3$. Hence for chaos it is necessary that $b \in (3,4]$. It is readily checked that f_b has a negative Schwarzian derivative:

$$sf_b(z) = \frac{-3}{2}b \left[\frac{2}{1-2z} \right]^2 < 0$$

By Theorem 3.6.1, for f_b to be both chaotic and ergodic¹¹⁵, there needs to exist some $n \geq 2$ such that the n^{th} iterate of the critical point is an unstable fixed point of f_b , i.e. there is $n \geq 2$ such that for $f_b'(z_2^*) = 0$ and $f_b(\bar{z}_2) = \bar{z}_2$, the following hold;

1. $f_b^n(z_2^*) = \bar{z}_2$
2. $|f_b'(\bar{z}_2)| > 1$

To begin, by the fact that $f_b'(\bar{z}_2) = 2 - b$ and $|f_b'(\bar{z}_2)| > 1$ if and only if $b \in (3,4]$ such that f_b is ergodic, it is required that $f_b^n(z_2^*) = \bar{z}_2$ for $z_2^* = 0.5$ and $b \in (3,4)$. This is equivalent to solving the following polynomial for some $n \geq 2$ and $b \in (3,4)$.

¹¹⁴ For $z_t \in [0, 1/2)$ the substitution effect dominates whereas for $z_t \in (1/2, 1]$ the income effect dominates.

¹¹⁵ The methodology here used is similar to that of Bala and Majumdar [6]. An analogous methodology is seen in Mukherji [68] in order to show that a one-dimensional tatonnement process can generate ergodic chaos. This latter argument in turn utilises the notion of ergodic chaos expounded in Boldrin and Woodford [16] and Majumdar and Mitra [53], [54], [55].

$$f_b^n(z_2^*) - (1 - (1/b)) = 0 \quad (3.6.14)$$

For $n=5$ the resultant polynomial of (3.6.17) is a 31 degree polynomial in b . Excluding complex roots and roots that do not belong to the interval $(3,4)$, the following three roots are obtained^{116,117}; $b_1 = 3.67857$, $b_2 = 3.92774$ and $b_3 = 3.98257$ written b_i , $i = 1, 2, 3$. It is also known that for $b=4$ f_b is ergodically chaotic. By using the expression

$\langle z_2 \rangle = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} f^n(z_2)$ the time averages of the backwards equilibrium map are obtained.

The average value of first period consumption is the derived from $\langle x_1 \rangle + \langle x_2 \rangle = \omega_1 + \omega_2$ given that $\langle z_2 \rangle = \langle x_2 - \omega_2 \rangle = \langle x_2 \rangle - \omega_2$. The value of the first period fixed point is then obtained by $\bar{x}_1 + \bar{x}_2 = \omega_1 + \omega_2$. The following table reports the value of the fixed points and the time average or expected value.

	$\bar{z}_1 = \bar{x}_1 - \omega_1$	$\bar{z}_2 = \bar{x}_2 - \omega_2$	$\langle z_1 \rangle = -\langle z_2 \rangle$	$\langle z_2 \rangle = \langle x_2 \rangle - \omega_2$
$b_1 = 3.67857$	-0.728155	0.728155	-0.6727337	0.6727337
$b_2 = 3.92774$	-0.745400663	0.745400	-0.6070116	0.6070116
$b_3 = 3.98257$	-0.728155	0.748905	-0.5609316	0.5609316
$b_3 = 4$	-0.75	0.75	-0.5	0.5

For each of the reported values one has that $\langle z_2 \rangle \leq \bar{z}_2$ and $\langle z_1 \rangle \geq \bar{z}_1$ as asserted by Theorem 3.3.1.

It is interesting to note that the same result could have been obtained by a manipulation of the dynamical system (3.6.3). By the fact that the variance of z_2 is positive then $\text{var}[z_2] = E[z_2^2] - [E[z_2]]^2 > 0$ so $E[z_2^2] > [E[z_2]]^2 > 0$. Consider that $f_b(z_2) = bz_2(1 - z_2) = bz_2 - bz_2^2$. Taking expectations of this last expression gives $E[f_b(z_2)] = bE[z_2] - bE[z_2^2]$. By ergodicity of the dynamical system $E[f_b(z_2)] = E[z_2]$ hence $E[z_2] = bE[z_2] - bE[z_2^2]$. Rearranging this last expression; $E[z_2^2] = E[z_2](1 - 1/b) = E[z_2]\bar{z}_2$. By the fact that $E[z_2^2] > [E[z_2]]^2$ then $E[z_2]\bar{z}_2 > [E[z_2]]^2$ hence $E[z_2] < \bar{z}_2$.

The linearity in the first period good implies that $\langle u_1(x_1) \rangle = u_1(\langle x_1 \rangle) = \langle x_1 \rangle$ and $u_1(\bar{x}_1) = \bar{x}_1$. In order to obtain the expected utility, $\langle u_2(x_2) \rangle = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} u_2(z_{2(n)})$ is utilised

¹¹⁶ All calculations were carried out on Mathematica and are correct up to 5 decimal places.

¹¹⁷ It is noted that for $n \geq 6$ iterations of $f_b^n(z_2^*)$ which determine the polynomial (6.14), the resultant expression is inordinately complex and intractable for the purposes of the analysis herein.

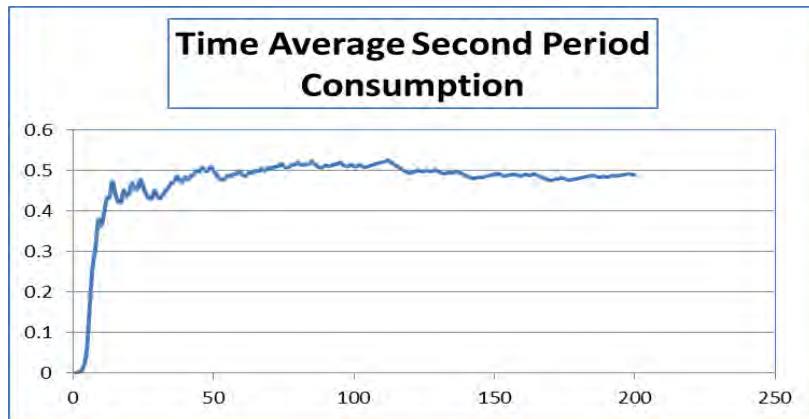
where $z_{2(n)}$ is the value of z_2 obtained from $z_{t-1} = f_b(z_t)$ (3.6.3). The following table reports the utility of the expected value and the expected utility as well as the utility at the fixed point in the second period good.

	$\langle u_2(x_2) \rangle$	$u_2(\langle x_2 \rangle)$	$u_2(\bar{x}_2)$
$b_1 = 3.67857$	1.5739	1.64229	1.70336
$b_2 = 3.92774$	1.49578	1.66057	1.83657
$b_3 = 3.98257$	1.397421	1.6074	1.8656
$b_4 = 4$	1.25	1.5	1.875

In each of the cases the result of Corollary 3.3.1 holds; $E[u(x)] \leq u(E[x]) \leq u(\bar{x})$

Figure 3.6.1 shows the time average for second period consumption given $b_4 = 4$. It is seen that the time average settles around 0.5 irrespective of the chaotic nature of the equilibrium sequence. Figure 3.6.2 shows the expected value of utility for $b_4 = 4$. This value settles at 1.25 as reported. Figures 3.6.3 – 3.6.5 report the invariant distribution for b_i , $i = 1, 2, 3$ respectively¹¹⁸.

Figure 3.6.1



¹¹⁸ The distributions were obtained by means of iterating f_b 50,000 times.

Figure 3.6.2

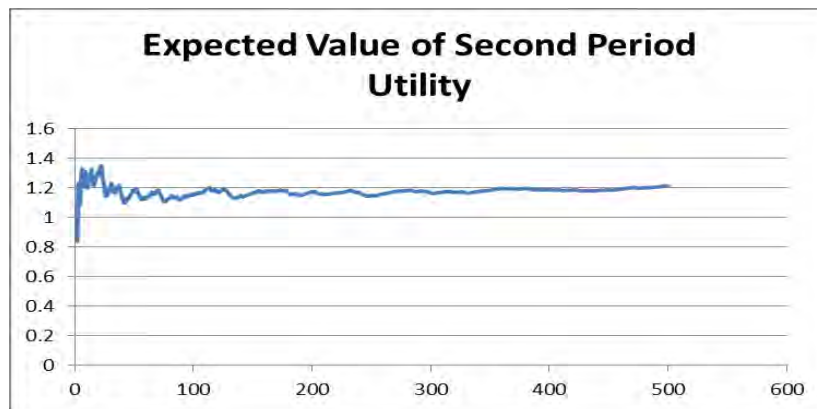


Figure 3.6.3

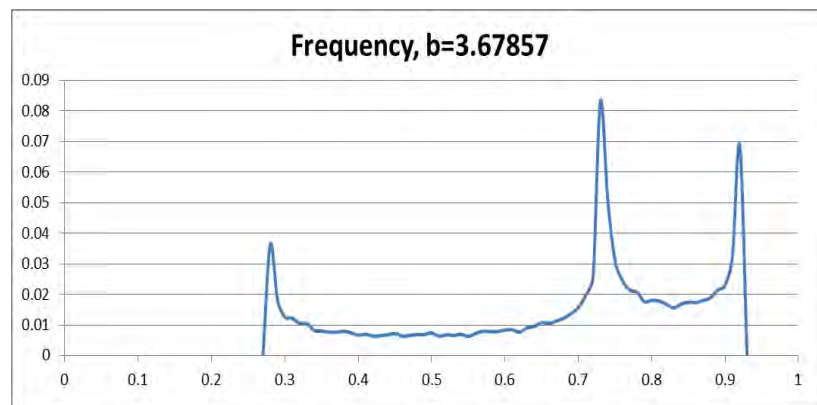


Figure 3.6.4

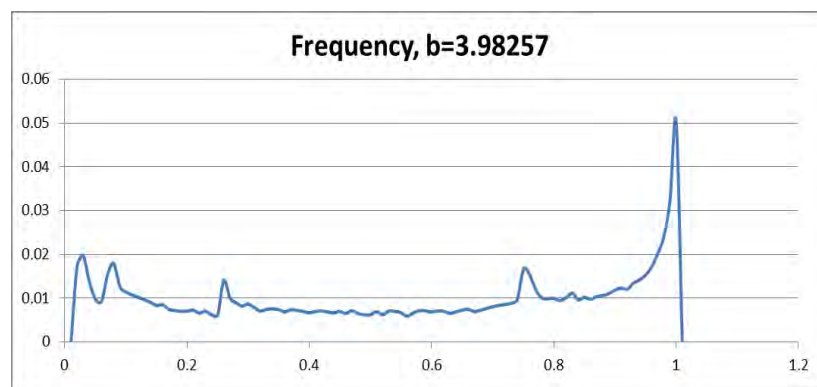
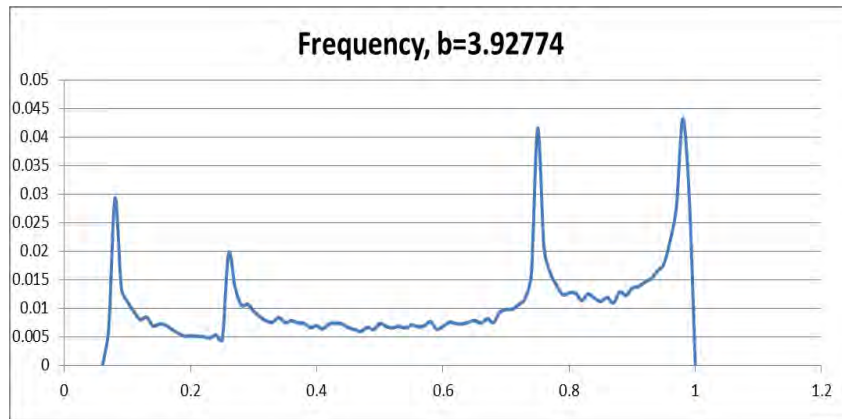


Figure 3.6.5



As a final note one may be inclined to disregard the relevance of ergodic chaos in that ergodic chaos may occur for a negligible set of $b \in (3, 4)$. In Chapter 4 of this Thesis, a Theorem of Jakobsen [48] is reported in which it is shown that for the map (3.6.3) if there is some $b \in (3, 4)$ such that f_b exhibits ergodic chaos then there is a set of positive Lebesgue measure in $(3, 4)$ every point of which generates a map which is ergodically chaotic. Hence, for $f_b(z_t) = bz_t(1 - z_t)$ ergodic chaos is robust in a measure theoretic sense¹¹⁹. The following corollary is stated.

Corollary 3.6.1 For the map defined by $f_b : [0, 1] \rightarrow [0, 1]$; $f_b(z_t) = bz_t(1 - z_t)$ there is a non-negligible set of $b \in (3, 4)$ such that there is an absolutely continuous measure invariant to f_b . ■

3.7. Conclusion

In this paper a model of equilibrium dynamics has been examined under the assumption that equilibria are ergodic and chaotic. A restriction of the form of the equilibrium dynamical system to be concave allowed the result that the expected value of consumption in the second period of life is less than consumption at the steady state and the expected value of the first period consumption is greater than consumption at the steady state. Welfare analysis then showed that the expected utility of representative agent is less than the level of utility at the

¹¹⁹ In this regard the fact that the parameters b_t are approximations to the true value does not alter the relevance of the discussion.

steady state. As a consequence a criterion was given by means of which stabilisation would be a benefit on average to society. Utilising the invariant probability measure of the chaotic equilibrium system, sunspot equilibria were discussed. Finally, a commonly studied dynamical system was explored in relation to the results. This paper fills a gap in the literature as to this author's knowledge there has been no discussion so far in which the welfare properties of ergodic chaos is treated. These arguments are taken up further in Chapter 4 of this thesis in which the equilibrium dynamics are analysed for an infinite lived agent model.

3.8. Appendix

Let $f: X \rightarrow X$ be a continuous map of the interval. *Topological chaos* is characterised by the set X possessing an uncountable and invariant subset $S \subset X$ for which any two points starting in S move arbitrarily close to each other¹²⁰ and far apart from each other¹²¹. This property is termed *sensitive dependence to initial conditions* and renders impossible the long-term forecasting of any trajectory. Moreover, if no trajectory is periodic of any (arbitrarily large) number, f is then said to be topologically chaotic. Li and Yorke (see [27]) showed that if $f: X \rightarrow X$ exhibits a trajectory that has a periodic orbit of prime period 3, then f is topologically chaotic. The defining characteristics of a topologically chaotic (economic) system are sensitive dependence to initial conditions, measurement accuracy and computation errors. Sensitivity to initial conditions means that if the initial starting value is inaccurately measured by an arbitrarily small amount, this measurement error will be amplified in finite time as the dynamical system evolves. As a consequence, long-term predictions are impossible. Similarly, if the dynamical system is parametrically defined, any measurement error in the parameter will generate asymptotic trajectories of the economic variables which are unrecognisable with respect to the each other. It is argued in Boldrin and Woodford [16], Day and Pianigiani [28] pg. 50 and Grandmont [39] pg. 62, that the notion of topological chaos may be unsatisfactory as the uncountable invariant set S over which the topologically chaotic dynamics persist may in fact have a negligible Lebesgue measure in X . The importance and relevance of topological chaos is then questionable, being of the variety of rare occurrence rather than a robust phenomenon.

To this end, a different definition of chaos is motivated; that of *ergodic chaos*. Intuitively, a dynamical system is ergodic and chaotic if the trajectories of almost all initial conditions visit infinitely often sets of non-negligible measure in X . This means that a dynamical system which is both ergodic and chaotic possesses a scrambled set S that is

¹²⁰ If $x, y \in S$ and $x \neq y$ then $\liminf_{n \rightarrow \infty} |f^n(x) - f^n(y)| = 0$.

¹²¹ If $x, y \in S$ and $x \neq y$, for $\varepsilon > 0$ then $\limsup_{n \rightarrow \infty} |f^n(x) - f^n(y)| \geq \varepsilon$.

'empirically observable'; which has a *positive Lebesgue measure*. In such a case, the set of all trajectories approximate an *ergodic and absolutely continuous distribution which is invariant under f* . A typical trajectory of f therefore approximates a limiting probability distribution and this distribution does not vary depending on the initial value of f ¹²². Hence, rather than concern oneself with an individual trajectory (a meaningless observation in a chaotic system) one can utilise the invariant probability distribution in order to make *probabilistic statements concerning the evolution of the state variable*. In order to define what it is for a chaotic system to be ergodic, one needs to define what it is for the invariant measure generated from f to be unique and absolutely continuous with respect to the Lebesgue measure¹²³.

Let X be a set and let B be the sigma-algebra defined over X . Let $\{X_n\}_{n=1}^{\infty}$ be a collection of disjoint sets in B and there be a measure μ defined over B where $\mu(X) = 1$. The triple (X, B, μ) defines a probability space. Let $f: X \rightarrow X$ be a continuous unimodal map of an interval where $X \subset \mathbb{R}$ is the compact set $X = [x', x'']$, $x' < x''$. If the measure μ attains the value of zero on points of X then μ is said to be *continuous*¹²⁴. That is to say, a measure is continuous if for any singleton $x \in X$, $\mu(\{x\}) = 0$. Furthermore, f is *measure preserving* and μ is *invariant* under f if $\mu(f^{-1}(X')) = \mu(X')$, for all $X' \in B$. $f: X \rightarrow X$ is *ergodic* if for $X' \in B$ and $f^{-1}(X') = X'$ then either $\mu(X') = 0$ or $\mu(X') = 1$ ^{125 126}. By this definition it can be seen that an ergodic process is one for which if x belongs to X' then, with probability 1, x remains in X' or with probability 0, x escapes from X' . In other words, the dynamical system cannot be broken into subsystems whereby the statistical properties are defined.

As a general property of one-dimensional discrete time dynamical maps which are ergodic, the following theorem states that if μ is an ergodic measure and f is measure

¹²² This need not be the case if f is topologically chaotic.

¹²³ The review of this part draws upon the works of Day [27], Day and Pianigiani [28], Day and Shafer [29] and Grandmont [39] as well as Bala and Mujamdu [6]. The definition of ergodic chaos appears throughout the economic literature, a good reference source being that of Barnett *et al.* [13] and Grandmont [39] and neat applications of ergodic chaos to economic models may be found in [17].

¹²⁴ Alternatively, μ is said to be non-atomic.

¹²⁵ The significance of f being a measure preserving transformation and μ an invariant measure is that almost every point of X returns to its vicinity infinitely often under action of f . In this sense the dynamical system can be interpreted as a stochastic system. However the set of points $x \in X$ which are visited (infinitely often) may in fact belong to a subset of X and the asymptotic trajectory of f may enter and not exit some subset, say X' of X . Whether or not the trajectory enters X' may in turn depend on the initial value of f . The invariant measure or the density of f then depends on the choice of initial values of f and any observed probability distribution will depend upon the particular choice of the initial value. The asymptotic dynamics of f can then be decomposed into subsets of X thereby rendering the study of the limiting distribution contingent upon the choice of initial values. If f is instead *ergodic* then the trajectories cannot be decomposed to belong to two or more subsets of X .

¹²⁶ By this definition it can be seen that an ergodic process is one for which if x belongs to X' then, with probability 1, x remains in X' or with probability 0, x escapes from X' . In other words, the

preserving and ergodic then there exists an equivalence between the time average of the trajectory and the expected value of the probability distribution.

Theorem 3.7.1. The Birkhoff Mean Ergodic Theorem Let (X, B, μ) be a probability space and let f be measure preserving and ergodic. Let g be any μ -integrable function. Then;

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} g(f^n(x)) = \int_X g(x) \mu(dx), \quad \text{for } \mu\text{-almost all } x \in X \quad (3.7.1)$$

The left hand side of (3.7.1) is the average (asymptotic) value of a trajectory for some $x \in X$ and integrable function g . The right hand side of (A.1) is the expected value or mean value of g evaluated along the trajectory. The Birkhoff Mean Ergodic Theorem states that if f has an ergodic and invariant measure μ , the time average is equal to the space average. Given the equality between the time average and space average, an invariant probability distribution can be constructed empirically from an observed trajectory $\{x_n\}_{n=0}^{\infty}$. Define χ as the characteristic function of f ,

$$\chi_{X'}(f^n(x)) = \begin{cases} 1 & \text{if } f^n(x) \in X' \\ 0 & \text{if } f^n(x) \notin X' \end{cases}$$

where $X' \in B$. $\sum \chi_{X'}(f^n(x))$ is the number of times which the trajectory enters $X' \in B$. The mean ergodic theorem then states that there exists a time average which is invariant to the starting point or initial choice of $x \in X$;

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \chi_{X'}(f^n(x)) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \chi_{X'}(x_n) = \mu(X') \quad (3.7.2)$$

(3.7.2) states that the trajectory of x will visit every measurable set in proportion to its measure. This can be seen by setting $g(x) = \chi_{X'}(x)$ in (A.1) from which;

$$\int_X g(x) \mu(dx) = \int_X \chi_{X'}(x) \mu(dx) = \mu(X') \quad (3.7.3)$$

For almost all initial conditions $x \in X$ (on a set of positive Lebesgue measure), if the measure μ is continuous, invariant and ergodic the trajectory of x will “fill up” the support of the measure μ . However, the existence of such a measure does not imply neither complex nor chaotic dynamics as the support of μ may be concentrated on a set of negligible measure, e.g. fixed or periodic points. In such a case f cannot exhibit complicated dynamics as if f returns infinitely often to every μ -measurable set then no set of points can ever be observed other than a finite set; complexity of trajectories is precluded. Conversely, the measure μ is

absolutely continuous with respect to the Lebesgue measure m if there exists an integrable function φ such that $\mu(X') = \int_{X'} \varphi dm$ for all measurable sets X' , where the function φ is the density of μ . Absolute continuity of the measure μ implies that for all $X' \in \mathcal{B}$, if $m(X') = 0$ then $\mu(X') = 0$ in which case the support of μ for any measurable set cannot be a set of measure zero. That is, if f exhibits complex dynamics which are ergodic the trajectories occur on a set which is large in the sense of Lebesgue measure. The measure μ is then absolutely continuous and invariant.

In sum, if a map is topologically chaotic any two values which start in the invariant scrambled set S will eventually move arbitrarily close together and far apart thereby becoming uncorrelated over time. However, the support of these erratic trajectories may be negligible in which case chaos may not be observable; chaotic trajectories, whilst existing, may belong to a set of Lebesgue measure zero. If instead the map is ergodic and chaotic, the trajectories will exhibit sensitive dependence to initial conditions and hence be uncorrelated, yet the frequencies of differing trajectories will converge to a stable density function the support of which has a positive measure. In this sense Li and Yorke's notion of chaos is observable and has a physical measure as the trajectories will 'fill up' the support of the set and hence be dense. Complex dynamics are then *empirically observable* and display *statistical regularities* in that the empirical distributions and time averages are asymptotically stable.

A Forward Looking Model of Money in the Utility Function and the Robustness of Ergodic Chaos

Abstract

An infinite horizon model in which money is an argument of the utility function is examined where equilibrium real money balances can display cycles and chaos. It is demonstrated that if ergodically chaotic real money balances are present there exists a relationship between the expected real money holding and the steady state value of real money balances where the latter are associated with high money supply growth rates. It is further demonstrated that the expected level of utility can be improved upon by a contraction in the money supply which yields a level of lifetime utility which is strictly larger than that obtained under a chaotic regime. A contractionary monetary policy can thereby stabilise the equilibrium system and improve welfare.

4.1. Introduction

In [31], [32] Fukuda examines a variant of Brock's [18] model of an infinitely lived agent in which real money balances appear in the utility function. In [31], [32] money demand is anticipatory (money demand one period hence is determined in the present period). Under the assumption of additively separable utility function, the resultant backwards looking map which defines the equilibrium balances of real money holdings, has the capacity to generate cyclical or periodic equilibria as well as equilibrium trajectories that are aperiodic or chaotic in a topological sense. Hand in hand with cyclical or periodic equilibria, stationary sunspot equilibria are shown to exist. The analyses of [31], [32] is however confined to establishing the link between sunspot equilibria and cycles [32] placing emphasis on the role of monetary policy as instrumental in the elimination of cycles (and by extension chaos) and thereby sunspot equilibria [12].

In Matsuyama [58], [59], [60], [61], [62], a model similar to Brock and not dissimilar to that of Fukuda [31], [32] is examined. Unlike [31], [32], in [58], [59], [60], [61], [62], it is assumed that money demand is not anticipatory and that the utility function is not additively separable. Given the specific functional form of the utility function, the model of Matsuyama [58], [59], [60], [62] generates a dynamical system of equilibrium real money holdings which are forward looking and, as pointed out in [59], [62] for certain parameters of the money supply growth rate, the equilibrium dynamics possess the capacity to generate cyclical, topological and ergodically chaotic equilibria. The structure of the equilibrium set is examined in [59], [62], wherein it is demonstrated for a non-negligible set of parameters, the equilibrium set has the structure of a fractal set. As in [31], [32], Matsuyama stops short of examining the ergodic chaotic properties of the dynamical equilibrium system, focusing instead on the topologically chaotic equilibrium dynamics and stationary sunspot equilibria.

The aim of this paper is to examine the equilibrium dynamics of the monetary model of [18] and [58], [59], [60], [62] as follows:

- i. Ergodically chaotic properties of the equilibrium dynamical system are examined.
- ii. Expected utility over the lifetime of a representative agent can be improved upon if the equilibrium system is stabilised away from ergodically chaotic equilibrium holdings of real money balances.
- iii. A contractionary monetary policy can both stabilise and improve lifetime welfare.

The paper is laid out as follows. Sections 4.2.1 – 4.2.3 introduce the monetary model of Matsuyama [59], [62] in which the utility function is specified and the dynamical system which defines the equilibrium path of real money holdings is discussed. The equilibrium map is a unimodal map of the interval in which bounded equilibrium fluctuations may occur for certain sets of parameter values. This part of the paper is based upon [59], [62], the techniques of which are standard in the one-dimensional discrete dynamical systems literature (see [6], [13], [15], [17], [24], [39] for instance). Whilst this section is technical in nature and replicates [59], [62], it is necessary to expound the concepts therein in order to facilitate subsequent discussions.

Sections 4.2.4 – 4.4 are the author's own contribution. In Section 4.2.4 the statistical properties of the equilibrium equation when ergodic chaos is present are discussed. It is shown that if the dynamical system is ergodically chaotic then the expected price level is strictly less than the stationary price level. In Section 4.2.5 it is shown that for a non-negligible set of money growth rates the dynamical system is ergodically chaotic. In this sense the equilibrium dynamical system is structurally robust with respect to ergodic chaos. In Section 4.3 the discussion turns to the welfare implications of the previous set of results showing that irrespective of the manner in which the ergodically chaotic system behaves, there exists a non-negligible set of monetary policies for which the equilibrium system is stabilised and lifetime indirect utility along the equilibrium path is increased. Section 4.4 concludes with a numerical evaluation of the dynamical system.

4.2. The Infinite Lived Agent Model and Equilibrium Dynamics

4.2.1. Introduction

In this section, the monetary model of Matsuyama [59], [62] is specified and discussed. The equilibrium equation in real money balances are transformed into an equation in prices. This transformation facilitates the analysis leaving unaltered the fundamental equilibrium properties of money demand. It is shown that the resultant dynamical system is a unimodal map of the interval which is globally convergent, i.e. the trajectories belong to a non-negligible compact set.

4.2.2. The Model

There is a fixed number of identical infinitely lived agents each of whom possesses perfect foresight. Time starts at $t = 0$ and extends indefinitely into the future; $t = 0, 1, 2, \dots$. At $t = 0$ each agent maximises the present discounted value of utility:

$$U = \sum_{t=0}^{\infty} \beta^t u(c_t, m_t^d) \quad (4.2.1)$$

subject to the constraint:

$$M_t^d = P_t(y - c_t) + M_{t-1}^d - T_t \quad (4.2.2)$$

where $\beta \in (0, 1)$ is the intertemporal discount factor, y is a constant endowment of the perishable consumption good, c_t is consumption, T_t is taxation and m_t^d is real money balances defined by $m_t^d = M_t^d / P_t$, where $P_t > 0 \quad \forall t$ is the price level of the consumption good. It is assumed that in each period taxation nets to zero; $T_t = 0 \quad \forall t$. The total supply of the commodity in the economy is fixed and equal to y . Standard concavity assumptions are made upon the utility function u : $u_{cc}'' < 0$, $u_{mm}'' < 0$, $u_{cm}'' < 0$, $u_{mc}'' < 0$ and $u_{cc}''u_{mm}'' - u_{cm}''u_{mc}'' > 0$. In the constraint (4.2.2), *nominal money demand is lagged*, dependent upon nominal money demand in the *previous* period. Hence *money demand is a present period decision variable*¹²⁷. The commodity market is competitive thus agents consider the value of P_t exogenously

¹²⁷ Money demand is hence not anticipatory as is the case of [31], [32].

determined and independent of money holdings. Writing (4.2.1) as $c_t = y + (M_t^d/P_t) - (M_t^d/P_t)$ over two time periods t and $t+1$:

$$\beta^t u(y + (M_t^d/P_t) - (M_t^d/P_t), (M_t^d/P_t)) + \beta^{t+1} u(y + (M_{t+1}^d/P_{t+1}) - (M_{t+1}^d/P_{t+1}), (M_{t+1}^d/P_{t+1}))$$

The derivative of the previous expression with respect to M_t^d yields the first order conditions:

$$\frac{-\beta^t u'_c(c_t, m_t^d)}{P_t} + \frac{\beta^t u'_m(c_t, m_t^d)}{P_t} + \frac{\beta^{t+1} u'_c(c_{t+1}, m_{t+1}^d)}{P_{t+1}} = 0,$$

which rearranged:

$$u'_c(c_t, m_t^d) = u'_m(c_t, m_t^d) + \beta u'_c(c_{t+1}, m_{t+1}^d) \frac{P_t}{P_{t+1}}. \quad (4.2.3)$$

The growth rate of fiat money from one period to the next is z . The period change in money supply is governed by the equation $M_{t+1} = zM_t$ where M_0 is given. Iteration shows that $M_t = z^t M_0$. It is noted that $P_t/P_{t+1} = P_t M_{t+1}/P_{t+1} M_{t+1} = m_{t+1}/zm_t$. It is assumed that $z > \beta$. Whilst this assumption is made for technical reasons, that is if $z \leq \beta$ then no stationary equilibrium exists, if the money supply is less than the discount rate then the equilibrium sequence is dampened as the rate at which agents discount future money demand is too large with respect to the money growth rate. Hence a friction between discounting and money growth exists, the latter dominating the former.

In equilibrium, markets clear when money demand is equal to money supply; $M_t^d = M_t = z^t M_0$ and when demand is equal to supply in the goods market; $c_t = y \quad \forall t \geq 0$. It is noted that equilibrium in the goods market is exogenously determined being imposed upon the equilibrium system. Consequently, in the absence of the specification of equilibrium formation in the goods market, the model must be treated as a partial equilibrium framework¹²⁸. Given equilibrium in the goods and money market, by (4.2.3) one has that:

$$\beta u'_c(y, m_{t+1}) m_{t+1} = z m_t [u'_c(y, m_t) - u'_m(y, m_t)] \quad (4.2.4)$$

where $m_t = M_t/P_t = z^t M_0/P_t$. Let Z be the set of monetary policy rules z where $Z = (\beta, \kappa)$ ¹²⁹ for some exogenously determined discount factor $\beta \in (0,1)$ and for some given $\kappa < \infty$. In following Matsuyama [59], [62], the utility function is specified as:

$$u(c, m) = \begin{cases} \frac{(c^\alpha m^{1-\alpha})^{1-\gamma}}{1-\gamma}, & \text{if } \gamma \neq 1, \gamma > 0 \\ \alpha \log c + (1-\alpha) \log m, & \text{if } \gamma = 1 \end{cases} \quad (4.2.5)$$

¹²⁸ Matsuyama [59] claims the converse; that the framework is general equilibrium. This author does not hold this view as a general equilibrium framework would necessitate the inclusion of the goods market dynamics.

¹²⁹ By the fact that $z > \beta$.

where the parameter $\alpha \in (0,1)$ is exogenously determined¹³⁰.

Combining (4.2.4) and (4.2.5) defines the dynamical system of equilibrium real money balance holdings:

$$(m_{t+1})^{-\eta} = (1 + \delta)(m_t)^{-\eta} \left[1 - \frac{1-\alpha}{\alpha} \frac{y}{m_t} \right] \quad (4.2.6)$$

where the parameterizations are:

$$\eta \equiv (1-\alpha)(\gamma-1)-1 \quad (4.2.7)$$

$$\delta = (z/\beta)-1 \quad (4.2.8)$$

The assumption that $z > \beta$ implies that $\delta = (z/\beta)-1 > 0$ in (4.2.8). In order to generate a well-defined dynamical system the following impositions on the parameter set are made.

$$\eta \in (0,1] \quad (4.2.9)^{131}$$

Define $\Delta(\eta) = \eta^{-\eta}(1+\eta)^{1+\eta} - 1$. It is assumed that¹³²:

$$0 < \eta < \delta < \Delta(\eta) \quad (4.2.10)$$

By (4.2.6) define the map:

$$g_z(m_{t+1}) = (1 + \delta)^{\frac{-1}{\eta}} m_t \left(1 - \frac{A}{m_t} \right)^{\frac{-1}{\eta}} \quad (4.2.12)$$

where $A \equiv \frac{(1-\alpha)}{\alpha} y > 0$. $g_z(m_{t+1})$ defines a one-dimensional discrete time dynamical system in real money balances which is parameterized by monetary policy $z \in Z$ where the choice of z determines the form of g_z and thereby the equilibrium path of real money balances. Consequently monetary policy impinges upon the dynamic stability or instability of the equilibrium system. It is noted that $g_z(m_{t+1}) > 0$ iff $m_t > A$ for all t .

To examine the manner in which g_z behaves, properties of g_z are stated. The critical and fixed point of g_z are respectively:

$$m^* = \frac{A(1+\eta)}{\eta} \quad (4.2.13)$$

¹³⁰ It is noted that consumption demand and money demand are gross substitutes. However, the role of gross substitution between the money market and the goods market should not be overstated as pointed out above, the framework is partial equilibrium as the goods market is *assumed* to be in equilibrium.

¹³¹ This assumption implies that $\gamma > 0$.

¹³² $0 < \eta < \Delta(\eta)$ is true if $\eta \ln \eta < \ln(1+\eta)$ which must be the case as $\eta \in (0,1]$. Hence $\eta \ln \eta < 0 < \ln(1+\eta)$. Thus the inequality $0 < \eta < \Delta(\eta)$ holds for all $\eta \in (0,1]$.

$$\bar{m}_z = \frac{A(1+\delta)}{\delta} = \frac{Az}{(z-\beta)} \quad (4.2.14)$$

where in (4.2.8) was used in (4.2.14). The critical point is not a function of z whereas the fixed point is a decreasing function of z ¹³³. By (4.2.10) $0 < \eta < \delta$ which implies that $\bar{m}_z < m^*$. The *trapping set* is the set of points which g_z asymptotically generates, defined by the iterate of the critical point given by $M_z = [m'_z, m''_z] = [g_z(m^*), g_z^2(m^*)]$ where

$$g_z(m^*) = A(1+\delta)^{\frac{-1}{\eta}} \eta^{-1} (1+\eta)^{\frac{1+\eta}{\eta}} \quad (4.2.15)$$

$$g_z(g_z(m^*)) = g_z^2(m^*) = A(1+\delta)^{\frac{-2}{\eta}} \eta^{-1} (1+\eta)^{\frac{1+\eta}{\eta}} [1 - (1+\delta)^{\frac{1}{\eta}} \eta (1+\eta)^{-\frac{(1+\eta)}{\eta}}]^{\frac{-1}{\eta}} \quad (4.2.16)$$

It is shown in the sequel¹³⁴ that M_z is the trapping set¹³⁵ of g_z . (g_z, M_z) defines the *dynamical system* of equilibrium real money demand. g_z is a *strictly convex function* over M_z ; since $g_z''(m) = \left[(1+\delta)^{\frac{-1}{\eta}} A^2 (1 - Am^{-1})^{-2-\frac{1}{\eta}} (1+\eta^{-1}) \right] [m^3 \eta]^{-1}$ then $g_z''(m) > 0$ as $m_t > A$ where this last inequality follows from the fact that if $m \in M_z$ then $m \geq g_z(m^*)$ and $g_z(m^*) > A$ ¹³⁶. Thus since $m_t > A$ then $g_z(m_{t+1}) > 0$.

In order to examine an ergodically chaotic equilibrium path of (g_z, M_z) , it is convenient to transform the function g_z into a function which is *unimodal* and *single peaked rather than convex*. This will permit the use of Theorem 4.3.1 in Section 4.3 which provides a criterion by which to evaluate whether equilibrium dynamics exhibit ergodic chaos.

The programme of the next part is to therefore transform the map g_z into a map f_z which has a critical point which is a maximum and thereby work in the *equilibrium price* space rather than in the equilibrium real money demand space. That is to say, it is required to transform m_t into a price variable p_t for which there exists a dynamical system f_z , the dynamics of which are equivalent to the dynamics of g_z . To that end the topological conjugate f_z to g_z is defined in Definition 4.2.1.

¹³³ $\partial_z \bar{m}_z = -A\beta(z-\beta)^{-2}$ as $z > \beta$.

¹³⁴ Proposition 4.2.1 in conjunction with Lemma 4.2.2.

¹³⁵ The trapping set is the set in the domain of the map for which all points in that set remain in that set under action of the map. Hence, M_z is the trapping set if $g_z(M_z) = M_z$ or $g_z : M_z \rightarrow M_z$.

¹³⁶ To show that this inequality holds true consider (4.2.10); that $\delta < \Delta(\eta)$ or $\eta^{-\eta} (1+\eta)^{1+\eta} - 1 > \delta$. Hence $\eta^{-1} (1+\eta)^{\frac{(1+\eta)}{\eta}} > (1+\delta)^{\frac{1}{\eta}}$ and $A(1+\delta)^{\frac{-1}{\eta}} \eta^{-1} (1+\eta)^{\frac{(1+\eta)}{\eta}} > A$ or $g_z(m^*) > A$.

Definition 4.2.1 Let $g_z : M_z \rightarrow M_z$ and $f_z : P_z \rightarrow P_z$ be two functions. g_z is topologically conjugate to f_z if and only if there exists a homeomorphism $h : M_z \rightarrow P_z$ such that $h \circ g_z = f_z \circ h$. h is then called a topological conjugacy.

To specify a topological conjugacy define the map $f_z : P_z \rightarrow P_z$ by:

$$p_{t+1} = f_z(p_t) = (1 + \delta)^{\frac{1}{\eta}} (p_t) (1 - p_t)^{\frac{1}{\eta}} \quad (4.2.17)$$

The critical point of f_z , given by $f'_z(p^*) = 0$, is¹³⁷:

$$p^* = \eta / (1 + \eta) \quad (4.2.18)$$

There are two fixed points of f_z :

$$\bar{\bar{p}}_z = 0 \quad (4.2.19)$$

$$\bar{p}_z = \delta(\delta + 1)^{-1} = (z - \beta)z^{-1} \quad (4.2.20)$$

(4.2.19) is termed the trivial steady state and (4.2.20) the (non-trivial) steady state. The trapping set of f_z is $P_z = [p'_z, p''_z] = [f_z^2(p^*), f_z(p^*)]$ where:

$$f_z(p^*) = (1 + \delta)^{\frac{1}{\eta}} \eta (1 + \eta)^{-\left(\frac{1+\eta}{\eta}\right)} \quad (4.2.21)$$

$$f_z(f_z(p^*)) = f_z^2(p^*) = (1 + \delta)^{\frac{2}{\eta}} \eta (1 + \eta)^{-\left(\frac{1+\eta}{\eta}\right)} (1 - (1 + \delta)^{\frac{1}{\eta}} \eta (1 + \eta)^{-\left(\frac{1+\eta}{\eta}\right)})^{\frac{1}{\eta}} \quad (4.2.22)$$

In the sequel (Proposition 4.2.1) it is demonstrated that f_z^t globally converges to P_z . As such, since $0 < f_z^2(p^*)$ then the equilibrium price dynamics do not obtain the non-trivial steady state¹³⁸. As a matter of definition, (f_z, P_z) is termed the dynamical system for $f_z : P_z \rightarrow P_z$.

Consider the map $h : M_z \rightarrow P_z$ defined by $h(m_t) = A(m_t)^{-1}$. This map transforms equilibrium real money demand into an equilibrium price variable; $p_t = h(m_t) = A(m_t)^{-1} = AP_t / M_t$ where $M_t = zM_{t-1} = z^t M_0$. By such a transformation there is a one-to-one relationship between p_t and P_t for some M_0 and $z \in Z$ and hence between p_t and m_t . The map defined by (4.2.17) is then an equilibrium map iff the dynamics of f_z and g_z are

¹³⁷ It is noted that $\partial_z \bar{p}_z = \beta z^{-2}$ and is hence increasing in z .

¹³⁸ This can also be seen by noting that $f'_z(0) > 1$ hence $\bar{\bar{p}}_z = 0$ is a repelling fixed point.

equivalent. The following lemma shows that h is a homeomorphism and f_z and g_z are topological conjugates¹³⁹.

Lemma 4.2.1 $g_z : M_z \rightarrow M_z$ as defined by (4.2.12) and $f_z : P_z \rightarrow P_z$ as defined by (4.2.17) are topological conjugates

Proof See Appendix 4.7. ■

By Lemma 4.2.1, $h \circ f_z^t = g_z^t \circ h$ for all $t \geq 0$ or equivalently $g_z^t = h \circ f_z^t \circ h^{-1}$ and the equilibrium price dynamics of f_z are homeomorphic to the dynamics of the equilibrium money balances of g_z . f_z is henceforth the object of study¹⁴⁰.

4.2.3. The Dynamical Equilibrium System

Since β is an exogenously determined parameter, $f_z : P_z \rightarrow P_z$ is a family of maps parameterized by the money supply rule $z \in Z$. It is now established that f_z is a unimodal map of the interval¹⁴¹. To begin, if $\eta = 0$, by (4.2.12); $p_{t+1}^0 = (1 + \delta)(p_t)^0(1 - p_t)$ or $1 = (1 + \delta)(1 - p_t)$ hence $p_t = \bar{p}_z \forall t \geq 0$ and the equilibrium dynamics collapse to the steady state and are hence trivial. Let $\eta \in (0, 1]$ as in (4.2.9). Define $I = [0, 1]$ and $I' = (0, 1)$. If $p_t \geq 1$ inspection of (4.2.17) reveals that $p_{t+1} > 0$ cannot hold. Therefore, if $p_t \leq 1$ then $0 \leq p_t \leq 1$ and $f_z(p_t) \in I \forall t$. Furthermore, if $p_t = 1$ then $p_{t+1} = 0$. Hence if $0 < p_{t+1} < 1$ then $f_z(p_t) \in I' \forall t$. The following set of properties which characterise f_z are reported in [58], [59] and are typical for unimodal one-dimensional discrete time dynamical systems (see [24] for example).

¹³⁹ Matsuyama [59] refers to the transformation $p_t = A(m_t)^{-1}$ as a price normalization. Strictly speaking, this is not a price normalization as to discuss the equilibrium dynamics in the equilibrium price space after having applied such a normalization, it is required to show an equivalence in the equilibrium money demand space.

¹⁴⁰ It is noted that by continuity of g_z , $\lim_{m \rightarrow 0} g_z(m) = +\infty$ so $h(\lim_{m \rightarrow 0} g_z(m)) = h(+\infty) = 0 = g_z(h(m)) = g(0) = \bar{p}_z$ by (4.2.14). Hence the trivial steady state of f_z is the 0 asymptote of g_z and does not obtain as $m_t > g_z(m^*) > A > 0, \forall t \geq 0$.

¹⁴¹ By Lemma 4.2.1, this implies that g_z is a unimodal map of the interval.

Properties A.1 – A.5

- A.1. $f_z(0) = f_z(1) = 0$.
- A.2¹⁴². f_z has a unique critical point p^* given by (2.13) such that $f'_z(p^*) = 0$. Also $f'_z(p) > 0 \quad \forall p \in [0, p^*)$ and $f'_z(p) < 0 \quad \forall p \in (p^*, 1]$ and:

$$f''_z(p^*) < 0 \quad (4.2.23)$$

- A.3. The gradients at the fixed points¹⁴³:

$$f'_z(0) = (1 + \delta)^{\frac{1}{\eta}} > 1 \quad (4.2.24)$$

$$f'_z(\bar{p}_z) = 1 - (\delta/\eta) \quad (4.2.25)$$

- A.4.
 - If $\delta < \Delta(\eta)$ then $f_z(p^*) < 1$.
 - If $\delta = \Delta(\eta)$ then $f_z(p^*) = 1$.
 - If $\delta > \Delta(\eta)$ then $f_z(p^*) > 1$ ¹⁴⁴.

Therefore if $\delta \leq \Delta(\eta)$ then $f_z : I \rightarrow I$ and if $\delta < \Delta(\eta)$ then $f_z : I' \rightarrow I'$.

- A.5¹⁴⁵.
 - $f_z(p) \geq p \Leftrightarrow p \leq \bar{p}_z$ (4.2.26)
 - $f_z(p) \leq p \Leftrightarrow p \geq \bar{p}_z$ (4.2.27)

It is noted that $\bar{p}_z$ is the unique fixed point in I' . Properties A.1 – A.5 imply that f_z is a unimodal map¹⁴⁶ and that if $\delta \leq \Delta(\eta)$ then $f'_z(p) \in I, \forall t$. Similarly if $\delta < \Delta(\eta)$ then $f'_z(p) \in I' \quad \forall t$. Hence the equilibrium dynamics belong to the unit interval and are thereby

¹⁴² $f'_z(p) > 0 \quad \forall p \in [0, p^*)$ iff $g'_z(m) < 0 \quad \forall m \in (m^*, \infty)$. To see this, by $h : M_z \rightarrow P_z, h(m^*) = p^*$ and $h((m^*, \infty)) = (0, p^*)$. By $h \circ g_z = f_z \circ h$ then $dh \circ dg_z = df_z \circ dh$ and $dg_z = dh^{-1} \circ df_z \circ dh$. Since $dh^{-1}, dh < 0$ and $df_z > 0$ for $p \in [0, p^*)$ then $dg_z < 0$ for $m \in (m^*, \infty)$. Similarly, $f'_z(p) < 0 \quad \forall p \in (p^*, 1]$ iff $g'_z(m) > 0 \quad \forall m \in (0, m^*)$.

¹⁴³ $f'_z(\bar{p}_z) = g'_z(\bar{m}_z)$. To see this $h(g_z(m)) = f_z(h(m))$ so $h'(g_z(m))g'_z(m) = f'_z(h(m))h'(m) \Leftrightarrow g'_z(m) = [h'(g_z(m))]^{-1}h'(m)f'_z(h(m))$. At the fixed point $g_z(\bar{m}_z) = \bar{m}_z$ so the last expression becomes $g'_z(\bar{m}_z) = [h'(\bar{m}_z)]^{-1}h'(\bar{m}_z)f'_z(h(\bar{m}_z)) \Leftrightarrow g'_z(\bar{m}_z) = f'_z(h(\bar{m}_z)) \Leftrightarrow f'_z(\bar{p}_z) = g'_z(\bar{m}_z)$.

¹⁴⁴ To see this rearrange (4.2.16) given the inequalities of A.4.

¹⁴⁵ $g'_z(m) \geq m \Leftrightarrow m \leq \bar{m}_z$ and $g'_z(m) \leq m \Leftrightarrow m \geq \bar{m}_z$.

¹⁴⁶ A map with a unique non-degenerate critical point being monotone on each side of the critical point.

globally bounded. In fact, a more refined statement can be made concerning the set within which the dynamics of f_z belong, i.e. f'_z globally converges to $P_z \subset I'$, $P_z \neq I'$. The following proposition establishes that P_z as defined by the interval (4.2.16) and (4.2.17) is the trapping set which is strictly contained in I . By Proposition 4.2.1 below, equilibrium sequences are contained in $P_z \subset I'$ and are hence bound away from 0 and 1.

Proposition 4.2.1¹⁴⁷ Let $0 < \eta < \delta < \Delta(\eta)$. Then i. $0 < f_z^2(p^*) < \bar{p}_z < f_z(p^*) < 1$ and $f_z : P_z \rightarrow P_z$ where $P_z = [f_z^2(p^*), f_z(p^*)]$. ii. For any $p_0 \in I'$, there exists some $T \geq 0$ such that for all $t \geq T$, $f_z^t(p_0) \in P_z$.

Proof See Appendix 4.7. ■

Having established Proposition 4.2.1 in which the dynamics are bounded in P_z , the stability and instability of f_z is defined in relation to the parameter $z \in Z$ for exogenously determined parameters α , β and γ . If $|f'_z(\bar{p}_z)| > 1$ then f_z is dynamically unstable and the equilibrium dynamics are contained in P_z . This condition is equivalent to $f'_z(\bar{p}_z) < -1$ which is the case if $1 - \delta\eta^{-1} < -1$ or alternatively $0 < 2\eta < \delta$ in which case f_z is unstable and cycles or erratic price movements are observed as $p_t \in P_z$ for all t and the fixed point is repelling. Conversely, if $0 < \delta \leq 2\eta$ then $|f'_z(\bar{p}_z)| \leq 1$ and the dynamical system is stable and the equilibrium trajectories are attracted to the steady state.

Given that erratic price movements are the object of study it is assumed that:

$$0 < 2\eta < \delta < \Delta(\eta) \quad (4.2.28)^{148}$$

To conclude, as δ increases, or equivalently z increases from $0 < \delta \leq 2\eta$ f_z at first exhibits stable equilibrium sequences. As z increases further such that $0 < 2\eta < \delta$ the eigenvalue of f_z at the fixed point passes through the unit circle, 2 cycles (or higher order cycles) or erratic fluctuations come into existence. This suggests that a higher money growth rate destabilises the equilibrium sequence of real money balances whilst a lower money growth rate stabilises the equilibrium sequence of real money balances. Finally, irrespective of whether f_z is stable or unstable, the equilibrium dynamics are bound within the set $P_z \subset I'$

¹⁴⁷ Proposition 4.2.1 is similar to Matsuyama, Proposition 3 pg 10 [59].

¹⁴⁸ It is noted that if $0 < 2\eta < \delta < \Delta(\eta)$ then the set of points $p \in I'$ such that $f'_z(p)$ converges has Lebesgue measure zero (Proposition 4, Matsuyama pg. 11 [59]) and cycles or erratic equilibrium price movements are typical.

and in the case of instability the equilibrium dynamics are bounded within the set $P_z \setminus \{\bar{p}_z\} \subset I' \setminus \{\bar{p}_z\}$.

To summarise this section, the dynamical systems (g_z, M_z) and (f_z, P_z) were shown to be topologically conjugate unimodal maps of the interval which define the equilibrium dynamical systems in the equilibrium money demand space and equilibrium price space respectively. Both maps are parameterized by the money supply growth rate $z \in Z$, have a unique critical point and negative Schwarzian derivative. The dynamical system (f_z, P_z) was then taken as the object of study. Given the exogenously determined parameters α , β , and γ the parameterization by z governs the stability or instability of f_z ; lower values of z are associated with a stable systems whilst higher values of z are associated with an unstable system.

4.3. Ergodic Chaos

In Section 4.2 the dynamical equilibrium system (f_z, P_z) was derived and the structure of the map was discussed. The evolution of the dynamical system is discussed in this section. In particular, the object of study is the properties of the ergodically chaotic equilibria and their relation to the fixed point are of interest. This section is set out as follows. First, ergodic chaos is discussed in a broad sense. Second, the relation between the expected value of f_z with respect to the value of the steady state is obtained under the *assumption* that equilibria are ergodic and chaotic. Finally, this assumption is justified by showing that there is a subset of $\bar{Z} \subset Z$ of positive measure for which (f_z, P_z) is ergodically chaotic.

Let $\bar{Z} \subset Z$ be the set of z for which (f_z, P_z) is ergodically chaotic¹⁴⁹ and let Σ_{f_z} be the set of all Lebesgue measurable subsets of P_z derived from f_z for some $z \in \bar{Z}$. Let μ_{f_z} be the measure defined on the measurable space (P_z, Σ_{f_z}) with $\mu_{f_z}(P_z) = 1$. $(P_z, \Sigma_{f_z}, \mu_{f_z})$ is constitutes a probability space. f_z is said to be ergodic on the interval P_z if:

C. 1. μ_{f_z} is absolutely continuous with respect to Lebesgue measure on M_z .

C. 2. For every $A \in \Sigma_{f_z}$, $\mu_{f_z}(f_z^{-1}(A)) = \mu_{f_z}(A)$.¹⁵⁰

¹⁴⁹ It is noted that $\bar{Z}$ is defined relative to β as β is the lower bound of Z and f_z is parameterized by z/β for a given β .

¹⁵⁰ This implies that the dynamical system cannot be decomposed into two subsystems which contain the dynamical process.

C. 3. For any $A \in \Sigma_{f_z}$ such that $f_z^{-1}(A) = A$, then either $\mu_{f_z}(A) = 0$ or $\mu_{f_z}(A) = 1$.

C. 4. For every μ_{f_z} -integrable real valued function l ;

$$\langle l(p_0) \rangle = \lim_{T \rightarrow \infty} T^{-1} \sum_{n=1}^T l(f_z^n(p_0)) \rightarrow \int_{P_z} l(p) \varphi_z(p) dp \quad (4.3.1)$$

for μ_{f_z} -a.e. initial condition $p_0 \in P_z$.

If μ_{f_z} is the unique probability measure which satisfies C.1. – C.4, then μ_{f_z} is the ergodic measure of the dynamical system (f_z, P_z) . Let $B \in \Sigma_{f_z}$ and consider a typical trajectory $\{p_t\}_{t=0}^\infty$, i.e. for a randomly chosen starting value. If (f_z, P_z) is an ergodic system the question arises as to how regularly does a typical trajectory visit the set B ? The characteristic function of points in the trajectory given a set A is:

$$\chi_B(f_z^t(p)) = \begin{cases} 1, & f_z^t(p) \in B \\ 0, & f_z^t(p) \notin B \end{cases}$$

Summing the number of times the trajectory enters the set A it is expected that this number is infinite if $\mu_{f_z}(B) > 0$ whereas if the average number of visits to A is finite then $\mu_{f_z}(B) = 0$.

The measure μ_{f_z} is thus invariant to f_z , i.e. does not change as the system evolves. (4.3.1) is the Birkhoff Mean Ergodic Theorem which implies that a typical trajectory will visit every measurable set in proportion to its measure¹⁵¹:

$$\lim_{T \rightarrow +\infty} T^{-1} \sum_{t=1}^T \chi_B(f_z^t(p)) = \mu_{f_z}(B) \text{ for almost all } p \in P_z.$$

Denote $\varphi_z(p)$ is the invariant density function for $z \in \bar{Z}$ and $\langle p_z \rangle = \lim_{T \rightarrow \infty} T^{-1} \sum_{n=1}^T f_z^n(p)$ as the time average of a trajectory. By the Birkhoff Mean Ergodic Theorem the time average is equal to the space average:

$$\langle p_z \rangle = \lim_{T \rightarrow \infty} T^{-1} \sum_{n=1}^T f_z^n(p_0) \rightarrow \int_{P_z} p \varphi_z(p) dp = E[p_z] \quad (4.3.2)$$

where $E[p_z]$ is the expected value of p and depends on $z \in \bar{Z}$. It is noted that if (f_z, P_z) is ergodically chaotic then the support of $\varphi_z(p)$ is non-negligible in the sense of Lebesgue measure and ergodic chaos is observable^{152 153}. It is also noted that the density function is

¹⁵¹ Day and Pianigiani [28] pg. 43.

¹⁵² This can be contrasted with topological chaos in which the set over which the topologically chaotic dynamics occur may be negligible in the sense of Lebesgue measure and thereby non-observable.

¹⁵³ Grandmont [39].

stable and does not depend on the choice of initial values p_0 ¹⁵⁴ (it is then said that the distribution is absolutely continuous and invariant with respect to the Lebesgue measure).

For the next part it is assumed that f_z is ergodically chaotic and thereby has density function $\varphi_z(p)$. This assumption is justified subsequently where it is shown that ergodic chaos does exist for a non-negligible subset of Z .

The following theorem defines a relationship between $E[p_z]$ and $\bar{p}_z$ for each $z \in \bar{Z}$.

Theorem 4.3.1¹⁵⁵ Let (f_z, P_z) be the dynamical system defined by (4.2.12). Suppose that (f_z, P_z) satisfies properties A.1 – A.5, has negative Schwarzian derivative and is ergodically chaotic. Then $E[p_z] < \bar{p}_z$ for all $z \in \bar{Z}$.

Proof See Appendix 4.7. ■

It is noteworthy that in establishing Theorem 4.3.1, no use was made of the strict concavity (or lack thereof) of the map f_z . Indeed, f_z need not be strictly concave for various choices of z , β and η yet Theorem 4.3.3 still holds¹⁵⁶. Theorem 3.1 will come into play in Section 4.4 when the analysis turns to the question of welfare.

In establishing Theorem 4.3.1, the question is begged whether ergodic chaos exists and whether it is a robust occurrence? That is to say, for some $\eta \in (0,1]$, is f_z ergodically chaotic for a non-negligible set of z in $\bar{Z}$? In order to examine the set of parameters $z \in Z$ for which $f_z : P_z \rightarrow P_z$ exhibits ergodic chaos, the following definition is stated:

Definition 4.3.1 Let $f \in C^3$. The Schwarzian derivative of f denoted $Sf(x)$ is defined as:

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2} \left[\frac{f''(x)}{f'(x)} \right]^2, \text{ s.t. } f'(x) \neq 0.$$

¹⁵⁴ Since the distribution is invariant the choice of initial value p_0 has no bearing in the determination of $\langle p_z \rangle$.

¹⁵⁵ Theorem 4.3.2 utilises the idea of Theorem 2 of Huang [43] in which it is shown that for a one-dimensional discrete time tatonnement process that is ergodically chaotic the long-run average inflation rate is positive.

¹⁵⁶ This differs to the result of Chapter 3 of this thesis in which the strict concavity of the dynamical system played a fundamental role in the obtaining of the result of that paper. In this sense the results of this paper are more general than those of Chapter 3 but such generality is obtained at the price of having to specify a utility function.

The following theorem reports sufficient conditions under which $f_z : P_z \rightarrow P_z$ exhibits ergodic chaos.

Theorem 4.3.2 (Grandmont [39] Theorem D.1.8 pg. 63) Let $f_z : P_z \rightarrow P_z$, $f_z \in C^3$ be an interval map which has a negative Schwarzian derivative $Sf_z(p) < 0$, $\forall p \in P_z$, $p \neq p^*$ and satisfies Property A.2 above. If there exists $k \geq 2$ such that $f_z^k(p^*) = \bar{p}_z$ is an unstable fixed point of f_z , i.e. $f_z(\bar{p}_z) = \bar{p}_z$, $|f'_z(\bar{p}_z)| > 1$, then f_z has a unique ergodic invariant measure that is absolutely continuous with respect to the Lebesgue measure. ■

Theorem 4.3.2 states that if f_z is a unimodal map that satisfies A.2 and if the choice of z (i.e. monetary policy) is such that the critical point falls on an unstable cycle, then f_z exhibits ergodic chaos. In order to utilise Theorem 4.3.1 it needs to be established that f_z has a negative Schwarzian derivative.

Lemma 4.3.1 If $\eta \in (0,1]$ then $f_z : P_z \rightarrow P_z$ has a negative Schwarzian derivative.

Proof See Appendix 4.7. ■

In order show that ergodic chaos exists for a non-negligible set of parameters, Definition 4.3.2 below defines a class of maps to which f_z belongs to which a Theorem of Jakobson [48] can be applied.

Definition 4.3.2. Quadratic Family of Maps The quadratic family of maps is a map which is a smooth function $f(x) : x \rightarrow \lambda x(1-x)$ where $0 \leq \lambda \leq 4$ and $f_\lambda(x) = \lambda f(x)$ with $f(x)$ sufficiently close to $x(1-x)$ in $C^3([0,1],[0,1])$.

Jakobson [48] considered the set of parameter values λ for which the class of quadratic maps f_λ of Definition 4.2.4 have an invariant measure absolutely continuous with respect to Lebesgue measure. Jakobson (Theorem B pg. 40 [48]) states the following Theorem.

Theorem 4.3.3. Jakobson (Theorem B pg. 40 [48]) Let $f_\lambda(x)$ be one of the maps in Definition 3.2. Then there is a positive measure Λ such that for $\lambda \in \Lambda$, f_λ admits an invariant measure μ_λ . ■

Jakobson remarks¹⁵⁷ that if f is a map which satisfies $f(x):[0,1] \rightarrow [0,1]$, $f(0)=f(1)=0$, $f'(c)=0$ (where $c \in (0,1)$ is the non-degenerate critical point), belong to a sufficiently small C^3 neighbourhood of $x(1-x)$. Then for a family of maps $\lambda f(x)$ there exists some λ_0 sufficiently close to 4 such that $\lambda_0 f(c)=1$. Considering some $\lambda \in [\lambda_0 - \varepsilon, \lambda_0]$, for the corresponding map, Theorem B holds. [...] for a family of maps $f_\lambda(x) = \lambda f(x)$ that are unimodal with $f(x):[0,1] \rightarrow [0,1]$ and $f(0)=f(1)=0$, having negative Schwarzian derivative, and for λ_0 such that $\lambda_0 f(c)$ falls into an unstable periodic orbit [...] f_λ admits an absolutely invariant continuous measure and λ_0 is a Lebesgue density point of this set.

As a matter of applying Theorem 4.3.3 in light of the above remark, it is noted that f_z belongs to the family of quadratic maps in Definition 4.3.2, is a unimodal interval map of the unit interval with $f_z:[0,1] \rightarrow [0,1]$ and $f_z(0)=f_z(1)=0$ and with non-degenerate critical point as well as possessing a negative Schwarzian derivative for all $\eta \in (0,1]$. Suppose that z_0 is a money supply rule and let n_0 be an integer such that $f_{z_0}^{n_0}(p^*) - \bar{p}_{z_0} = 0$, and $|f'_z(\bar{p}_{z_0})| > 1$ thereby satisfying Theorem 4.3.2. Then, by Theorem 4.3.3, there exists $\hat{\rho} > 0$ such that for any $\varepsilon > 0$ there exists the set of one-sided Lebesgue points of $\bar{Z}$ for which f_z admits an absolutely continuous and invariant measure and is ergodically chaotic:

$$\hat{\lambda}\{z \in \bar{Z} : z_0 \geq z > z_0 - \hat{\rho}\} > \hat{\rho}(1 - \varepsilon)$$

where $\hat{\lambda}$ is the Lebesgue measure^{158 159}. Consequently, if there is some $z \in Z$ such that f_z is ergodically chaotic then there exists a non-negligible subset of Z such that every z in this set belongs to $\bar{Z}$. Alternatively if $z \in \bar{Z}$, then z can be perturbed slightly to z' with $z' < z$ and $z' \in \bar{Z}$.

As a matter of establishing that $\bar{Z}$ is non-empty let $\eta = 1$ and $z\beta^{-1} = 4$. Then f_z is ergodically chaotic [24] and by Theorem 4.3.3 $\bar{Z}$ has a non-negligible measure. Furthermore, since f_z is a smooth function of η for $\eta \in (0,1]$, and since $p(1-p)^{\frac{1}{\eta}}$ is close to $p(1-p)$ for η

¹⁵⁷ Remark XIII/5 pg. 87 [48]. See also Boldrin *et al.* [17].

¹⁵⁸ See de Melo and van Strien [64] pg. 390.

¹⁵⁹ See Boldrin *et al.* [17] pg. 19 for application of Theorem 4.3.3 in a model in which equilibrium is defined by the dynamic accumulation of capital in an ergodically chaotic equilibrium path.

close to 1 then there is an open and dense set of η such that f_z is ergodically chaotic. Other parameters values for which f_z is ergodically chaotic are exhibited in Section 4.5.

Corollary 4.3.1 Let $0 < 2\eta < \delta < \Delta(\eta)$ and $\eta \in (0,1]$. Assume that $z \in \bar{Z}$. Then,

- i. For every $z \in \bar{Z}$, $E[p_z] < \bar{p}_z$.
- ii. $\bar{Z}$ is a non-negligible set for which $f_z : P_z \rightarrow P_z$ is ergodically chaotic. Furthermore there is an open and dense set of η for which $f_z : P_z \rightarrow P_z$ is ergodically chaotic for some $z \in \bar{Z}$.

Proof Part i follows from Proposition 4.2.2. Part ii follows from Theorem 4.2.3 and the subsequent discussion. ■

Corollary 4.3.1 establishes that *ergodic chaos is robust over non-negligible sets of Z* and $\eta \in (0,1]$. From the viewpoint of observability of chaos, this result bodes well. After all, the principle reason for analysing ergodic chaos rather than topological chaos is due to the argument that topological chaos is found lacking inasmuch as the set of points over which chaotic dynamics are observed may be negligible in a Lebesgue measure sense^{160 161} and hence unobservable. Instead, ergodic chaos, is characterised by an absolutely invariant continuous distribution with support of non-negligible measure and is in this generates sense a physical measure. This notion of chaos is thus observable in that the set of points over which the chaotic dynamics are defined has a positive measure and the resultant probability distribution is both unique and stable. Given the robustness of the dynamical system with respect to the existence of ergodic chaos, one must necessarily take into account the manner in which monetary policy impinges upon the behaviour of the economic system and in particular the effect that monetary policy has on welfare if ergodic chaos is present¹⁶². This is the argument of Section 4.4.

4.4. Welfare Analysis

The welfare implications of ergodic chaos are now evaluated. It is found that for every $z \in \bar{Z}$ there exists a non-negligible set of $z' \in \tilde{Z} \subset Z$ with $z' < z$ and $\bar{Z} \cap \tilde{Z} = \emptyset$ such that *lifetime*

¹⁶⁰ See Appendix 3.8 in Chapter 3 for a discussion of the differences between topological and ergodic chaos.

¹⁶¹ See Grandmont pg. 66 [39].

¹⁶² If instead chaos were unobservable occurring rarely then monetary policy would not have any bearing as such a policy would be operating on a set of negligible measure and thereby redundant in a measure theoretic sense.

indirect utility at the steady state of z' is strictly greater than the expected lifetime indirect utility given z .

In equilibrium $c_t = y$, $\forall t$ and $m_t^d = m_t$ $\forall t$ where the sequence $\{m_t\}_{t=0}^{\infty}$ is determined by (4.2.4). Indirect utility along the equilibrium path each t is:

$$u(y, m) = \frac{(y^\alpha m^{(1-\alpha)})^{(1-\gamma)}}{1-\gamma} = \frac{-y^{\alpha(1-\gamma)} m^{(1-\alpha)(1-\gamma)}}{(\gamma-1)} \quad (4.4.1)$$

where $\gamma > 1$ (as $\eta \in (0, 1]$ ¹⁶³) and $\alpha \in (0, 1)$. By definition $\eta = (1-\alpha)(\gamma-1)-1$, so $-(\eta+1) = (1-\alpha)(1-\gamma)$. Hence write (4.4.1) as:

$$\begin{aligned} u(y, m) &= \frac{-y^{\alpha(1-\gamma)}}{(\gamma-1)} (AA^{-1}m)^{-(\eta+1)} = \frac{-y^{\alpha(1-\gamma)} A^{-(\eta+1)}}{(\gamma-1)} (A^{-1}m)^{-(\eta+1)} \\ &= -B(Am^{-1})^{(\eta+1)} \end{aligned} \quad (4.4.2)$$

where $B \equiv (y^{\alpha(1-\gamma)} A^{-(\eta+1)}) (\gamma-1)^{-1} > 0$ and $A = (1-\alpha)\alpha^{-1}y > 0$ and is independent of z , $u(m)$ is a negative function increasing in m as well as strictly concave. Thus higher equilibrium money demand yields a higher level of indirect utility. Applying now the transformation $p = h(m) = Am^{-1}$ to (4.4.2):

$$u(y, h(m)) = -Bp^{(\eta+1)} \quad (4.4.3)$$

By applying the transformation h , $u(h(m))$ becomes a negative function, concave and decreasing in p ; a higher equilibrium price level is associated with a lower level of indirect utility.

Suppose that $z \in \bar{Z}$. By Theorem 4.3.1 $E[p_z] < \bar{p}_z$ and by (4.4.3):

$$\begin{aligned} u\left(\int_{p_z} p \varphi_z(p) dp\right) &= u(E[p_z]) \\ &= -B(E[p_z])^{(\eta+1)} > -B(\bar{p}_z)^{(\eta+1)} \\ &= u(\bar{p}_z) \end{aligned}$$

Hence $u(E[p_z]) > u(\bar{p}_z)$. Lifetime indirect utility of expected price is:

$$\sum_{t=0}^{\infty} \beta^t u(E[p_z]) = (1-\beta)^{-1} u(E[p_z]) \quad (4.4.4)$$

Lifetime indirect utility at the steady state is:

¹⁶³ $(\gamma-1)(1-\alpha)-1 > 0 \Leftrightarrow (\gamma-1) > (1-\alpha)^{-1} \Leftrightarrow \gamma > 1+(1-\alpha)^{-1} > 1, \forall \alpha \in (0, 1)$.

$$\sum_{t=0}^{\infty} \beta^t u(\bar{p}_z) = (1 - \beta)^{-1} u(\bar{p}_z) \quad (4.4.5)$$

By the fact that $u(E[p_z]) > u(\bar{p}_z)$, (4.4.4) and (4.4.5) imply:

$$(1 - \beta)^{-1} u(E[p_z]) > (1 - \beta)^{-1} u(\bar{p}_z) \quad (4.4.6)$$

Turning now to expected lifetime utility. Since the equilibrium sequence of prices is chaotic it is unpredictable. Hence the best measure of the welfare associated with the behaviour of the equilibrium price is the expected indirect utility along an equilibrium price path. This measure is thus derived.

Consider two periods t and $t+1$. The equilibrium real price level at t is p_t whereas the equilibrium price level at $t+1$ is p_{t+1} which is a function of the present equilibrium price level p_t inflated by the discount rate $\beta \in (0,1)$, i.e. $p_{t+1} = f_z(\beta; p_t) = (z\beta^{-1})^{\frac{1}{\eta}} (p_t)(1 - p_t)^{\frac{1}{\eta}}$. Hence indirect utility in equilibrium at time t is simply $\beta^t u(p_t)$ and indirect utility in equilibrium at time $t+1$ is the discounted utility of a function of inflated present period equilibrium real price:

$$\beta^{t+1} u(p_{t+1}) = \beta^{t+1} u(f_z(\beta; p_t)) = \beta^{t+1} u((z\beta^{-1})^{\frac{1}{\eta}} (p_t)(1 - p_t)^{\frac{1}{\eta}})$$

The sum of indirect utilities over time periods t and $t+1$ at time t is therefore:

$$\beta^t u(p_t) + \beta^{t+1} u(f_z(\beta; p_t))$$

Summing indirect utility in this manner is consistent with the manner in which a rational agent views the sequence of future real prices and discounts the utility of such prices with respect to the present period. Thus, the substitution of expression (4.2.17) into the sequence of discounted utility over the infinite horizon generates lifetime indirect utility; $\beta^0 u(y, p_0) + \beta u(y, p_1) + \beta^2 u(y, p_2) + \dots$. Taking the expectation of discounted utility over the infinite horizon yields:

$$\begin{aligned} E \left[\sum_{t=0}^{\infty} \beta^t u(y, p_t) \right] &= E \left[\beta^0 u(y, p_0) + \beta u(y, p_1) + \beta^2 u(y, p_2) + \dots \right] \\ &= E \left[\beta^0 u(p_0) + \beta u(f_z(p_0)) + \beta^2 u(f_z^2(p_0)) + \dots \right] \\ &= \int_{p_z} \left(\beta^0 u(p_0) + \beta u(f_z(p_0)) + \beta^2 u(f_z^2(p_0)) + \dots \right) \varphi_z(p) dp \end{aligned} \quad (4.4.7)$$

$$= \int_{p_z} \left(\beta^0 u(p) + \beta u(p) + \beta^2 u(p) + \dots \right) \varphi_z(p) dp \quad (4.4.8)$$

$$= (1 - \beta)^{-1} \int_{p_z} u(p) \varphi_z(p) dp \quad (4.4.9)$$

where the equality in (4.4.8) comes from the fact that by the Birkhoff Ergodic Theorem ergodicity implies that $\int_{P_z} f_z(p) \varphi_z(p) dp = \int_{P_z} p \varphi_z(p) dp$ hence $\int_{P_z} u(f_z(p)) \varphi_z(p) dp = \int_{P_z} u(p) \varphi_z(p) dp$. To compare (4.4.9) to (4.4.5) it is noted that since $u(\cdot)$ is strictly concave, Jensen's inequality implies that $\int_{P_z} u(p) \varphi_z(p) dp < u\left(\int_{P_z} p \varphi_z(p) dp\right)$ or equivalently $E[u(p_z)] < u(E[p_z])$. Thereby it follows that:

$$(1 - \beta)^{-1} E[u(p_z)] < (1 - \beta)^{-1} u(E[p_z]) \quad (4.4.10)$$

A comparison between (4.4.6) and (4.4.10) reveals that the value of $(1 - \beta)^{-1} E[u(p_z)]$ relative to the value of $(1 - \beta)^{-1} u(\bar{p}_z)$ is indeterminate in that one cannot state whether lifetime expected utility is greater than or less than lifetime utility at the steady state. Two cases thus present:

$$(1 - \beta)^{-1} u(\bar{p}_z) < (1 - \beta)^{-1} E[u(p_z)] \quad (4.4.11)$$

$$(1 - \beta)^{-1} E[u(p_z)] < (1 - \beta)^{-1} u(\bar{p}_z) \quad (4.4.12)$$

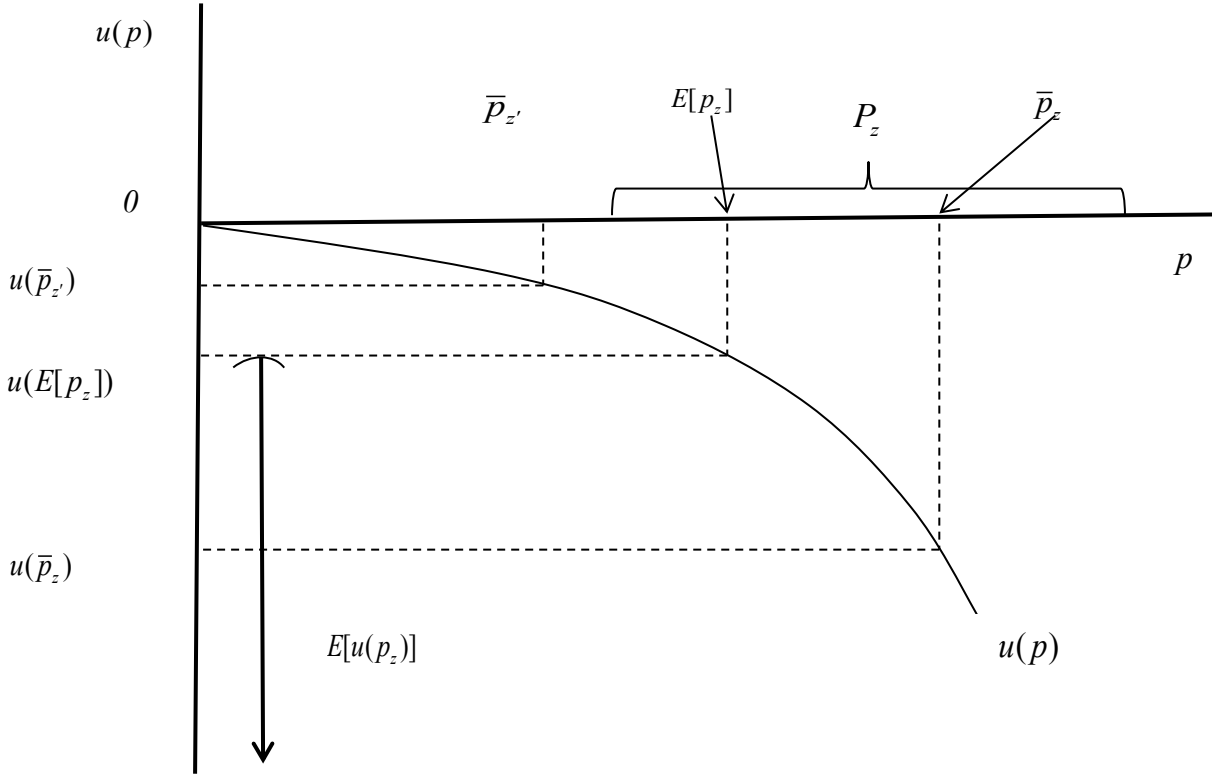
This indeterminacy does not however imply that the role of monetary policy is also indeterminate. Suppose that $z \in \bar{Z}$ where, by Corollary 4.3.1.ii, $\bar{Z}$ is a set of non-negligible Lebesgue measure. Suppose also that there exists a subset $\tilde{Z}$ of Z with $z' \in \tilde{Z}$ and $z' < z$ for which $f_{z'}$ is dynamically stable such that $u(\bar{p}_{z'}) > u(E[p_z])$. Since $u(E[p_z]) > E[u(p_z)]$ and $u(E[p_z]) > u(\bar{p}_z)$ by (4.4.6) and (4.4.10) respectively, by (4.4.15) and (4.4.16) it then follows that either:

$$u(\bar{p}_{z'}) > u(E[p_z]) > E[u(p_z)] > u(\bar{p}_z) \text{ or} \quad (4.4.13)$$

$$u(\bar{p}_{z'}) > u(E[p_z]) > u(\bar{p}_z) > E[u(p_z)] \quad (4.4.14)$$

In either case, the monetary policy z' generates a level of lifetime utility which is strictly greater than expected lifetime utility of z and the lifetime utility at the monetary steady state associated with z . Figure 4.4.1 illustrates these principles. It is noted that since $\bar{Z}$ is a compact set, it is not difficult to characterise the optimal monetary policy as the choice of $z \in \bar{Z}$ such that expected lifetime indirect utility is maximised.

Figure 4.4.1



To show that there exists a set $\tilde{Z}$ which contains some z' such that either (4.4.13) or (4.4.14) hold, it is recalled that $\bar{p}_z = 1 - \beta z^{-1}$ and $\partial_z \bar{p}_z = \beta z^{-2} > 0$ hence the steady state is an increasing function of z . The gradient at the steady state is given by (4.2.20); $f'_z(\bar{p}_z) = 1 - (\delta/\eta)$ where $\partial_z f'_z(\bar{p}_z) = -(\beta\eta)^{-1} < 0$ and is a decreasing function of z . Indirect utility at the steady state is:

$$\begin{aligned} u(\bar{p}_z) &= -B(\bar{p}_z)^{(\eta+1)} \\ &= -B(1 - \beta z^{-1})^{(\eta+1)} \end{aligned} \quad (4.4.15)$$

where $\partial_z u(\bar{p}_z) = -\beta B(1 + \eta)z^{-2}(1 - \beta z^{-1})^{(\eta+1)} < 0$ and is hence a decreasing function of z . Thus lower levels of z , i.e. a contractionary monetary policy, are associated with a lower monetary steady state which in turn is associated with a higher level of indirect utility. Furthermore, since $\partial_z f'_z(\bar{p}_z) < 0$ a fall in z increases $f'_z(\bar{p}_z)$ and thereby renders stable the dynamical system f_z ¹⁶⁴.

The programme for this next part is to show that there exists a non-negligible subset $\tilde{Z}$ of Z such that for any $z \in \tilde{Z}$ and any $z \in \bar{Z}$:

¹⁶⁴ It is recalled that $f'_z(\bar{p}_z) < 0$.

$$\bar{p}_{z'} < E[p_z] \Leftrightarrow u(\bar{p}_{z'}) > E[u(p_z)] \quad (4.4.16)$$

(4.4.16) implies either (4.4.13) or (4.4.14). Recall that:

- i. If $0 < \eta < \delta \leq 2\eta$ then p_t converges to p_z .
- ii. If $0 < 2\eta < \delta$ then p_t exhibits bounded fluctuations in P_z .

The following lemma shows that if $z \in \bar{Z}$ then $E[p_z] \in (p'_z, p''_z)$

Lemma 4.4.1 Let $z \in \bar{Z}$ then $E[p_z] \in (p'_z, p''_z)$ and $p'_z < E[p_z] < \bar{p}_z < p''_z$

Proof See Appendix 4.7. ■

Proposition 4.4.1 can now be proven.

Proposition 4.4.1 There exists a subset $\tilde{Z}$ of z which excludes $\bar{Z}$ such that:

- i. For every $z' \in \tilde{Z}$ and $z \in \bar{Z}$; $u(\bar{p}_{z'}) > E[u(p_z)]$.
- ii. $\tilde{Z}$ has positive Lebesgue measure.
- iii. For every $z' \in \tilde{Z}$, $|f_{z'}(\bar{p}_{z'})| < 1$.

Proof See Appendix 4.7. ■

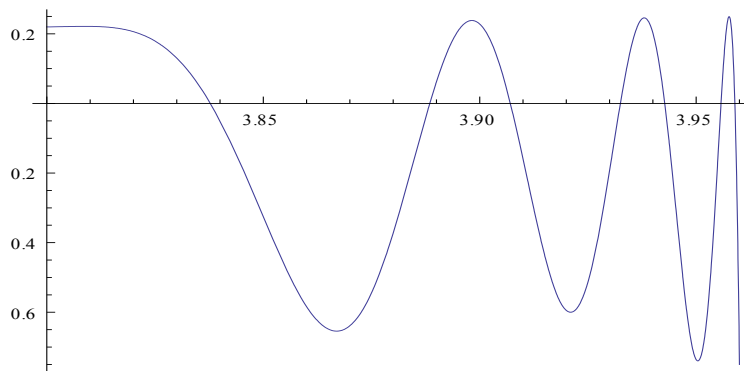
Proposition 4.4.1 can be extended to the discounted infinite horizon utility function. Therefore, from the point of view of policy, whilst ergodic chaos may or may not be associated with a higher level of expected lifetime utility than the lifetime utility at the steady state, for a non-negligible set $\bar{Z}$ there exists a non-negligible set of money growth rates which are associated with a stable dynamical system and which yields a higher level of indirect utility. This implies that chaos can be improved upon by choosing a money growth rate that is sufficiently low.

4.5. Numerical Analysis

In order to examine the dynamical system Theorem 4.3.2 is utilised to obtain values of $z \in \bar{Z}$ for which g_z is ergodically chaotic. These values are contrasted amongst each other and with their respective steady states as well as the values of indirect utility and expected utility.

Let $\eta = 1$. $f_z : P_z \rightarrow P_z$ is then defined by $p_{t+1} = f_z(p_t) = (z/\beta)p_t(1-p_t)$. Assume further that $\beta = 0.99$, $\gamma = 1$ and $\alpha = 0.3$ ¹⁶⁵. The fixed point is $\bar{p}_z = (z - 0.99)z^{-1}$ and the critical point is $p^* = 0.5$ for every $z \in Z$. The Schwarzian Derivative is negative for every $z \in Z$ and every $p \neq p^*$ by Lemma 4.3.1¹⁶⁶. Such that f_z is dynamically unstable, by Corollary 4.3.1.i, it is required that $0 < 2\eta < \delta \leq \Delta(\eta)$. Given that $\eta = 1$, $\Delta(\eta) = 3$ then $2.97 < z \leq 3.96$. Conversely, if $0 < \delta < 2\eta = 2$ or $0 < z < 2.97$ then f_z is dynamically stable and $\bar{p}_z$ is reached either monotonically or in an oscillatory manner. It is noted that $Z = (0.99, 3.96]$ and $\bar{Z} = (2.97, 3.96]$. In order to obtain those $z \in Z$ for which f_z is ergodically chaotic, by Theorem 4.3.2 it is required to find some $k \geq 2$ such that $f_z^k(p^*) = \bar{p}_z$ is an unstable fixed point, i.e. $|f'_z(\bar{p}_z)| > 1$. Figure 4.4.1 shows the plot of $f_z^k(p^*) - \bar{p}_z$ for $k = 7$ as a function of z . The resultant function $f_z^k(p^*) - \bar{p}_z : Z \rightarrow \mathbb{R}$ is a polynomial of order 128. The set of z for which the polynomial obtains its zeroes is a subset of $z \in \bar{Z}$.

Figure 4.5.1



¹⁶⁵ Any value of $\beta \in (0,1)$ can be chosen as it is the *ratio* of z and β that parameterizes f_z . Hence chaos can be generated with the imposition of reasonable discount parameter values.

¹⁶⁶ Calculation shows that $Sf_z(p) = -7.40741(1.1111 - 2.2222p)^{-2}$.

The following table presents some approximations¹⁶⁷ of the polynomial to its zeroes.

Table 4.5.1

$z_1 = 3.64178777$	$\bar{p}_{z_1} = 0.728155$	$E[\bar{p}_{z_1}] = 0.671875$
$z_2 = 3.7531858$	$\bar{p}_{z_2} = 0.7362241$	$E[\bar{p}_{z_2}] = 0.649376$
$z_3 = 3.837774284$	$\bar{p}_{z_3} = 0.742038$	$E[\bar{p}_{z_3}] = 0.60753836$
$z_4 = 3.884596317$	$\bar{p}_{z_4} = 0.7451473$	$E[\bar{p}_{z_4}] = 0.5894378$
$z_5 = 3.90708417$	$\bar{p}_{z_5} = 0.7466141$	$E[\bar{p}_{z_5}] = 0.5940590$
$z_6 = 3.9324889$	$\bar{p}_{z_6} = 0.748251038$	$E[\bar{p}_{z_6}] = 0.5549860$

For each z_i in Table 4.5.1 $E[p_z] < \bar{p}_z$ in accordance with Theorem 4.3.1. Whilst the set of z_i in Table 4.5.1 constitutes a discrete set and is hence of measure zero, by Theorem 4.3.3 and Corollary 4.3.1 there is a set of z in the vicinity of each z_i , $i=1,...,6$ for which f_z is ergodically chaotic. Therefore, $\bar{Z}$ has non-negligible Lebesgue measure. Given the utility function (4.4.3) and the specified parameters then $A = 2\frac{1}{3}$, $B = 9/140$ and $u(p) = -(9/140)p^2$. Table 4.5.2 reports the expected indirect utility of prices, indirect utility of expected prices and indirect utility of prices at the steady state.

Table 4.5.2

	$E[u(p_{z_i})]$	$u(E[p_{z_i}])$	$u(\bar{p}_{z_i})$
$z_1 = 3.64178777$	-0.0314260	-0.02902	-0.03408
$z_2 = 3.7531858$	-0.0307270	-0.0271088	-0.0348445
$z_3 = 3.837774284$	-0.0289686	-0.0237316	-0.0353970
$z_4 = 3.884596317$	-0.0282405	-0.0223352	-0.0356943
$z_5 = 3.90708417$	-0.0285118	-0.0226868	-0.0358350
$z_6 = 3.9324889$	-0.0267045	-0.0198006	-0.0359923

Table 4.5.3 reports lifetime expected indirect utility, lifetime indirect utility of expected prices and indirect lifetime utility of prices at the steady state as well as the empirical sum of discounted indirect utility of prices. It is noteworthy that in Table 4.5.3:

¹⁶⁷ By way of the following remark below, the fact that z yields an approximation of $f_z^k(p^*) - \bar{p}_z = 0$ is inconsequential as by Theorem 4.3.3, ergodic chaos is structurally robust from which it may be concluded that within a small distance of an approximation there is some z' for which f_z is ergodically chaotic.

$$(1-\beta)^{-1}u(\bar{p}_{z_i}) < (1-\beta)^{-1}E[u(p_{z_i})] < (1-\beta)^{-1}u(E[p_{z_i}]), \forall i.$$

Hence, for the few observations reported, expected lifetime utility is higher than lifetime utility at the steady state

Table 4.5.3

	$(1-\beta)^{-1}E[u(p_{z_i})]$	$(1-\beta)^{-1}u(E[p_{z_i}])$	$(1-\beta)^{-1}u(\bar{p}_{z_i})$	$\sum_{t=0}^{\infty} \beta^t u(f_{z_i}^t(p))$
$z_1 = 3.64178777$	-3.14260	-2.902	-3.408	-3.0749417
$z_2 = 3.7531858$	-3.07270	-2.71088	-3.48445	-3.0626409
$z_3 = 3.837774284$	-2.89686	-2.37316	-3.53970	-2.8863443
$z_4 = 3.884596317$	-2.82405	-2.23352	-3.56943	-2.7984181
$z_5 = 3.90708417$	-2.85118	-2.26868	-3.58350	-2.8369480
$z_6 = 3.9324889$	-2.67045	-1.98006	-3.59923	-2.5821699

In consideration of Proposition 4.4.1, of interest is the set $\tilde{Z}$ such that for every $z' \in \tilde{Z}$, the dynamical system is stable and the value of lifetime indirect utility is strictly greater than the expected indirect utility for z_i , $i=1, \dots, 6$. To induce stability on the map f_z , it is required that $0 < z < 2.97$. Let $\tilde{Z} = (0.99, 1.4)$, for $z' \in \tilde{Z}$, $|f_{z'}(\bar{p}_{z'})| < 1$. By the fact that $\bar{p}_z$ is a continuous (increasing) function of z the corresponding set of steady state prices is $(0, 41/140)$ and the corresponding interval of indirect utilities is $(1-\beta)^{-1}u((0, 41/140)) = (0, (1-\beta)^{-1}u(41/140)) = (0, 0.005513)$. Thus for any $z' \in (0, 41/140)$:

$$(1-\beta)^{-1}u(\bar{p}_{z'}) > (1-\beta)^{-1}u(E[p_{z_i}]) > (1-\beta)^{-1}E[u(p_{z_i})] > (1-\beta)^{-1}u(\bar{p}_{z_i})$$

which accords with Proposition 4.4.1.

Turning to the question of whether *ergodic chaos is robust* in an analytical sense, it is required to be established that $\bar{Z}$ is composed of a set of non-negligible measure which would imply that *ergodic chaos is a structurally stable phenomenon* and hence observable for a significant set of parameters. In such a case small *deviations in the money growth rate do not disrupt the ergodic nature of the equilibrium dynamics*. The following theorem of Keller reported in [64]¹⁶⁸ is utilised.

¹⁶⁸ Corollary 3.1, pg. 355, [64]

Theorem 4.5.1 Let $f_z \in C^3$, $f_z : P_z \rightarrow P_z$ be a unimodal map M_z with a non-flat critical point p^* that has a negative Schwartzian derivative. Then there exists a constant λ_{f_z} such that for almost all p :

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \log |Df_z^n(p)| = \lambda_{f_z} \quad (4.5.1)$$

Moreover:

- i. $\lambda_{f_z} > 0$ if and only if f_z has an absolutely continuous invariant probability measure. In such a case

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \log |Df_z^n(p)| = \lambda_{f_z} \text{ a.e. } m \quad (4.5.2)$$

- ii. $\lambda_{f_z} < 0$ if and only if f_z has a hyperbolic periodic attractor. ■

λ_{f_z} defines the Lyapunov exponent – a measure of the exponential rate at which orbits which start close to each other separate under action of the map f_z . A positive Lyapunov exponent is indicative of sensitive dependence to initial conditions and hence chaos, whereas a negative Lyapunov exponent is indicative of stability and convergence to a fixed point or periodic orbits [24]. In order to apply Theorem 4.4.1, if f_z has a positive Lyapunov exponent, i.e. (4.5.2) is positive, then there exists an absolutely continuous invariant probability measure and f_z is ergodically chaotic. Figures 4.5.2, 4.5.3, and 4.5.4 report the plot of the Lyapunov exponent for $\eta = 0.9$, $\eta = 0.95$ and $\eta = 1$ respectively against the value of z utilising the relationship

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \log |Df_z^n(m)| = \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=1}^n \log |Df_z^i(m)|.$$

For each η reported below, it appears that there is a set of z of positive measure such that $\lambda_{f_z} > 0$ thereby suggesting that ergodic chaos is structurally stable for a non-negligible set of parameter values $\bar{Z}$.

Figure 4.5.2 $\eta = 0.9$

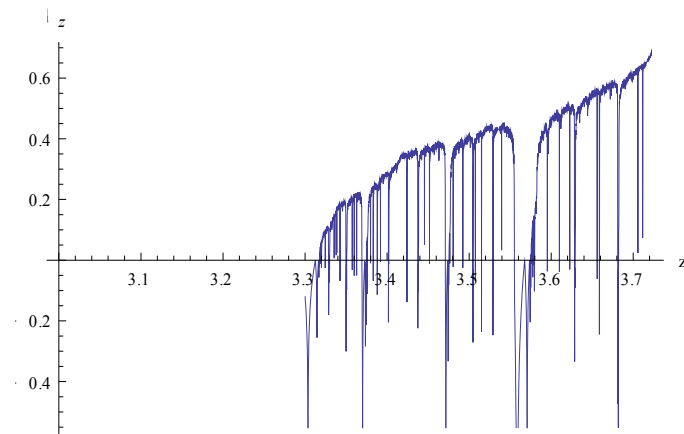


Figure 4.5.3 $\eta = 0.95$

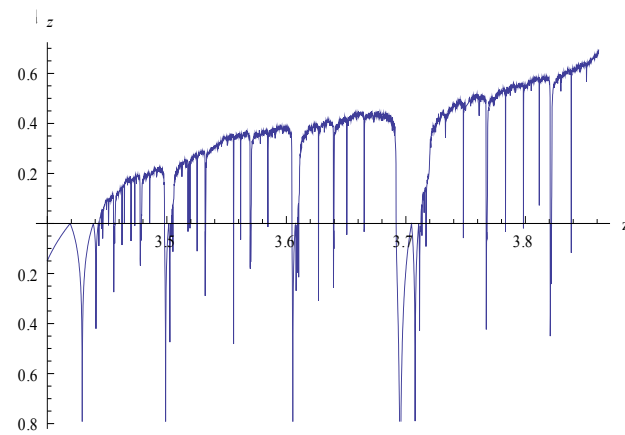
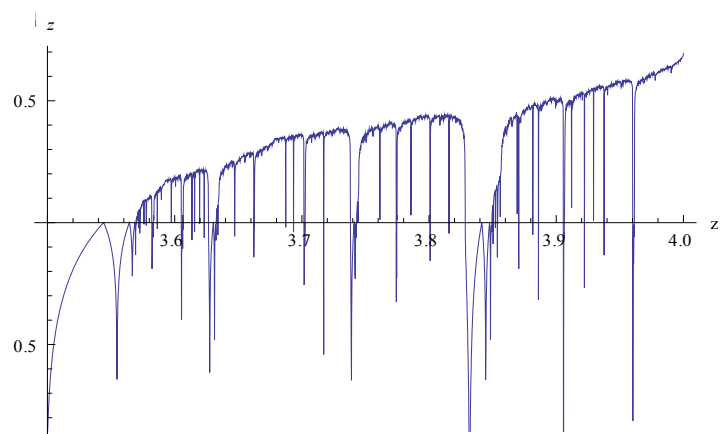


Figure 4.5.4 $\eta = 1$



In Figures 4.5.2 – 4.5.4, for each z such that $\lambda_{f_z} > 0$, there exists an absolutely continuous invariant probability distribution $\varphi_z(p)$ which satisfies $E[p_z] < \bar{p}_z$ and for which $(1-\beta)^{-1}u(\bar{p}_z) < (1-\beta)^{-1}u(E[p_z])$. By Proposition 4.4.1 for each such z there exists a subset $\tilde{Z}$ such that for any $z' \in \tilde{Z}$ one has that $(1-\beta)^{-1}u(E[p_z]) < (1-\beta)^{-1}u(\bar{p}_{z'})$. Furthermore, by inspection of Figures 4.5.2 – 4.5.4 it appears that the set of $\bar{Z}$ for which $\lambda_{f_z} > 0$ has non-negligible Lebesgue measure thereby validating Corollary 4.3.1.ii. It is concluded that ergodic chaos appears to be structurally stable as are the set of stabilising monetary policies which yield a value of lifetime indirect utility which is strictly greater than the lifetime expected utility of prices under ergodic chaotic price sequences.

4.6. Conclusion

A model of an infinite horizon money in the utility function has been analysed. The convex equilibrium map which determines real money balances was transformed into a map which defines the sequence of equilibrium prices which was shown to be a unimodal map of the interval. For certain money supply levels the equilibrium system had the capacity to generate ergodic and chaotic dynamics where these dynamics are characterised by the expected value being strictly less than the value at the fixed point. By application of a Theorem of Jakobsen [48] it was concluded that the set of money growth rates for which this inequality held had Lebesgue measure non-zero. This implied that the set of money growth rate rules which exhibit ergodic chaos cannot be dismissed as unimportant and the implications of ergodic chaos must be considered. To this end a comparison between the expected lifetime utility under ergodic chaos and lifetime utility under a stabilised regime was carried out. It was shown that for every money growth rate rule which supported an ergodic and chaotic equilibrium sequence there existed a non-negligible set of money growth rate rules which stabilised the dynamical system and for which lifetime utility at the steady state was strictly greater than expected lifetime utility. Consequently, a contractionary monetary policy can not only stabilise the equilibrium system but also improve welfare along the stabilised path. Furthermore since there is always a choice of $z' \in \tilde{Z}$ such that $\bar{p}_{z'} < E[p_z] \Leftrightarrow u(\bar{p}_{z'}) > E[u(p_z)]$ with $\bar{p}_{z'} < \inf P_z$ then welfare is always improved in moving from chaos to stability, not simply improved on average.

On a similar note, in the model presented here, this author was unable to establish whether for some given $z \in Z$ lifetime expected utility was greater than or less than lifetime utility at the steady state; i.e. whether either $(1-\beta)^{-1}u(\bar{p}_z) < (1-\beta)^{-1}E[u(p_z)]$ or $(1-\beta)^{-1}E[u(p_z)] < (1-\beta)^{-1}u(\bar{p}_z)$. On the one hand this ambiguity need not be deemed of any import as the fixed point is never reached due to the instability of the dynamical

equilibrium system. On the other hand the value of the fixed point with respect to an expected value is relevant in determining the welfare improving property of a monetary contraction.

That this point of indeterminacy obtains should not be surprising given that the convex dynamical system g_z was transformed into a quasi-concave dynamical system f_z by means of the homeomorphism h . Hence whilst $E[p_z] < \bar{p}_z$ for f_z one has that $E[m_z] > \bar{m}_z$ for g_z ¹⁶⁹. The basic problem faced in the analysis here is that g_z is convex and $u(\cdot)$ is concave and *increasing* in m . Hence $u(E[m_z]) > u(\bar{m}_z)$ and $u(E[m_z]) > E[u(m_z)]$ yet it is not clear whether $u(\bar{m}_z) > E[u(m_z)]$ or $E[u(m_z)] > u(\bar{m}_z)$. By applying the homeomorphism h does not yield one inequality over the other¹⁷⁰.

It is noteworthy that the results of this paper extend to the case in which f_z is cyclical. In such a case the measure derived from the forward iterates of the dynamical system has support concentrated on a negligible set.

4.7. Appendix

Lemma 4.2.1 $g_z : M_z \rightarrow M_z$ as defined by (4.2.12) and $f_z : P_z \rightarrow P_z$ as defined by (2.17) are topological conjugates

Proof Define the homeomorphism $h : M_z \rightarrow P_z$ by $h(m) = A(m)^{-1}$. The following shows that $h \circ g_z = f_z \circ h$

$$\begin{aligned} h(g_z(m)) &= A[g_z(m)]^{-1} \\ &= A(1 + \delta)^{\frac{1}{\eta}} m^{-1} (1 - Am^{-1})^{\frac{1}{\eta}} \end{aligned}$$

¹⁶⁹ To see that $E[m_z] > \bar{m}_z$ suppose that (g_z, M_z) is ergodically chaotic. Since g_z is ergodically chaotic then there exists an absolutely invariant continuous probability distribution $\tilde{\varphi}_z(m)$ over M_z . Since g_z is strictly convex then Jensen's inequality for convex functions implies: $g_z\left(\int_{M_z} m \tilde{\varphi}_z(m) dm\right) < \int_{M_z} g_z(m) \tilde{\varphi}_z(m) dm$. Ergodicity of g_z implies $\int_{M_z} g_z(m) \tilde{\varphi}_z(m) dm = \int_{M_z} m \tilde{\varphi}_z(m) dm$. It then follows that $g_z\left(\int_{M_z} m \tilde{\varphi}_z(m) dm\right) < \int_{M_z} m \tilde{\varphi}_z(m) dm$ or equivalently $g_z(E[m_z]) < E[m_z] \Leftrightarrow E[m_z] > \bar{m}_z$ by applying h to Property A.5.

¹⁷⁰ This differs to the result of Chapter 3 in which expected utility was less than the utility at the fixed point. This result was obtained for a concave dynamical system and concave utility function unlike the case here in which the result is obtained for a convex dynamical system and a concave utility function.

$$= (1 + \delta)^{\frac{1}{\eta}} p(1 - p)^{\frac{1}{\eta}} = f_z(h(m))$$

Similarly $h(m^*) = p^*$, $h(\bar{m}_z) = \bar{p}_z$ and $h(M_z) = P_z$. h being a continuous bijection with continuous inverse, is a homeomorphism and f_z and g_z are topologically conjugate to each other. ■

Proposition 4.2.1 Let $0 < \eta < \delta < \Delta(\eta)$. Then i. $0 < f_z^2(p^*) < \bar{p}_z < f_z(p^*) < 1$ and $f_z : P_z \rightarrow P_z$ where $P_z = [f_z^2(p^*), f_z(p^*)]$. ii. For any $p_0 \in I'$, there exists some $T \geq 0$ such that for all $t \geq T$, $f_z^t(p_0) \in P_z$.

Proof. i. Since $0 < \eta < \delta$ then $p^* < \bar{p}_z = f_z(\bar{p}_z) < f_z(p^*)$. Since $\delta < \Delta(\eta)$ then $f_z(p^*) < 1$. Now, divide P_z into two non-negligible sub-intervals; $P_{z,1} = [f_z^2(p^*), p^*]$ and $P_{z,2} = [p^*, f_z(p^*)]$ and note that $f_z^2(p^*) < f_z^3(p^*) < f_z(p^*)$. Then:

$$f_z(P_{z,1}) = f_z([f_z^2(p^*), p^*]) = [f_z^2(p^*), f_z^3(p^*)] \subset P_z \text{ and}$$

$$f_z(P_{z,2}) = f_z([p^*, f_z(p^*)]) = [f_z^2(p^*), f_z(p^*)] = P_z$$

it follows that $f_z(P_{z,1} \cup P_{z,2}) = P_z$. To see ii assume otherwise. Then $f_z^t(p_0) \in I' \setminus P_z = (0, f_z^2(p^*)) \cup (f_z(p^*), 1)$, $\forall t$ which contradicts part i of this lemma. ■

Lemma 4.3.1 If $\eta \in (0, 1]$ then $f_z : P_z \rightarrow P_z$ has a negative Schwarzian derivative.

Proof It is first noted that if $\ln|f'_z|$ is concave then $Sf'_z(p) < 0$.¹⁷¹ Let $r(p) = \ln|f'_z(p)|$. Manipulation yields:

$$r''(p) = \frac{-[p^2 + \eta^2(p^2 - 4p + 3) + \eta(2p^2 - 4p + 1)]}{\eta(p-1)^2(\eta(p-1) + p)^2}$$

For $r(p)$ to be concave it is sufficient that $r''(p) < 0$ for all $p \in P_z \subset I'$. The denominator is clearly positive for all $p \in I'$ and $\eta > 0$. By inspection, the numerator is positive for all $p \in I'$ and $0 < \eta \leq 1$. ■

¹⁷¹ See Grandmont [39]

Theorem 4.3.1 Let (f_z, P_z) be the dynamical system defined by (4.2.12). Suppose that (f_z, P_z) satisfies properties A.1 – A.6, has negative Schwarzian derivative and is ergodically chaotic. Then $E[p_z] < \bar{p}_z$ for all $z \in \bar{Z}$.

Proof Rewrite (4.2.12):

$$\left[\frac{f_z(p)}{p} \right]^\eta = (z/\beta)(1-p) \quad (4.3.3)$$

Take the time average of both sides of (4.2.26):

$$\left\langle \left[\frac{f_z(p)}{p} \right]^\eta \right\rangle = \langle (z/\beta)(1-p) \rangle \quad (4.3.4)$$

Suppose that

$$\langle [f_z(p)/p]^\eta \rangle > 1 \quad (4.3.5)$$

(4.3.5) implies that $(z/\beta)(1 - E[p_z]) > 1$. Rearranging and using the expression for the fixed point (4.2.20), one has that $E[p_z] < (z - \beta)/z = \bar{p}_z$ ¹⁷². Hence, in order to show the proposition, it is sufficient to show that (4.2.31) holds. To show this it is recalled that f_z satisfies (4.2.26) and (4.2.27) of property A.5. Given that $0 < \eta \leq 1$, if $f_z(p) > p \Leftrightarrow p < \bar{p}_z$ then $[f_z(p)]^\eta > p^\eta \Leftrightarrow p^\eta < \bar{p}_z^\eta$. Hence:

$$\int_{p'_z}^{\bar{p}_z} \left(\frac{[f_z(p)]^\eta - p^\eta}{p^\eta} \right) \varphi_z(p) dp > \int_{p'_z}^{\bar{p}_z} \left(\frac{[f_z(p)]^\eta - p^\eta}{\bar{p}_z^\eta} \right) \varphi_z(p) dp \quad (4.3.6)$$

On the other hand, if $f_z(p) < p \Leftrightarrow p > \bar{p}_z$ then $[f_z(p)]^\eta < p^\eta \Leftrightarrow p^\eta > \bar{p}_z^\eta$, hence:

$$\int_{\bar{p}_z}^{p'_z} \left(\frac{[f_z(p)]^\eta - p^\eta}{p^\eta} \right) \varphi_z(p) dp > \int_{\bar{p}_z}^{p'_z} \left(\frac{[f_z(p)]^\eta - p^\eta}{\bar{p}_z^\eta} \right) \varphi_z(p) dp \quad (4.3.7)$$

Summing (4.3.5) and (4.3.6) gives:

$$\int_{p'_z}^{\bar{p}_z} \left(\frac{[f_z(p)]^\eta - p^\eta}{p^\eta} \right) \varphi_z(p) dp + \int_{\bar{p}_z}^{p'_z} \left(\frac{[f_z(p)]^\eta - p^\eta}{p^\eta} \right) \varphi_z(p) dp$$

¹⁷² $\left\langle [f_z(p)p^{-1}]^\eta \right\rangle = \langle (z/\beta)(1-p) \rangle = E[(z/\beta)(1-p)] = (z/\beta)(1 - E[p]) > 1 \Leftrightarrow E[p_z] < \bar{p}_z$.

$$\begin{aligned}
& > \int_{p'_z}^{\bar{p}_z} \left(\frac{[f_z(p)]^\eta - p^\eta}{\bar{p}_z^\eta} \right) \varphi_z(p) dp + \int_{\bar{p}_z}^{p''_z} \left(\frac{[f_z(p)]^\eta - p^\eta}{\bar{p}_z^\eta} \right) \varphi_z(p) dp \\
& = \frac{1}{\bar{p}_z^\eta} \int_{p'_z}^{p''_z} ([f_z(p)]^\eta - p^\eta) \varphi_z(p) dp = 0
\end{aligned} \tag{4.3.7}$$

where the last equality in (4.3.7) follows from the fact that ergodicity implies:

$$\int_{P_z} f_z(f_z(p)) \varphi_z(p) dp = \int_{P_z} f_z(p) \varphi_z(p) dp = \int_{P_z} p \varphi_z(p) dp = \langle p_z \rangle = E[p_z].$$

Hence $\int_{p'_z}^{p''_z} [f_z(p)]^\eta \varphi_z(p) dp = \int_{p'_z}^{p''_z} p^\eta \varphi_z(p) dp$. Therefore:

$$\int_{p'_z}^{p''_z} \left(\frac{[f_z(p)]^\eta - p^\eta}{p^\eta} \right) \varphi_z(p) dp = \left\langle \frac{[f_z(p)]^\eta - p^\eta}{p^\eta} \right\rangle > 0$$

which can be written as $\left\langle \frac{[f_z(p)]^\eta}{p^\eta} \right\rangle > 1$ which implies $E[p_z] < \bar{p}_z$. Since z is arbitrary then this holds for all $z \in \bar{Z}$. This ends the proof¹⁷³. ■

Lemma 4.4.1 Let $z \in \bar{Z}$ then $E[p_z] \in (p'_z, p''_z)$ and $p'_z < E[p_z] < \bar{p}_z < p''_z$

Proof Suppose that $z \in \bar{Z}$. Then z satisfies $0 < 2\eta < \delta$ where $\eta \in (0, 1]$ and $0 < \eta < \delta \leq \Delta(\eta)$ and η is not a function of z . The trapping set is $P_z = [p'_z, p''_z] = [f_z^2(p^*), f_z(p^*)]$ where by Proposition 4.2.1 $f_z^2(p^*) < p^* < \bar{p}_z < f_z(p^*)$. Since $0 < \delta \leq \Delta(\eta)$, or by Proposition 4.2.1, then $P_z \subset I'$ which implies that $0 < f_z^2(p^*)$. By the same proposition, $\lim_{t \rightarrow \infty} f_z^t(p) \in P_z$ hence $E[p_z] \in P_z$ ¹⁷⁴. Suppose that either $E[p_z] = p'_z$ or $E[p_z] = p''_z$. then the probability distribution $\varphi_z(p)$ derived from f_z would be concentrated at a point in which case the probability measure would not be absolutely continuous and invariant with respect to the Lebesgue measure and would visit a negligible set of points in a measure theoretic sense centred at either p'_z or p''_z . Therefore, $E[p_z] \in (p'_z, p''_z)$. By Theorem 4.3.1, $E[p_z] < \bar{p}_z$ and by Proposition 4.2.1.i $\bar{p}_z < p''_z$ hence $p'_z < E[p_z] < \bar{p}_z < p''_z$. ■

¹⁷³ Theorem 4.3.1 holds whenever $\varphi_z(p)$ and $E[p_z]$ are well-defined. It is not necessary that f_z is ergodically chaotic but simply that there exists a non-degenerate probability distribution with expected value. This may be the case if f_z is periodic for instance.

¹⁷⁴ $E[p_z] = \langle p_z \rangle$ and is the time average of asymptotic values of f_z . Since the asymptotic values of f_z belong to P_z then $E[p_z] \in P_z$.

Proposition 4.4.1 There exists a subset $\tilde{Z}$ of z which excludes $\bar{Z}$ such that:

- i. For every $z' \in \tilde{Z}$ and $z \in \bar{Z}$; $u(\bar{p}_{z'}) > E[u(p_z)]$.
- ii. $\tilde{Z}$ has positive Lebesgue measure.
- iii. For every $z' \in \tilde{Z}$, $|f_{z'}(\bar{p}_{z'})| < 1$.

Proof Suppose that $z \in \bar{Z}$. Then $E[p_z] < \bar{p}_z$ where $0 < p'_z < E[p_z] < \bar{p}_z < p''_z$ by Lemma 4.4.1. $u(\cdot)$, being a strictly concave function and decreasing in p , is strictly decreasing in p . Then:

$$u(P_z) = u([p'_z, p''_z]) = u([f_z^2(p^*), f_z(p^*)]) = [u(f_z(p^*)), u(f_z^2(p^*))]$$

Since $p'_z < E[p_z] < \bar{p}_z < p''_z$ it follows that:

$$0 > u(f_z(p^*)) > u(E[p_z]) > u(\bar{p}_z) > u(f_z^2(p^*))$$

Let $\varepsilon > 0$ such that $0 < p'_z - \varepsilon$. By continuity of $u(\cdot)$, $u(0) > u(p'_z - \varepsilon)$ with $0 > u(p'_z - \varepsilon) > u(p'_z)$. Moreover, since $u(\cdot)$ is a bijection then the preimage of the interval $(u(0), u(p'_z - \varepsilon))$ is $u^{-1}((u(0), u(p'_z - \varepsilon))) = (0, p'_z - \varepsilon)$. The open interval $(0, p'_z - \varepsilon)$ then sits to the left of P_z with $(0, p'_z - \varepsilon) \cap P_z = \emptyset$. Let $\bar{p}_{z'} \equiv p'_z - \varepsilon$. Such an $\bar{p}_{z'}$ always exists for some $\varepsilon > 0$ as $\bar{p}_z$ is a continuous (increasing) function of z and is bound away from 0 as $z > \beta$. Then $0 < \bar{p}_{z'} < p'_z < E[p_z]$ and $0 > u(\bar{p}_{z'}) > u(p'_z) > u(E[p_z])$. By (4.4.14) one has that $u(E[p_z]) > E[u(p_z)]$ hence $0 > u(\bar{p}_{z'}) > E[u(p_z)]$. This shows part i. To show part ii, simply set $\tilde{Z} = (\beta, z']$. $\tilde{Z}$ clearly has non-negligible Lebesgue measure and exists for every $z \in \bar{Z}$. Then for any $z'' \in \text{int}(\tilde{Z})$ one has that $u(\bar{p}_{z'}) > u(\bar{p}_{z'}) > u(E[p_z]) > E[u(p_z)]$. To show part iii, let $|z' - \beta| < \varepsilon'$ where $\varepsilon' > 0$ and $z' > \beta > 0$ and $0 < \delta < 2\eta$. Since $0 < \delta < 2\eta$ then $|f_{z'}(\bar{p}_{z'})| < 1$ and the equilibrium system associated with the lower money growth rate z' is dynamically stable, i.e. $|p_t - \bar{p}_{z'}| < 0$ as $t \rightarrow \infty$. ■

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