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The role of expository writing in mathematical problem solving

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Abstract

Mathematical problem-solving is notoriously difficult to teach in a standard university mathematics classroom. The project on which this article reports aimed to investigate the effect of the writing of explanatory strategies in the context of mathematical problem solving on problem-solving behaviour. This article serves to describe the effectiveness of using writing as a tool for deeper engagement with mathematical problems. Students' claims about, and tutor observations of, problem-solving behaviour were analysed through the lens of Piaget's theory of cognitive development. Examples of enhanced problem-solving behaviour are presented as well as reports from student interviews that writing "forces" deeper engagement. The analysis of students' work and reflections indicated that writing about problem-solving processes potentially resulted in a cognitive perturbation when students were forced to confront their incomplete understanding (and hence their unstable knowledge structures) and therefore had to achieve a deeper level of understanding in order to adequately describe the solution process.

Keywords: problem solving; writing; mathematical understanding; cognitive conflict

Introduction

This article reports on an exploratory study to determine the effect of writing explanatory strategies in mathematical problem solving. The overarching aim of the project was to investigate to what extent writing explanatory paragraphs on the mathematical problem-solving process in the context of specific problems could positively influence problem-solving behaviour, without having to restructure the course within which the project was located.

The definition of a mathematical problem varies widely across the literature, from (implicitly) any mathematical question (Alibali, Phillips & Fischer, 2009), through word problems (Kilpatrick, 1983) to Schoenfeld's (1985, p.74) definition of a non-routine problem:

> … a task that is difficult for the individual who is trying to solve it. Moreover, that difficulty should be an intellectual impasse rather than a computational one….To state things more formally, if one has ready access to a solution schema for a mathematical task, that task is an exercise and not a problem.

In general, in literature focussed specifically on mathematical problem solving, it is definitions similar to Schoenfeld's that are accepted (Yee & Bostik, 2014), rather than the

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narrow definition of word problems or the too general definition of any mathematical question.

Much of the work carried out on mathematical problem solving is grounded in or draws upon Pólya's (1945) list of "heuristics" or "heuristic strategies" for successful problem solving. Pólya's broad-brush heuristics for the problem-solving process are (1) understand the problem, (2) devise a plan, (3) carry out the plan and (4) look back. It is the vital first step of *understanding the problem* that is of interest in this article and is referred to in student reflections on the writing process. Pólya's work on problem solving forms the firm foundation for much of the problem-solving research carried out in the last few decades.

Writing for learning has been studied for decades and is encouraged in many national curricula, including in South Africa, yet it is not often practised in mathematics classrooms, particularly university mathematics classrooms (Latulippe & Latulippe, 2014). Early in the literature linking writing and problem solving, it was argued that the same processes are involved in thoughtful expository writing and in mathematical problem solving and hence engaging in either develops competencies in related types of critical thinking (Bell & Bell, 1985; Kenyon, 1989). Many descriptions of the writing process even use the same or similar terms to Pólya's problem-solving steps of understand the problem – devise a plan – carry out the plan – look back. For example, Kenyon (1989) defines a writing exercise as requiring the following stages: planning (attempting to understand, ideas are generated and organised), composition (ideas are translated into extended text), and revising (text is reviewed, redundancies are removed, clarity is increased). Examples of writing in mathematics reported on in the literature take on various forms, such as essays on the history of mathematics, explaining mathematical concepts in one's own words, explicit reference to a given list of heuristic strategies, and so forth. The writing asked of the students in the project discussed in this article was the *writing of explanatory paragraphs on problem-solving strategies* employed in specifically chosen problems (see Appendix A for selected problems).

Examples of the use of writing in mathematics classrooms are easily found in the mathematics education literature. Studies focussing on the theoretical underpinnings of the effect of writing in a mathematical context are rarer and this article hopes to add to that body. Theoretical work in the field of mathematical writing which broadly influenced this project includes work on the metacognitive effects of writing (Pugalee, 2001, 2004), and classification and categorisation schemes (Waywood, 1992, 1994). When investigating the effect of explanatory writing on students' understanding of calculus Porter and Masingila (2000) conclude that writing is related to a deeper understanding of the mathematical concepts, but they question whether such learning occurs during mathematical writing because of the instrumental role of the actual writing, or perhaps because of time spent on task during the writing. In contrast, van Dyke and Malloy (2014) take it as given that writing proves effective in mathematics learning through a process of resolution of cognitive conflict. In this article I suggest that the role writing plays in learning mathematics could be described using Piaget's theory of cognitive development and that it is not simply time on task which proves effective. The mechanism by which the act of writing about problem-solving processes deepens engagement can be described using a theory of learning based on perturbations as cognitive conflict.

Theoretical framework

A theory of learning formulated initially by Piaget (1985) and developed by others (see for example Dubinsky & Lewin, 1986; Lewin, 1991) describes the process of learning beginning when a person, the "epistemic subject", encounters something new, a novel item, or "aliment" (Dubinsky & Lewin, 1986; von Glasersfeld, 2007). This encounter triggers an activity, which, if it leads to an expected result, causes the person to not differentiate the item from those previously encountered. However, if the activity does not lead to an expected result, then that contrast results in a perturbation (Steffe, 2011) or disequilibration (Dubinsky & Lewin, 1986), and the subject discriminates the item from those previously encountered. Thereafter, a cyclic process of assimilation and accommodation ideally occurs, resulting in equilibration (Čadež & Kolar, 2015; Harel & Koichu, 2010; Sepeng & Madzorera, 2014). "When a student's extant schemes are inadequate for a task, the student is in a perturbed state because the student cannot act to restore equilibrium using the extant schemes." (Steffe, 2011, p. 257)

The epistemic subject can exhibit a number of different types of behaviour when encountering a novel item. (1) The novelty of the item could be ignored or an incomplete process of equilibration could occur, leaving the understanding of the novel item in an unstable state, (2) the novel item could be successfully assimilated into a new cognitive structure through a robust process of assimilation and accommodation, or (3) the cognitive structures could be sufficiently well developed that, despite the novelty of the item, it can still be assimilated without cognitive restructuring. Piaget (1985) codified these different behaviours into three categories, Type-α, Type-β and Type-γ behaviour. These three behavioural categories are hierarchical in that they display increasing success in equilibration and greater anticipation within the cognitive system of variations on experiences already encountered. Dubinsky and Lewin (1986) consider "beta behaviour … the paradigm case of successful learning." (p. 64). The *Cambridge Companion to Piaget* describes alpha and beta reactions:

> Alpha reactions are characterised by the absence of any attempt to integrate the perturbations into the system in question… Alpha reactions involve a centration on the affirmations and a total neglect of negations. Beta reactions integrate the perturbing element that has sprung up into the system… It involves partial compensations, superior to alpha reactions, through the reworking of the conceptualizations involved.
> (Bloom, 2009, p. 141-142)

It could be argued that, pedagogically speaking, what teachers are trying to achieve is beta behaviour. If the entire class consists of students whose cognitive systems are sufficiently well developed that all variations upon the mathematical topics encountered in the classroom have been anticipated, then they are not going to learn much that is new for them. The usual aim of the teacher is to teach students wholly new concepts and processes, for which the cognitive structures are, almost by definition, not yet sufficient to encompass the new material without accommodation. For instance, the first time university students encounter complex numbers, a relatively abstract number system generally encountered for the first time in post-secondary studies, as rich as their understanding of real numbers might be, there are going to be entirely new concepts for which they will have to do some cognitive restructuring. While gamma behaviour is admirable, it is not what the teacher of mathematics can fairly expect, in general, to see in her students. Instead, rich and complex beta behaviour indicates, as Dubinsky and Lewin have stated, successful learning, rather than the encountering of information for which the student was already prepared.

The cognitive response to the encountering of a perturbation is the distinguishing factor between alpha and beta behaviour. “Perturbations may lead to a disequilibrated scheme whose internal operations can be relatively easily modified to restore equilibrium” (Steffe, 2011, p. 257).

Method

Context and Rationale

The context of the writing project was a first-year mathematics course at the author’s institution. The course consisted primarily of calculus (differential calculus with applications, integral calculus with applications, basic ordinary differential equations) as well as introductory complex numbers, linear algebra, vector geometry and infinite series. The course was a required course for all students majoring in mathematical and physical sciences as well as actuarial science and, as such, contained a large amount of content required by concurrent courses in applied mathematics, physics, chemistry, computer science, and statistics, as well as future courses in mathematics, applied mathematics, physics, chemistry, computer science, statistics, and economics. The focus of the course was traditionally on algorithms and recipes for solving mathematical problems. While there was an emphasis on the need to understand the underlying mathematics and to not apply algorithms blindly, there was historically little explicit emphasis on problem solving per se, as experienced by the author as a lecturer on the course for several years prior to this study. Contrary to the pedagogy of the course, the occasional true problem was encountered by the students in tutorials, tests and examinations and the students inevitably exhibited (as observed by the author) extremely poor problem-solving behaviour in attempting to solve the problems. It was the aim of the writing project to investigate the potential effects on students’ problem-solving behaviour of writing explanatory strategies to mathematical problems, without any need to restructure the course.

The requirement of not changing the course in structure or content was important, as none of the courses relying on this first-year course would accept any real or perceived decrease or change in content. The logistic organisation of the course included one 45-minute traditional (“chalk and talk”) lecture per day as well as one two-hour afternoon tutorial per week for each student. There were multiple times and venues for tutorials during the week, one of which it was compulsory for every student to attend. In each week’s tutorial, the students would receive a page of exercises and problems based on the work covered in the previous week's lectures. The students were encouraged to work in groups on the tutorial exercises, with a tutor present to provide help if necessary. The course was a full-year course, consisting of two semesters. The writing intervention was carried out in the 12-week second semester. The author, as a lecturer on the course, was assigned two tutorial classes under ordinary course administrative procedure. Each of the two tutorial groups contained participants and non-participants in the writing study. Ethical clearance was obtained from the institution for the running of this study and students were given the choice whether to participate or not in the writing intervention. The cohort of study participants did not differ in any obvious way from the entire class on measures of gender, race, language or degree programme. All students worked on the same problems. Non-participants are reported on in this article in broad terms only and the observations reported on were part of the standard tutoring process of observing students' progress and do not indicate infringement of the students' privacy.

Research Methodology

A total of 39 students took part in the study, producing a total of 155 written submissions. Following instructions, the students wrote explanatory paragraphs on their problem-solving procedures during the tutorials and submitted them to the tutor at the end of the session. While students could feasibly have completed more than one submission, in practice this did not occur. No formal training in problem solving occurred other than that routinely present in the course. The students received brief formative feedback in the form of the author's comments on their returned submissions. The feedback included positive comments on anything well done or clearly explained and also included mention of some aspects which could be improved or extended. The writing exercises were not graded. No differences in approach or results were observed across language (30 English, 9 non-English main language) or gender (14 female, 25 male) subgroups; this lack of difference is quite possibly due to the small numbers involved, particularly since Sepeng and Madzorera (2014) did find a relationship between "vocabulary knowledge" and proficiency at word problems in a South African context. The degree programmes of the students in the participant group were varied. The cohort included students registered in the Commerce Faculty (3 different degree programmes), the Engineering Faculty (5 degree programmes), the Science Faculty (5 degree programmes) and 1 degree programme in the Humanities Faculty. Seventeen students consented to be interviewed.

Data

For this study three forms of data were collected during the study project, namely interviews, field notes and the students' written exercises. Analysis of the written submissions is unrelated to the research question addressed in this article and has been published elsewhere (Craig, 2011). A process of "meaning condensation" (Kvale, 1996, p. 193) of the interviews, supported by observations in the field notes, revealed potential evidence of learning which can be described using Piaget's learning theory (1985). Analysis of the field notes and interviews is discussed in this article.

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Results and Discussion

The interviews

While writing about mathematics (problem solving or otherwise) is neither a necessary nor sufficient condition for deeper engagement or increased mathematical understanding, data from student interviews provides evidence that writing increases the probability of beta behaviour and its associated enhanced understanding. Themes emerged from the student reports on the effect of the writing tasks and are discussed here as (1) expected outcome of process, (2) multiple solution strategies, (3) self-confrontation (perturbation), (4) awareness of mathematical requirements and (4) responsibility towards audience.

Theme 1: The usefulness of the writing tasks depends on the expected outcome of mathematical problem solving

Several students express the view that the usual intended outcome for problem solving is to obtain a result of the mathematical problem. This problem-solving strategy favours alpha behaviour – engaging with the mathematical work at a shallow level. However, some students recognise post hoc that the writing tasks increase understanding and thereafter value the role that writing plays in their learning – forcing the deeper cognitive engagement associated with beta behaviour. This view is illustrated by the two quotes below.

> [The aim of the writing exercises was] to force insight, I suppose. If you didn't have the writing, then people wouldn't care what's going on with the problem. They would just do it. (DC)
>
> [My answer to my colleague] depends on what the person wants. Do you want to understand what you're doing, or do you just want to do it? If he wants to do it, I'd tell him to go to a normal tutorial, but if you want to understand it more go to the writing. (DL)

Theme 2: The writing tasks can encourage multiple solution strategies, whereas the direct approach has a risk of 'dead ends' and at best a single solution strategy

Several students point to the effect of the reflection of the writing tasks. Instead of beginning a calculation process immediately and without pause for thought (alpha behaviour), the writing task invites the student to consider various options and signals hurdles in advance of calculation. Recognising the existence of multiple strategies and reflecting on their relative strengths and weaknesses is indicative of beta behaviour, particularly if such reflection would not have occurred in the absence of the writing tasks. The following two quotes illustrate these views.

> You see a question and write whatever comes to your mind, and, halfway through, you realise it is not right. Then you start again. So by having to think over the problem [in the writing task] you know what you're supposed to do and you go straight to doing it. (TJ)
>
> If you're not used to [a difficult mathematical process], it helps to be forced to try and see it in another light. Because that can help you get through the block and then … you can find another perspective, which, especially if you're finding one [problem]

> hard, if you came at it from another angle. I think it's more useful than if you already figured it out. (AJ)

Theme 3: The writing tasks are self-confrontational, and force you to query your existing problem-solving strategy

A large number of students describe the experience of having to justify the thinking behind their existing solution strategy. Confronting their own understanding of the problem "forced" a perturbation, indicating that beta behaviour is often (always?) non-spontaneous. The quotes below are only a few of many in a similar vein.

> If you have a challenged understanding of the work, then it [the writing task] pushes you to understand the processes, and see links between different ideas. (CP)
>
> Well, I have to admit, if I didn't really understand it, I would probably have left it out. But because I knew I had to submit something, it forced me to think about, maybe read up from the textbook. I'd like to always produce an answer. (IS)
>
> They [the writing exercises] kind of force you to [think about the problems]. (CP)
>
> but it really helps when you're looking at the harder stuff, to be forced to go through the process … Because when the mathematics is easy then at least you can think through that; if you're not used to it, it helps to be forced to try and see it in another light. (AJ)

Theme 4: The writing tasks encourage an awareness of the mathematical requirements of the problem.

Writing about the mathematical problems required the students to become aware of the requirements of the problem context and hence to engage in deeper thinking about the problems. Reflecting on their own knowledge and coming to greater understanding is indicative of beta behaviour.

> With one or two of the writing exercises, because you had to think about it and take a guess, you could actually take an educated guess. And *that was surprising*, because you didn't think you would be able to know without working it out, but because you had to think about it, you can predict what will happen, at least sort of. (DC, emphasis added. Context: In this case an initial approximation of the solution was required before calculation)
>
> Well, for me, it helped in that you go a lot deeper into the actual question than you do in a normal tut where it is just do it, and get an answer. You had to sit and think about it (RG)
>
> I remember on two occasions I had calculated the wrong argument[1]. That was definitely a weakness, so it [the writing task] alerted me to my weakness, which was really crippling me. (AS1)

Theme 5: The responsibility (self-)imposed by the writing exercises demands that the student submit a completed assignment of a high standard.

[1] "Argument" here is used in the mathematical sense. In this context it can be understood to mean "angle".

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One student expressed the need to submit a written assignment of a satisfactory standard, with the implication that the level of understanding thereby achieved might not have occurred in the absence of the writing task.

> If you say *explain*, then I really have to explain it, I can't make a mess of it, and if I'm going to explain something then I really have to understand it. And then I have to explain it clearly enough that it's clear enough for me and it will also be clear enough for someone else (NW)

In the interviews there were few negative comments about the writing tasks. Five (of 17) students (both first language English and not) found the tasks relatively difficult to do, struggling to express mathematical ideas in words. Three of the students found the feedback unhelpful and three found that the tasks took too much time, although five students reported a perception that the writing saved time overall, by avoiding "dead end" procedures.

The interviews included student claims regarding their experience suggesting two levels of engagement with the mathematical content of the tutorials, which can be framed as a surface approach and a deep approach (Chi, Feltovich and Glaser, 1981; Marton, Hounsell and Entwhistle, 1997), which in turn resonate with Piaget's alpha and beta behaviour. Analysis of student utterances during the interviews as well as observations made in the field notes suggest surface and deep approaches to engaging with mathematical problems.

A question which needs to be asked and which this study was not designed to answer is whether that perceived deeper engagement is an indicator of beta behaviour and its associated greater understanding through resolution of cognitive conflict. Is a surface approach to problem solving equivalent to alpha behaviour? Further research is needed to interrogate the links between alpha behaviour and a surface approach, and beta behaviour and a deep approach to problem solving. There are, however, intriguing parallels which can be drawn between how the students express their engagement with the problems and the theory on alpha and beta cognitive behaviour.

The field notes

The students participating in the writing project appeared to engage with the problems about which they were writing at a deeper or more detailed level than the students who were not participating. Three illustrative examples are discussed here, recorded in the field notes made both during and after the tutorial classes.

In week 7 the students were called upon to solve for the orthogonal trajectories of a family of ellipses. The concept of orthogonal trajectories refers to the method of finding, through the methods of differential equations, a family of curves which is at all points perpendicular to another given family of curves. All of the writing students drew a sketch with their written explanation, and all of the students in the same tutorial group not participating in the writing activity omitted drawing a sketch (even though the question required it, see Appendix A). In addition, the calculation was tricky in that the solution required modulus signs to be inserted

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which were not acquired through rote calculation. Seventeen students were in attendance in that tutorial group at the time of addressing the orthogonal trajectories problem, ten participating in the writing study project, seven not participating. Five of the ten writing students who completed the problem inserted the modulus signs or made an equivalent symmetry argument, while the seven non-participating students reached a (merely partially correct) algebraic solution and considered the problem complete, a paradigm case of alpha behaviour.

In week 8 students encountered improper integrals. In brief, an improper integral is an integral with one or both limits of integration infinite or, alternately, which has an integrand that approaches infinity within the interval of integration. The students were presented with several integrals, all potentially improper, and had to determine which integrals converged and which diverged (see Appendix A). The writing exercise for that week required the students to describe what they were doing and why. The students who chose not to participate in the project simply applied an algorithm which was not appropriate in all given cases, giving an erroneous result in one case. The students who completed the writing exercises, first described the characteristics of an integral to cause it to be “improper” and thus recognised which integrals were improper and which not and why (which was not obvious in every case). This recognition resulted in those students correctly solving all the integrals.

The orthogonal trajectories example and the improper integrals example both illustrate how the students participating in the writing project engaged with the problems more deeply than those students in the class who were not participating. In the case of the orthogonal trajectories the algebraic solution was at odds with a diagrammatic representation of the problem requirement. The algebraic solution (without modulus signs) accounted for only half of the diagram. All the students doing the writing drew a diagram, in contrast to none of the students not writing, and half of those students, recognising the conflict between the symbolic solution and the diagram, went on to either insert the modulus signs at the correct place in the solution or make a post-hoc (and mathematically satisfying) argument from symmetry. In the case of the improper integrals, one of the integrands was not immediately apparent as being discontinuous within the interval of interest. In such a case it is possible to apply a standard algorithm and achieve an incorrect, but apparently straightforward, solution. It appears that, in having to write an explanatory paragraph addressing what characteristics made each given integral improper or not, the hidden discontinuity became sufficiently apparent for those students to realise that a more careful approach would be needed in that case.

In Week 10, the students were completing a section on linear algebra. The tutorial problem assigned to the writing project that day was one involving inverses and determinants of matrices (see Appendix A). The problem was designed so that the students could make observations about the inverses and determinants of matrix transposes. The transpose terminology and notation were made available but were not familiar to them. Of the students not taking part in the project, none of them were observed to notice the transpose-related results. Those students carried out the required calculations and proceeded to the next question. Of the 12 students taking part in the writing project in that particular class, ten observed the patterns, six of these discussed correct transpose notation with the tutor and two students delved deeper to look for reasons for the patterns.

Conclusion

In Piagetian learning theory, in the absence of disequilibration beta behaviour is less likely to be invoked than alpha behaviour. Beta behaviour characterises successful learning through accommodation and reflective abstraction while alpha behaviour signifies partial or no learning. The student-reported interview data support field note observations of the existence of the two types of engagement, deep and shallow, which resonate with Piaget's beta and alpha behaviour. Moreover, the requirement of disequilibration through a perturbation to invoke beta behaviour in the Piagetian model is endorsed by the repeated insistence in the interviews that the deeper engagement was "forced" by the writing exercises.

The study set out to investigate the effect of the writing of explanatory strategies on problem-solving behaviour. The results suggest that writing increased understanding of the underlying mathematics of the problem, where understanding is recognised as being the crucial first step in Polya's (1945) stages of effective problem solving: (1) understand the problem, (2) devise a plan, (3) carry out the plan and (4) look back. The mechanism by which understanding was increased has been explained using Piaget's theory of cognitive development. In particular, I argue that writing provides a perturbation causing cognitive disequilibration, requiring the student to re-establish equilibrium through the reflective process described by beta behaviour. The students report on how writing "forced" them to greater understanding of the underlying mathematics, a phrasing which resonates with Piaget's beta behaviour and how such behaviour is unlikely to be invoked without a perturbation. It is possible, when performing a calculation, to not realise that one's understanding is faulty or incomplete, or to not admit that instability to oneself. By writing about a solution process, this study suggests that one is more likely to encounter the cognitive instability or incompleteness and to rectify that deficient situation. Providing a mechanism for this process of learning through writing builds on the work of others in the field, such as metacognitive theorists, and responds to suggestions that writing increases understanding solely through spending time on task.

The study reported on in this article provides a good starting point for further, finer grained, research to investigate whether writing exercises encourage beta behaviour by simultaneously encouraging a deep approach to engagement (and hence encouraging beta behaviour from the outset) and providing a challenge, a perturbation, to the unstable knowledge structures created by or associated with alpha behaviour (and hence encouraging a switch to beta behaviour from initial alpha behaviour). Admittedly, the students taking part were volunteers and therefore perhaps were students with a greater interest in improving their mathematical skill; this would be a concern to be dealt with in future research in this vein. While the writing exercises alone are neither a necessary nor sufficient condition for comprehensive understanding of any mathematical idea, their inclusion in a mathematical learning environment perhaps makes understanding a more likely outcome of engagement due to associated beta behaviour.

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References

Alibali M.W., Phillips, K.M.O. & Fischer, A.D. (2009). Learning new problem-solving strategies leads to changes in problem representation. *Cognitive Development*, *24*, 89-101.

Bell, E.S. & Bell, R.N. (1985). Writing and mathematical problem solving: Arguments in favour of synthesis. *School Science and Mathematics*, *8*(3), 210-221.

Bloom, J. (2009). Piaget on Equilibration. In U. Müller, J.I.M. Carpendale & L. Smith (Eds.), *The Cambridge Companion to Piaget* (pp. 132-149). Cambridge, UK: Cambridge University Press.

Čadež, T.H. & Kolar, V.M. (2015). Comparison of types of generalizations and problem-solving schemas used to solve a mathematical problem. *Educational Studies in Mathematics*, *89*, 283-306.

Chi, M.T.H., Feltovich, P.J. & Glaser, R. (1981). Categorization and representation of physics problems by experts and novices. *Cognitive Science*, *5*, 121-152.

Craig, T.S. (2011). Categorisation and analysis of explanatory writing in mathematics, *International Journal of Mathematics Education in Science and Technology*, *42*(7), 867-878.

Dubinsky, E. & Lewin, P. (1986). Reflective abstraction and mathematics education: The genetic decomposition of induction and compactness. *Journal of Mathematical Behavior*, *5*, 55-92.

Harel, G. & Koichu, B. (2010). An operational definition of learning. *The Journal of Mathematical Behavior*, *29*, 115-124.

Kenyon, R.W. (1989). Writing is problem solving. In P. Connolly & T. Vilardi (Eds.), *Writing to Learn Mathematics and Science* (pp. 73-87). New York: Teachers College Press.

Kilpatrick, J. (1983). A retrospective account of the past 25 years of research on teaching mathematical problem solving. In E.A. Silver (Ed.), *Teaching and Learning Mathematical Problem Solving: Multiple Research Perspectives* (pp. 1-15). Hillsdale, New Jersey: Lawrence Erlbaum Associates.

Kvale, S. (1996). *InterViews: An Introduction to Qualitative Research Interviewing*. Thousand Oaks, California, US: SAGE Publications.

Latulippe, C. & Latulippe, J. (2014). Student perceptions of writing projects in a university differential-equations course. *International Journal of Mathematical Education in Science and Technology*, *45*(1), 1-11.

Lewin, P. (1991). Reflective abstraction in humanities education: thematic images and personal schemes. In L.P. Steffe (Ed.), *Epistemological Foundations of Mathematical Experience* (pp. 203-237). New York: Springer-Verlag.

Marton, F., Hounsell, D., & Entwistle, N. (Eds.), (1997). *The Experience of Learning* (2nd edition). Edinburgh, UK: Scottish Academic Press.

Piaget, J. (1985). *The Equilibration of Cognitive Structures – The Central Problem of Intellectual Development* (T. Brown & K.J. Thampy Trans.). Chicago: University of Chicago Press. (Original work published 1975).

Pólya, G. (1945). *How to Solve It*. Second edition. Princeton, New Jersey: Princeton University Press.

Porter, M.K. & Masingila, J.O. (2000). Examining the effects of writing on conceptual and procedural knowledge in calculus. *Educational Studies in Mathematics, 42*, 165-177.

Pugalee, D.K. (2001). Writing, mathematics, and metacognition: looking for connections through students' work in mathematical problem solving. *School Science and Mathematics, 101*(5), 236-245.

Pugalee, D.K. (2004). A comparison of verbal and written descriptions of students' problem solving processes. *Educational Studies in Mathematics*, *55*, 27-47.

Schoenfeld, A.H. (1985). *Mathematical Problem Solving*. Orlando, Florida, USA: Academic Press, inc.

Sepeng, P. & Madzorera, A. (2014). Sources of difficulty in comprehending and solving mathematical word problems. *International Journal of Educational Sciences*, *6*(2), 217-225.

Steffe, L.P. (2011). A theory bite on learning through mathematical activity. *Cognition and Instruction*, *29*(2), 256-263.

van Dyke, F. & Malloy, E.J. (2014). An activity to encourage writing in mathematics. *Canadian Journal of Science, Mathematics and Technology*, *14*(4), 371-387.

von Glasersfeld, E. (2007). *Key Works in Radical Constructivism*. Rotterdam, The Netherlands: Sense Publishers.

Waywood, A. (1992). Journal writing and learning mathematics. *For the Learning of Mathematics*, *12*(2), 34-43.

Waywood, A. (1994). Informal writing-to-learn as a dimension of a student profile. *Educational Studies in Mathematics*, *27*, 321-340.

Yee, S.P. & Bostik, J.D. (2014). Developing a contextualisation of students' mathematical problem solving. *The Journal of Mathematical Behavior*, *36*, 1-19.

Appendix A

For each of the cases below, every student saw "the question", only the students participating in the writing study saw "the writing task".

Week 7: A differential equations problem

The question

Find the orthogonal trajectories of the family of ellipses $2x^2 + 5y^2 = C$, and sketch several members of each family.

The writing task

Write a short paragraph on your solution process. Be sure to include what you did, why you did it, and whether your answer looked as expected.

Week 8: An improper integral problem

The question

Which one of the following improper integrals converges?

$$(A)\int_0^1 \frac{1}{(x-1)^2}dx \quad (B)\int_{-1}^1 \frac{1}{(x+1)^2}dx \quad (C)\int_0^\infty \frac{1}{(x+1)^2}dx \quad (D)\int_0^\infty \frac{1}{(x-1)^2}dx \quad (E)\int_{-2}^2 \frac{1}{x^2}dx$$

The writing task

Solve the problem, however you wish.

Pretend you are explaining the problem to a puzzled fellow student. Write out (in words, as much as possible) how you solved the problem.

Was your final answer the one you expected (if you did)? If not, can you explain why?

Week 10: A linear algebra problem

The question(s)

1. Find the inverses of the following matrices:

$$\begin{pmatrix} 2 & 1 \\ 7 & 4 \end{pmatrix} \quad \begin{pmatrix} 2 & 7 \\ 1 & 4 \end{pmatrix} \quad \begin{pmatrix} 2 & 1 & 0 \\ 7 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 2 & 0 & 7 \\ 0 & 1 & 0 \\ 1 & 0 & 4 \end{pmatrix}$$

4. Find the determinants of the following matrices:

$$\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} \quad \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & 3 & 1 \\ 4 & 0 & -2 \\ 0 & -1 & 1 \end{pmatrix} \quad \begin{pmatrix} 2 & 4 & 0 \\ 3 & 0 & -1 \\ 1 & -2 & 1 \end{pmatrix}$$

The writing task

Try the following steps in both question 1 and question 4:

Say what you notice about the given matrices.

Carry out the calculations.

Describe what you notice.