PRICING 2-COLOUR RAINBOWS: NONPARAMETRIC METHODS USING COPULAE

By

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Abstract

This paper investigates the use of copulae for nonparametric pricing of multivariate contingent claims. Price estimates and no-arbitrage bounds for various types of two-colour rainbow options on the South African equity and bond markets were calculated.

Implied marginal risk-neutral distributions were derived nonparametrically from each assets option price spread. This was achieved in a very simple manner by assuming that, for each of the underlying assets in question, a continuum of option prices exist. Cubic splines were used to fit this continuum to the implied volatilities of the actual options available.

Two nonparametric copulae were considered: an empirical copula based directly upon the data and a kernel copula derived from a smooth two-dimensional kernel approximation of the historic density function. In addition, various parametric copulae were considered for comparison purposes. The differences between each of these approaches was found to vary from one type of rainbow to another.

Chapter 1

Introduction

A rainbow option is a multivariate contingent claim, i.e. a claim whose payoff is derived from more than one underlying market variable. To price these instruments, it is necessary to estimate the *joint* distribution of these underlying variables - knowledge of the univariate distributions of each of the underlying assets is not enough. Moreover, one needs to estimate this joint distribution in the *risk-neutral* world. This involves a change of probability measure from the real world, as discussed in section 3. Real world distributions will generally lead to arbitrage.

Estimation of the joint risk-neutral distribution can be done *parametrically* or *nonparametrically.* Parametric methods, of which the Black-Scholes [4] framework is the most famous example in derivatives pricing, involve making specific parametric assumptions about the underlying processes (section 3.3). The advantage of this method is that closed form solutions for prices can often be found. However, these assumptions might be too restrictive and not describe the data accurately enough. Nonparametric methods do not impose such restrictions. The distribution is derived from variables that are observed in the market.

The task of estimating the joint distribution is made simpler by the use of *copulae*, which are, in essence, multivariate distribution functions with uniform *marginal distributions* (or *marginals*). The formal definition is given in section 2.1. Sklar's theorem (theorem

2.1.1) allows us to construct a multivariate distribution from its component marginals (univariate) and a copula which binds them together - defining their *dependence structure.*

There is a wealth of literature available (see section 1.1.1) on how to estimate the implied risk-neutral marginal distributions nonparametrically from each assets vertical spread of options prices. Most of these methods are derived from the pioneering result of Breeden and Litzenberger [6] in 1978, which states that the implied risk-neutral distribution can be calculated given a continuum of call option prices across strike. This result is presented in section 3.2.

In each individual market, deviations from normality are well-known and have been written about extensively. The most obvious difference is that markets crash far more often than is implied by the normal distribution. This implies the existence of fat tails. Almost every historical price process exhibits negative *skewness* and excess *kurtosis* in comparison to its normal counterpart. The existence of the *volatility smile* (or *skew*) gives evidence that this phenomenon is priced into the options markets. If the Black-Scholes model (which uses Gaussian assumptions) was accurate, the prices of the options would translate into a flat volatility skew. This is not the case. Implied volatilities of most markets exhibit the smile effect which implies that these markets are more concerned about the downside than they would be if they were using a Gaussian pricing model.

When multivariate derivatives are considered, practitioners often return to normal assumptions (extended into higher dimensions) by using the *multivariate normal distribution* which relies on the *linear correlation coefficient* alone to describe the dependence structure. However, in so doing, the information in each assets implied volatility smile is lost. The correlation coefficient (section 2.2.2) does not hold enough information to join two non-normal distributions accurately. A more versatile dependence structure is required.

Copula methods provide this flexibility.

It will be shown that, if we admit no arbitrage, the copula functions are bounded from above and below. These bounds, known as the Fréchet-Hoeffding bounds and discussed in section 2.1.2, define the cases of perfect positive and negative dependence.

Within these bounds exist various families of *parametric copulae* which can have very useful properties. Some of these copulae are discussed in section 2.3. Arguably the most important is the family of *normal copulae* which is defined by one parameter - the linear correlation coefficient. It is the copula that is implicit in the multivariate normal distribution. The *product copula* specifies the case of independence.

Two nonparametric copulae were considered: an *empirical* copula based directly upon the data and a *kernel* copula derived from a smooth two-dimensional kernel approximation of the historic density function of South African equities and bonds. Unlike the empirical approach, the kernel approach allows for a degree of control over the trade-off between smoothness and bias, as discussed in section 2.4.2.

The remainder of this chapter discusses some of the important literature that has been written about copulae (section 1.1.1) and pricing of multivariate contingent claims (section 1.1.2) . Chapter 2 provides the background theory and mathematics that is used in pricing rainbow options using copulae. The most well-known of these options are discussed in chapter 3. Chapter 4 gives details of methodology used in a practical application of the above theory. Two types of nonparametric copulae, based on historic returns, as well as a variety of parametric copulae, are used together with the implied risk-neutral distributions from option spreads to price 2-colour rainbows. The no-arbitrage bounds are also calculated using the maximum and minimum copulae. Results are presented in chapter 5, and conclusions are set out in chapter 6.

1.1 Literature Review

This section discusses some of the important literature related to the subject of nonparametric pricing of rainbows with copulae. It is by no means a comprehensive literature review. The literature has been broken down into two broad groups: univariate risk-neutral density estimation and pricing rainbow options using copulae.

1.1.1 Univariate Risk-neutral Density Estimation

Jackwerth's literature review [32] comprehensively covers the literature that describes how to derive risk-neutral density functions in the univariate setting. Although the multivariate setting is the primary focus of this paper, a few of the methods for the univariate setting are discussed in the following section.

There are two main approaches: parametric and nonparametric. The parametric approach involves assumptions about the underlying process and the resulting return distribution. The most well known example of this approach is the Black-Scholes framework [4]. If the underlying process is assumed to be a geometric Brownian motion, the resulting distribution is indeed lognormal and the implied volatility is constant.

Breeden and Litzenberger prove a very important result in [6]. They show that assuming a continuum across strike of European call option prices, the state-price density (SPD) can be calculated from the second derivative of the option pricing function with respect to strike, irrespective of the underlying process:

$$f(x) = e^{r\tau} \frac{\partial C^2}{\partial^2 K}\bigg|_{K=x}$$

where f is the SPD, or risk-neutral density, r is the risk-free rate, and τ is the time to expiry. It uniquely characterizes the equivalent martingale measure under which every asset price discounted at the risk-free rate are martingales. This result is discussed in more detail in section 3.2.

Derman and Kani [17] propose an algorithm for calculating an implied risk-neutral binomial tree which incorporates the volatility surface. Their methods reproduce the volatility smile accurately in some circumstances, but according to Barle and Cakici in [2], the methods fail in a high interest rate environment. The latter propose a correction to the Derman Kani algorithm, and allow for dividends.

Placing parametric assumptions on the underlying process is limiting. This paper concentrates on nonparametric methods where the data defines the distribution. Most papers use option prices as their data. Stutzer [49] uses empirical return data. He does not need to use smoothing methods (to get rid of discontinuities) for density estimation since he uses rolling periods which have implicit smoothing characteristics. The maximum entropy principle is then used to estimate an equivalent martingale measure - basically transforming the historical distribution to fit current market conditions.

Shimko [45] fits a parabola to the implied volatility curve, then uses Black-Scholes in reverse to calculate a continuum of option prices from which he derives the SPD employing the main result of [6]. Lognormal probabilities are grafted onto the tails of the distribution. Although this method seems to employ Black-Scholes, it doesn't really. Market prices are quoted in implied volatility terms. These have been calculated using Black-Scholes. By using Black-Scholes in reverse, we transform the implied volatilities back into actual market prices. This method simply does the curve fitting to implied volatilities, rather than to the option prices. The resulting distribution is more general than the lognormal distribution implied by Black-Scholes. Methods for deriving the first four moments of a distribution (mean, standard deviation, skewness and kurtosis) are discussed thoroughly.

Rubinstein [43] starts with a lognormal prior distribution consistent with a standard binomial tree (which is equivalent to the discrete form of the Black-Scholes framework) using the average of the Black-Scholes implied volatilities of the two nearest-the-money call options. This prior distribution is then used to calculate call option prices which, by construction, will not fit observed options prices unless these prices have flat implied volatilities. A posterior distribution which fits the observed options prices is derived by changing the original distribution such that the least-squares difference between the prior distribution and the posterior distribution is minimized under the constraint that the calculated options prices fit between the bid-offer spread in the market.

In [31], Jackwerth and Rubinstein discuss several different ways of recovering the risk-neutral density function from option prices, investigating the use of different objective functions: least-squares as used in [43], goodness-of-fit, absolute differences and entropy. They also briefly cover the choice of prior distribution and find that neither of these choices make a significant difference to the resulting density function.

Ait-Sahalia and Lo [1] use nonparametric kernel regression and dimension reduction techniques to estimate an option pricing formula and then use the main result of [6] to get the SPD.

Cakici and Foster [7] use kernel methods (see section 2.4.2) to derive an historical prior distribution and then use minimum entropy techniques to find the "closest" distribution that satisfies the risk-neutral constraints: the mean is equal to the risk-free rate, and the volatility is determined by the at-the-money option price.

1.1.2 Pricing Rainbows using Copulae: Relevant Literature

Nelsen [38] offers the most comprehensive reference to the mathematics of copulae. This is a very popular reference that was used extensively throughout this dissertation.

Rosenberg [41] uses a semiparametric approach for options pricing. Options spreads are used to fit a Plackett density function to the empirical density. The Plackett density method allows for general marginal distributions to be considered.

Rosenberg [42] uses nonparametric techniques for pricing bivariate underperformance and outperformance options on the Dax and the S&P500. A copula is calculated from historical returns using Nadaraya-Watson kernel estimators. This copula, together with the marginal risk-neutral density functions from options prices (using the methods described in [1]) is used to estimate the joint density function. It should be noted that the copula that is calculated is the historical copula and this is not to be confused with the risk-neutral copula. He compares his results to those obtained from standard multivariate diffusion-based methods and finds the differences to be significant.

Embrechts, Lindskog and McNeil [20] investigate the concepts of correlation, dependence and concordance in detail. A wide range of parametric copulae are discussed with reference to the above. The usefulness of copulae in the areas of insurance and market risk is examined.

Cherubini and Luciano [8] use various parametric copulae as well as an historical copula together with semi-parametric marginals to price digital binary options, options on the minimum of two assets, and exchange options. Quadratic functions are fitted to the implied volatility skews of four individual market indices (MIB30, S&P500, FTSE, DAX) in much the same way as was presented in [45]. Black-Scholes call option prices are then calculated from the fitted quadratic functions. By fitting a quadratic skew, it is possible to derive closed form solutions for the implied marginal distribution functions using elementary probability theory and the Breeden and Litzenberger result. These marginals were then embedded into three different Archimedean copulae (Gumbel, Clayton, Franks) and a copula based on historical returns. Franks copula was found to be the most suitable of the parametric copulae, based on inspection and mean-squared error efficiency.

Fermanian and Scaillet [22] use kernel estimators to approximate both the historical pdf and cdf. Products of univariate Gaussian kernels are used. A stationary vector autoregressive process is used for Monte Carlo simulation and statistical measures of dependence are applied to the resulting kernels. Various measures of association (Kendall's tau, Gini's gamma, Spearman's rho, Blomqvist's beta, positive quadrant dependence and left tail decreasing behaviour) are considered. These measures, some of which are discussed in section 2.2, are then applied to the CAC and DAX, as well as the S&P500 and DJI.

In [50], van den Goorbergh and Genest investigate the effects of allowing the copula to change shape over time by using a *dynamic copula* as opposed to a static one. The marginal distributions of both the S&P500 and Nasdaq composite index are modelled as GARCH processes from which their risk-neutral counterparts are derived. The use of a wide range of parametric copula families is investigated. The dependence parameters of the dynamic copula are allowed to change over time as the conditional volatilities of the underlying asset price processes change. The valuation scheme is applied to best-of-two-markets and worst-of-two-markets options. They find results obtained from a dynamic copula differ substantially from static copula methods, particularly when the conditional volatilities are high. It is suggested that increased dependence occurs in times of high volatility.

Coutant *et al* [12] consider basket options on two or more underlying assets, spread options, double digitals, best-of, worst-of, min and max options. Pricing equations are given using the risk-neutral copula framework. Forward-looking indicators of dependence are defined in terms of the market prices of multi-asset derivatives.

Chapter 2

Copulae and Random Variables

2.1 Mathematical Background

Copulae can be thought of as multivariate distribution functions on the interval [0,1] that have uniform marginals. They are functions which join or *couple* two or more marginal distribution functions to form their joint distribution function.

Before formally stating the definition, some notation and mathematical concepts must be introduced. All theorems and definitions in this chapter are quoted from Nelsen's comprehensive introduction to copulae [38].

Let $\mathbf{R}$ denote the ordinary real line $(-\infty, \infty)$, $\bar{\mathbf{R}}$ denote the extended real line $[-\infty, \infty]$, and $\bar{\mathbf{R}}^2$ denote the extended real plane $\bar{\mathbf{R}} \times \bar{\mathbf{R}}$. A *rectangle* in $\bar{\mathbf{R}}^2$ is the Cartesian product B of two closed intervals: $B = [x_1, x_2] \times [y_1, y_2]$. The *vertices* of a rectangle B are the points $(x_1, y_1), (x_1, y_2), (x_2, y_1)$ and (x_2, y_2). The *unit square* $\mathbf{I}^2$ is the product $\mathbf{I} \times \mathbf{I}$ where $\mathbf{I} = [0, 1]$. A *2-place real function* H is a function whose domain, Dom$H$, is a subset of $\bar{\mathbf{R}}^2$ and whose range, Ran$H$, is a subset of $\mathbf{R}$.

Definition 2.1.1. Let S_1 and S_2 be nonempty subsets of $\bar{\mathbf{R}}$, and let H be a function such that Dom$H = S_1 \times S_2$. Let $B = [x_1, x_2] \times [y_1, y_2]$ be a rectangle, all of whose vertices are in Dom$H$. Then the *H-volume* of B is given by

$$V_H(B) = H(x_2, y_2) - H(x_2, y_1) - H(x_1, y_2) + H(x_1, y_1)$$

□

Definition 2.1.2. A 2-place real function H is *2-increasing* if $V_H(B) \geq 0$ for all rectangles B whose vertices lie in Dom$H$. □

Suppose that S_1 has a least element a_1 and that S_2 has a least element a_2. A function H from $S_1 \times S_2$ into $\mathbf{R}$ is said to be *grounded* if $H(x, a_2) = 0 = H(a_1, y)$ for all (x, y) in $S_1 \times S_2$.

Definition 2.1.3. A *two-dimensional subcopula* is a function C' with the following properties:

1. Dom$C' = S_1 \times S_2$, where S_1 and S_2 are subsets of $\mathbf{I}$ containing 0 and 1
2. C' is grounded and 2-increasing
3. For every u in S_1 and every v in S_2, $C'(u, 1) = u$ and $C'(1, v) = v$ □

The formal definition of a copula follows:

Definition 2.1.4. A *two-dimensional copula* is a 2-subcopula C whose domain is $\mathbf{I}^2$. □

Alternatively, a copula can be defined to be a function C from $\mathbf{I}^2$ to $\mathbf{I}$ with the following properties:

1. For every (u, v) in $\mathbf{I}$, $C(u, 0) = 0 = C(0, v)$
2. For every (u, v) in $\mathbf{I}$, $C(u, 1) = u$, $C(1, v) = v$
3. For every u_1, u_2, v_1, v_2 in $\mathbf{I}$, with $u_1 \leq u_2$ and $v_1 \leq v_2$,

$C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0$.

2.1.1 Sklar's Theorem

Definition 2.1.5. A *distribution function* is a function F with domain $\bar{\mathbf{R}}$ such that
1. F is nondecreasing
2. $F(-\infty) = 0$, and $F(\infty) = 1$ □

Definition 2.1.6. A *bivariate joint distribution function* is a function H with domain $\bar{\mathbf{R}}^2$ such that
1. H is 2-increasing
2. $H(x, -\infty) = H(-\infty, y) = 0$, and $H(\infty, \infty) = 1$ □

H has *margins* F and G defined as $F(x) = H(x, \infty)$ and $G(x) = H(\infty, y)$. Following from the definition of a joint distribution function, these margins are distribution functions.

Suppose H is a joint distribution function with marginals F and G, and let (X, Y) be a random vector with distribution H. It is easy to check that $F(X)$ and $G(Y)$ are uniformly distributed random variables. If the joint distribution of $(F(X), G(y))$ is C, then

$$\begin{aligned} C(x,y) &= \mathbb{P}(F(X) \leq x, G(Y) \leq y) \\ &= \mathbb{P}(X \leq F^{-1}(x), Y \leq G^{-1}(y)) \\ &= H\left(F^{-1}(x), G^{-1}(y)\right) \end{aligned}$$

and then

$$C\left(F(x), G(y)\right) = H(x,y)$$

Hence C can be thought of as a function which couples the margins together to yield the joint distribution. In the above derivation, we assumed that F and G were invertible. Sklar's theorem removes this assumption.

Theorem 2.1.1 (Sklar's Theorem [47]). *Let H be an 2-dimensional distribution function with margins F and G. Then there exists a copula C such that for all x, y in $\bar{\mathbf{R}}$,*

$$H(x,y) = C(F(x), G(y)) \tag{2.1.1}$$

If F and G are both continuous then C is uniquely defined. Otherwise, C is uniquely determined on $RanF \times RanG$. Conversely, if C is a copula and F and G are distribution functions, then the function H defined by equation 2.1.1 is a joint distribution function with margins F and G.$\square$

The formal proof of the theorem is given in [38].

This important result was proved by Sklar [47]. It allows us to express a joint distribution function in terms of a copula and the marginal distributions. This process can be reversed to express the copula in terms of a joint distribution function and the marginals.

However, if the margins are not *strictly* increasing then it does not have an inverse in the usual sense. We must first define the *quasi-inverse* of a distribution: [38]

Definition 2.1.7 (Quasi-inverse). Let F be a distribution function. Then a *quasi-inverse* of F is any function $F^{(-1)}$ with domain $\mathbf{I}$ such that

1. if t is in Ran$F$, then $F^{(-1)}(t)$ is any number x in $\bar{\mathbf{R}}$ such that $F(x) = t$
2. if t is not in Ran$F$, then

$$F^{(-1)}(t) = \inf\{x|F(x) \geq t\} = \sup\{x|F(x) \leq t\}$$

□

If F is *strictly* increasing, then it has a unique quasi-inverse function, which is the same as the usual inverse function.

Corollary 2.1.2. *Let H, F, G and C be as in theorem 2.1.1, and let $F^{(-1)}$ and $G^{(-1)}$ be quasi-inverses of F and G respectively. Then for any (u, v) in $\mathbf{I}^2$,*

$$C(u, v) = H(F^{(-1)}(u), G^{(-1)}(v)) \tag{2.1.2}$$

□

Let X and Y be random variables with distribution functions F and G respectively. If $H(x, y) = C(F(x), G(y))$, then we call C the copula of X and Y and denote it as C_{XY} where convenient.

Theorem 2.1.3 (Copula Invariance). *Let X and Y be continuous random variables with copula C_{XY}. If α and β are strictly increasing on Ran$X$ and Ran$Y$ respectively, then $C_{\alpha(X)\beta(Y)} = C_{XY}$. So C_{XY} is invariant under strictly increasing transformations of X and Y.*

Proof: Let F_1, G_1, F_2, and G_2 denote the distribution functions of $X, Y. \alpha(X)$ and $\beta(Y)$ respectively. Since α and β are strictly increasing, $F_2(X) = \mathbb{P}[\alpha(X) \leq x] = \mathbb{P}[X \leq \alpha^{-1}(x)] = F_1(\alpha^{-1}(x))$ and $G_2(X) = G_1(\beta^{-1}(y))$.

Thus for any x, y in $\bar{\mathbf{R}}$,

$$\begin{aligned} C_{\alpha(X)\beta(Y)}(F_2(x), G_2(y)) &= \mathbb{P}[\alpha(X) \leq x, \beta(Y) \leq y] \\ &= \mathbb{P}[X \leq \alpha^{-1}(x), Y \leq \beta^{-1}(y)] \\ &= C_{XY}(F_1(\alpha^{-1}(x)), G_1(\beta^{-1}(y))) \\ &= C_{XY}(F_2(x), G_2(y)) \end{aligned}$$

Since X and Y are continuous, $\text{Ran}F_2 = \text{Ran}G_2 = \mathbf{I}$, from which it follows that $C_{\alpha(X)\beta(Y)} = C_{XY}$ on $\mathbf{I}^2$. □

In the case where at least one of α and β is strictly decreasing, we obtain results in which the copula of the random variables $\alpha(X)$ and $\beta(Y)$ is a simple transformation of C_{XY}. We have

Theorem 2.1.4. *Let X and Y be continuous random variables with copula C_{XY}. Let α and β be strictly monotone on RanX and RanY respectively.*

1. *If α is strictly increasing and β is strictly decreasing, then*

$$C_{\alpha(X)\beta(Y)}(u, v) = u - C_{XY}(u, 1 - v)$$

2. *If α is strictly decreasing and β is strictly increasing, then*

$$C_{\alpha(X)\beta(Y)}(u, v) = v - C_{XY}(1 - u, v)$$

3. *If α and β are strictly decreasing, then*

$$C_{\alpha(X)\beta(Y)}(u, v) = u + v - 1 + C_{XY}(1 - u, 1 - v)$$

□

2.1.2 Fréchet-Hoeffding Bounds

The following result is known as the copula version of the *Fréchet-Hoeffding bounds* inequality in a bivariate setting [38].

Theorem 2.1.5. *Let C be a copula. Then for every (u, v) in the DomC,*

$$\max(u + v - 1, 0) \leq C(u, v) \leq \min(u, v) \tag{2.1.3}$$

Proof: Let (u, v) be an arbitrary point in the domain of C. Now $C(u, v) \leq C(u, 1) = u$ and $C(u, v) \leq C(1, v) = v$ give you that $C(u, v) \leq \min(u, v)$. Furthermore, following from point 3 of definition 2.1.4, for the rectangle $[u, 1] \times [1, v]$, we have that $C(1, 1) - C(1, v) - C(u, 1) + C(u, v) \geq 0$ which implies that $C(u, v) \geq u + v - 1$. Then, since $C(u, v) \geq 0$, we have that $C(u, v) \geq \max(u + v - 1, 0)$. □

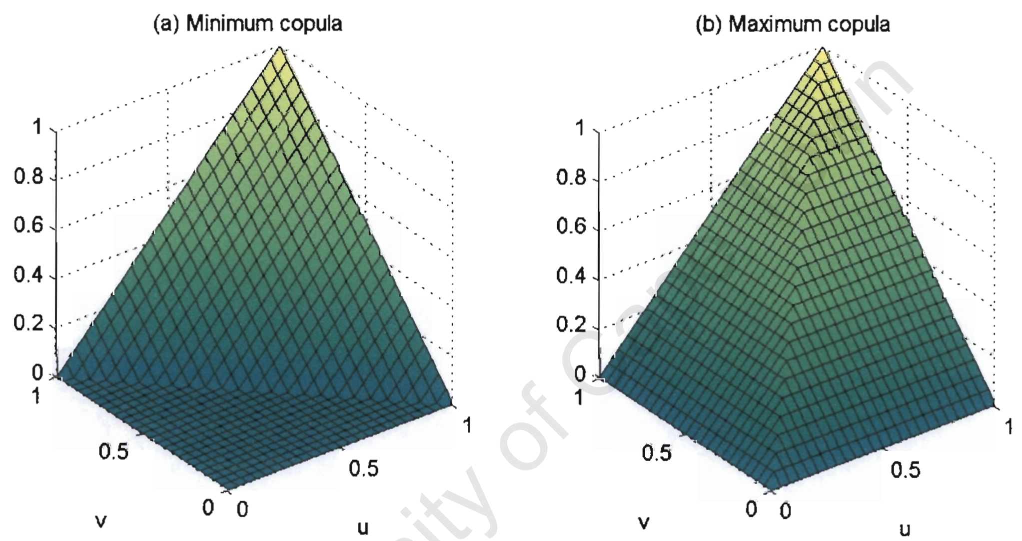

Figure 2.1: (a) The minimum copula $W(u, v)$. (b) The maximum copula $M(u, v)$. All copulae lie within these bounds.

Define the *minimum copula*

$$W(u, v) = \max(u + v - 1, 0)$$

and the *maximum copula*

$$M(u, v) = \min(u, v)$$

.

Inequality 2.1.3 tells us that all copulae lie between W and M. These bounds can be seen in figure 2.1.

It can easily be verified that W and M are copulae in the bivariate setting. However it should be noted that inequality 2.1.3 is not true in higher dimensions. See for example [20].

If X and Y are random variables, with a joint distribution function H and margins F and G respectively, then as a result of inequality 2.1.3 and theorem 2.1.1, for all x, y in $\bar{\mathbf{R}}$, we have

$$\max(F(x) + G(y) - 1, 0) \leq H(x, y) \leq \min(F(x), G(y))$$

The converse of theorem 2.1.1, used with the fact that M and W are copulae, ensures that the above bounds are joint distribution functions. They are known as the *Fréchet-Hoeffding bounds* for a joint distribution. This result was independently derived by both Fréchet [23] and Hoeffding [28].

Definition 2.1.8. A subset of S of $\bar{\mathbf{R}}^2$ is *nondecreasing* if, for any (x, y) and (u, v) in S, $x < u$ implies $y \leq v$. Similarly, a subset of S of $\bar{\mathbf{R}}^2$ is *nonincreasing* if, for any (x, y) and (u, v) in S, $x < u$ implies $y \geq v$. □

Theorem 2.1.6. *Let X and Y be random variables with joint distribution function H. Then H is identically equal to its Fréchet-Hoeffding upper (lower) bound if and only if the support of H is a nondecreasing (nonincreasing) subset of $\bar{\mathbf{R}}^2$.* □

The proof is given in [38].

If X and Y are continuous random variables, then Y is almost surely an increasing function of X if and only if $C_{XY} = M$. Y is almost surely a decreasing function of X if and only if $C_{XY} = W$. So each random variable is a deterministic function of the other.

In the first case, the variables X and Y are said to be *comonotonic* and they have perfect positive dependence whereas in the second case, they are *counter-comonotonic* and have perfect negative dependence.

Definition 2.1.9. The set $A \subset \bar{\mathbf{R}}^2$ is said to be *comonotic* if and only if, for any (x_1, y_1), (x_2, y_2) in A, either

$$x_1 \leq y_1 \text{ and } x_2 \leq y_2$$

or

$$x_1 \geq y_1 \text{ and } x_2 \geq y_2$$

□

Definition 2.1.10. The random vector (X, Y) is *comonotonic* or *perfectly positively dependent* if and only if there exists a comonotic set $A \subset \bar{\mathbf{R}}^2$ such that

$$\mathbb{P}((X,Y) \in A) = 1$$

□

Definition 2.1.11 (Concordance ordering). The copula C_1 is *smaller* than copula C_2 - written as $C_1 \prec C_2$ - if and only if, for all $(u, v) \in \mathbf{I}^2$

$$C_1(u,v) \leq C_2(u,v)$$

□

2.1.3 Product Copula

Define the *product copula* $\Pi(u, v) = uv$. This copula can be seen in figure 2.2.

Theorem 2.1.7. *Let X and Y be continuous random variables. Then X and Y are independent if and only if $C_{XY} = \Pi$* □

This result follows from corollary 2.1.2 and the fact that X and Y are independent if and only if $H(x, y) = F(x)G(y)$ for all x, y in $\bar{\mathbf{R}}^2$.

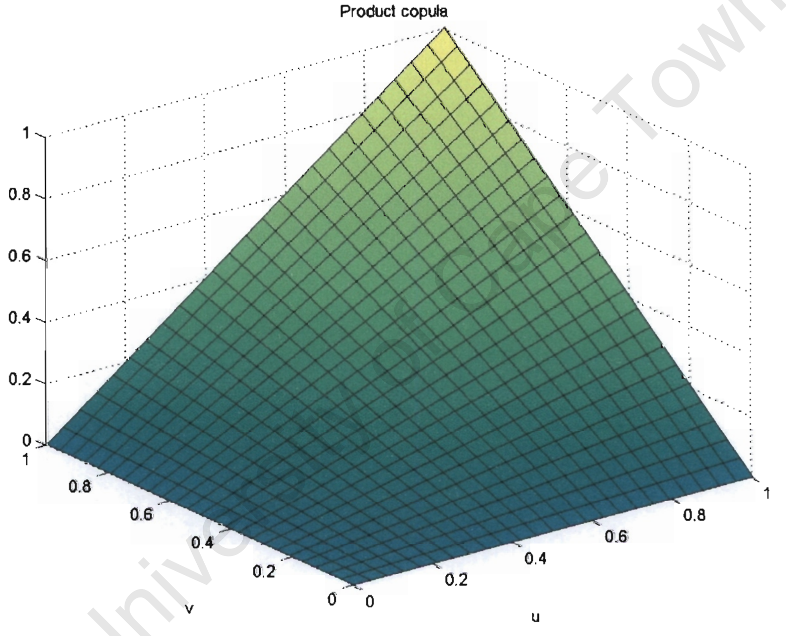

Figure 2.2: The product copula $\Pi(u, v)$.

2.2 Measures of Association

Association is the term used for describing the relationship between two random variables. A variety of different concepts and measures can be used to describe this relationship. The remainder of this section discusses some of these, namely: concordance, correlation, Kendall's τ and Spearman's ρ.

2.2.1 Concordance

Informally, if X and Y are random variables, concordance concepts give us an idea of the probability of X being high or low, given that Y is high or low.

The following axiomatic properties for a *measure of concordance* ($M_{X,Y}$ or M_C) between the random variables X and Y, with copula C were formally characterized by Scarsini [44]:

Definition 2.2.1. $M_{X,Y} = M_C$ is a *measure of concordance* between the random variables X and Y - with copula C - iff

1. it is defined for every pair of random variables (completeness)

2. it is a relative (or normalized) measure: $M_{X,Y} \in [-1, 1]$

3. it is symmetric: $M_{X,Y} = M_{Y,X}$

4. if X and Y are independent, then $M_{X,Y} = 0$

5. $M_{-X,Y} = M_{X,-Y} = -M_{X,Y}$

6. it converges (pointwise) when the copula does: if $[(X_n, Y_n)]$ is a sequence of continuous random variables with copula C_n, and

$$\lim_{n\to+\infty} C_n(u, v) = C(u, v) \text{ for every } (u, v) \in \mathbf{I}^2$$

then

$$\lim_{n\to+\infty} M_{X_n,Y_n} = M_{X,Y}$$

7. it respects concordance order: if $C_1 \prec C_2$, then $M_{C_1} \leq M_{C_2}$. See definition 2.1.11.

□

It should be noted that the converse of axiom (4) in definition 2.2.1 is not true. In other words, independence of the random variables X and Y is a sufficient, but not a necessary condition for $M_{X,Y}$ to be equal to zero.

The above characterization implies that M is bounded, and invariant under strictly increasing transformations:

Theorem 2.2.1. *If X and Y are comonotonic then $M_{X,Y} = 1$. If they are countermonotonic, then $M_{X,Y} = -1$* □

Theorem 2.2.2. *If α_1, α_2 are a.s. increasing on $RanF_1$, $RanF_2$, then* $M_{X,Y} = M_{\alpha_1(X),\alpha_2(Y)}$ □

2.2.2 Linear Correlation

Also known as *Pearson's product moment* or *Pearson's r*, the well-known *linear correlation coefficient* is defined in [21] as follows:

Definition 2.2.2. Let $(X, Y)^T$ be vector of random variables having non-zero finite differences. The *linear correlation coefficient* for $(X, Y)^T$ is defined as

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} \tag{2.2.1}$$

where $\text{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$ is the covariance of $(X, Y)^T$ and Var$X$ and Var$Y$ are the variances of X and Y. □

In the case of perfect linear dependence where $Y = aX + b$, we have almost surely $|\rho(X, Y)| = 1$. The converse is also true. If $\rho = +1$, we have *perfect positive correlation* and the data, when plotted against each other, lies on a straight line with a positive slope. Similarly, when $\rho = -1$, the data lies on a straight line with a negative slope. In this case, there is said to be *perfect negative correlation*. When $\rho = 0$, the variables are *uncorrelated*.

If the data is known to be significantly correlated, then ρ is an effective measure of the strength of association and can in fact be used to describe the residuals that would be expected if a straight line was fitted to the data. Unfortunately, in general we do not know *a priori* if there is significant correlation. The linear correlation coefficient does not decide whether it is significant or not since it does not take into account the individual distributions of X and Y. If they have elliptical distributions (definition 2.3.1), then ρ is always significant [21].

The use of linear correlation has a number of serious 'pitfalls' when the distributions are not elliptical. Many examples are discussed in [21] and [9], amongst others.

It is shown in [9] that the linear correlation coefficient does not satisfy axiom (6) in definition 2.2.1 and it is thus not a concordance measure.

The difficulties in interpreting the significance of the linear correlation coefficient have led to the concept of nonparametric measures of association or *rank correlations*. By replacing each observation by its rank, the resulting vectors are uniformly distributed by construction. There is a loss of information about the individual distributions, however when correlation is detected nonparametrically, there is definitely correlation. This type of measure is more robust than the linear correlation coefficient. The two most well-known nonparametric correlation measures are Kendall's τ and Spearman's ρ_S.

2.2.3 Spearman's ρ_S

The following definition is given in [21]:

Definition 2.2.3. Let X and Y be random variables with distribution functions F and G, and joint distribution function H. *Spearman's rank correlation* is given by

$$\rho_S(X, Y) = \rho(F(X), G(Y))$$

where ρ is the linear correlation. □

Theorem 2.2.3. *Let X and Y be continuous random variables whose copula is C. Then Spearman's ρ for X and Y is given by*

$$\rho_S = 12 \int\int_{I^2} C(u, v)\, du\, dv - 3 \tag{2.2.2}$$

Proof: It follows from definitions 2.2.2 and 2.2.3 that

$$\rho_S = \frac{Cov(F(X), G(Y))}{\sqrt{Var(F(X))Var(G(Y))}}$$

Now

$$Cov(F(X), G(Y)) = \mathbb{E}\left[F(X)G(Y)\right] - \mathbb{E}\left[F(X)\right]\mathbb{E}\left[G(Y)\right]$$

and $F(X)$ and $G(Y)$ are standard uniform distributions with mean $\frac{1}{2}$ and variance $\frac{1}{12}$. Furthermore,

$$\mathbb{E}\left[F(X)G(Y)\right] = \int\int_{I^2} uv\, dC(u,v) = \int\int_{I^2} C(u,v)\, du\, dv$$

The result follows. □

Theorem 2.2.4. *Let (X_1, Y_1), (X_2, Y_2) and (X_3, Y_3) be three independent and identically distributed random vectors with a common joint distribution function H. The population version of Spearman's ρ is proportional to the probability of concordance minus the probability of discordance for the two vectors (X_1, Y_1) and (X_2, Y_3). I.e. a pair of vectors with the same margins, but one vector has distribution function H, while the components of the other are independent:*

$$\rho_S = 3\left\{\mathbb{P}[(X_1 - X_2)(Y_1 - Y_3) > 0] - \mathbb{P}[(X_1 - X_2)(Y_1 - Y_3) < 0]\right\} \tag{2.2.3}$$

□

The proof is given in [38].

It can be shown that Spearman's ρ_S satisfies all the axioms of definition 2.2.1 and is thus a concordance measure. See [44] or [21] for proof.

2.2.4 Kendall's τ

The random vectors (X_1, Y_1) and (X_2, Y_2) are said to be concordant if $X_1 > X_2$ whenever $Y_1 > Y_2$; and discordant in the opposite case.

Definition 2.2.4. Let (X_1, Y_1) and (X_2, Y_2) be independent and identically distributed random vectors, each with joint distribution function H. Then Kendall's τ is equal to the probability of concordance minus the probability of discordance:

$$\tau = \mathbb{P}[(X_1 - X_2)(Y_1 - Y_2) > 0] - \mathbb{P}[(X_1 - X_2)(Y_1 - Y_2) < 0] \tag{2.2.4}$$

□

Theorem 2.2.5. *Let X and Y be continuous random variables whose copula is C. Then Kendall's τ for X and Y is given by*

$$\tau = 4 \int\int_{I^2} C(u,v)dC(u,v) - 1$$

□

The proof can be found in [38].

It can be shown that Kendall's τ satisfies all the axioms of definition 2.2.1 and is thus a concordance measure. See [44] or [21] for proof.

2.3 Parametric Copulae

The focus of this paper is on nonparametric techniques. In section 2.4 the methods for estimating the required copula nonparametrically will be outlined. It is, however, instructive to examine the various classes and families since they are simple to compute, most of the well-known measures of association (section 2.2) can easily be calculated in closed form from them, and they provide useful reference cases for nonparametric solutions which were discussed in section 2.4. Parametric copulae can be divided into two main classes: elliptical and Archimedean.

2.3.1 Elliptical Copulae

Elliptical copulae are those extracted from the class of *elliptical distributions* which includes the families of *normal* (or *Gaussian*) distributions and *t-distributions* as a subset. Elliptical distributions share many of the tractable properties of the Gaussian distribution since they are *radially symmetric.*

Definition 2.3.1. If $\mathbf{X}$ is an n-dimensional random vector and, for some $\mu \in \Re^n$ and some $n \times n$ non-negative definite symmetric matrix Σ, the characteristic function $\varphi_{\mathbf{X}-\mu}(\mathbf{t})$ of $\mathbf{X} - \mu$ is a function of the quadratic form $\mathbf{t}^T\Sigma\mathbf{t}$, $\varphi_{\mathbf{X}-\mu}(\mathbf{t}) = \phi(\mathbf{t}^T\Sigma\mathbf{t})$ then X has an *elliptical distribution* with parameters μ, Σ and ϕ. We write $\mathbf{X} \sim E_n(\mu, \Sigma, \phi)$. $\square$

Normal Copula

The bivariate *normal* or *Gaussian* copula with linear correlation ρ (as discussed in section 2.2.2) is defined as:

$$C(u,v) = \Phi(\Phi^{-1}(u), \Phi^{-1}(v)) \tag{2.3.1}$$

where Φ is the bivariate standard normal distribution with linear correlation ρ and Φ^{-1} is the standard normal inverse function. This copula is implicitly used to construct a bivariate normal distribution from two univariate normal distributions. We have

$$C(u,v) = \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi(1-\rho^2)^{\frac{1}{2}}} e^{-\frac{1}{2}\frac{u^2-2\rho uv+v^2}{1-\rho^2}} \, du dv \tag{2.3.2}$$

Note that when $\rho = 0$, this reduces to the product copula Π.

$$\begin{aligned} C(u,v) &= \frac{1}{2\pi}\int_{-\infty}^{\Phi^{-1}(u)} e^{-\frac{1}{2}u^2}\,du \int_{-\infty}^{\Phi^{-1}(v)} e^{-\frac{1}{2}v^2}\,du_2 \\ &= uv \end{aligned}$$

2.3.2 Archimedean Copulae

This class of copulae is very popular due to the ease with which they can be related to measures of association. They are constructed from a continuous, strictly decreasing, convex generating function $\phi : [0,1] \rightarrow [0,\infty]$ such that $\phi(1) = 0$. A copula can be computed, given such a function ϕ, by using the following formula: [38]

$$C(u,v) = \phi^{-1}(\phi(u) + \phi(v))$$

Three of most well-known families of Archimedean copulae are the Gumbel family, the Clayton family and the Frank family.

Gumbel Family

Gumbel copulae are constructed using a generating function of the form $\phi_\alpha(t) = (-\ln t)^\alpha$. α is restricted to the range $[1, +\infty)$, and hence the Gumbel copula does not allow for negative dependence. The copula can be computed using the following formula:

$$C(u,v) = \exp\left\{-\left[(-\ln u)^\alpha + (-\ln v)^\alpha\right]^{\frac{1}{\alpha}}\right\} \tag{2.3.3}$$

It becomes the upper Fréchet bound when $\alpha \rightarrow +\infty$, and the product copula when $\alpha = 1$.

τ can be computed directly from the parameter α: [9]

$$\tau = 1 - \alpha^{-1} \tag{2.3.4}$$

ρ_S can only be computed numerically by using definition 2.2.5.

Clayton Family

Clayton copulae are constructed in a similar manner using a *generating function* of the form $\phi_\alpha(t) = \frac{1}{\alpha}(t^{-\alpha} - 1)$. α is restricted to the range $[-1, 0) \cup (0, +\infty)$. The copula can be computed using the following formula:

$$C(u, v) = \max\left[(u^{-\alpha} + v^{-\alpha} - 1)^{\frac{-1}{\alpha}}, 0\right] \tag{2.3.5}$$

The upper and lower Fréchet bounds are reached when $\alpha \to +\infty$ and $\alpha = -1$ respectively. It is the same as the product copula when $\alpha = 0$.

τ can be computed directly from the parameter α: [9]

$$\tau = \frac{\alpha}{\alpha + 2} \tag{2.3.6}$$

The closed form solution for ρ_S is complicated. It may be quicker to solve numerically.

Frank Family

Frank's copula uses a generating function $\phi_\alpha(t) = -\ln \frac{\exp(-\alpha t)-1}{\exp(-\alpha)-1}$ and is defined for $\alpha \in (-\infty, 0) \cup (0, +\infty)$. The formula for computing the copula is:

$$C(u, v) = -\frac{1}{\alpha} \ln\left(1 + \frac{(\exp(-\alpha u) - 1)(\exp(-\alpha v) - 1)}{\exp(-\alpha) - 1}\right) \tag{2.3.7}$$

This family of copulae reduces to the product copula when $\alpha = 0$ and reaches the upper and lower Fréchet bounds when $\alpha \to +\infty$ and $\alpha \to -\infty$ respectively.

Both τ and ρ_S can be computed directly from the parameter α: [9]

$$\begin{aligned} \tau &= 1 + \frac{4\left[D_1(\alpha) - 1\right]}{\alpha} \\ \rho_S &= 1 - \frac{12\left[D_2(-\alpha) - D_1(-\alpha)\right]}{\alpha} \end{aligned} \tag{2.3.8}$$

where the *Debye functions* $D_k(\alpha)$ are defined as follows

$$D_k(\alpha) = \frac{k}{\alpha^k} \int_0^\alpha \frac{t^k}{\exp(t) - 1}\, dt$$

for $k = 1, 2$.

2.4 Nonparametric Copulae

2.4.1 Empirical Copula

Deheuvels [15] introduced empirical copulae. The following definition is given in [38]:

Definition 2.4.1. Let $\{(x_k, y_k)\}_{k=1}^n$ denote a sample of size n from a continuous bivariate distribution. The *empirical copula* is the function C_n given by

$$C_n\left(\frac{i}{n}, \frac{j}{n}\right) = \frac{\text{number of pairs } (x, y) \text{ in the sample s.t. } x \leq x_{(i)} \text{ and } y \leq y_{(j)}}{n} \tag{2.4.1}$$

where $x_{(i)}$ and $y_{(j)}$, $1 \leq i, j \leq n$, denote order statistics from the sample. □

The above definition is not everywhere defined. The following extension defines the empirical copula everywhere on $\mathbf{I}^2$.

Definition 2.4.2. Let $\{(x_k, y_k)\}_{k=1}^n$ be defined as above. The *empirical copula* is the function C_n given by

$$\begin{aligned} C_n(a, b) &= C_n\left(\frac{i}{n}, \frac{j}{n}\right) \\ &= \frac{\text{no. of pairs } (x, y) \text{ in the sample s.t. } x \leq x_{(i)} \text{ and } y \leq y_{(j)}}{n} \end{aligned}$$

where $i = [a \cdot n]$ and $j = [b \cdot n]$, and $x_{(i)}$, $y_{(j)}$ denote order statistics from the sample. □

2.4.2 Kernel Copula

The starting point for calculating the kernel copula is the estimation of the joint density function of the empirical return data. See section 2.4.2. This density function can then be numerically integrated with respect to each asset to give the joint cumulative distribution, from which, using the corollary to Sklar's theorem (corollary 2.1.2), we can extract the historic marginal distributions, leaving behind the kernel copula.

Density Estimation

Suppose we have a random variable X with probability density function f. By definition, the probabilities associated with X can be calculated using the following relationship:

$$P[a < X < b] = \int_a^b f(x)\,dx \tag{2.4.2}$$

In most cases however, we only have a finite data set which is assumed to be a sample from the unknown probability density f. Density estimation, as the name implies, is the construction of an estimate of the density function from the limited data available. [46] is an excellent source of information in this area of statistics and data analysis.

Parametric density estimation assumes that the data is drawn from a known family of distributions. The data is then fitted to this distribution by solving for the relevant parameters, for example, the lognormal distribution with parameters μ and σ. Parametric techniques are often used in statistical exercises. Nonparametric techniques are more representative of the data since they do not explicitly constrain the density function to some shape or form.

The most well known density estimator is the *histogram*. It is a simple. effective method of visualizing random data - hence its popularity. A histogram is defined by the following equation:

$$\hat{f}(x) = \frac{1}{nh}\,[\text{number of } X_i \text{ in same bin as } x] \tag{2.4.3}$$

where $X_1, ..., X_n$ is the data set, $n =$ number of observations and $h =$ bin size.

From a mathematical point of view, histograms are not very accurate (leading to inefficient use of the data), they do not easily lend themselves to differentiation (due to discontinuity) and they are not perfectly objective: by changing the position of the origin or the bin size, two histograms representing the same data set can look very different. These problems are compounded when trying to analyse multivariate data.

The *naive estimator* is another method of density estimation which is basically a histogram where every data point is the centre of a bin. This adaptation is a step in the right direction. It eliminates the problem of choice of origin. The bin width is the only choice remaining. This parameter controls the degree of smoothing. However the naive estimator still results in a discontinuous estimation of the density function.

$$\hat{f}(x) = \frac{1}{2nh}\left[\text{number of } X_i \text{ falling in } (x-h, x+h)\right]$$

or if we define a weight function as

$$w(x) = \begin{cases} \frac{1}{2} & \text{if } |x| < 1 \\ 0 & \text{else} \end{cases}$$

then

$$\hat{f}(x) = \frac{1}{nh}\sum_{i=1}^{n} w\left(\frac{x-X_i}{h}\right) \tag{2.4.4}$$

Kernel estimators generalize the above approach further. We can replace the weight function with a kernel function K which satisfies

$$\int_{-\infty}^{\infty} K(x)\,dx = 1 \tag{2.4.5}$$

and is usually a symmetric density function itself. The kernel estimator then becomes

$$\hat{f}(x) = \frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{x-X_i}{h}\right) \tag{2.4.6}$$

The bandwidth h controls the amount of smoothing. As h decreases, the density estimate becomes more bumpy, and as it increases, the estimate starts to look more and more like the kernel function K. This is the trade-off between bias and smoothness.

Two of the most common choices of kernel are the Gaussian and *Epanechnikov* kernel: K_e and K_n. these are defined as follows:

$K_g(t) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}t^2\right)$ and

$K_e(t) = \frac{3}{4\sqrt{5}}\left(1-\frac{1}{5}t^2\right)$ for $|t| < \sqrt{5}$; 0 otherwise.

The problems associated with histograms are compounded when examining multivariate distributions. The kernel approach can easily be extended into higher dimensions by constructing a kernel that has the same number of dimensions as the data. The d-dimensional density estimate then becomes

$$\hat{f}(\mathbf{x}) = \frac{1}{nh^d} \sum_{i=1}^{n} K \left\{ \frac{1}{h}(\mathbf{x} - \mathbf{X}_i) \right\} \tag{2.4.7}$$

where bold text denotes a point in d-space.

In order to accommodate different ranges, the kernel should be scaled across each of its dimensions according to the range of the distribution of each underlying univariate distribution. Furthermore, the kernel should be transformed to represent the correlations between each of its components. In this way, each kernel is the same shape as the data sample itself. Equivalently, we could pre-whiten the data by linearly transforming it to have a unit covariance matrix and then using a radially symmetric kernel of some form to smooth the data. This smooth density can then by transformed back to its original form. This bias-reduction technique, known as *pre-whitening* or *sphering* the data, was suggested by Fukunaga in [24]. The approach is implemented by using a density estimate of the following form:

$$\hat{f}(\mathbf{x}) = \frac{(\det S)^{-1/2}}{nh^d} \sum_{i=1}^{n} k \left\{ h^{-2}(\mathbf{x} - \mathbf{X}_i)^T S^{-1} (\mathbf{x} - \mathbf{X}_i) \right\} \tag{2.4.8}$$

where S is the covariance matrix of the sample data. The function k is given by

$$k(\mathbf{x}^T\mathbf{x}) = K(\mathbf{x})$$

In the case where K is the normal kernel, then

$$k(u) = \frac{1}{(2\pi)^{d/2}} \exp\left(-\frac{1}{2}u\right)$$

Chapter 3

Derivatives Pricing

3.1 Risk-neutral Valuation

Harrison and Pliska [27] showed that if a market admits no *arbitrage* opportunities, then there exists a *risk-neutral measure*, modulo some technical conditions. Under such a measure $\mathbb{Q}$, every *discounted* price process is a *martingale*. This means that its future expected value is the same as its current value. Equivalently, every asset is expected to grow at the risk-free rate under $\mathbb{Q}$.

Let $S(t)$ be the price at time t of any basic asset in the market and let A denote the riskless bank account. If $\bar{S}_t$ is the *discounted value* of S, we have

$$\bar{S}_t = \frac{S_t}{A_t}$$

and

$$\mathbb{E}_t^{\mathbb{Q}}\left[\bar{S}_T\right] = \bar{S}_t$$

where $\mathbb{E}_t^{\mathbb{Q}}$ denotes the expectation at time t under the *risk-neutral measure* $\mathbb{Q}$.

Now suppose that X is a European-style contingent claim expiring at T. If the payoff of X can be perfectly replicated by some (probably dynamic) self-financing portfolio V consisting of basic assets, then in the absence of arbitrage, we must have $V(t) = X(t)$ and V is made up of underlying assets which are $\mathbb{Q}$-martingales, therefore V itself must be a

$\mathbb{Q}$-martingale. So the discounted value of X at time t is given by the following *risk-neutral valuation* formula:

$$\bar{X}_t = \mathbb{E}_t^{\mathbb{Q}}\left[\bar{X}_T\right] \tag{3.1.1}$$

where $\bar{X}_t = \frac{X_t}{A_t}$.

In the above formulation, the discounted asset price was given by $\bar{S}(t) = \frac{S(t)}{A(t)}$. This is the value of S in units of A. In [25] it was shown that we do not have to use the riskless bank account. Given an asset $\hat{A}$, we can "discount" any underlying asset S using $\hat{A}$:

$$\hat{S}_t = \frac{S_t}{\hat{A}_t}$$

In this case, $\hat{S}$ is the price of asset S in units of $\hat{A}$. We say that asset $\hat{A}$ (it could be an asset or a portfolio of assets) is the *numeraire.*

It was shown in [25] that in the absence of arbitrage opportunities, modulo some technical conditions, for each numeraire $\hat{A}$, there exists a measure $\hat{\mathbb{Q}}$ under which every numeraire-deflated process is a $\hat{\mathbb{Q}}$-martingale. So

$$\mathbb{E}_t^{\hat{\mathbb{Q}}}\left[\hat{S}_T\right] = \hat{S}_t$$

$\hat{\mathbb{Q}}$ is referred to as the *equivalent martingale measure* for $\hat{A}$.

Using the exact same argument as for the bank account, if X can be perfectly replicated, then

$$\hat{X}_t = \mathbb{E}_t^{\hat{\mathbb{Q}}}\left[\hat{X}_T\right]$$

and

$$X_t = \hat{A}_t \mathbb{E}_t^{\hat{\mathbb{Q}}}\left[\frac{X_T}{\hat{A}_T}\right]$$

This technique is known as a *change of numeraire* and was formally stated in [25]:

Theorem 3.1.1. *Suppose that $A_1(t)$ and $A_2(t)$ are numeraires, and that $\mathbb{Q}_1$ and $\mathbb{Q}_2$ are their respective associated equivalent martingale measures, then for any European-style derivative X that can be perfectly hedged, we have*

$$A_1(t)\mathbb{E}_t^{\mathbb{Q}_1}\left[\frac{X(T)}{A_1(T)}\right] = A_2(t)\mathbb{E}_t^{\mathbb{Q}_2}\left[\frac{X(T)}{A_2(T)}\right]$$

□

Now consider two assets, S_1 and S_2, and a derivative security whose payoff is contingent upon the price of these two assets. $S_1(t)$ and $S_2(t)$ are assumed to be non-negative continuous random variables. For a European-style derivative whose payoff can be written as $G(S_1(T), S_2(T))$, we need to calculate a pricing function $g(S_1(t), S_2(t))$ which rules out arbitrage. By the risk-neutral valuation formula (equation 3.1),

$$g(S_1(t), S_2(t), t) = e^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}\left[G(S_1(T), S_2(T))\right]$$

Then applying basic probability theory we have

$$g(S_1(t), S_2(t), t) = e^{-r(T-t)} \int_0^\infty \int_0^\infty G(S_1, S_2) q(S_1, S_2)\, dS_1\, dS_2 \qquad (3.1.2)$$

where $q(S_1, S_2)$ is the risk-neutral probability density function. This is sometimes referred to as the *pricing kernel of the market*.

From equation 3.1.2, it can be seen that if the joint distribution of the underlyings is known, we can price these contingent claims. Often we have a good idea of each marginal distribution (see section 3.2) and need to put them together to give the joint distribution. Copulae enable us to do just this.

3.2 Marginal Risk-neutral Distributions

In the pioneering work of Breeden and Litzenberger [6], it was shown that under perfect-market assumptions[1], given a continuum of call option prices across strike price, the implied risk-neutral density function of the underlying asset can be computed from the second derivative of the option prices with respect to strike:

$$f_{S(T)}(X) = e^{rT}\frac{\partial^2 c(X,T)}{\partial X^2} \qquad (3.2.1)$$

The argument used is as follows. Let $S(t)$ be the price at time t of any security or portfolios of securities. Let $c(X_i, T)$ be a set of call options on S, with strike prices X_i, expiring at time T.

Initially assume that $S(T)$ has a discrete distribution where the step size is 1. Define an *elementary claim* $P(M,T)$ to be one which pays \$1 at time T if $S(T) = M$ and nothing otherwise. The payoff of $P(M,T)$ is replicated exactly by the following portfolio of call options:

$$\begin{aligned} P(M,T) &= [c(M-1,T) - c(M,T)] - [c(M,T) - c(M+1,T)] \\ &= [c(M-1,T) - 2c(M,T) + c(M+1,T)] \end{aligned}$$

This portfolio has the same payoff as P and must therefore have the same price. In more general terms, if the step size is ΔM,

$$P(M,T;\Delta M) = \frac{1}{\Delta M}\left[c(M-\Delta M,T) - 2c(M,T) + c(M+\Delta M,T)\right]$$

Division by ΔM yields

$$\frac{P(M,T;\Delta M)}{\Delta M} = \frac{1}{(\Delta M)^2}\left[c(M-\Delta M,T) - 2c(M,T) + c(M+\Delta M,T)\right]$$

and in the limit, as ΔM tends to zero

$$\lim_{\Delta M \to 0} \frac{P(M,T;\Delta M)}{\Delta M} = \frac{\partial^2 c(M,T)}{\partial M^2}$$

Note that strict convexity of $c(M,T)$ across strike ensures that the price of an elementary claim is always positive. Assuming a continuous distribution of S, the pricing function $P(M,T)$ is analogous to the density function of this distribution.

$$P(M,T) = c_{XX}(X = M, T)\, dM_T$$

The value of any derivative asset with S as its underlying can be determined from the pricing functions of the elementary claims. That is, if a security f has payoffs q_T^f over

[1]Perfect-market assumptions: no restrictions on short sales, no transaction costs or taxes and investors can borrow at the risk-free rate of interest.

time that are known functions of only the level of S at each time, then its price must be

$$\begin{aligned} V^f &= \int_T \int_{M_T} q_T^f(M_T)\, P(M,T) dT \\ V^f &= \int_T \int_{M_T} q_T^f(M_T)\, c_{XX}(X = M, T)\, dM_T\, dT \end{aligned}$$

When all the payoffs are at one time only, as is the case for European style options, this reduces to

$$V^f = \int_M q^f(M)\, c_{XX}(X = M, T)\, dM$$

It is important to note that no assumptions were made about the underlying process. The only requirement was the continuum of options across strike.

The above result can easily be verified using the principle of risk-neutral valuation (section 3.1):

Theorem 3.2.1. *Let S be a traded security. The risk-neutral distribution function F of $S(T)$ can be determined from the prices $C(K)$ of European call options as follows:*

$$F(s) = 1 + e^{rT} \frac{\partial C(K)}{\partial K}\Big|_{K=s} \tag{3.2.2}$$

Proof: From the risk-neutral valuation formula (equation 3.1):

$$\begin{aligned} C(K) &= e^{-rT} \mathbb{E}_{\mathbb{Q}} \left[(S(T) - K)^+ \right] \\ &= e^{-rT} \int_0^\infty \mathbb{Q}\,(S(T) - K > u)\, du \\ &= e^{-rT} \int_K^\infty \mathbb{Q}\,(S(T) > v)\, dv \end{aligned}$$

where $v = u + K$. So then

$$\frac{\partial C(K)}{\partial K} = -e^{-rT} \mathbb{Q}\,(S(T) > K)$$

and the result follows.□

As a consequence we have

$$f(s) = e^{rT} \frac{\partial^2 C(K)}{\partial K^2}\Big|_{K=s} \tag{3.2.3}$$

where f is the risk-neutral density function of $S(T)$.

When prices are calculated from a flat volatility smile, the above approach yields a lognormal risk-neutral distribution. In this case, the result is analogous to the Black-Scholes [4] model. As a downward slope is introduced, the distribution becomes negatively skewed. As it becomes curved, the distribution becomes leptokurtotic. Figure 3.1 shows the implied density functions for a linear skew and a quadratic skew with reference to that of a flat skew.

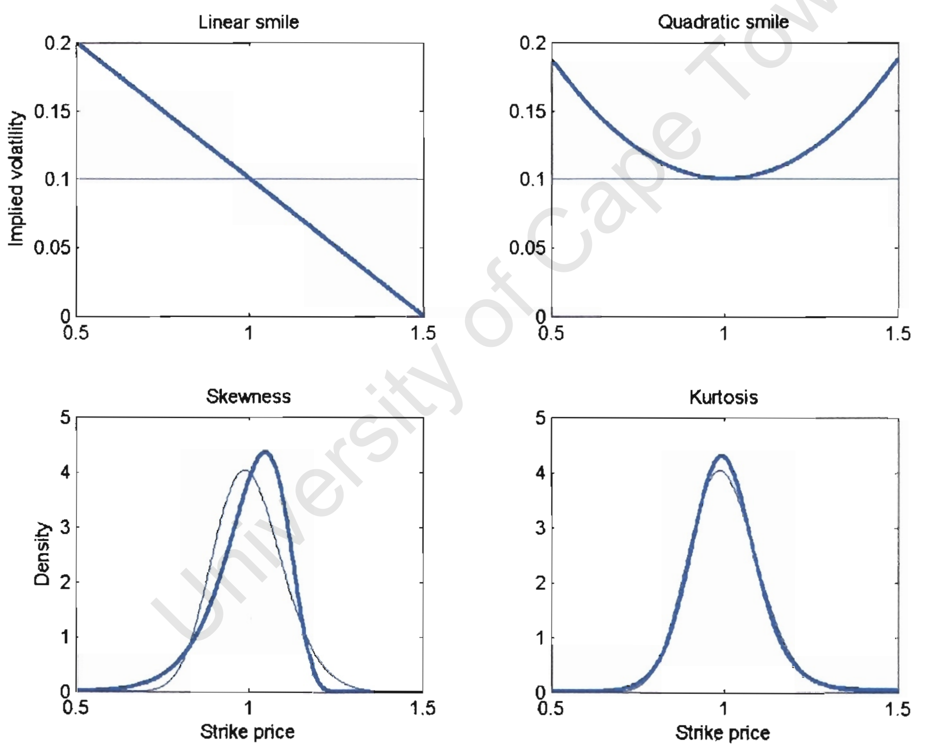

Figure 3.1: Schematic of volatility smiles and their implied distributions. The top-left graph shows a flat volatility skew and a downward sloping linear skew. The risk-neutral densities implied by each of the above are shown in the graph on the bottom left. Similarly, the top right graph shows a flat skew and a quadratic skew. The bottom right graph shows their implied distributions.

3.3 Black-Scholes Model

Black and Scholes [4] assume that the risk-neutral price process of the underlying asset S follows a *geometric Brownian motion*:

$$dS = rSdt + \sigma Sdz$$

where r is the risk-free rate and dz is a standard Brownian motion.

When this formulation is extended into two dimensions, we have [48]

$$\begin{aligned} dS_1 &= rS_1dt + \sigma_1 S_1 d\bar{z}_1 \\ dS_2 &= rS_2dt + \sigma_2 S_2 d\bar{z}_2 \end{aligned}$$

where $\bar{z}_1$ and $\bar{z}_2$ are correlated one-dimensional Brownian motions with correlation coefficient ρ.

This implies that the joint distribution of logreturns is a bivariate normal distribution. Under these assumptions, the price for many rainbow options can be solved for in closed form. See, for example, [48].

An alternative method of pricing under this paradigm is to estimate the bivariate normal distribution function that is implicit in the formulation. This can be done by using a normal copula (as defined by equation 2.3.1) to combine the two normal return distributions.

3.4 Price Bounds

The following proposition is stated and proved in [40].

Proposition 3.4.1. *Let $G(x, y)$ be a continuous payoff function. If $\frac{\partial^2 G}{\partial x \partial y}$ is a nonnegative (nonpositive) measure, then the option price is nondecreasing (nonincreasing) with respect to the concordance order.* □

This means that if the payoff of an option satisfies either one of the above conditions, given any two distributions for the underlying assets, the option price bounds must correspond to the maximum and minimum copulae (section 2.1.2).

Assume the univariate distributions for the two underlying assets are given. Let V^- and V^+ be the minimum and maximum price of an option with payoff function G. Let M and W represent the minimum and maximum copulae.

Proposition 3.4.2. *If $\frac{\partial^2 G}{\partial x \partial y}$ is a nonpositive (nonnegative) measure, then V^- and V^+ correspond to the cases $C = W$ $(C = M)$ and $C = M$ $(C = W)$ respectively.* □

The proof is given in [40].

It is obvious that for the payoffs of the rainbow options discussed in section 3.5. $\frac{\partial^2 G}{\partial x \partial y}$ are either nonnegative or nonpositive.

3.5 Rainbow Options

In contrast to a standard option, the payoff of a rainbow option is a function of several underlyings. A rainbow option whose payoff depends on n underlying assets is referred to as an n-colour rainbow. Some examples of 2-colour rainbows are listed below:

- Options on the minimum of two assets
- Options on the maximum of two assets
- Spread options
- Digital binary options

3.5.1 Call Option on the Minimum of Two Assets

The holder of a *call option on the minimum of two assets* has the right to buy the minimum asset at the strike price at expiry. The payoff function is therefore

$$\text{Call}^{-}(S_1(T), S_2(T), K) = \max(\min(S_1(T), S_2(T)) - K, 0)$$

and can be seen in figure 3.2(a).

Stulz [48] derives the pricing formulae for these options under lognormal assumptions:

$$\begin{aligned}
\text{Call}^{-}(S_1(T), S_2(T), K, \tau) &= S_1 N_2\left(\gamma_1 + \sigma_1\sqrt{\tau}, \frac{\ln\left(\frac{S_2}{S_1}\right) - \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}, \frac{\rho\sigma_2 - \sigma_1}{\sigma}\right) \\
&+ S_2 N_2\left(\gamma_2 + \sigma_2\sqrt{\tau}, \frac{\ln\left(\frac{S_2}{S_1}\right) - \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}, \frac{\rho\sigma_1 - \sigma_2}{\sigma}\right) \\
&- Ke^{-r\tau} N_2(\gamma_1, \gamma_2, \rho) \qquad (3.5.1)
\end{aligned}$$

where τ is the time to expiry, $N_2(\alpha, \beta, \rho)$ is the standard bivariate normal distribution (see remark 3.5.1) with correlation ρ, evaluated at (α, β), and

$$\begin{aligned}
\gamma_1 &= \frac{\ln\left(\frac{S_1}{K}\right) + (r - \frac{1}{2}\sigma_1^2)\tau}{\sigma_1\sqrt{\tau}} \\
\gamma_2 &= \frac{\ln\left(\frac{S_2}{K}\right) + (r - \frac{1}{2}\sigma_2^2)\tau}{\sigma_2\sqrt{\tau}} \\
\sigma^2 &= \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2
\end{aligned}$$

Remark 3.5.1. There is no closed form solution for the standard bivariate normal distribution. It must be numerically approximated using an appropriate algorithm. A common algorithm used was derived by Drezner [18], but this method can yield negative prices for call options on the minimum of two assets, under fairly realistic correlation assumptions. A more accurate version, is suggested by West [51]. It is a slightly modified version of Drezner and Wesolowsky's algorithm [19].

3.5.2 Put Option on the Minimum of Two Assets

The holder of a *put option on the minimum of two assets* has the right to sell the minimum asset at the strike price at expiry. The payoff function is therefore

$$\text{Put}^{-}(S_1(T), S_2(T), K) = \max(K - \min(S_1(T), S_2(T)), 0)$$

and can be seen in figure 3.2(b).

Stulz [48] derives the price of these options from using arbitrage and parity arguments

$$\text{Put}^{-}(S_1, S_2, K, \tau) = \text{Call}^{-}(S_1, S_2, K, \tau) - \text{Call}^{-}(S_1, S_2, 0, \tau) + KB$$

where B is the risk-free bank account.

3.5.3 Call Option on the Maximum of Two Assets

The holder of a *call option on the maximum of two assets* has the right to buy the maximum asset at the strike price at expiry. The payoff function is therefore

$$\text{Call}^{+}(S_1(T), S_2(T), K) = \max(\max(S_1(T), S_2(T)) - K, 0)$$

and can be seen in figure 3.2(c).

Stulz [48] derives the price of these options from, once again using arbitrage and parity arguments.

$$\text{Call}^{+}(S_1, S_2, K, \tau) = \text{C}(S_1, K, \tau) + \text{C}(S_2, K, \tau) - \text{Call}^{-}(S_1, S_2, K, \tau)$$

where the first two terms are standard (univariate) call options on S_1 and S_2 respectively.

3.5.4 Put Option on the Maximum of Two Assets

The holder of a *put option on the maximum of two assets* has the right to sell the maximum asset at the strike price at expiry. The payoff function is therefore

$$\text{Put}^{+}(S_1(T), S_2(T), K) = \max(K - \max(S_1(T), S_2(T)), 0)$$

and can be seen in figure 3.2(d).

Stulz [48] derives the price of these options

$$\text{Put}^+(S_1, S_2, K, \tau) = \text{Call}^+(S_1, S_2, K, \tau) - \text{Call}^+(S_1, S_2, 0, \tau) + KB$$

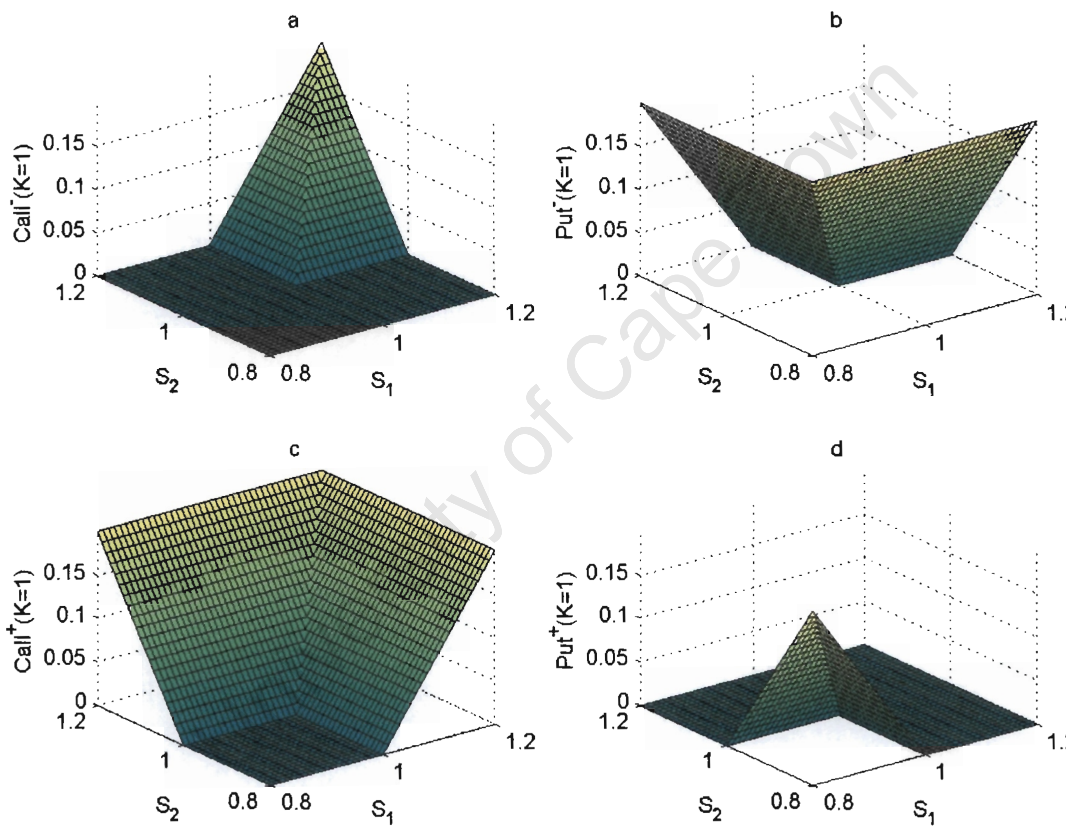

Figure 3.2: Payoff functions for a call on the minimum (a), put on the minimum (b), call on the maximum (c), put on the maximum (d). The strike is 1 for all.

3.5.5 Spread Options

Spread options have the following payoff function:

$$G(S_1(T), S_2(T), K) = \max(S_1(T) - S_2(T) - K, 0) \tag{3.5.2}$$

as seen in figure 3.3(b).

At present, no closed form solution exists for these instruments, except in the special case when $K = 0$. In this case, it is known as an *exchange option.*

Exchange Options

An *exchange option*, or *outperformance option*, gives the holder the right to exchange one unit of S_2 for a unit of S_1 at a specific time in the future. Equivalently, an exchange option can be thought of as a spread options with a strike of zero. The payoff function, as seen in figure 3.3(a), is

$$E(S_1(T), S_2(T)) = G(S_1(T), S_2(T), 0) = \max(S_1(T) - S_2(T), 0) \tag{3.5.3}$$

These instruments were first priced by Margrabe [35] under lognormal assumptions. It was shown that the price is given by

$$E(S_1, S_2, \tau) = S_1 N(d_1) - S_2 N(d_2) \tag{3.5.4}$$

where

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{S_1}{S_2}\right) + \frac{1}{2}\hat{\sigma}^2\tau}{\hat{\sigma}\sqrt{\tau}} \\ d_2 &= d_1 - \hat{\sigma}\sqrt{\tau} \\ \hat{\sigma} &= \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} \end{aligned}$$

ρ is the linear correlation coefficient between the returns of S_1 and S_2.

It can easily be shown that the payoff of a call option on the minimum of two assets with a strike of zero can be derived from the price of an exchange option on the same two assets, and the price of the first underlying asset.

$$\text{Call}^{-}(S_1, S_2, 0, \tau) = S_1 - E(S_1, S_2, \tau)$$

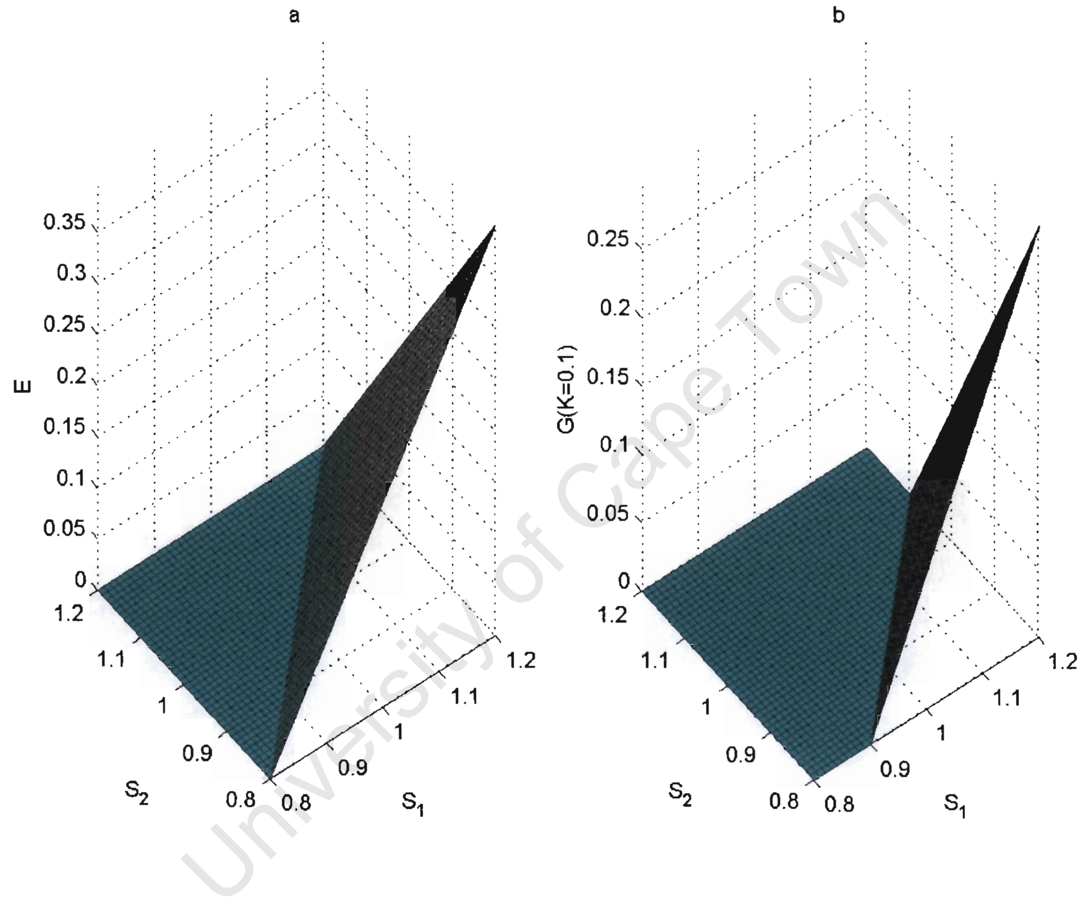

Figure 3.3: Payoff functions for an exchange option (a) and a spread option with $K = 0.1$ (b).

3.5.6 Digital Binary Options

Digital binary options are one of the simplest bivariate contingent claims, paying either one or nothing at expiry, depending on the state of two underlying variables. The *standard digital binary option* (which we will call a *digital binary call*) D_C pays one if each variable is greater than its particular strike.

Define a *digital binary put* D_P to be an instrument that pays one if the prices of the two assets are less than their respective strikes at expiry, and nothing otherwise.

Under Black-Scholes assumptions,

$$\begin{aligned} D_P(K_1, K_2) &= e^{-r\tau}\mathbb{Q}\left[S_1(T) \leq K_1, S_2(T) \leq K_2\right] \\ &= e^{-r\tau}\mathbb{Q}\left[\ln\left(\frac{S_1(T)}{S_1(0)}\right) \leq \ln\left(\frac{K_1}{S_1(0)}\right), \ln\left(\frac{S_2(T)}{S_2(0)}\right) \leq \ln\left(\frac{K_2}{S_2(0)}\right)\right] \end{aligned} \tag{3.5.5}$$

Now

$$\ln\left(\frac{S_1(T)}{S_1(0)}\right) \sim N\left(\left(r - \frac{1}{2}\sigma_1^2\right)T, \sigma_1\sqrt{T}\right)$$

and

$$\ln\left(\frac{S_2(T)}{S_2(0)}\right) \sim N\left(\left(r - \frac{1}{2}\sigma_2^2\right)T, \sigma_2\sqrt{T}\right)$$

Simple manipulations of the above normal distributions lead to the price

$$D_P(K_1, K_2) = e^{-r\tau}N_2(-d_{12}, -d_{22}, \rho) \tag{3.5.6}$$

where N_2 is the standard bivariate normal distribution function with correlation ρ and

$$d_{12} = \frac{\ln\left(\frac{S_1(0)}{K_1}\right) + \left(r - \frac{1}{2}\sigma_1^2\right)T}{\sigma_1\sqrt{T}}$$

$$d_{22} = \frac{\ln\left(\frac{S_2(0)}{K_2}\right) + \left(r - \frac{1}{2}\sigma_2^2\right)T}{\sigma_2\sqrt{T}}$$

The price of D_C follows from no arbitrage:

$$D_C(K_1, K_2) = D_P(K_1, K_2) + D_1(K_1) + D_2(K_2) - e^{-rT} \tag{3.5.7}$$

where D_1 and D_2 are univariate digital call options on S_1 and S_2 respectively.

It follows from equation 3.5.7 that under no-arbitrage, the price of a digital binary call D_C is bounded by the inequality

$$\max(D_1(K_1) + D_2(K_2) - e^{-rT}, 0) \leq D_C(K_1, K_2) \leq \min(D_1(K_1), D_2(K_2)) \tag{3.5.8}$$

Chapter 4

Application Methodology

In chapter 3 it was shown that any 2-colour rainbow can be priced from the implied bivariate risk-neutral density function. This chapter presents the methods and results of a practical implementation of the theory discussed in chapters 2 and 3. The computations were performed in Matlab. The functions and routines used are included on Disk1 and a basic explanation to the code and a list of Matlab variables and their descriptions can be found in appendix C. Exact results from any of the calculations can be extracted from Matlab once the main routine has been executed.

4.1 Data

The two underlying assets were chosen to be South African equity and bond indices. The inputs required are implied volatility skews for each of the underlying assets, and a historical set of returns. The data used in this exercise can be found in *ThesisData.xls* on the enclosed Disk1.

The implied volatility skew (from SAFEX) of one-month-to-expiry call options on Top 40 futures (expiring 16 September 2004) were used for recovering the implied risk-neutral distributions for equity. A quadratic skew was formulated for bonds in the absence of a realistic (and readily available) market skew[1], although any skew could have been used.

Monthly total return index prices for the all-share and all-bond indices were used to generate a set of monthly logreturns from which to calibrate the copulae. The particular sample used in this exercise is from 30 September 1995 to 30 September 2000. This five year period exhibits extremely strong positive dependence (see section 2.2) between the two underlying assets, as seen in figure 4.1. The red dot indicates the particular period used in this exercise.

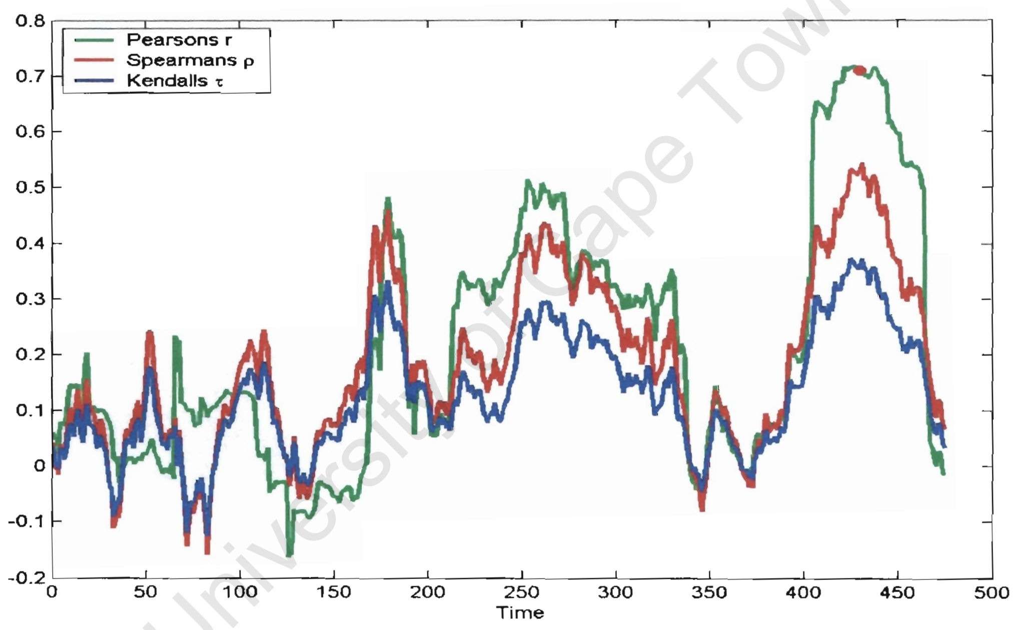

Figure 4.1: Measures of dependence between the South African equity and bond markets for rolling 60 month periods. The red dot indicates the particular period used in this exercise.

[1]Following the work of Shimko [45], Luciano and Cherubini [8] use quadratic skews and then compute the implied risk-neutral distributions from the closed form solution for the derivative of the call options prices with respect to strike. This research differs from the above papers in that the implied marginal risk-neutral distributions are completely nonparametric. So any two skews of a general nature can be used for input.

4.2 Risk-neutral Marginals

The risk-neutral marginals are calculated from the spread of traded call option prices. The method used depends on the important result of Breeden and Litzenberger [6] which is detailed in section 3.2. In the above formulation, it is assumed that a continuum of options exist across strike price. This is not the case. For this reason, *cubic splines* were used to interpolate the two implied volatility skews. This interpolation technique requires that the interpolated curve passes through all the given data points, is smooth in the first derivative and continuous in the second derivative [39]. A smooth skew translates into a smooth density function.

At this point it should be noted that the input skews are for options on *futures*, not the underlying indices themselves. So the option prices are given by the SAFEX-Black formula (appendix A), rather than the traditional Black-Scholes [4] formula. This has some important implications. Instead of equations 3.2.2 and 3.2.3, we have

$$F(s) = 1 + \frac{\partial C(K)}{\partial K}|_{K=s} \tag{4.2.1}$$

and

$$f(s) = \frac{\partial^2 C(K)}{\partial K^2}|_{K=s} \tag{4.2.2}$$

Note that any expectation under this risk-neutral measure must still be discounted at the risk-free rate to arrive at a price.

After interpolation of the implied volatility skews at the required data points, call option prices are calculated using the SAFEX-Black formula as detailed in appendix A. The volatility skews and call option prices are shown in figure 4.2. Without loss of generality, the underlying indices were both assumed to be 1. The risk-free continuously compounded interest rate was assumed to be $r = 8\%$, the time to expiry $\tau = \frac{1}{12}$. The futures prices were thus calculated as $e^{r\tau} = 1.0067$.

The implied risk-neutral distributions and densities were then approximated from the

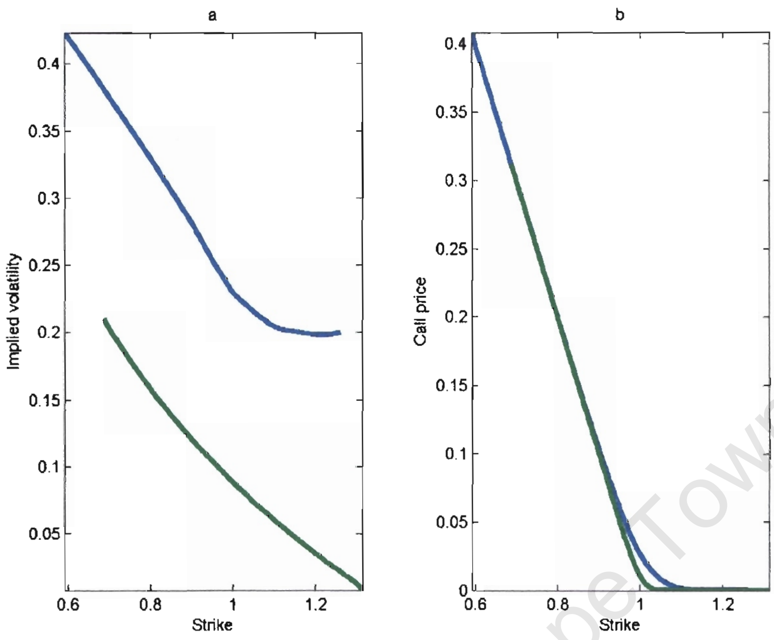

Figure 4.2: Call option implied volatilities (a) and prices (b). Blue lines represent equity options, and green lines represent bond options.

prices above using equations 4.2.1 and 4.2.2. The method of *central differences* was used to numerically differentiate these curves. These results are shown in figure 4.3.

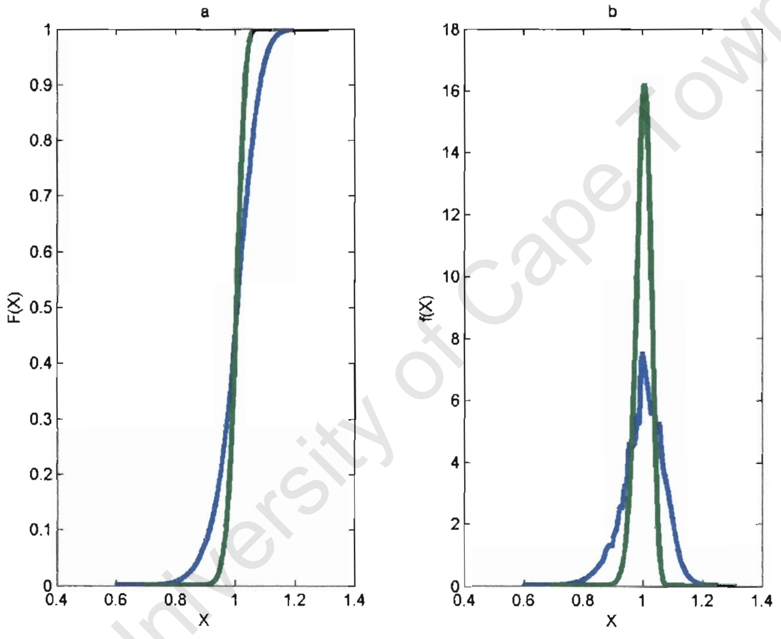

Figure 4.3: Risk-neutral marginal distributions (a) and densities (b) implied by options prices. Blue lines represent equity, and green lines represent bonds.

4.3 Copulae

The risk-neutral marginal distributions from option prices can then be combined, using Sklar's theorem (theorem 2.1.1), with any given copula to give us a joint risk-neutral distribution with the required marginals. We assume that the dependence structure of the past is likely to remain similar going forward. All copulae for pricing are therefore calibrated from the historical return data, either directly (nonparametric) or indirectly (parametric).

4.3.1 Nonparametric Copulae

Empirical Copula

This copula, shown in figure 4.4 was computed using the method described in section 2.4.1.

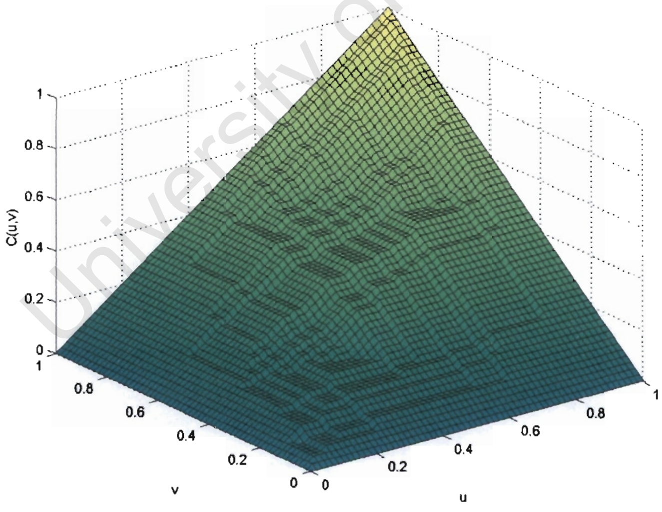

Figure 4.4: Empirical copula estimated from historical monthly returns of South African equities and bonds.

Kernel Copula

Kernel methods were used to approximate the bivariate density of historical returns. Equation 2.4.8 gives the particular equation used. In effect, this method, "spheres" the data before employing a bivariate Gaussian kernel to approximate the resulting normalized density function. The approximated historical density function obtained, using a kernel bandwidth of 0.7278, can be seen in figure 4.5. This bandwidth was chosen by inspection. The resulting density function appears to exhibit a reasonable trade-off between smoothness and kernel bias as discussed in section 2.4.2.

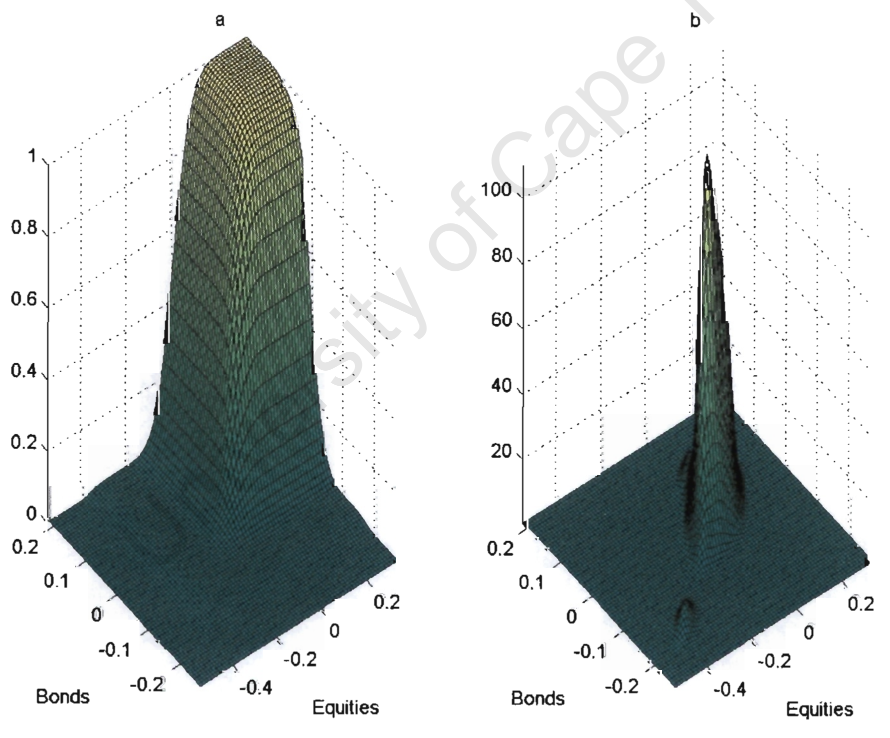

Figure 4.5: Kernel estimates of the historical bivariate distribution (a) and density (b) functions.

The bivariate historical probability density function was numerically integrated to give the cumulative distribution from which we remove the effects of the two marginal distributions of the historic data using corollary 2.1.2. The resulting distribution has uniform marginals and is thus the empirical *kernel copula* which can be seen in figure 4.6. Notice how smooth the kernel copula is in comparison to the empirical copula shown in figure 4.4.

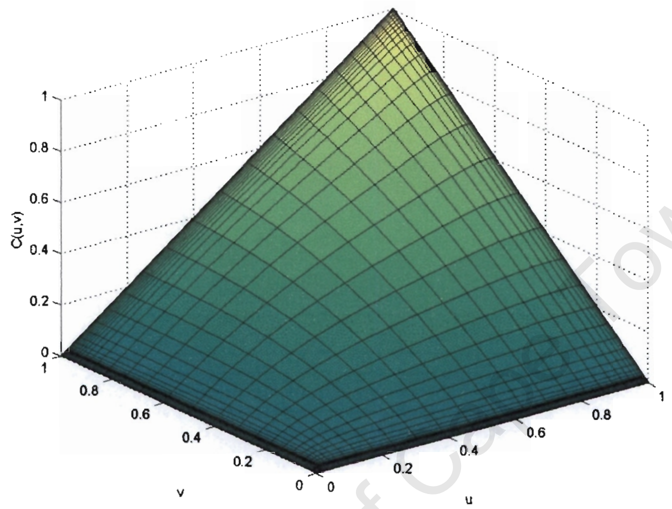

Figure 4.6: Kernel copula estimated from historical monthly returns of South African equities and bonds.

The procedure above requires the *quasi-inverse* (definition 2.1.7) of each historical marginal distribution to be calculated.

Theorem 2.1.1 is then used to construct the implied joint risk-neutral distribution function H from the kernel copula (figure 4.6) and the marginal risk-neutral distributions F and G that were discussed in section 4.2. We have

$$H(x,y) = \bar{H}(\bar{F}^{(-1)}(F(x)), \bar{G}^{(-1)}(G(x)))$$

where $\bar{H}$ is the historical joint distribution, and $\bar{F}$ and $\bar{G}$ are the historical marginals. Differentiation with respect to each asset yields the joint risk-neutral density function h implied by the market.

$$h(x,y) = \frac{\partial^2 H(x,y)}{\partial x \partial y}$$

$H(x, y)$ and $h(x, y)$ are shown in figure 4.7.

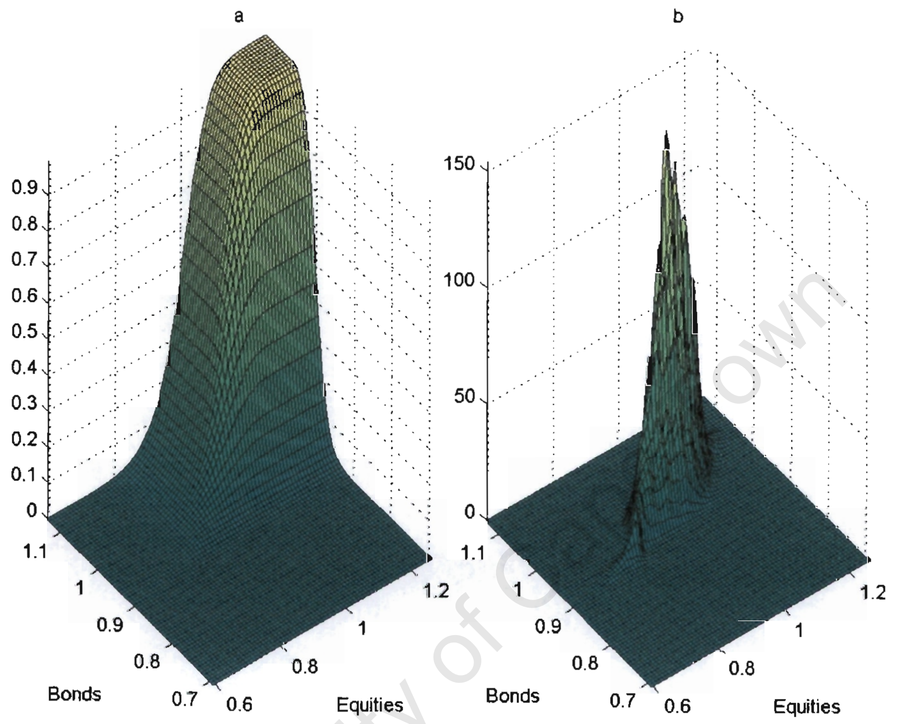

Figure 4.7: Joint risk-neutral distribution (a) and density (b) implied by options prices and kernel copula.

4.3.2 Parametric Copulae

A variety of parametric copulae, as discussed in section 2.3, were computed and used for estimating the bivariate risk-neutral distribution function H. These copulae were calibrated from the same data set as the nonparametric copulae described in section 4.3.1.

$$H(x,y) = C(F^{(-1)}(x), G^{(-1)}(y))$$

Normal copula

The linear correlation coefficient of the data sample was calculated to be $\rho = 0.710$. This single parameter completely defines the normal copula to be used. Equation 2.3.1, in two dimensions, becomes

$$C(u,v) = \Phi(\Phi^{-1}(u), \Phi^{-1}(v), \rho)$$

where $\Phi(x,y,\rho)$ is the bivariate standard cumulative normal distribution with correlation $\rho = 0.710$, evaluated at x and y and $\Phi^{-1}(u)$ is the inverse of the standard normal distribution function.

Remark 4.3.1. Throughout this exercise, where an approximation for the bivariate standard normal distribution is required, the algorithm suggested by West [51] will be used. See remark 3.5.1 for more details about this matter.

Archimedean Copulae

Kendall's rank correlation coefficient ($\tau = 0.256$) was used to calculate the parameters for the Gumbel ($\alpha = 1.34$), Clayton ($\alpha = 0.687$) and Frank ($\alpha = 2.52$) copulae. Equations 2.3.4, 2.3.6 and 2.3.8, when used in reverse, yield the above results.

Equations 2.3.3, 2.3.5 and 2.3.7 were used to compute the Gumbel, Clayton and Frank copulae respectively, from the parameters above.

Maximum and Minimum Copulae

When using historic returns to infer the dependence structure, the frequency of the data, the number of data points considered and the kernel bandwidth used in the density estimation procedure will have an effect. On the other hand, the minimum, maximum and product copulae are completely objective, leaving nothing to choice. The minimum and maximum copulae (see section 2.1.2) correspond to perfect positive and negative dependence. They represent the range of probability in which no arbitrage opportunities exist. If a price fell outside these bounds, a super-replicating strategy could be constructed to take advantage of this "free lunch".

Product Copula

The product copula, which is discussed in section 2.1.3, represents the case of perfect independence. It is instructive to calculate prices using this copula to get an idea of where the point of independence lies. It is not necessary for these prices to lie in the middle of the upper and lower bounds. The relationship is generally not symmetric.

4.4 Black-Scholes Price

Since the Black-Scholes model (as discussed in section 3.3) has closed form solutions for pricing the most well-known 2-colour rainbows, it is the obvious reference paradigm. Prices from the copula models can be compared with those from the Black-Scholes model. It is not necessary to use the closed form solutions. We can assume flat volatility skews from the at-the-money options, and use the lognormal marginal distributions that are implied by the Breeden and Litzenberger result (equation 3.2.1), together with the normal copula with the appropriate correlation coefficient ($\rho = 0.710$) to generate the same bivariate normal distribution which is implicit in the Black-Scholes formulation. In this way we can price any derivative using the same numerical procedures that were used in the copula models.

In this exercise, the at-the-money volatilities of equities and bonds (interpolated from the input volatility skews), were $\sigma_1 = 22.978\%$ and $\sigma_2 = 8.8305\%$ respectively.

Where possible, closed form solutions were calculated to get an idea of the inaccuracies that are due to the numerical procedures alone. The formulas for these solutions are given in section 3.5 and the solutions are presented in the price tables in appendix B. The input parameters throughout were $S_1 = S_2 = 1$, $\sigma_1 = 22.978\%$, $\sigma_2 = 8.8305\%$, $r = 8\%$, $\tau = \frac{1}{12}$, and $\rho = 0.710$.

4.5 Pricing

Sections 4.2 and 4.3 set out the methods used for calculating risk-neutral marginal distributions from option prices, and various forms of copulae. The bivariate risk-neutral distributions follow directly from theorem 2.1.1. Differentiating with respect to each asset (using a 2-dimensional central difference algorithm) yields the risk-neutral density $q(S_1, S_2)$.

Given any European-style bivariate payoff function $G(S_1, S_2)$, prices can be calculated using equation 3.1.2. The double integral is approximated using a very basic numerical integration procedure.

Chapter 5

Results

The prices of the following instruments are calculated for a range of strike prices:

1. Options on the minimum (sections 3.5.1 and 3.5.2)
2. Options on the maximum (sections 3.5.3 and 3.5.4)
3. Spread options (section 3.5.5)
4. Digital binary options (section 3.5.6)

Detailed results are presented in appendix B. In each of the pricing figures that follow, three graphs are shown. Each one has the range of strike prices on its horizontal axis. Appendix B gives more detailed descriptions of the abbreviations used in the graphs.

The top graph shows the no-arbitrage bounds (using the maximum and minimum copulae) and the point of independence (product copula). The solid lines represent the prices obtained from the implied risk-neutral marginals (section 4.2) and the dotted lines show those that are implied by the Black-Scholes model (section 4.4).

The middle graph shows the solid lines mentioned above as reference points and the prices calculated from historical dependence functions.

The bottom graph shows the above prices relative to the prices obtained from the Black-Scholes model.

The following codes are used in the graphs:

BS	Normal copula	Lognormal marginals (based on at-the-money options)
Emp	Empirical copula	Market implied marginals
Kernel	Kernel copula	Market implied marginals
Normal	Normal copula	Market implied marginals
Frank	Frank's copula	Market implied marginals
Gumbel	Gumbel copula	Market implied marginals
Clayton	Clayton copula	Market implied marginals
CF	Closed form solution	Same assumptions as BS

5.1 Options on the Minimum of Two Assets

The prices of call options on the minimum of the two assets can be seen in figure 5.1. The no-arbitrage bounds, shown in the top graph, were found to converge to zero quite quickly as the strike was increased. The range of possible prices gets wider at lower strikes. The BS model consistently underestimates these price bounds at lower strikes.

It can be seen from the bottom graph below that, at higher strikes, the BS model overstates the price in comparison to the copula models - particularly the empirical copula model.

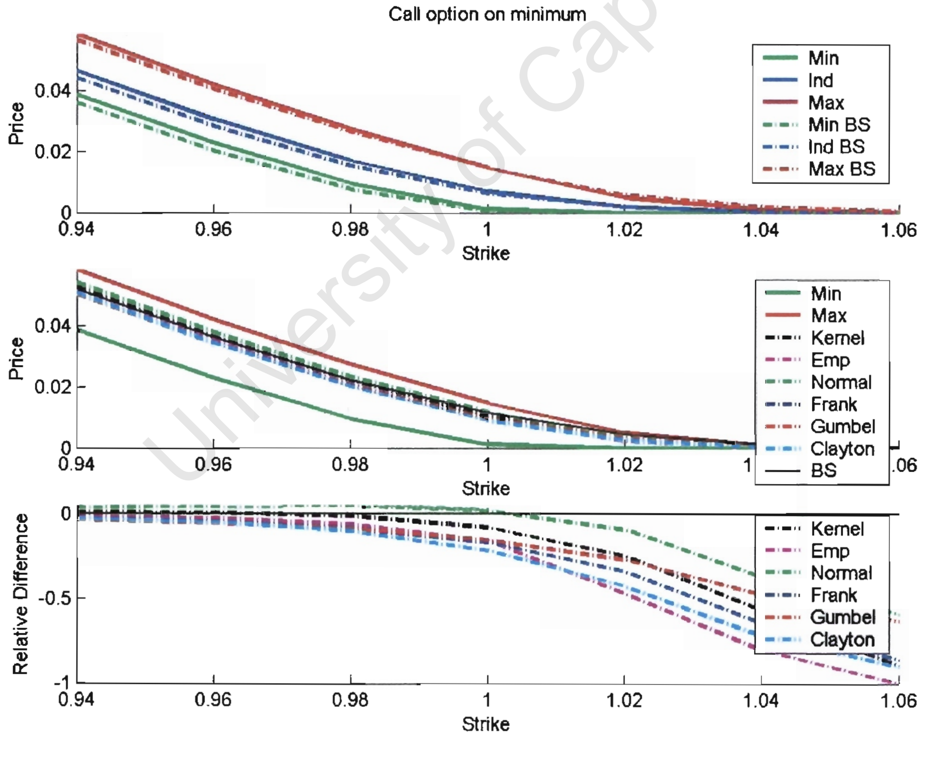

Figure 5.1: Call option on the minimum of two assets

The prices of put options on the minimum of the two assets can be seen in figure 5.2. The no-arbitrage bounds, shown in the top graph, were both found to converge to zero quite quickly as the strike was increased. The range of possible prices gets wider at lower strikes. Upon inspection, the BS model appears to yield similar price bounds as the copula model.

The relative difference graph shows the empirical copula model to be an outlier once again. The BS model consistently underprices these options, particularly at lower strikes.

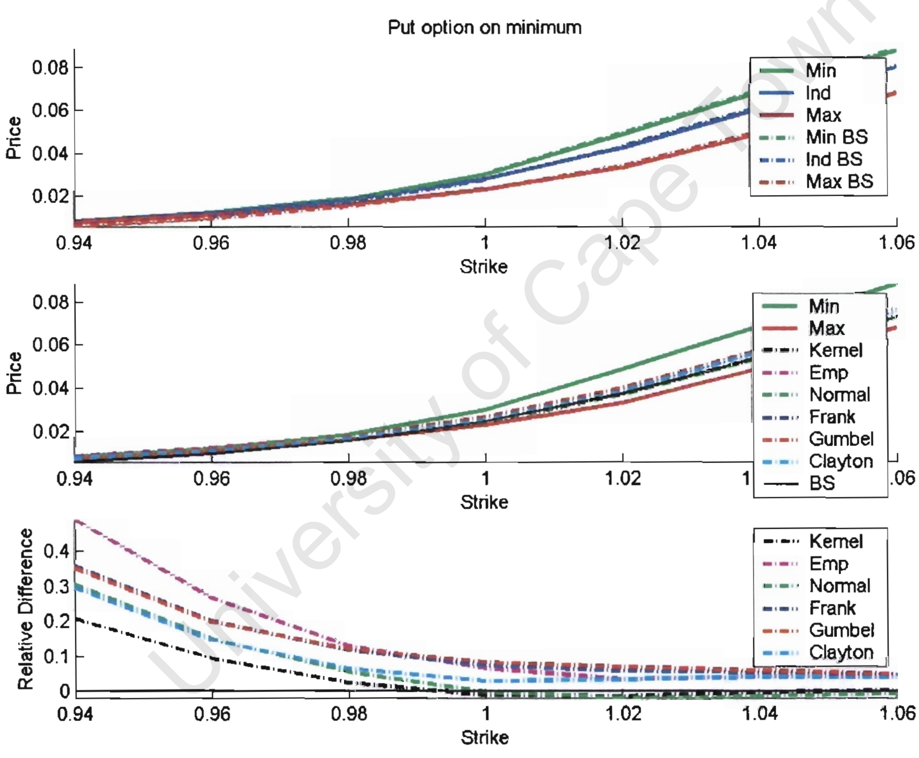

Figure 5.2: Put option on the minimum of two assets

5.2 Options on the Maximum of Two Assets

The prices of call options on the maximum of the two assets can be seen in figure 5.3. The no-arbitrage bounds (shown in the top graph) were both found to converge to zero quite quickly as the strike was increased. The range of possible prices gets wider at lower strikes. Once again, the BS model appears to replicate those of the copula model fairly accurately.

The empirical copula method is again the outlier in graph three. For this option, the BS model under-estimates prices at low strikes, and over-estimates them at high strikes.

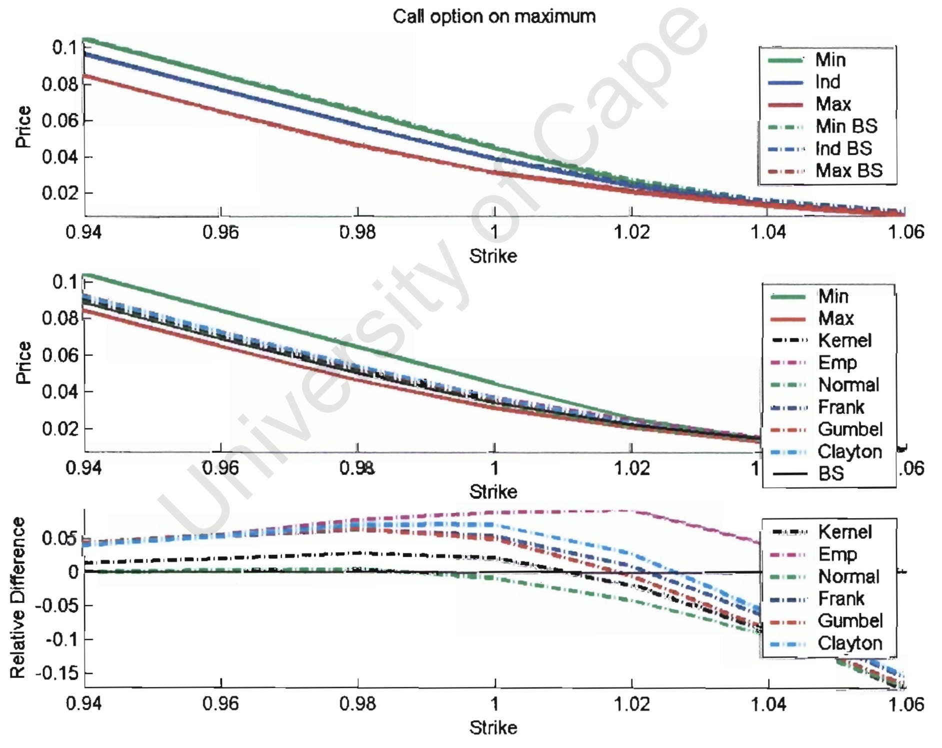

Figure 5.3: Call option on the maximum of two assets

The prices of put options on the maximum of the two assets can be seen in figure 5.4. The no-arbitrage bounds were both found to converge to zero quite quickly as the strike was decreased. The range of possible prices gets wider at higher strikes. The BS model slightly over-estimates the price bounds at higher strikes.

For this option, the BS price seems to be fairly accurate. At low strikes, the normal and Clayton copula models yield prices in excess of BS, whereas the Kernel, Frank and Gumbel copulae yield lower prices. The empirical copula method is a distant outlier in this case. At higher strikes the BS model consistently overprices the options.

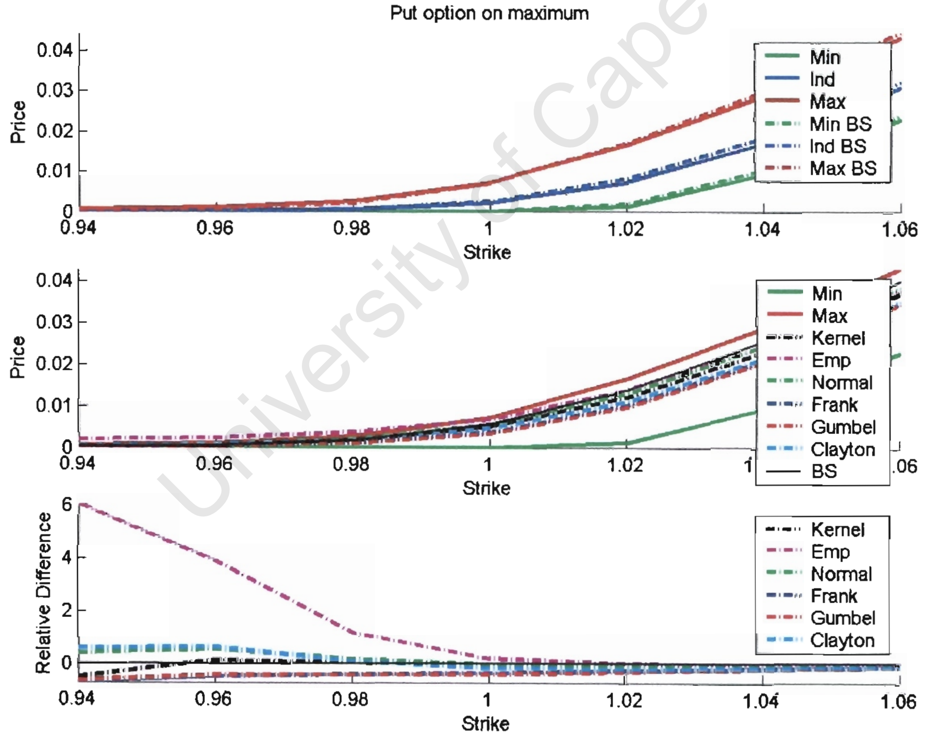

Figure 5.4: Put option on the maximum of two assets

5.3 Spread Options

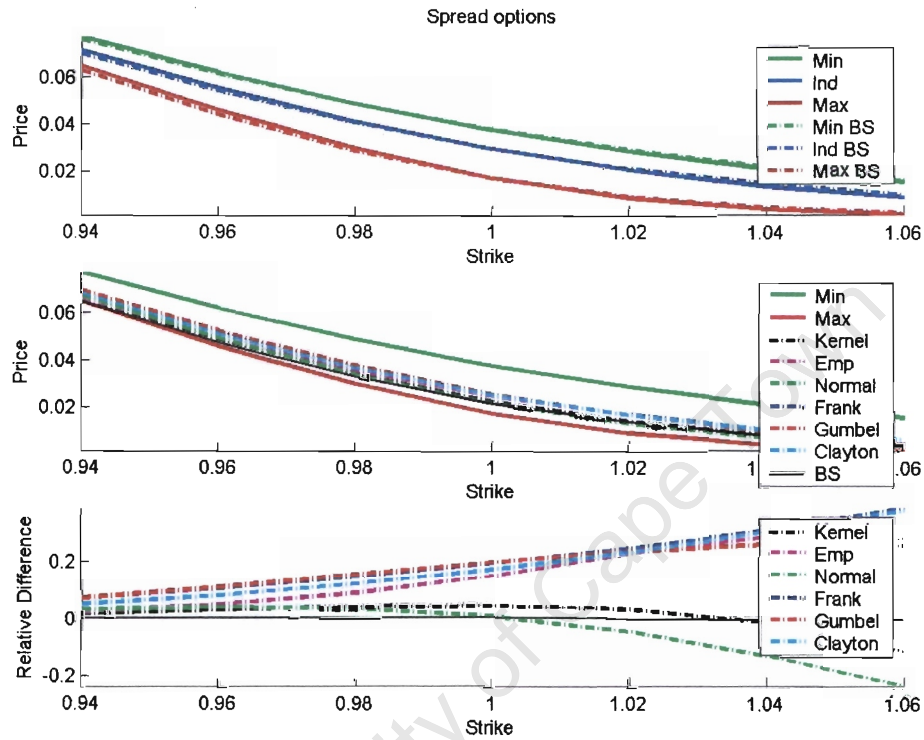

Figure 5.5: Spread option no-arbitrage bounds and prices for a range of strike prices.

The top graph in figure 5.5 shows the no-arbitrage bounds of spread option prices for a range of strike prices. The prices obtained by using the product copula are also shown. All these prices decrease as the strike increases in much the same manner as a call option does. Furthermore, the no-arbitrage bound for maximum positive dependence is lower than for that of maximum negative dependence. This follows from the fact that $\frac{\partial^2 G}{\partial S_1 \partial S_2}$ is nonpositive. It can be seen that, at low strikes, the BS model tends to under-estimate the fair price of the spread option; whereas at high strikes, these prices are over-estimated.

The BS model gives roughly the same prices as the kernel copula model. The normal copula gives lower prices and the other copula models give higher prices. This time the empirical copula model gives very similar prices as the Archimedean copulae.

5.4 Digital Binary Options

The prices of digital binary calls can be seen in figure 5.6. The no-arbitrage bounds (shown in the top graph) were both found to converge to zero as the strike increased. The range of possible prices is extremely wide, particularly when the strike is close to 1. The BS model consistently under-estimates these bounds, particularly when the strike is close to 1.

The BS model appears to replicate the prices of the other models fairly accurately while the strike is less than 1. At higher strikes, the BS model considerably over-estimates prices.

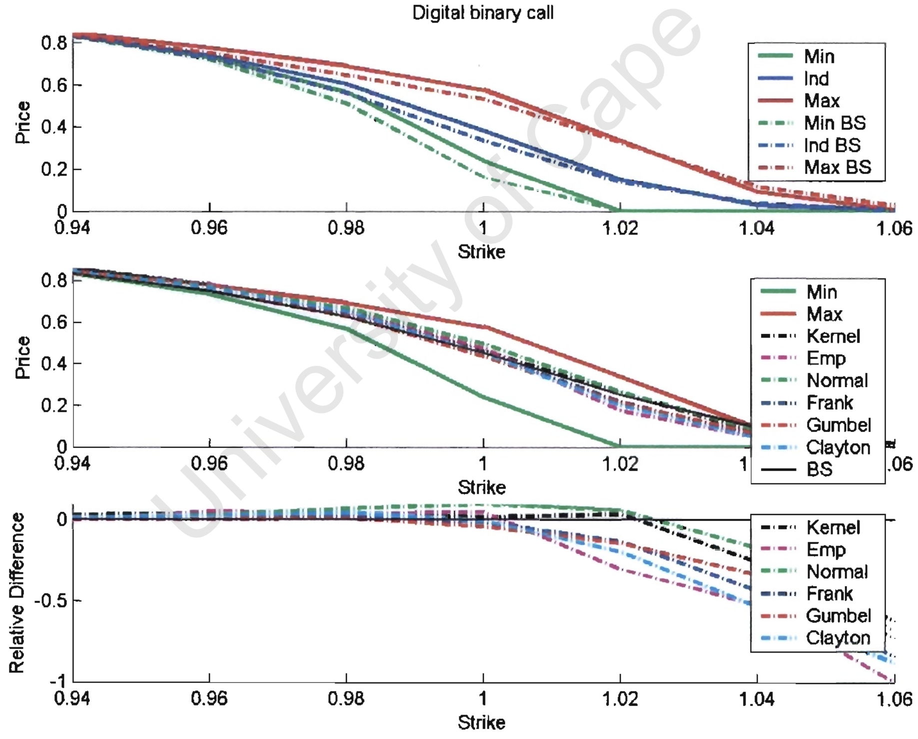

Figure 5.6: Digital binary call pays 1 if both equities and bonds are greater than strike.

The prices of digital binary puts can be seen in figure 5.7. The no-arbitrage bounds (shown in the top graph) were both found to converge to zero as the strike decreased. The range of possible prices is extremely wide, particularly when the strike is close to 1. The BS model consistently over-estimates these bounds, particularly when the strike is close to 1.

The BS model appears to replicate the prices of the other models fairly accurately while the strike is greater than 1. At lower strikes, the BS model under-estimates prices from the empirical, kernel,normal and Clayton models, but over-estimates those from the Gumbel and Frank models.

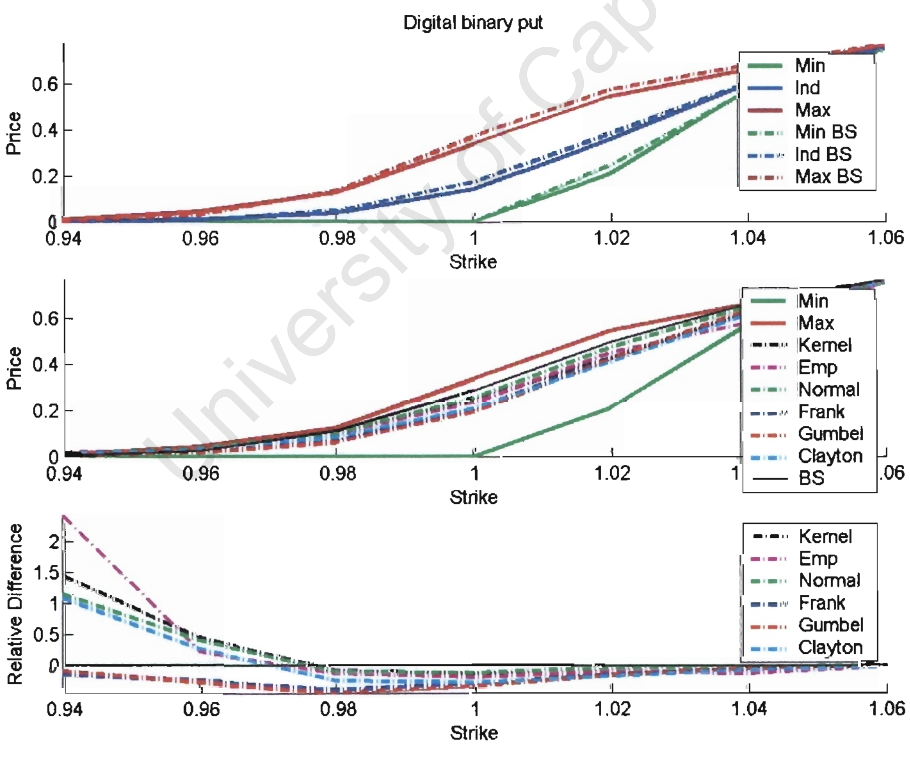

Figure 5.7: Digital binary put pays 1 if both equities and bonds are less than strike.

5.5 Informal Analysis

The results present a number of interesting aspects.

Firstly the trade off between *numerical error* and *model error* should be considered. The copula models, particularly the nonparametric copula models, will have considerable numerical error. The integration method (and others) used in this exercise was archaic - to say the least. When the variables are highly dependent, the shape that must be integrated becomes very thin - tending towards a plane. One would expect numerical error to be higher in these regions. One can get an idea of the numerical errors involved by examining the closed form solutions (**CF**) in comparison to the **BS** solutions in the result tables in appendix B. Both cases have the same underlying assumptions, so we can ignore model error. The **BS** prices have considerable numerical error however.

However, the underlying model of **BS** and **CF** is wrong (to some degree). We know this from the existence of the implied volatility skews. We can get an idea of how wrong by looking at the differences between the **BS** prices and the **Normal** prices which have roughly the same numerical errors, but different marginals which lead to different models. Assuming that the **Normal** model is correct, we can get an idea of how wrong flat skews are.

Model error can be decomposed into *marginal error* and *copula error*. The discussion in the previous paragraph covers marginal error. Viewing prices of the copula models in comparison to each other, we can get an idea of the copula error involved. Of course the copulae are bounded by the maximum and minimum copulae, at which points the copula error becomes zero, but these would hardly ever be used for pricing. The range of prices obtained using different copula models, all calibrated from the same data set, will be referred to as *copula spread*, the bounds of which can be regarded as robust estimates of the option prices.

The bottom graph of each of the preceding result figures elucidates the amount of model error in total. The range for all of the copula models (seen most clearly in the bottom graph) is the copula spread.

The results show that the copula spread can be very significant (in comparison to model error) for some instruments at some strikes, in which case it may be safer to use the closed form solutions under Black-Scholes assumptions. See, for example, the results for spread options (figure 5.5) at high strikes. For other instruments, the **BS** price differs significantly (in the same direction) from those of all the copula models. In which case any of the copula models would be preferable. See, for example, the results for call options on the minimum (figure 5.1) at high strikes.

The **Frank** prices and **Gumbel** prices are always quite similar. Most of the time the **Clayton** prices are also roughly equal to these, but with a few exceptions. See figure 5.4, where the **Clayton** prices differ significantly from **Frank** and **Gumbel**.

The **Emp** prices are almost always outliers. An example of an exception to this rule can be seen in figure 5.5.

Chapter 6

Conclusions

Copula methods were found to present an elegant solution to the problem of pricing 2-colour rainbows - the essence of which is to estimate the joint risk-neutral probability distribution of the two underlying assets. This distribution should be consistent with the implied marginal risk-neutral distributions of each asset in isolation. These marginals can be recovered from the spread of standard call options available on each asset in a number of ways. Copulae allow us to combine these marginal distributions into a joint distribution.

The maximum and minimum copulae define the probability bounds of a multivariate distribution. The results of this work show that prices of rainbows associated with these bounds can differ, to varying extents, when calculated from normal marginals or implied risk-neutral marginals.

The copula can be calibrated from historical return data in many ways. A variety of one-parameter copula families can be completely defined by Kendall's rank correlation coefficient τ, amongst others. A normal copula can be computed from the linear correlation coefficient. These parametric copulae allow us to easily combine any two marginal distributions.

Copulae can also be approximated from the data nonparametrically. Two of these methods were outlined in this dissertation. The empirical and kernel copulae include information from the data that is overlooked by parametric copulae.

Further research needs to be carried out in a number of areas. This research uses just one set of input volatility skews. How do these change over time, particularly with reference to the second asset? Does a steepening of the one skew imply a steepening of the other etc? Furthermore, how do the copulae evolve over time? Dynamic copula models are considered in [50].

Very basic methods, based on Kendall's τ alone, were used to calibrate the parametric copulae. Other methods, such as maximum likelihood estimation, could be used to calibrate these copulae more accurately. See, for example, [9].

It seems that there are times (in terms of instrument and strike) when any copula model is superior to the Black-Scholes model, and other times when the copula spread is so great that it may be safer to use the latter model. Further research could be done into why and when the copula models diverge.

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Appendix A

The SAFEX-Black Options Pricing Formula

An important fact about the South African Futures Exchange (SAFEX) is that options, as well as futures, are fully margined. This leads to a modified Black-Scholes pricing formula that includes neither the underlying's dividend yield nor the risk-free rate. The pricing formula is known as the *SAFEX-Black formula*. The derivation can be found in [52].

Let F be the futures price, and σ its volatility. A SAFEX call option on this future, with a strike of K and time to expiry τ, is

$$C = FN(d_1) - KN(d_2) \tag{A.0.1}$$

The price of the corresponding put is

$$P = KN(-d_2) - FN(-d_1) \tag{A.0.2}$$

where

$$d_1 = \frac{\ln\left(\frac{F}{K}\right) + \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}$$

and

$$d_2 = d_1 - \sigma\sqrt{\tau}$$

Appendix B

Result Tables

The tables that follow contain detailed results, including closed form solutions where they exist. Each table represents a particular instrument. Each column represents a particular strike price and the rows display sets of prices obtained for a variety of methods and dependence assumptions.

An explanation of the codes in the leftmost column follows:

BS	Normal copula	Lognormal marginals (based on at-the-money options)
Emp	Empirical copula	Market implied marginals
Kernel	Kernel copula	Market implied marginals
Normal	Normal copula	Market implied marginals
Frank	Frank's copula	Market implied marginals
Gumbel	Gumbel copula	Market implied marginals
Clayton	Clayton copula	Market implied marginals
CF	Closed form solution	Same assumptions as BS

BS and **CF** are both based on the same Black-Scholes assumptions. The difference between these two sets of results gives a useful indication of the amount of numerical error that is inherent in the pricing methodology.

Min, **Ind** and **Max** represent prices obtained under market implied marginals, using the minimum, product and maximum copula respectively. **Min BS**, **Ind BS** and **Max BS** represent prices obtained under lognormal marginal assumptions, using the minimum, product and maximum copula respectively. **Min CF**, **Ind CF** and **Max CF** are the closed form solutions, under Black-Scholes assumptions, for correlations of -1, 0 and 1 respectively.

Each table consists of three frames. The first frame holds four blocks of data on top of each other:

The first block shows prices that are based on the assumption that future dependencies will be the same as those of the past. These are, in effect, the "best-view" prices. The second block represents prices obtained under the assumption that the two assets returns will be perfectly negatively dependant. The third block represents independence, and the fourth - perfect positive dependence.

The third and fourth frames show the absolute and relative differences between the prices in the first block and the **BS** price.

Price	Call option on the minimum						
	Strike						
	0.94	**0.96**	**0.98**	**1**	**1.02**	**1.04**	**1.06**
BS	0.052	0.036	0.022	0.012	0.005	0.001	0.000
Emp	0.051	0.035	0.021	0.010	0.002	0.000	0.000
Kernel	**0.053**	**0.036**	**0.022**	**0.011**	**0.003**	**0.001**	**0.000**
Normal	0.054	0.038	0.023	0.012	0.004	0.001	0.000
Frank	0.050	0.034	0.020	0.010	0.003	0.001	0.000
Gumbel	0.050	0.034	0.020	0.010	0.003	0.001	0.000
Clayton	0.050	0.034	0.020	0.009	0.003	0.000	0.000
CF	0.051	0.035	0.021	0.011	0.004	0.001	0.000
Min	0.039	0.023	0.010	0.001	0.000	0.000	0.000
Min BS	0.036	0.020	0.008	0.001	0.000	0.000	0.000
Min CF	0.035	0.019	0.007	0.000	0.000	0.000	0.000
Ind	0.046	0.030	0.017	0.007	0.002	0.000	0.000
Ind BS	0.044	0.028	0.015	0.006	0.002	0.000	0.000
Ind CF	0.043	0.027	0.014	0.006	0.002	0.000	0.000
Max	0.058	0.042	0.027	0.015	0.005	0.001	0.000
Max BS	0.056	0.040	0.026	0.015	0.006	0.002	0.001
Max CF	0.055	0.039	0.025	0.014	0.005	0.001	0.000

Price relative to BS price

Emp	-0.001	-0.001	-0.001	-0.002	-0.002	-0.001	0.000
Kernel	0.001	0.000	0.000	-0.001	-0.001	-0.001	0.000
Normal	0.002	0.002	0.001	0.000	0.000	-0.001	0.000
Frank	-0.002	-0.002	-0.002	-0.002	-0.002	-0.001	0.000
Gumbel	-0.002	-0.002	-0.002	-0.002	-0.001	-0.001	0.000
Clayton	-0.002	-0.002	-0.002	-0.003	-0.002	-0.001	0.000
CF	-0.001	-0.001	-0.001	-0.001	-0.001	0.000	0.000

Percentage change relative to BS

Emp	-1%	-3%	-7%	-16%	-48%	-81%	-100%
Kernel	1%	1%	-1%	-8%	-25%	-57%	-89%
Normal	4%	5%	5%	2%	-10%	-38%	-59%
Frank	-3%	-5%	-8%	-17%	-34%	-65%	-86%
Gumbel	-3%	-5%	-9%	-16%	-27%	-48%	-63%
Clayton	-3%	-5%	-10%	-22%	-43%	-72%	-90%
CF	-2%	-3%	-5%	-8%	-13%	-26%	-58%

Figure B.1: Prices of call options on the minimum of two assets.

	Put option on the minimum						
	Strike						
Price	**0.94**	**0.96**	**0.98**	**1**	**1.02**	**1.04**	**1.06**
BS	0.006	0.010	0.015	0.024	0.037	0.054	0.073
Emp	0.009	0.012	0.018	0.026	0.038	0.056	0.075
Kernel	**0.007**	**0.010**	**0.016**	**0.024**	**0.037**	**0.054**	**0.073**
Normal	0.007	0.011	0.016	0.024	0.037	0.053	0.072
Frank	0.008	0.011	0.017	0.026	0.039	0.057	0.076
Gumbel	0.008	0.011	0.017	0.026	0.040	0.057	0.076
Clayton	0.007	0.011	0.016	0.025	0.038	0.056	0.075
CF	0.005	0.009	0.015	0.025	0.038	0.055	0.074
Min	0.008	0.012	0.018	0.030	0.048	0.068	0.088
Min BS	0.006	0.010	0.017	0.030	0.049	0.069	0.089
Min CF	0.005	0.009	0.017	0.030	0.050	0.070	0.090
Ind	0.008	0.012	0.018	0.028	0.042	0.060	0.080
Ind BS	0.006	0.010	0.017	0.027	0.043	0.061	0.080
Ind CF	0.005	0.009	0.016	0.028	0.043	0.062	0.081
Max	0.007	0.011	0.016	0.023	0.033	0.049	0.068
Max BS	0.005	0.009	0.015	0.023	0.034	0.050	0.068
Max CF	0.005	0.009	0.015	0.023	0.034	0.051	0.069

Price relative to BS price

Emp	0.003	0.003	0.002	0.002	0.001	0.002	0.003
Kernel	0.001	0.001	0.000	0.000	0.000	0.000	0.000
Normal	0.002	0.001	0.001	0.000	-0.001	-0.001	-0.001
Frank	0.002	0.002	0.002	0.002	0.002	0.003	0.003
Gumbel	0.002	0.002	0.002	0.002	0.003	0.003	0.004
Clayton	0.002	0.001	0.001	0.001	0.001	0.002	0.003
CF	-0.001	-0.001	0.000	0.000	0.001	0.001	0.001

Percentage change relative to BS

Emp	49%	27%	13%	7%	3%	4%	4%
Kernel	21%	9%	2%	-1%	-1%	0%	0%
Normal	31%	15%	6%	0%	-2%	-1%	-1%
Frank	36%	20%	12%	7%	6%	5%	5%
Gumbel	35%	20%	12%	8%	7%	6%	5%
Clayton	30%	15%	6%	3%	3%	4%	4%
CF	-12%	-6%	-2%	0%	1%	2%	2%

Figure B.2: Prices of put options on the minimum of two assets.

Call option on the maximum

Price	Strike						
	0.94	**0.96**	**0.98**	**1**	**1.02**	**1.04**	**1.06**
BS	0.088	0.069	0.050	0.034	0.023	0.015	0.009
Emp	0.092	0.073	0.054	0.038	0.025	0.015	0.009
Kernel	**0.090**	**0.070**	**0.052**	**0.035**	**0.022**	**0.013**	**0.008**
Normal	0.088	0.069	0.051	0.034	0.022	0.013	0.008
Frank	0.092	0.072	0.053	0.036	0.023	0.014	0.008
Gumbel	0.092	0.073	0.054	0.036	0.023	0.013	0.008
Clayton	0.092	0.072	0.054	0.037	0.023	0.014	0.008
CF	0.087	0.067	0.049	0.033	0.021	0.014	0.009
Min	0.104	0.084	0.064	0.045	0.026	0.014	0.008
Min BS	0.105	0.085	0.065	0.045	0.027	0.016	0.010
Min CF	0.103	0.083	0.063	0.043	0.025	0.015	0.009
Ind	0.096	0.076	0.057	0.039	0.024	0.014	0.008
Ind BS	0.096	0.077	0.057	0.040	0.025	0.016	0.010
Ind CF	0.095	0.075	0.055	0.038	0.024	0.014	0.009
Max	0.084	0.065	0.047	0.031	0.021	0.013	0.008
Max BS	0.084	0.065	0.046	0.031	0.021	0.014	0.009
Max CF	0.083	0.063	0.045	0.030	0.021	0.014	0.009

Price relative to BS price

Emp	0.003	0.004	0.004	0.003	0.002	0.001	0.000
Kernel	0.001	0.001	0.001	0.001	0.000	-0.001	-0.002
Normal	0.000	0.000	0.000	0.000	-0.001	-0.001	-0.002
Frank	0.004	0.004	0.003	0.002	0.000	-0.001	-0.001
Gumbel	0.004	0.004	0.003	0.002	0.000	-0.001	-0.002
Clayton	0.004	0.004	0.004	0.002	0.001	-0.001	-0.001
CF	-0.002	-0.002	-0.002	-0.001	-0.001	-0.001	-0.001

Percentage change relative to BS

Emp	4%	5%	8%	9%	9%	4%	0%
Kernel	1%	2%	3%	2%	-2%	-8%	-17%
Normal	0%	0%	0%	-1%	-4%	-9%	-17%
Frank	4%	5%	6%	5%	1%	-6%	-15%
Gumbel	4%	5%	6%	5%	0%	-8%	-16%
Clayton	4%	5%	7%	7%	3%	-6%	-15%
CF	-2%	-2%	-3%	-4%	-5%	-6%	-8%

Figure B.3: Prices of call options on the maximum of two assets.

Put option on the maximum							
	Strike						
Price	0.94	0.96	0.98	1	1.02	1.04	1.06
BS	0.000	0.000	0.002	0.006	0.013	0.025	0.040
Emp	0.002	0.002	0.004	0.007	0.014	0.024	0.038
Kernel	**0.000**	**0.001**	**0.002**	**0.005**	**0.012**	**0.023**	**0.037**
Normal	0.000	0.001	0.002	0.005	0.013	0.024	0.038
Frank	0.000	0.000	0.001	0.004	0.010	0.020	0.034
Gumbel	0.000	0.000	0.001	0.003	0.009	0.020	0.034
Clayton	0.000	0.001	0.002	0.005	0.011	0.021	0.035
CF	0.000	0.000	0.002	0.006	0.014	0.026	0.041
Min	0.000	0.000	0.000	0.000	0.001	0.009	0.022
Min BS	0.000	0.000	0.000	0.000	0.002	0.010	0.024
Min CF	0.000	0.000	0.000	0.000	0.002	0.011	0.025
Ind	0.000	0.000	0.001	0.002	0.007	0.017	0.030
Ind BS	0.000	0.000	0.001	0.003	0.008	0.018	0.032
Ind CF	0.000	0.000	0.001	0.003	0.009	0.019	0.033
Max	0.001	0.001	0.003	0.007	0.016	0.028	0.043
Max BS	0.000	0.001	0.002	0.007	0.017	0.029	0.044
Max CF	0.000	0.000	0.002	0.007	0.017	0.030	0.045

Price relative to BS price

Emp	0.002	0.002	0.002	0.001	0.000	-0.001	-0.002
Kernel	0.000	0.000	0.000	-0.001	-0.002	-0.003	-0.003
Normal	0.000	0.000	0.000	0.000	-0.001	-0.001	-0.001
Frank	0.000	0.000	-0.001	-0.002	-0.004	-0.005	-0.005
Gumbel	0.000	0.000	-0.001	-0.002	-0.004	-0.005	-0.005
Clayton	0.000	0.000	0.000	-0.001	-0.003	-0.004	-0.005
CF	0.000	0.000	0.000	0.000	0.001	0.001	0.001

Percentage change relative to BS

Emp	603%	387%	116%	20%	1%	-5%	-5%
Kernel	-50%	16%	6%	-10%	-13%	-10%	-7%
Normal	39%	55%	18%	-3%	-6%	-5%	-4%
Frank	-71%	-48%	-39%	-35%	-27%	-19%	-13%
Gumbel	-62%	-43%	-41%	-39%	-30%	-20%	-14%
Clayton	58%	60%	10%	-16%	-20%	-17%	-12%
CF	-90%	-42%	-4%	5%	5%	4%	4%

Figure B.4: Prices of put options on the maximum of two assets.

Price	Spread options						
	Strike						
	0.94	**0.96**	**0.98**	**1**	**1.02**	**1.04**	**1.06**
BS	0.064	0.047	0.032	0.021	0.013	0.007	0.004
Emp	0.066	0.049	0.035	0.024	0.016	0.010	0.005
Kernel	**0.065**	**0.048**	**0.034**	**0.022**	**0.013**	**0.007**	**0.004**
Normal	0.066	0.049	0.033	0.021	0.012	0.006	0.003
Frank	0.069	0.052	0.037	0.025	0.016	0.010	0.006
Gumbel	0.069	0.052	0.037	0.025	0.016	0.009	0.005
Clayton	0.067	0.050	0.036	0.024	0.016	0.010	0.006
CF				0.021			
Min	0.076	0.061	0.048	0.037	0.028	0.021	0.015
Min BS	0.075	0.061	0.048	0.037	0.029	0.021	0.016
Min CF				0.037			
Ind	0.071	0.055	0.041	0.029	0.020	0.013	0.008
Ind BS	0.069	0.053	0.040	0.029	0.021	0.014	0.009
Ind CF				0.028			
Max	0.064	0.045	0.029	0.017	0.008	0.004	0.001
Max BS	0.062	0.044	0.028	0.016	0.009	0.004	0.002
Max CF				0.016			

Price relative to BS price

BS	0.001	0.002	0.003	0.003	0.003	0.002	0.001
Emp	0.001	0.001	0.001	0.001	0.000	0.000	0.000
Normal	0.002	0.002	0.001	0.000	-0.001	-0.001	-0.001
Frank	0.004	0.005	0.005	0.004	0.003	0.002	0.002
Gumbel	0.005	0.005	0.005	0.004	0.003	0.002	0.001
Clayton	0.003	0.004	0.004	0.004	0.003	0.002	0.002
CF				0.000			

Percentage error relative to BS price

Emp	2%	5%	9%	15%	23%	29%	29%
Kernel	2%	3%	4%	5%	4%	-1%	-11%
Normal	3%	4%	3%	0%	-5%	-14%	-24%
Frank	7%	10%	15%	19%	25%	31%	39%
Gumbel	7%	11%	15%	20%	24%	26%	26%
Clayton	5%	8%	12%	17%	23%	30%	38%
CF				-2%			

Figure B.5: Spread option prices.

	Digital binary call option						
	Strike						
Price	0.94	0.96	0.98	1	1.02	1.04	1.06
BS	0.828	0.742	0.620	0.448	0.248	0.095	0.023
Emp	0.825	0.776	0.645	0.468	0.172	0.045	0.000
Kernel	**0.851**	**0.771**	**0.644**	**0.455**	**0.256**	**0.070**	**0.009**
Normal	0.837	0.763	0.661	0.491	0.262	0.078	0.007
Frank	0.831	0.746	0.629	0.442	0.215	0.053	0.004
Gumbel	0.832	0.745	0.624	0.429	0.212	0.063	0.006
Clayton	0.837	0.759	0.646	0.445	0.198	0.044	0.003
CF	0.836	0.744	0.616	0.434	0.229	0.081	0.018
Min	0.828	0.727	0.562	0.237	0.000	0.000	0.000
Min BS	0.824	0.716	0.509	0.161	0.000	0.000	0.000
Min CF	0.832	0.716	0.493	0.124	0.000	0.000	0.000
Ind	0.829	0.736	0.600	0.378	0.150	0.030	0.002
Ind BS	0.825	0.723	0.556	0.331	0.137	0.036	0.006
Ind CF	0.833	0.724	0.545	0.313	0.121	0.029	0.004
Max	0.839	0.770	0.686	0.573	0.334	0.090	0.007
Max BS	0.829	0.745	0.642	0.528	0.325	0.117	0.026
Max CF	0.836	0.748	0.641	0.523	0.297	0.098	0.021
Price relative to BS price							
Emp	-0.003	0.035	0.025	0.020	-0.076	-0.050	-0.023
Kernel	0.022	0.029	0.024	0.007	0.008	-0.025	-0.014
Normal	0.009	0.021	0.041	0.043	0.014	-0.017	-0.016
Frank	0.003	0.004	0.009	-0.006	-0.033	-0.042	-0.019
Gumbel	0.003	0.004	0.004	-0.019	-0.036	-0.032	-0.016
Clayton	0.009	0.017	0.026	-0.003	-0.050	-0.051	-0.020
CF	0.007	0.002	-0.005	-0.014	-0.019	-0.014	-0.004
Percentage change relative to BS							
Emp	0%	5%	4%	4%	-31%	-53%	-100%
Kernel	3%	4%	4%	2%	3%	-26%	-62%
Normal	1%	3%	7%	10%	6%	-18%	-69%
Frank	0%	1%	1%	-1%	-13%	-44%	-84%
Gumbel	0%	0%	1%	-4%	-15%	-34%	-73%
Clayton	1%	2%	4%	-1%	-20%	-54%	-88%
CF	1%	0%	-1%	-3%	-8%	-15%	-19%

Figure B.6: Prices of digital binary call options.

Price	Digital binary put option						
	Strike						
	0.94	**0.96**	**0.98**	**1**	**1.02**	**1.04**	**1.06**
BS	0.005	0.026	0.111	0.287	0.495	0.657	0.766
Emp	0.016	0.031	0.098	0.233	0.449	0.576	0.754
Kernel	**0.011**	**0.037**	**0.102**	**0.247**	**0.425**	**0.620**	**0.762**
Normal	0.010	0.036	0.100	0.254	0.472	0.645	0.763
Frank	0.004	0.019	0.068	0.206	0.424	0.620	0.760
Gumbel	0.004	0.019	0.062	0.193	0.421	0.630	0.762
Clayton	0.009	0.032	0.084	0.208	0.408	0.611	0.759
CF	0.003	0.028	0.123	0.309	0.520	0.679	0.784
Min	0.000	0.000	0.000	0.000	0.210	0.567	0.756
Min BS	0.000	0.000	0.000	0.000	0.247	0.562	0.743
Min CF	0.000	0.000	0.000	0.000	0.291	0.597	0.765
Ind	0.002	0.009	0.038	0.141	0.360	0.597	0.758
Ind BS	0.001	0.007	0.046	0.171	0.383	0.598	0.749
Ind CF	0.001	0.008	0.052	0.189	0.412	0.627	0.770
Max	0.011	0.043	0.124	0.337	0.543	0.657	0.764
Max BS	0.005	0.029	0.133	0.368	0.572	0.679	0.769
Max CF	0.004	0.032	0.148	0.399	0.587	0.695	0.786

Price relative to BS price

Emp	0.011	0.006	-0.013	-0.055	-0.046	-0.081	-0.012
Kernel	0.006	0.011	-0.009	-0.040	-0.070	-0.037	-0.004
Normal	0.005	0.010	-0.011	-0.034	-0.023	-0.012	-0.002
Frank	-0.001	-0.006	-0.044	-0.082	-0.070	-0.037	-0.006
Gumbel	0.000	-0.007	-0.049	-0.095	-0.073	-0.028	-0.003
Clayton	0.005	0.007	-0.027	-0.079	-0.087	-0.047	-0.007
CF	-0.001	0.003	0.012	0.022	0.025	0.021	0.018

Percentage change relative to BS

Emp	244%	22%	-12%	-19%	-9%	-12%	-2%
Kernel	143%	45%	-9%	-14%	-14%	-6%	-1%
Normal	114%	40%	-10%	-12%	-5%	-2%	0%
Frank	-16%	-25%	-39%	-28%	-14%	-6%	-1%
Gumbel	-10%	-28%	-44%	-33%	-15%	-4%	0%
Clayton	108%	26%	-24%	-28%	-18%	-7%	-1%
CF	-23%	10%	10%	8%	5%	3%	2%

Figure B.7: Prices of digital binary put options.

Appendix C

Matlab User Manual

Disk1 contains an Excel spreadsheet (*ThesisData.xls*) with the actual input data, an electronic version of this paper (*Thesis.pdf*), and a folder (*Matlab*) with a Matlab "*mat*" file (from which the required input data is loaded) and an assortment of Matlab "m-file" routines and functions. In order to run the application, copy the folder *Matlab* on the disk to a destination on the C drive. Open Matlab (not included on disk) and point the current directory box to the destination above.

In the command window, type "Copula" and press "Enter". The application could take as long as 5 minutes to run.

The "*mat*" file *matlabinput* holds two matrices. *optionsdata* is a 41-by-4 matrix containing the two input volatility skews. The four columns are (in order): strike (asset 1), vol (asset 1), strike (asset 2), vol (asset 2). This matrix can (in theory) be modified to hold any two skews. The number of rows (strikes) must be greater than 10. The two assets must have the same number of strikes. **data** is a 534-by-2 matrix containing monthly log-return data of asset 1 and 2 respectively. The entire sample is not used in the application. In the main routine (**Copula**), by choosing an appropriate *RowStart*, the user can shift a 60-day window called *DataSample* up or down the larger set (*data*). It is recommended that the value of *RowStart* remains above 250, since earlier data becomes less reliable and there are periods where bonds (asset 2) have the same return for many months on end.

This could cause problems when calculating standard deviations, correlation etc. The particular window ($RowStart = 430$) used in this application corresponds to the period from 30-September-1995 to 30-September 2000.

A few parameters can easily be changed in the main file *Copula*. The vector *Strike* can be modified to the user's specifications. It is recommended that the range of strikes doesn't get too much wider. The interest rate R, and the time to expiry T can also be changed to correspond to the options provided in *optionsdata*.

The bandwidth of the kernel used in the density estimation algorithm can be changed by setting the *kernel correction factor* c in the *kernel2* function. The larger the bandwidth becomes, the smoother the density that results, and the greater the bias (towards the kernel function). As the bandwidth decreases, the density approximation represents the data more accurately - becoming more bumpy. The number of points on each axis can be changed by setting N in *Copula* to the desired level.

List of Matlab Variables:

data	full set of historical logreturns
DataSample	60 month window of historical return data used
optionsdata	implied volatility skews
cp	SAFEX call option prices
FmO	risk-neutral marginal distributions
xymO	the strike prices associated with FmO
Gumbel	Market implied marginals
Clayton	Market implied marginals
CF	Same assumptions as BS